Optimal Portfolio Choice and the Capital Asset Pricing Model

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The Expected Return of a Portfolio

Calculating Portfolio Returns

- To find an optimal portfolio, we need a method to define a portfolio and analyze its return.
- We can describe a portfolio by its portfolio weights, the fraction of the total investment in the portfolio held in each individual investment in the portfolio:

$$\omega_i = \frac{\text{Value of investment } i}{\text{Total value of portfolio}}$$

- These portfolio weights add up to 1 (that is, $\sum_{i=1}^{N} \omega_i = 1$), so that they represent the way we have divided our money between the different individual investments in the portfolio.
- As an example, consider a portfolio with 200 shares of Dolby Laboratories worth \$30 per share and 100 shares of Coca-Cola worth \$40 per share.
- The total value of the portfolio is $200 \times \$30 + 100 \times \$40 = \$10{,}000$, and the corresponding portfolio weights ω_D and ω_C are :

$$\omega_D = \frac{200 \cdot \$30}{\$10,000} = 60\%, \qquad \omega_C = \frac{100 \cdot \$40}{\$10,000} = 40\%$$

The Expected Return of a Portfolio

Calculating Portfolio Returns

- Given the portfolio weights, we can calculate the return on the portfolio.
- Suppose $\omega_1, \ldots, \omega_N$ are the portfolio weights of the N investments in a portfolio, and these investments have returns $r_1, \ldots, r_N$.
- Then the return on the portfolio, r_p, is the weighted average of the returns on the investments in the portfolio, where the weights correspond to portfolio weights:

$$r_p = \omega_1 \cdot r_1 + \omega_2 \cdot r_2 + \ldots + \omega_N \cdot r_N = \sum_{i=1}^{N} \omega_i \cdot r_i$$

- The return of a portfolio is straightforward to compute if we know the returns of the individual stocks and the portfolio weights.

The Expected Return of a Portfolio

Problem

Calculating Portfolio Returns

- Suppose you buy 200 shares of Dolby Laboratories at $30 per share and 100 shares of Coca-Cola stock at $40 per share.
- If Dolby's share price goes up to $36 and Coca-Cola's falls to $38, what is the new value of the portfolio, and what return did it earn?
- After the price change, what are the new portfolio weights?

The Expected Return of a Portfolio

Solution

Calculating Portfolio Returns

- The new value of the portfolio is 200 x \$36 + 100 x \$38 = \$11,000, for a gain of \$1,000 or, a 10% return on your \$10,000 investment.
- Dolby's return was $^{36}/_{30} - 1 = 20\%$, and Coca-Cola's was $^{38}/_{40} - 1 = -5\%$.
- Given the initial portfolio weights of 60% Dolby's and 40% Coca-Cola, we can also compute the portfolio's return :

$$r_p = \omega_1 \cdot r_1 + \omega_2 \cdot r_2 + \ldots + \omega_N \cdot r_N = \sum_{i=1}^{N} \omega_i \cdot r_i$$

$$r_p = \omega_D \cdot r_D + \omega_C \cdot r_C$$

$$r_p = 60\% \cdot 20\% + 40\% \cdot (-5\%) = 10\%$$

- After the price change, the new portfolio weights are :

$$\omega_D = \frac{200 \cdot \$36}{\$11,000} = 65.45\%, \qquad \omega_C = \frac{100 \cdot \$38}{\$11,000} = 34.55\%$$

- Without trading, the weights increase for those stocks whose returns exceed the portfolio's return.

The Expected Return of a Portfolio

Calculating Portfolio Returns

- Using the facts that the expectation of a sum is just the sum of the expectations and that the expectation of a known multiple is just the multiple of its expectation, we arrive at the following formula for a portfolio's expected return:

$$\mathbb{E}(r_p) = \mathbb{E}\left(\sum_{i=1}^{N} \omega_i \cdot r_i\right) = \sum_{i=1}^{N} \mathbb{E}\left(\omega_i \cdot r_i\right) = \sum_{i=1}^{N} \omega_i \cdot \mathbb{E}\left(r_i\right)$$

- That is, the expected return of a portfolio is simply the weighted average of the expected returns of the investments within it, using the portfolio weights.

The Expected Return of a Portfolio

Problem

Calculating Portfolio Returns

- Suppose you invest $10,000 in Ford stock, and $30,000 in Tyco International stock.
- You expect a return of 10% for Ford and 16% for Tyco.
- What is your portfolio's expected return?

The Expected Return of a Portfolio

Solution

Calculating Portfolio Returns

- You invested $40,000 in total, so your portfolio weights are 10,000/40,000 = 25% in Ford and 30,000/40,000 = 75% in Tyco.
- Therefore, your portfolio's expected return is :

$$\mathbb{E}(r_p) = \omega_F \cdot \mathbb{E}(r_F) + \omega_T \cdot \mathbb{E}(r_T)$$

$$\mathbb{E}(r_p) = 25\% \cdot 10\% + 75\% \cdot 16\% = 14.5\%$$

The Volatility of a Two-Stock Portfolio

Calculating Portfolio Returns

- Combining stocks in a portfolio eliminates some of their risk through diversification.
- The amount of risk that will remain depends on the degree to which the stocks are exposed to common risks.
- Let's begin with a simple example of how risk changes when stocks are combined in a portfolio. Table 11.1 shows returns for three hypothetical stocks, along with their average returns and volatilities.

TABLE 11.1 Returns for Three Stocks, and Portfolios of Pairs of Stocks

	Stock Returns			Portfolio Returns	
Year	North Air	West Air	Tex Oil	$1/2R_N + 1/2R_W$	$1/2R_W + 1/2R_T$
2010	21%	9%	−2%	15.0%	3.5%
2011	30%	21%	−5%	25.5%	8.0%
2012	7%	7%	9%	7.0%	8.0%
2013	−5%	−2%	21%	−3.5%	9.5%
2014	−2%	−5%	30%	−3.5%	12.5%
2015	9%	30%	7%	19.5%	18.5%
Average Return	10.0%	10.0%	10.0%	10.0%	10.0%
Volatility	13.4%	13.4%	13.4%	12.1%	5.1%

The Volatility of a Two-Stock Portfolio

Calculating Portfolio Returns

- While the three stocks have the same volatility and average return, the pattern of their returns differs.
- When the airline stocks performed well, the oil stock tended to do poorly (see 2010–2011), and when the airlines did poorly, the oil stock tended to do well (2013–2014).
- Table 11.1 shows the returns for two portfolios of the stocks.
    1. The first portfolio consists of equal investments in the two airlines, North Air and West Air.
    2. The second portfolio includes equal investments in West Air and Tex Oil.
- The average return of both portfolios is equal to the average return of the stocks. However, their volatilities (12.1% and 5.1%) are very different from the individual stocks and from each other.

The Volatility of a Two-Stock Portfolio

Calculating Portfolio Returns

- This example demonstrates two important phenomena :
    1. Combining stocks into a portfolio, we reduce risk through diversification. Because the prices of the stocks do not move identically, some of the risk is averaged out in a portfolio. As a result, both portfolios have lower risk than the individual stocks.
    2. The amount of risk that is eliminated in a portfolio depends on the degree to which the stocks face common risks and their prices move together.
- Because the two airline stocks tend to perform well or poorly at the same time, the portfolio of airline stocks has a volatility that is only slightly lower than that of the individual stocks.
- The airline and oil stocks, by contrast, do not move together; indeed, they tend to move in opposite directions. As a result, additional risk is canceled out, making that portfolio much less risky. This benefit of diversification is obtained costlessly without any reduction in the average return.

Determining Covariance and Correlation

Calculating covariance & correlation

- To find the risk of a portfolio, we need to know more than the risk and return of the component stocks: We need to know the degree to which the stocks face common risks and their returns move together.
- We introduce two statistical measures, covariance and correlation, that allow us to measure the co-movement of returns between r_i and r_j:

$$cov(r_i, r_j) = \sigma_{i,j} = \frac{1}{T}\sum_{i=1}^{T}(r_{i,t} - \mu_i)(r_{j,t} - \mu_j) \tag{1}$$

- The covariance between r_i and r_i written $\sigma_{i,i}$ is the variance of r_i (σ_i^2) because:

$$\sigma_{i,i} = \frac{1}{T}\sum_{i=1}^{T}(r_{i,t} - \mu_i)(r_{i,t} - \mu_i) \tag{2}$$

$$\sigma_i^2 = \frac{1}{T}\sum_{i=1}^{T}(r_{i,t} - \mu_i)^2 \tag{3}$$

Determining Covariance and Correlation

Calculating covariance & correlation

- Let's consider covariance between r_i and r_j:

$$\sigma_{i,j} = \frac{1}{T} \sum_{i=1}^{T} (r_{i,t} - \mu_i)(r_{j,t} - \mu_j) \tag{4}$$

- Intuitively, if two stocks move together, their returns will tend to be above or below average at the same time, and the covariance will be positive.
- If the stocks move in opposite directions, one will tend to be above average when the other is below average, and the covariance will be negative.
- While the sign of the covariance is easy to interpret, its magnitude is not. It will be larger if the stocks are more volatile (and so have larger deviations from their expected returns), and it will be larger the more closely the stocks move in relation to each other.

Determining Covariance and Correlation

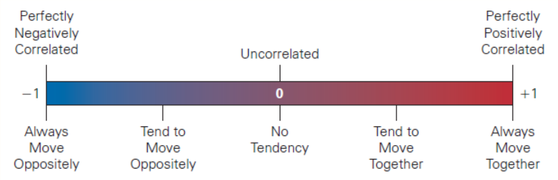
Calculating covariance & correlation

- In order to control for the volatility of each stock and quantify the strength of the relationship between them, we can calculate the correlation between two stock returns, defined as the covariance of the returns divided by the standard deviation of each return:

$$\rho_{i,j} = \frac{\frac{1}{T}\sum_{t=1}^{T}\left(r_{i_t}-\mu_i\right)\left(r_{j_t}-\mu_j\right)}{\frac{1}{T}\sqrt[2]{\sum_{t=1}^{T}\left(r_{i_t}-\mu_i\right)^2} \times \frac{1}{T}\sqrt[2]{\sum_{t=1}^{T}\left(r_{j_t}-\mu_j\right)^2}} = \frac{\sigma_{i,j}}{\sigma_i\sigma_j} \quad (5)$$

FIGURE 11.1 Correlation

Correlation measures how returns move in relation to each other. It is between $+1$ (returns always move together) and -1 (returns always move oppositely). Independent risks have no tendency to move together and have zero correlation.



Determining Covariance and Correlation

Calculating correlation

- The correlation between two stocks has the same sign as their covariance, so it has a similar interpretation.
- Dividing by the volatilities ensures that correlation is always between -1 and +1, which allows us to gauge the strength of the relationship between the stocks.
- As Figure 11.1 shows, correlation is a barometer of the degree to which the returns share common risk and tend to move together.
- The closer the correlation is to +1, the more the returns tend to move together as a result of common risk. When the correlation (and thus the covariance) equals 0, the returns are uncorrelated; that is, they have no tendency to move either together or in opposition to one another.
- Independent risks are uncorrelated.
- Finally, the closer the correlation is to -1, the more the returns tend to move in opposite directions.

Computing a Portfolio's Variance and Volatility

Let's consider a N-stock portfolio

Variance & Volatility

- Let's write Variance-Covariance matrix of the portfolio p with N stocks as Ω_p such as:

$$\Omega_p = \begin{pmatrix} \sigma_1^2 & \dots & \sigma_{1,j} & \dots & \sigma_{1,N} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{i,1} & \dots & \sigma_i^2 & \dots & \sigma_{i,N} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{N,1} & \dots & \sigma_{N,j} & \dots & \sigma_N^2 \end{pmatrix} \tag{6}$$

$$\sigma_p^2 = (\omega_1, \dots, \omega_i, \dots, \omega_N) \begin{pmatrix} \sigma_1^2 & \dots & \sigma_{1,j} & \dots & \sigma_{1,N} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{i,1} & \dots & \sigma_i^2 & \dots & \sigma_{i,N} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{N,1} & \dots & \sigma_{N,j} & \dots & \sigma_N^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_i \\ \vdots \\ \omega_N \end{pmatrix} \tag{7}$$

$$\sigma_p^2 = \omega^\top \Omega \omega = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j \sigma_{i,j} \tag{8}$$

Computing a Portfolio's Variance and Volatility

Let's consider a two-stock portfolio

Variance & Volatility

- Consider the variance of a two-stock portfolio :

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_1\sigma_2\rho_{1,2} \qquad \omega_2 = 1 - \omega_1 \tag{9}$$

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + (1-\omega_1)^2\sigma_2^2 + 2\omega_1(1-\omega_1)\sigma_1\sigma_2\rho_{1,2} \tag{10}$$

TABLE 11.1 Returns for Three Stocks, and Portfolios of Pairs of Stocks

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2013	−5%	−2%	21%	−3.5%	9.5%
2014	−2%	−5%	30%	−3.5%	12.5%
2015	9%	30%	7%	19.5%	18.5%
Average Return	10.0%	10.0%	10.0%	10.0%	10.0%
Volatility	13.4%	13.4%	13.4%	12.1%	5.1%

Computing a Portfolio's Variance and Volatility

Let's consider a two-stock portfolio

Variance & Volatility

- As always, the volatility is the square root of the variance.
- Let's check this formula for the airline and oil stocks in Table 11.1.
- Consider the portfolio containing shares of West Air and Tex Oil. The variance of each stock is equal to the square of its volatility, $(0.134)^2 = 0.018$.
- Suppose the covariance between the stocks is -0.0128. Therefore, the variance of a portfolio with 50% invested in each stock is :

$$\sigma_p^2 = \frac{1}{2}^2 \cdot 0.018 + \frac{1}{2}^2 \cdot 0.018 + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot (-0.0128) = 0.0026 \tag{11}$$

- The volatility of the portfolio is $\sqrt{0.0026} = 5.1\%$, which corresponds to the calculation in Table 11.1.
- For the North Air and West Air portfolio, the calculation is the same except for the stocks' higher covariance of 0.0112, resulting in a higher volatility of 12.1%.

The Volatility of a Large Portfolio

Let's consider a large portfolio

Variance & Volatility

- We can gain additional benefits of diversification by holding more than two stocks in our portfolio.
- While these calculations are best done on a computer, by understanding them we can obtain important intuition regarding the amount of diversification that is possible if we hold many stocks.
- Recall that the return on a portfolio of N stocks is simply the weighted average of the returns of the stocks in the portfolio where :

$$r_{i,t} = \frac{[P_{i,t} - P_{i,t-1}] + D_{i,t}}{P_{i,t-1}} \tag{12}$$

and $P_{i,t}$ is the price of the asset i, $D_{i,t}$ is the dividend of the asset i at time t then :

$$r_{p,t} = \sum_{t=1}^{N} \omega_{i,t} r_{i,t} \tag{13}$$

The Volatility vs. Diversification

Volatility & the number of stocks

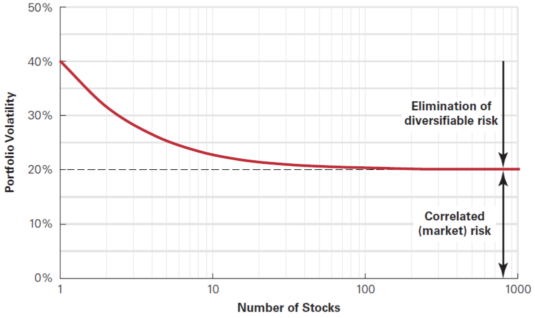
- The volatility declines as the number of stocks in the portfolio grows. In fact, nearly half of the volatility of the individual stocks is eliminated in a large portfolio as the result of diversification.
- The benefit of diversification is most dramatic initially: The decrease in volatility when going from one to two stocks is much larger than the decrease when going from 100 to 101 stocks (indeed, almost all of the benefit of diversification can be achieved with about 30 stocks).
- Even for a very large portfolio, however, we cannot eliminate all of the risk.

The Volatility vs. Diversification

FIGURE 11.2

Volatility of an Equally Weighted Portfolio Versus the Number of Stocks

The volatility declines as the number of stocks in the portfolio increases. Even in a very large portfolio, however, market risk remains.



Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

What does efficient portfolio means?

- Consider a portfolio of Intel and Coca-Cola stock.
- Suppose an investor believes these stocks are uncorrelated and will perform as follows:

Stock	Expected return	Volatility
Intel	26%	50%
Coca-Cola	6%	25%

- How should the investor choose a portfolio of these two stocks?
- Are some portfolios preferable to others?

Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

What does efficient portfolio means?

- Consider a portfolio of Intel and Coca-Cola stock.

Stock	Expected return	Volatility
Intel (I)	26%	50%
Coca-Cola (CC)	6%	25%

- Let's compute the expected return and volatility for different combinations of the stocks.
- Consider a portfolio with 40% invested in Intel stock and 60% invested in Coca-Cola stock.
- We can compute the expected return from equation 14:

$$\mathbb{E}(r_p) = \mathbb{E}\left(\sum_{i=1}^{N} \omega_i \cdot r_i\right) = \sum_{i=1}^{N} \mathbb{E}\left(\omega_i \cdot r_i\right) = \sum_{i=1}^{N} \omega_i \cdot \mathbb{E}\left(r_i\right) \tag{14}$$

$$\mathbb{E}(r_p) = \omega_I \cdot \mathbb{E}(r_I) + \omega_{CC} \cdot \mathbb{E}(r_{CC})$$

$$\mathbb{E}(r_p) = 40\% \cdot 26\% + 60\% \cdot 6\% = 14\%$$

Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

What does efficient portfolio means?

- Suppose an investor believes these stocks are uncorrelated.

Stock	Expected return	Volatility
Intel	26%	50%
Coca-Cola	6%	25%

- We can compute the variance using equation 15:

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + (1-\omega_1)^2\sigma_2^2 + 2\omega_1(1-\omega_1)\sigma_1\sigma_2\rho_{1,2} \tag{15}$$

$$\sigma_p^2 = 40\%^2 \cdot 50\%^2 + (1-40\%)^2 \cdot 25\%^2 + 2 \cdot 40\% \cdot (1-40\%) \cdot 50\% \cdot 25\% \cdot 0 = 0.0625 \tag{16}$$

- Based on that result, the volatility measured by the standard deviation is the square root of variance : $\sqrt{0.0625} = 25\%$.

Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

Diversification impact

- Due to diversification, it is possible to find a portfolio with even lower volatility than either stock.
- Investing 20% in Intel stock and 80% in Coca-Cola stock, for example, has a volatility of only 22.4%.
- But knowing that investors care about volatility and expected return, we must consider both simultaneously. To do so, we plot the volatility and expected return of each portfolio.

Intel	Coca-Cola	Expected return	Volatility
ω_I	ω_{CC}	$\mathbb{E}(r_p)$	σ_p
100%	0%	26%	50,0%
80%	20%	22%	40,3%
60%	40%	18%	31,6%
40%	60%	14%	25,0%
20%	80%	10%	22,4%
0%	100%	6%	25,0%

Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

Diversification impact

- Suppose the investor considers investing 100% in Coca-Cola stock. Other portfolios, such as the portfolio with 20% in Intel stock and 80% in Coca-Cola stock, make the investor better off in both ways:
    1. They have a higher expected return, and
    2. they have lower volatility.
- As a result, investing solely in Coca-Cola stock is not a good idea.

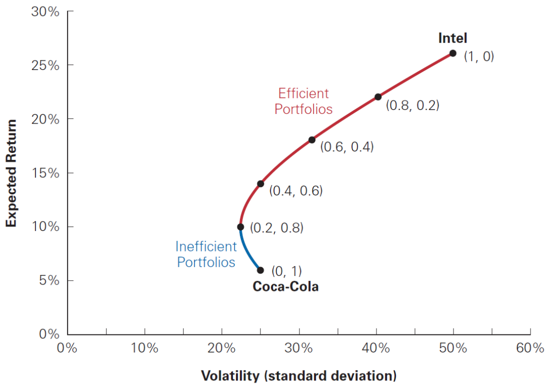
Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

FIGURE 11.3

Volatility Versus Expected Return for Portfolios of Intel and Coca-Cola Stock

Labels indicate portfolio weights (x_I, x_C) for Intel and Coca-Cola stocks. Portfolios on the red portion of the curve, with at least 20% invested in Intel stock, are efficient. Those on the blue portion of the curve, with less than 20% invested in Intel stock, are inefficient—an investor can earn a higher expected return with lower risk by choosing an alternative portfolio.



Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

Identifying Inefficient Portfolios

- We say a portfolio is an inefficient portfolio whenever it is possible to find another portfolio that is better in terms of both expected return and volatility. Looking at Figure 11.3, a portfolio is inefficient if there are other portfolios above and to the left (that is, to the northwest) of it.
- Investing solely in Coca-Cola stock is inefficient, and the same is true of all portfolios with more than 80% in Coca-Cola stock (the blue part of the curve).
- Inefficient portfolios are not optimal for an investor seeking high returns and low volatility.

Identifying Efficient Portfolios

- By contrast, portfolios with at least 20% in Intel stock are efficient (the red part of the curve): There is no other portfolio of the two stocks that offers a higher expected return with lower volatility.
- But while we can rule out inefficient portfolios as inferior investment choices, we cannot easily rank the efficient ones: investors will choose among them based on their own preferences for return versus risk.
- An extremely conservative investor who cares only about minimizing risk would choose the lowest-volatility portfolio (20% Intel, 80% Coca-Cola).
- An aggressive investor might choose to invest 100% in Intel stock (even though that approach is riskier, the investor may be willing to take that chance to earn a higher expected return).

Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

The Effect of Correlation

- In Figure 11.3, we assumed that the returns of Intel and Coca-Cola stocks are uncorrelated.
- Let's consider how the risk and return combinations would change if the correlations were different.
- Correlation has no effect on the expected return of a portfolio. For example, a 40–60 portfolio will still have an expected return of 14%. However, the volatility of the portfolio will differ depending on the correlation.
- In particular, the lower the correlation, the lower the volatility we can obtain.
- In terms of Figure 11.3, as we lower the correlation and therefore the volatility of the portfolios, the curve showing the portfolios will bend to the left to a greater degree, as illustrated in Figure 11.4 (next slide).

Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

The Effect of Correlation

- When the stocks are perfectly positively correlated, we can identify the set of portfolios by the straight line between them. In this extreme case (the red line in Figure 11.4), the volatility of the portfolio is equal to the weighted average volatility of the two stocks (there is no diversification).
- When the correlation is less than 1, however, the volatility of the portfolios is reduced due to diversification, and the curve bends to the left. The reduction in risk (and the bending of the curve) becomes greater as the correlation decreases.
- At the other extreme of perfect negative correlation (blue line), the line again becomes straight, this time reflecting off the vertical axis. In particular, when the two stocks are perfectly negatively correlated, it becomes possible to hold a portfolio that bears absolutely no risk.

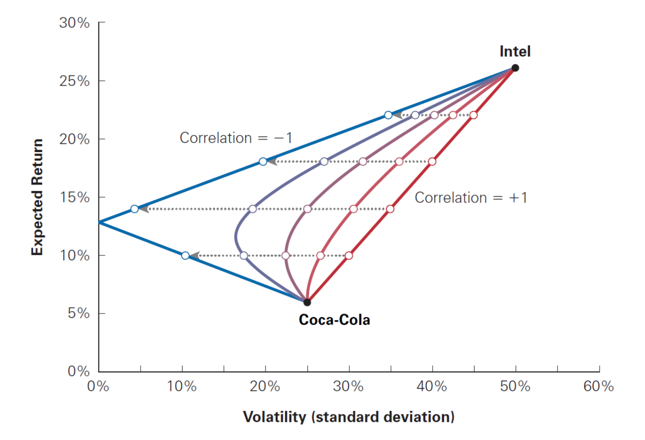
Efficient Portfolios with Two Stocks

Let's consider a two-stock portfolio

FIGURE 11.4

Effect on Volatility and Expected Return of Changing the Correlation between Intel and Coca-Cola Stock

This figure illustrates correlations of 1, 0.5, 0, −0.5 and −1 The lower the correlation, the lower the risk of the portfolios.



Expected Return and Volatility with a Short Sale

Let's consider a two-stock portfolio

Short selling effect

- We have considered only portfolios in which we invest a positive amount in each stock. We refer to a positive investment in a security as a long position in the security.
- But it is also possible to invest a negative amount in a stock, called a short position, by engaging in a short sale, a transaction in which you sell a stock today that you do not own, with the obligation to buy it back in the future.
- As the next example demonstrates, we can include a short position as part of a portfolio by assigning that stock a negative portfolio weight.

Expected Return and Volatility with a Short Sale

Problem

Short selling effect

- Suppose you have \$20,000 in cash to invest.
- You decide to short sell \$10,000 worth of Coca-Cola stock and invest the proceeds from your short sale, plus your \$20,000, in Intel.
- What is the expected return and volatility of your portfolio?

Expected Return and Volatility with a Short Sale

Solution

Short selling effect

- We can think of our short sale as a negative investment of -$10,000 in Coca-Cola stock. In addition, we invested +$30,000 in Intel stock, for a total net investment of $30,000 - $10,000 = $20,000 cash.
- The corresponding portfolio weights are :

$$\omega_I = \frac{\text{value of investment in Intel}}{\text{Total value of portfolio}} = \frac{30,000}{20,000} = 150\%$$

$$\omega_{CC} = \frac{\text{value of investment in Coca-Cola}}{\text{Total value of portfolio}} = \frac{-10,000}{20,000} = -50\%$$

- Note that the portfolio weights still add up to 100%.
- Using these portfolio weights, we can calculate the expected return and volatility of the portfolio such as :

$$\mathbb{E}(r_p) = \omega_I \cdot \mathbb{E}(r_I) + \omega_{CC} \cdot \mathbb{E}(r_{CC})$$

$$\mathbb{E}(r_p) = 150\% \cdot 26\% - 50\% \cdot 6\% = 36\%$$

Expected Return and Volatility with a Short Sale

Solution

Short selling effect

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + (1-\omega_1)^2\sigma_2^2 + 2\omega_1(1-\omega_1)\sigma_1\sigma_2\rho_{1,2}$$

$$\sigma_p = \sqrt{150\%^2 \cdot 50\%^2 + (1-150\%)^2 \cdot 25\%^2 + 2 \cdot 150\% \cdot (1-150\%) \cdot 50\% \cdot 25\% \cdot 0} = 76\%$$

- Note that in this case, short selling increases the expected return of your portfolio, but also its volatility, above those of the individual stocks.

Remember

- Short selling is profitable if you expect a stock's price to decline in the future.
- Recall that when you borrow a stock to short sell it, you are obligated to buy and return it in the future. So when the stock price declines, you receive more upfront for the shares than the cost to replace them in the future.
- But as the preceding example shows, short selling can be advantageous even if you expect the stock's price to rise, as long as you invest the proceeds in another stock with an even higher expected return. That said, and as the example also shows, short selling can greatly increase the risk of the portfolio.

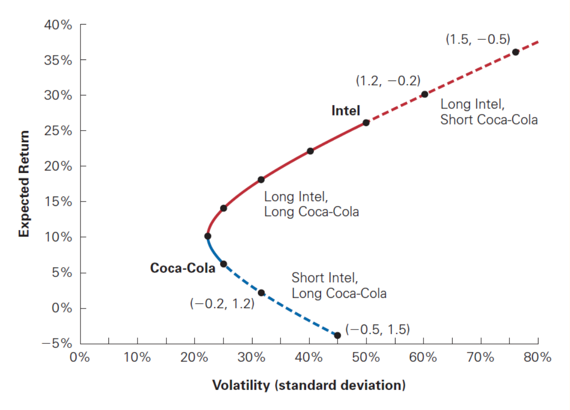
Expected Return and Volatility with a Short Sale

Let's consider a two-stock portfolio

FIGURE 11.5

Portfolios of Intel and Coca-Cola Allowing for Short Sales

Labels indicate portfolio weights (X_I, X_C) for Intel and Coca-Cola stocks. Red indicates efficient portfolios, blue indicates inefficient portfolios. The dashed curves indicate positions that require shorting either Coca-Cola (red) or Intel (blue). Shorting Intel to invest in Coca-Cola is inefficient. Shorting Coca-Cola to invest in Intel is efficient and might be attractive to an aggressive investor who is seeking high expected returns.



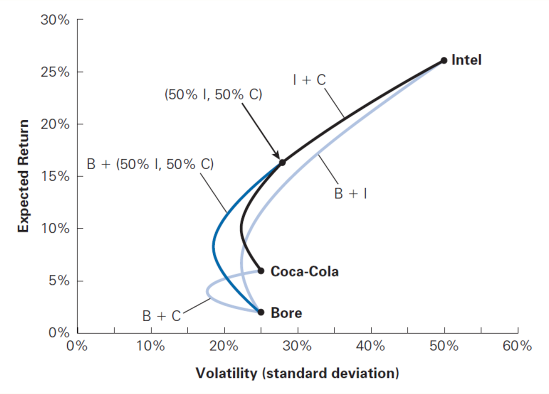
Efficient Portfolios with Many Stocks

Let's consider a three-stock portfolio

FIGURE 11.6

Expected Return and Volatility for Selected Portfolios of Intel, Coca-Cola, and Bore Industries Stocks

By combining Bore (B) with Intel (I), Coca-Cola (C), and portfolios of Intel and Coca-Cola, we introduce new risk and return possibilities. We can also do better than with just Coca-Cola and Intel alone (the black curve). Portfolios of Bore and Coca-Cola (B + C) and Bore and Intel (B + I) are shown in light blue in the figure. The dark blue curve is a combination of Bore with a portfolio of Intel and Coca-Cola.



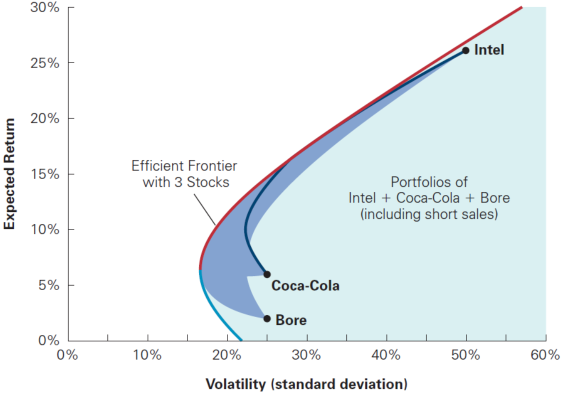
Efficient Portfolios with Many Stocks

Let's consider a three-stock portfolio

FIGURE 11.7

The Volatility and Expected Return for All Portfolios of Intel, Coca-Cola, and Bore Stock

Portfolios of all three stocks are shown, with the dark blue area showing portfolios without short sales, and the light blue area showing portfolios that include short sales. The best risk–return combinations are on the efficient frontier (red curve). The efficient frontier improves (has a higher return for each level of risk) when we move from two to three stocks.



Efficient Portfolios with Many Stocks

Let's consider a *N*-stock portfolio

Diversification with *N* stocks

- Adding more stocks to a portfolio reduces risk through diversification.
- Let's consider the effect of adding to our portfolio a third stock, Bore Industries, which is uncorrelated with Intel and Coca-Cola but is expected to have a very low return of 2%, and the same volatility as Coca-Cola (25%).
- Figure 11.6 illustrates the portfolios that we can construct using these three stocks. Because Bore stock is inferior to Coca-Cola stock (it has the same volatility but a lower return) you might guess that no investor would want to hold a long position in Bore.
- However, that conclusion ignores the diversification opportunities that Bore provides.
- Figure 11.6 shows the results of combining Bore with Coca-Cola or with Intel (light blue curves), or combining Bore with a 50–50 portfolio of Coca-Cola and Intel (dark blue curve).
- Notice that some of the portfolios we obtained by combining only Intel and Coca-Cola (black curve) are inferior to these new possibilities.

Efficient Portfolios with Many Stocks

Let's consider a *N*-stock portfolio

Diversification with *N* stocks

- When we combine Bore stock with every portfolio of Intel and Coca-Cola, and allow for short sales as well, we get an entire region of risk and return possibilities rather than just a single curve.
- This region is shown in the shaded area in Figure 11.7. But note that most of these portfolios are inefficient.
- The efficient portfolios (those offering the highest possible expected return for a given level of volatility) are those on the northwest edge of the shaded region, which we call the efficient frontier for these three stocks.
- In this case none of the stocks, on its own, is on the efficient frontier, so it would not be efficient to put all our money in a single stock.

Efficient Portfolios with Many Stocks

Let's consider a N-stock portfolio

Diversification with N stocks

- When the set of investment opportunities increases from two to three stocks, the efficient frontier improves.
- Visually, the old frontier with any two stocks is located inside the new frontier. In general, adding new investment opportunities allows for greater diversification and improves the efficient frontier. Figure 11.8 (next slide) uses historical data to show the effect of increasing the set from three stocks (Amazon, GE, and McDonald's) to ten stocks.
- Even though the added stocks appear to offer inferior risk–return combinations on their own, because they allow for additional diversification, the efficient frontier improves with their inclusion.
- Thus, to arrive at the best possible set of risk and return opportunities, we should keep adding stocks until all investment opportunities are represented.
- Ultimately, based on our estimates of returns, volatilities, and correlations, we can construct the efficient frontier for all available risky investments showing the best possible risk and return combinations that we can obtain by optimal diversification.

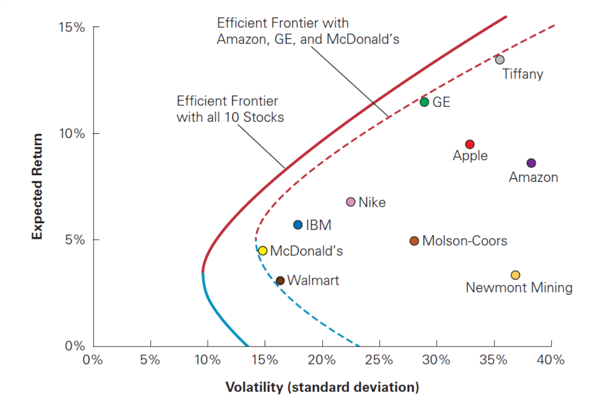
Efficient Portfolios with Many Stocks

Let's consider a *N*-stock portfolio

FIGURE 11.8

Efficient Frontier with Three Stocks Versus Ten Stocks

The efficient frontier expands as new investments are added. (Volatilities and correlations based on monthly returns, 2005–2015, expected returns based on forecasts.)



Risk-Free Saving and Borrowing

Let's consider a *N*-stock portfolio

Introducing the risk free asset called r_f

- We have considered the risk and return possibilities that result from combining risky investments into portfolios. By including all risky investments in the construction of the efficient frontier, we achieve maximum diversification.
- There is another way besides diversification to reduce risk that we have not yet considered: We can keep some of our money in a safe, no-risk investment like Treasury bills. Of course, doing so will reduce our expected return.
- Conversely, if we are an aggressive investor who is seeking high expected returns, we might decide to borrow money to invest even more in the stock market.
- In this section we will see that the ability to choose the amount to invest in risky versus risk-free securities allows us to determine the optimal portfolio of risky securities for an investor.

Risk-Free Saving and Borrowing

Let's consider a N-stock portfolio

Introducing the risk free asset called r_f

- Consider an arbitrary risky portfolio with returns r_p. Let's look at the effect on risk and return of putting a fraction ω of our money in the portfolio p, while leaving the remaining fraction $(1-\omega)$ in risk-free Treasury bills with a yield of r_f. The portfolio that comprises p and r_f is called ψ.
- Using our previous equations, we calculate the expected return and variance of this portfolio.

$$\mathbb{E}(r_\psi) = \omega \cdot \mathbb{E}(r_p) + (1-\omega) \cdot r_f \tag{17}$$

$$\mathbb{E}(r_\psi) = \omega \cdot \mathbb{E}(r_p) + rf - \omega \cdot r_f \tag{18}$$

$$\mathbb{E}(r_\psi) = r_f + \omega \cdot (\mathbb{E}(r_p) - r_f) \tag{19}$$

- The first equation simply states that the expected return is the weighted average of the expected returns of Treasury bills and the portfolio. Because we know up front the current interest rate paid on Treasury bills, we do not need to compute an expected return for them.
- The second equation rearranges the first to give a useful interpretation: our expected return is equal to the risk-free rate plus a fraction of the portfolio's risk premium, $\mathbb{E}(r_p) - r_f$, based on the fraction ω that we invest in it.

Risk-Free Saving and Borrowing

Let's consider a N-stock portfolio

Introducing the risk free asset called r_f

- Next, let's compute the volatility. Because the risk-free rate r_f is fixed and does not move with (or against) our portfolio, its volatility and covariance with the portfolio are both zero.

$$\sigma_\psi = \sqrt{(1-\omega)^2 \cdot \sigma_{r_f}^2 + \omega^2 \cdot \sigma_{r_p}^2 + 2 \cdot (1-\omega) \cdot \omega \cdot \sigma_{r_f, r_p}} \tag{20}$$

$$\sigma_\psi = \sqrt{\color{blue}{(1-\omega)^2 \cdot \sigma_{r_f}^2} + \omega^2 \cdot \sigma_{r_p}^2 + \color{blue}{2 \cdot (1-\omega) \cdot \omega \cdot \sigma_{r_f, r_p}}} \tag{21}$$

- The blue terms equal zero,

$$\sigma_\psi = \sqrt{\omega^2 \cdot \sigma_{r_p}^2} \tag{22}$$

$$\sigma_\psi = \omega \cdot \sigma_{r_p} \tag{23}$$

- That is, the volatility is only a fraction of the volatility of the portfolio, based on the amount we invest in it.
- The blue line in Figure 11.9 illustrates combinations of volatility and expected return for different choices of ω. Looking at Eq. 23, as we increase the fraction ω invested in p, we increase both our risk and our risk premium proportionally.
- Hence the line is straight from the risk-free investment through p.

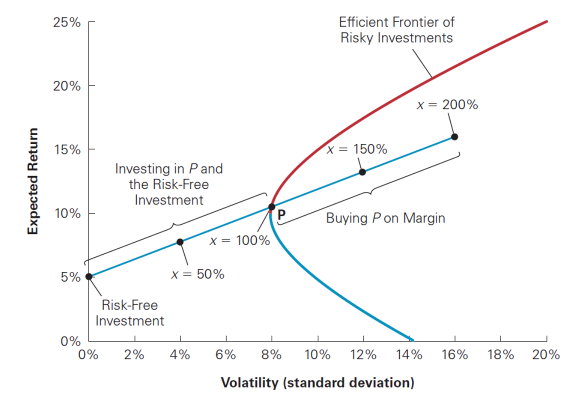
Risk-Free Saving and Borrowing

Let's consider a *N*-stock portfolio

FIGURE 11.9

The Risk–Return Combinations from Combining a Risk-Free Investment and a Risky Portfolio

Given a risk-free rate of 5%, the point with 0% volatility and an expected return of 5% represents the risk-free investment. The blue line shows the portfolios we obtained by investing x in portfolio P and $(1 - x)$ in the risk-free investment. Investments with weight $(x > 100\%)$ in portfolio P require borrowing at the risk-free interest rate.



Borrowing and Buying Stocks on Margin

Let's consider a N-stock portfolio

Borrowing and Buying Stocks on Margin

- As we increase the fraction ω invested in the portfolio p from 0 to 100%, we move along the line in Figure 11.9 from the risk-free investment to p.
- If we increase ω beyond 100%, we get points beyond p in the graph.
- In this case, we are short selling the risk-free investment, so we must pay the risk-free return; in other words, we are borrowing money at the risk-free interest rate.
- Borrowing money to invest in stocks is referred to as buying stocks on margin or using leverage.
- A portfolio that consists of a short position in the risk-free investment is known as a *levered portfolio*. As you might expect, margin investing is a risky investment strategy.
- Note that the region of the blue line in Figure 11.9 with $\omega > 100\%$ has higher risk than the portfolio p itself. At the same time, margin investing can provide higher expected returns than investing in p using only the funds we have available.

Margin Investing

Problem

Introducing the risk free asset called r_f

- Suppose you have \$10,000 in cash, and you decide to borrow another \$10,000 at a 5% interest rate in order to invest \$20,000 in portfolio Q, which has a 10% expected return and a 20% volatility.

1. What is the expected return and volatility of your investment?
2. What is your realized return if Q goes up 30% over the year?
3. What if Q falls by 10%?

Margin Investing

Solution

Introducing the risk free asset called r_f

1. You have doubled your investment in Q using margin, so $\omega = 200\%$. From our last equations, we see that you have increased both your expected return and your risk relative to the portfolio Q:

$$\mathbb{E}(r_\psi) = r_f + \omega \cdot (\mathbb{E}(r_Q) - r_f) \tag{24}$$

$$\mathbb{E}(r_\psi) = 5\% + 200\% \cdot (10\% - 5\%) = 15\% \tag{25}$$

$$\sigma_\psi = \omega \cdot \sigma_Q = 200\% \cdot 20\% = 40\% \tag{26}$$

2. If Q goes up 30%, your investment will be worth \$26,000, but you will owe \$10,000 x (1+5%) = \$10,500 on your loan, for a net payoff of \$15,500 or a 55% return on your \$10,000 initial investment.
3. If Q drops by -10%, you are left with \$18,000 - \$10,500 = \$7500, and your return is -25%. Thus the use of margin doubled the range of your returns (55% - (-25%) = 80% vs. 30% - (-10%) = 40%), corresponding to the doubling of the volatility of the portfolio.

Identifying the Tangent Portfolio

Let's consider a N-stock portfolio

Introducing the risk free asset called r_f

- Looking back at Figure 11.9, we can see that portfolio p is not the best portfolio to combine with the risk-free investment.
- By combining the risk-free asset with a portfolio somewhat higher on the efficient frontier than portfolio p, we will get a line that is steeper than the line through p.
- If the line is steeper, then for any level of volatility, we will earn a higher expected return.
- To earn the highest possible expected return for any level of volatility we must find the portfolio that generates the steepest possible line when combined with the risk-free investment.
- The slope of the line through a given portfolio p is often referred to as the Sharpe ratio of the portfolio:

$$\text{Sharpe ratio} = \frac{\text{Portfolio Excess Return}}{\text{Portfolio volatility}} = \frac{\mathbb{E}(r_p) - r_f}{\sigma}$$

Identifying the Tangent Portfolio

Let's consider a N-stock portfolio

Introducing the risk free asset called r_f

- The Sharpe ratio measures the ratio of reward-to-volatility provided by a portfolio.
- The optimal portfolio to combine with the risk-free asset will be the one with the highest Sharpe ratio, where the line with the risk-free investment just touches, and so is tangent to, the efficient frontier of risky investments, as shown in Figure 11.10 (next slide).
- The portfolio that generates this tangent line is known as the tangent portfolio.
- All other portfolios of risky assets lie below this line. Because the tangent portfolio has the highest Sharpe ratio of any portfolio in the economy, the tangent portfolio provides the biggest reward per unit of volatility of any portfolio available.
- As is evident from Figure 11.10, combinations of the risk-free asset and the tangent portfolio provide the best risk and return trade-off available to an investor. This observation has a striking consequence: The tangent portfolio is efficient and, once we include the risk-free investment, all efficient portfolios are combinations of the risk-free investment and the tangent portfolio.

Identifying the Tangent Portfolio

Let's consider a N-stock portfolio

Introducing the risk free asset called r_f

- Therefore, the optimal portfolio of risky investments no longer depends on how conservative or aggressive the investor is; every investor should invest in the tangent portfolio independent of his or her taste for risk.
- The investor's preferences will determine only how much to invest in the tangent portfolio versus the risk-free investment.
- Conservative investors will invest a small amount, choosing a portfolio on the line near the risk-free investment.
- Aggressive investors will invest more, choosing a portfolio that is near the tangent portfolio or even beyond it by buying stocks on margin.
- But both types of investors will choose to hold the same portfolio of risky assets, the *tangent portfolio*.

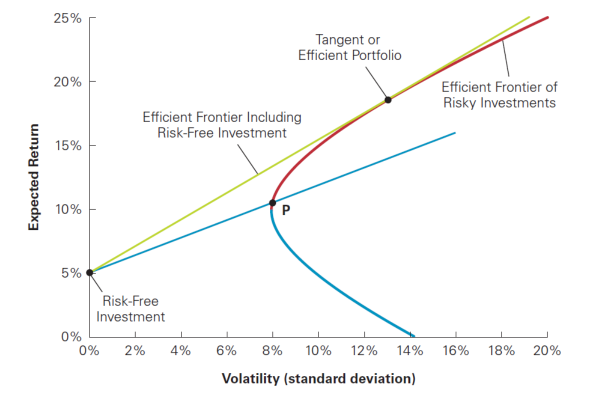
Identifying the Tangent Portfolio

Let's consider a *N*-stock portfolio

FIGURE 11.10

The Tangent or Efficient Portfolio

The tangent portfolio is the portfolio with the highest Sharpe ratio. Investments on the green line connecting the risk-free investment and the tangent portfolio provide the best risk and return trade-off available to an investor. As a result, we also refer to the tangent portfolio as *the* efficient portfolio.



Portfolio Improvement: Beta and the Required Return

Let's consider a N-stock portfolio

Introducing the risk free asset called r_f

- Take an arbitrary portfolio p, and let's consider whether we could raise its Sharpe ratio by selling some of our risk-free assets (or borrowing money) and investing the proceeds in an investment i. If we do so, there are two consequences:

1. Expected return: Because we are giving up the risk-free return and replacing it with i's return, our expected return will increase by i's excess return, $\mathbb{E}(r_i) - r_f$
2. Volatility: We will add the risk that i has in common with our portfolio (the rest of i's risk will be diversified). Incremental risk is measured by i's volatility multiplied by its correlation with p : $\sigma_i \cdot \rho_{i,p}$

- Is the gain in return from investing in i adequate to make up for the increase in risk?
- Another way we could have increased our risk would have been to invest more in portfolio p itself. In that case, p's Sharpe ratio computed as $\frac{\mathbb{E}(r_p)-r_f}{\sigma_p}$ tells us how much the return would increase for a given increase in risk.
- Because the investment in i increases risk by $\sigma_i \cdot \rho_{i,p}$, it offers a larger increase in return than we could have gotten from p alone.

Portfolio Improvement: Beta and the Required Return

Let's consider a N-stock portfolio

Introducing the risk free asset called r_f

$$\mathbb{E}(r_i) - r_f > \sigma_i \cdot \rho_{i,p} \cdot \frac{\mathbb{E}(r_p) - r_f}{\sigma_p} \quad (27)$$

- To provide a further interpretation for this condition, let's combine the volatility and correlation terms to define the β of investment i with portfolio p:

$$\beta_i^p = \frac{\sigma_i \cdot \rho_{i,p}}{\sigma_p} \quad (28)$$

- β_i^p measures the sensitivity of the investment i to the fluctuations of the portfolio p. That is, for each 1% change in the portfolio's return, investment i's return is expected to change by β_i^p% due to risks that i has in common with p.
- With this definition, we can pose :

$$\mathbb{E}(r_i) = r_f + \beta_i^p \cdot (\mathbb{E}(r_p) - r_f) \quad (29)$$

- That is, increasing the amount invested in i will increase the Sharpe ratio of portfolio p if its expected return $\mathbb{E}(r_i)$ exceeds its required return given portfolio p.

Portfolio Improvement: Beta and the Required Return

Let's consider a N-stock portfolio

Introducing the risk free asset called r_f

- The required return is the expected return that is necessary to compensate for the risk investment i will contribute to the portfolio.
- The required return for an investment i is equal to the risk-free interest rate plus the risk premium of the current portfolio, p, scaled by i's sensitivity to p, β_i^p.
- If i's expected return exceeds this required return, then adding more of it will improve the performance of the portfolio.

Expected Returns and the Efficient Portfolio

Problem

Problem: Expected Returns and the Efficient Portfolio

- You are currently invested in the Omega Fund, a broad-based fund with an expected return of 15% and a volatility of 20%, as well as in risk-free Treasuries paying 3%.
- Your broker suggests that you add a real estate fund to your portfolio. The real estate fund has an expected return of 9%, a volatility of 35%, and a correlation (ρ) of 0.10 with the Omega Fund.
- Will adding the real estate fund improve your portfolio?

Expected Returns and the Efficient Portfolio

Solution

Solution: Expected Returns and the Efficient Portfolio

- Let r_{re} be the return of the real estate fund and r_{of} be the return of the Omega Fund.
- From our last equations, the β of the real estate fund with the Omega Fund is :

$$\beta_{re}^{of} = \frac{\sigma_{re} \cdot \rho_{re,of}}{\sigma_{of}} = \frac{35\% x 10\%}{20\%} = 0.175 \tag{30}$$

  We can now determine the required return that makes the real estate fund an attractive addition to our portfolio :

$$r_{re} = r_f + \beta_{re}^{of} \cdot (\mathbb{E}(r_{of}) - r_f) = 3\% + 0.175 \cdot (15\% - 3\%) = 5.1\% \tag{31}$$

- Because its expected return of 9% exceeds the required return of 5.1%, investing some amount in the real estate fund will improve our portfolio's Sharpe ratio.

Expected Returns and the Efficient Portfolio

The famous Efficient Porfolio

Introducing the true Efficient Portfolio

- If a security's expected return exceeds its required return, then we can improve the performance of portfolio p by adding more of the security. But how much more should we add?
- As we buy shares of security i, its correlation (and therefore its β) with our portfolio will increase, ultimately raising its required return until $\mathbb{E}(r_i) = r_i$.
- At this point, our holdings of security i are optimal. Similarly, if security i's expected return is less than the required return r_i, we should reduce our holdings of i. As we do so the correlation and the required return r_i will fall until $\mathbb{E}(r_i) = r_i$.
- Thus, if we have no restrictions on our ability to buy or sell securities that are traded in the market, we will continue to trade until the expected return of each security equals its required return (that is, until $\mathbb{E}(r_i) = r_i$ holds for all i).
- At this point, no trade can possibly improve the risk-reward ratio of the portfolio, so our portfolio is the optimal, efficient portfolio. A portfolio is efficient if and only if the expected return of every available security equals its required return.

$$\mathbb{E}(r_i) = r_i = r_f + \beta_i^{eff} \left(\mathbb{E}(r_{eff}) - r_f\right) \tag{32}$$

- where r_{eff} is the return of the efficient portfolio, the portfolio with the highest Sharpe ratio of any portfolio in the economy.

The Capital Asset Pricing Model

Introduction

The three main CAPM Assumptions

1. Investors can buy and sell all securities at competitive market prices (without incurring taxes or transactions costs) and can borrow and lend at the risk-free interest rate.
2. All investors choose a portfolio of traded securities that offers the highest possible expected return given the level of volatility they are willing to accept: "*Investors hold only efficient portfolios of traded securities-portfolios that yield the maximum expected return for a given level of volatility.*"
   - Investors hold only efficient portfolios of traded securities-portfolios that yield the maximum expected return for a given level of volatility.
   - There are many investors in the world, and each may have his or her own estimates of the volatilities, correlations, and expected returns of the available securities.
   - Investors don't come up with these estimates arbitrarily; they base them on historical patterns and other information (including market prices) that is widely available to the public.
   - If all investors use publicly available information sources, then their estimates are likely to be similar. Consequently, it is not unreasonable to consider a special case in which all investors have the same estimates concerning future investments and returns, called *homogeneous expectations*.

The Capital Asset Pricing Model

Introduction

The three main CAPM Assumptions

- Although investors' expectations are not completely identical in reality, assuming homogeneous expectations should be a reasonable approximation in many markets, and represents the third simplifying assumption of the CAPM:

3. "*Investors have homogeneous expectations regarding the volatilities, correlations, and expected returns of securities*".

The Capital Asset Pricing Model

Optimal Investing: The Capital Market Line

The Capital Market Line (CML)

- When the CAPM assumptions hold, the market portfolio is efficient, so the tangent portfolio in Figure 11.10 (next slide) is actually the market portfolio (M).
- The tangent line graphs the highest possible expected return we can achieve for any level of volatility. When the tangent line goes through the market portfolio, it is called the capital market line (CML).
- According to the CAPM, all investors should choose a portfolio on the capital market line, by holding some combination of the risk-free security and the market portfolio.

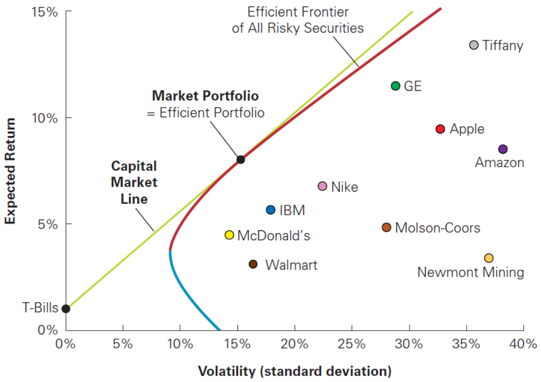
The Capital Asset Pricing Model

Optimal Investing: The Capital Market Line

FIGURE 11.11

The Capital Market Line

When investors have homogeneous expectations, the market portfolio and the efficient portfolio coincide. Therefore, the capital market line (CML), which is the line from the risk-free investment through the market portfolio, represents the highest-expected return available for any level of volatility. (Data from Figure 11.8.)



The Capital Asset Pricing Model

Determining the Risk Premium

Market Risk and β

- Under the CAPM assumptions, we can identify the efficient portfolio: It is equal to the market portfolio. Thus, if we don't know the expected return of a security or the cost of capital of an investment, we can use the CAPM to find it by using the market portfolio as a benchmark.

$$\mathbb{E}(r_i) = r_f + \beta_i \left(\mathbb{E}(r_M) - r_f\right) \tag{33}$$

$$\beta_i = \frac{\sigma_i \cdot \rho_{i,M}}{\sigma_M} = \frac{\sigma_{i,M}}{\sigma_M^2} \tag{34}$$

- The β of a security measures its volatility due to market risk relative to the market as a whole, and thus captures the security's sensitivity to market risk.
- In order to determine the appropriate risk premium for any investment, we must rescale the market risk premium (the amount by which the market's expected return exceeds the risk-free rate) by the amount of market risk present in the security's returns, measured by its β with the market.
- We can interpret the CAPM equation as follows. Following the Law of One Price, in a competitive market, investments with similar risk should have the same expected return.
- Because investors can eliminate firm-specific risk by diversifying their portfolios, the right measure of risk is the investment's beta with the market portfolio, β_i.

The Capital Asset Pricing Model

Determining the Risk Premium: Problem

Market Risk and β

- Suppose the risk-free return is 4% and the market portfolio has an expected return of 10% and a volatility of 16%.
- 3M stock has a 22% volatility and a correlation with the market of 0.50.

1. What is 3M's beta with the market?
2. What capital market line portfolio has equivalent market risk, and what is its expected return?

The Capital Asset Pricing Model

Determining the Risk Premium: Solution

Market Risk and β

1. We can use our previous formula :

$$\beta_i = \frac{\sigma_i \cdot \rho_{i,M}}{\sigma_M} = \frac{\sigma_{i,M}}{\sigma_M^2} \tag{35}$$

$$\beta_i = \frac{22\% \cdot 0.5}{16\%} = 0.69 \tag{36}$$

- That is, for each 1% move of the market portfolio, 3M stock tends to move 0.69%.
- We could obtain the same sensitivity to market risk by investing 69% in the market portfolio, and 31% in the risk-free security.

2. Because it has the same market risk, 3M's stock should have the same expected return as this portfolio, which is :

$$\mathbb{E}(r_i) = r_f + \beta_i \left(\mathbb{E}(r_M) - r_f\right) \tag{37}$$

$$\mathbb{E}(r_{3M}) = 4\% + 0.69 \cdot (10\% - 4\%) = 8.1\% \tag{38}$$

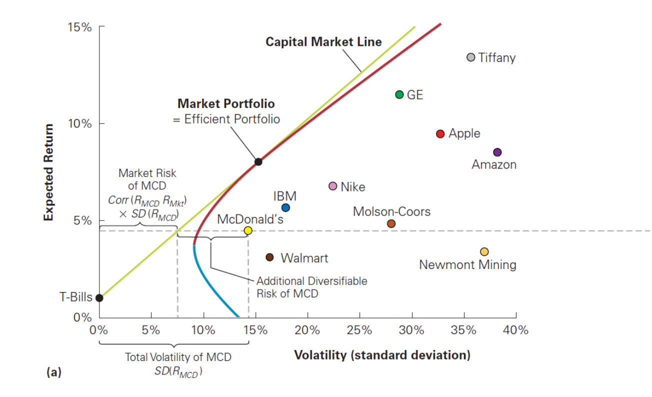
- Investors will require an expected return of 8.1% to compensate for the risk associated with 3M stock.

The Capital Asset Pricing Model

Determining the Risk Premium

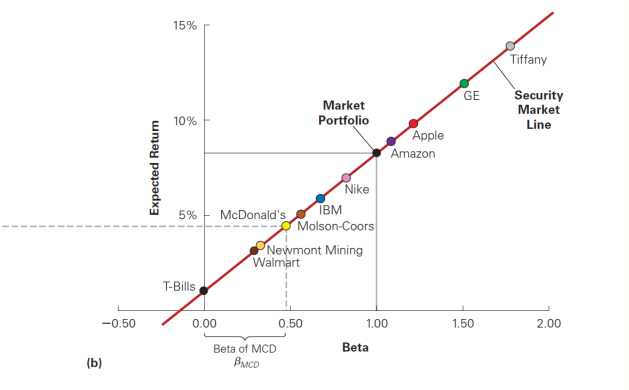
FIGURE 11.12 **The Capital Market Line and the Security Market Line**

(a) The CML depicts portfolios combining the risk-free investment and the efficient portfolio, and shows the highest expected return that we can attain for each level of volatility. According to the CAPM, the market portfolio is on the CML and all other stocks and portfolios contain diversifiable risk and lie to the right of the CML, as illustrated for McDonald's (MCD).



The Capital Asset Pricing Model

The Security Market Line



(b) The SML shows the expected return for each security as a function of its beta with the market. According to the CAPM, the market portfolio is efficient, so all stocks and portfolios should lie on the SML.

The Capital Asset Pricing Model

The Security Market Line

$\mathbb{E}(r_i)$ vs. β

$$\mathbb{E}(r_i) = r_f + \beta_i \left(\mathbb{E}(r_M) - r_f\right) \tag{39}$$

- Equation 39 implies that there is a linear relationship between a stock's β and its expected return. The last figure graphs this line through the risk-free investment (with a β of 0) and the market (with a $\beta = 1$); it is called the security market line (SML).
- Under the CAPM assumptions, the security market line (SML) is the line along which all individual securities should lie when plotted according to their expected return and β.
- As we illustrate for McDonald's (MCD), a stock's expected return is due only to the fraction of its volatility that is common with the market;
- The distance of each stock to the right of the capital market line is due to its diversifiable risk.
- The relationship between risk and return for individual securities becomes evident only when we measure market risk rather than total risk.

The Capital Asset Pricing Model

Beta of a Portfolio

$\mathbb{E}(r_i)$ vs. β

- Because the security market line (SML) applies to all tradable investment opportunities, we can apply it to portfolios as well.
- Consequently, the expected return of a portfolio is given by $\mathbb{E}(r_p) = r_f + \beta_p \left(\mathbb{E}(r_M) - r_f\right)$ and therefore depends on the portfolio's β.

$$\beta_p = \frac{\sigma_p \cdot \rho_{p,M}}{\sigma_M} = \frac{\sigma_{p,M}}{\sigma_M^2} = \frac{cov(r_p, r_M)}{var(r_M)} \tag{40}$$

- Using Eq. 40, we calculate the β of a portfolio where:

$$\mathbb{E}(r_p) = \mathbb{E}\left(\sum_{i=1}^{N} \omega_i \cdot r_i\right) = \sum_{i=1}^{N} \mathbb{E}\left(\omega_i \cdot r_i\right) = \sum_{i=1}^{N} \omega_i \cdot \mathbb{E}\left(r_i\right) \tag{41}$$

$$\beta_p = \frac{cov\left(\sum_{i=1}^{N} \omega_i \cdot r_i, r_M\right)}{var(r_M)} = \sum_{i=1}^{N} \omega_i \cdot \frac{cov\left(r_i, r_M\right)}{var(r_M)} = \sum_{i=1}^{N} \omega_i \cdot \beta_i \tag{42}$$

- In other words, the β of a portfolio is the weighted average β of the securities in the portfolio.