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Prediction of α-stable GARCH and ARMA-GARCH-M models

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Abstract

The best prediction of generalized autoregressive conditional heteroskedasticity (GARCH) models with α-stable innovations, α-stable power-GARCH models and autoregressive moving average (ARMA) models with GARCH in mean effects (ARMA-GARCH-M) are proposed. We present a sufficient condition for stationarity of α-stable GARCH models. The prediction methods are easy to implement in practice. The proposed prediction methods are applied for predicting future values of the daily SP500 stock market and wind speed data.

KEYWORDS

GARCH-M model, α-stable distribution, conditional expectation, prediction, volatility

1 | INTRODUCTION

Generalized autoregressive conditional heteroskedasticity (GARCH) processes are widely used for modeling financial data at regular intervals on stocks, currency investments, and other assets. The GARCH model was first introduced by Bollerslev (1986) as a generalization of the ARCH model, which was introduced by Engle (1982). There are typical variants of GARCH models such as exponential GARCH (Nelson, 1991), Glosten–Jagannathan–Runkle GARCH (Glosten, Jagannathan, & David, 1993), nonlinear GARCH (Engle & Ng. V. K., 1993), long memory GARCH (Conrad & Karanasos, 2006), and stable mixture GARCH (Broda, Haas, Krause, Paolella, & Steude, 2013), to name a few. Typical variants of GARCH models have been used for modeling data arising in different fields, such as econometrics, signal processing, and hydrology. See Bollerslev (2008) for typical variants of GARCH models and their applications. Regarding the wide application of typical variants of GARCH models, prediction of such models is an important issue.

The process $\{Z_t, t \in \mathbb{Z}\}$ is called GARCH(r, s) if it satisfies the following systems:

$$Z_t = \epsilon_t \nu_t \tag{1}$$

and

$$\epsilon_t^2 = \omega + \sum_{i=1}^{r} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{s} \beta_i Z_{t-i}^2, \tag{2}$$

where $\{\nu_t\}$ is an independent and identically distributed (i.i.d.) sequence, $\omega > 0$, and $\{\alpha_i\}$ and $\{\beta_i\}$ are two sequences of non-negative numbers with at leat one nonzero β_i. In financial contexts, $\{Z_t\}$ and $\{\epsilon_t\}$ are referred as return and volatility processes. Bollerslev (1986) shows that the GARCH process in Equations 1 and 2 is stationary if $\sum_{i=1}^{r}\alpha_i+E(\nu_0^2)\sum_{i=1}^{s}\beta_i<1$. Also, a necessary and sufficient condition for stationarity of GARCH processes is given in Bougerol and Picard (1992). We say that $\{Z_t\}$ is an α-stable GARCH process with $\alpha \in (1, 2]$ if it satisfies Equations 1 and 2, where $\{\nu_t\}$ is an i.i.d. sequence of α-stable random variables. In Section 2, we present a sufficient condition for stationarity of α-stable GARCH models. For an application of α-stable GARCH models in econometrics, see Rachev, Stoyanov, Biglova and Fabozzi (2005). When $\alpha = 2$, $\{\nu_t\}$ is an i.i.d. sequence of Gaussian random variables. For elementary properties of α-stable distributions see Samorodnitsky and Taqqu (1994). The α-stable GARCH models are different from α-stable power-GARCH models, which were considered by Panorska, Mittnik, and Rachev (1995) and Mittnik, Paolella, and Rachev (1998), among many others.

Using the same procedure as in Baillie and Bollerslev (1992) one can find prediction of an ARMA model with α-stable power-GARCH innovations. Since $E(\nu_t^2) = \infty$ for $\alpha \in (0, 2)$ the method presented in Baillie and Bollerslev cannot be used for predicting α-stable GARCH models. In this paper, we present a method for estimating the conditional mean of volatilities following α-stable GARCH models. Also, we show that our method works for predicting volatilities of α-stable power-GARCH models.

One of the most common mean equations for excess asset returns is the GARCH-M model, which was initially introduced by Engle, Lilien, and Robins (1987). This model is widely used in stock returns, industrial production, and interest rates. For further applications see Elyasiani and Mansur (1998), Karanasos and Kim (2000), and Panait and Slavescu (2012), among many others. Moreover, for an application of ARMA models with GARCH in mean effects (ARMA-GARCH-M) for modeling the mean and volatility of wind speed, see Ewing, Kruse, and Schroeder (2006) and Liu, Erden, and Shi (2011).

Prediction of an ARMA process with GARCH errors has been initially considered by Engle and Bollerslev (1986) and Baillie and Bollerslev (1992). We refer to De Gooijer and Hyndman (2006) for a survey on the prediction of time series.

There are two different representations for ARMA-GARCH-M models, which we state in the following. We say that the process $\{X_t, t \in \mathbb{Z}\}$ is an ARMA-GARCH-M-I(p, q, r, s) process if it satisfies the following system:

$$\Phi(B)X_t = \varphi + \delta\epsilon_t + \Theta(B)Z_t, \tag{3}$$

and an ARMA-GARCH-M-II(p, q, r, s) process if $\{X_t, t \in \mathbb{Z}\}$ satisfies the following system:

$$\Phi(B)X_t = \varphi + \delta\epsilon_t^2 + \Theta(B)Z_t, \tag{4}$$

where $\{Z_t\}$ and $\{\epsilon_t\}$ satisfy Equations 1 and 2, $\Phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$, $\Theta(B) = 1 + \sum_{i=1}^{q} \theta_i B^i$, and B is the backward shift operator. We assume that $E(v_1) = 0$ and $E(v_1^2) = 1$. The ARMA-GARCH-M$(0, 0, r, s)$ process is called an GARCH-M(r, s) process. The difference between the two models in Equations 3 and 4 is in terms of their mean equations. The model in Equation 3 is the most common form of ARMA-GARCH-M model.

In this paper, we find $E_t(X_{t+h})$ for $h > 0$ from a general ARMA-GARCH-M model. To our knowledge, the best prediction of an ARMA-GARCH-M-I model in terms of mean square error has not yet been considered. In this paper, we attempt to fill this gap in the literature. The best prediction of ARMA-GARCH-M-II models is considered in Karanasos (2001). The goal of his method, however, is theoretical purity rather than the prediction of expressions intended for practical use (Karanasos, 2001, p. 556). Moreover, his method works under five assumptions on the coefficients of the model. Using the same procedure for predicting ARMA-GARCH-M-I model stated in this paper, we find a new formula for h-step-ahead predictions in terms of some past values such that it is easy to implement in practice, rather than the method in Karanasos (2001) by making use of existing statistical software packages. The obtained formula works without any assumptions on the coefficients, and it uses the prediction formula stated in Baillie and Bollerslev (1992).

The rest of the paper is organized as follows. In Section 2, we present optimal predictions for α-stable GARCH models and α-stable power-GARCH models. Section 3 is devoted to the best prediction of future values from ARMA-GARCH-M models. In Section 4, we apply the proposed prediction method for predicting daily returns of SP500 stock market and wind speed data.

2 | PREDICTION OF α-STABLE GARCH MODELS

Let $\{\epsilon_t\}$ be the volatility process as in Equation 1. When $\{v_t\}$ is an i.i.d. sequence of finite variance random variables the exact formula for the conditional expectation of squares, that is, $E_t(\epsilon_{t+h}^2)$, as an optimal predictor of ϵ_{t+h}^2 is given in Baillie and Bollerslev (1992). When $\{v_t\}$ is an i.i.d. sequence such that $E(v_t^2) = \infty$, then $E_t(\epsilon_{t+h}^2)$ does not exist, and the method in Baillie and Bollerslev (1992) fails. In this section, we describe how we can find optimal predictors of α-stable GARCH and α-stable power-GARCH processes in practice. Also, the presented method works for GARCH models with finite variance innovations.

First, we present a sufficient condition for strictly stationary solution of α-stable GARCH models. Let $m = \max(r, s)$, $\alpha_i = 0$ for $i = r + 1, \ldots, m$ and $\beta_i = 0$ for $i = s + 1, \ldots, m$.

Theorem 1. *Let $\{Z_t\}$ and $\{\epsilon_t\}$ be an α-stable GARCH(r, s), $1 < \alpha < 2$, process and its corresponding volatility process, respectively, satisfying Equations 1 and 2. Let*

$$\sum_{i=1}^{m} E(\alpha_i + \beta_i v_i^2)^{1/2} < 1. \tag{5}$$

Then $\{Z_t\}$ has a strictly stationary solution and $E(\epsilon_t) < \infty$.

Proof. *Since $\{v_t\}$ is an i.i.d. sequence of α-stable random variables with $\alpha \in (1, 2)$, we have $E(|v_1|) < \infty$. By subsequent substitution in Equation 2 and using the proof of Theorem 1 in Bollerslev (1986) page 323, we have*

$$\begin{aligned}
\epsilon_t^2 &= \omega + \sum_{i=1}^{m}(\alpha_i + \beta_i v_{t-i}^2)\epsilon_{t-i}^2 \\
&= \omega + \omega\sum_{i=1}^{m}(\alpha_i + \beta_i v_{t-i}^2) \\
&\quad + \sum_{i=1}^{m}(\alpha_i + \beta_i v_{t-i}^2)\sum_{j=1}^{m}(\alpha_i + \beta_i v_{t-i-j}^2)\epsilon_{t-i-j}^2 \\
&\quad \vdots \\
&= \omega + \omega\sum_{j=1}^{\infty}\Pi_{k=1}^{j}\sum_{t_k=1}^{m}(\alpha_{t_k} + \beta_{t_k} v_{t-\sum_{e=1}^{k} t_e}^2).
\end{aligned} \tag{6}$$

Therefore,

$$\epsilon_t \leqslant \omega^{1/2} \sum_{j=1}^{\infty} \Pi_{k=1}^{j} \sum_{t_k=1}^{m} \left(\alpha_{t_k} + \beta_{t_k} \nu^2_{t-\sum_{e=1}^{k} t_e} \right)^{1/2} + \omega^{1/2}.$$

Using the condition in Equation 5 we have

$$E\Big(\sum_{j=1}^{\infty} \Pi_{k=1}^{j} \sum_{t_k=1}^{m} \left(\alpha_{t_k} + \beta_{t_k} \nu^2_{t-\sum_{e=1}^{k} t_e} \right)^{1/2} \Big) < \infty, \tag{7}$$

and this implies that the right-hand side of Equation 6 is finite, almost surely. Therefore, $\{\epsilon_t^2\}$ has a strictly stationary solution as in Equation 6. This shows that $\{Z_t\}$ has a strictly stationary solution too, and Equation 7 implies that $E(\epsilon_t) < \infty$. □

Let $Z_1, \dots, Z_t$ be a realization from an α-stable GARCH(r, s) process. We want to find an approximation for $E_t(\epsilon_{t+h})$ in terms of the past values $Z_1, \dots, Z_t$. Let $s > 1$. Using Equation 2 we have the following matrix representation:

$$\begin{pmatrix} \epsilon^2_{t+h} \\ \epsilon^2_{t+h-1} \\ . \\ \epsilon^2_{t+h-r+1} \\ Z^2_{t+h-1} \\ Z^2_{t+h-2} \\ . \\ Z^2_{t+h-s} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \nu^2_{t+h-1} & \alpha_2 & \dots & & \alpha_r & \beta_2 & \dots & & \beta_s & \omega \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & & 0 & 0 \\ . & & & 0 & 0 & & & & & \\ 0 & & \dots & 1 & 0 & 0 & \dots & & 0 & 0 \\ \nu^2_{t+h-1} & 0 & \dots & & & & \dots & & 0 & 0 \\ 0 & 0 & \dots & & 0 & 1 & \dots & 0 & 0 & 0 \\ . & & \dots & & & & \dots & & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \epsilon^2_{t+h-1} \\ \epsilon^2_{t+h-2} \\ . \\ \epsilon^2_{t+h-r} \\ Z^2_{t+h-2} \\ Z^2_{t+h-3} \\ . \\ Z^2_{t+h-s-1} \\ 1 \end{pmatrix}$$

or, more compactly,

$$W_{t+h} = \Gamma_{t+h} W_{t+h-1}, \tag{8}$$

where e_j refers to a column vector of zeros, except for 1 in the jth element. By repeated substitution in Equation 8 we have

$$W_{t+h} = (\Pi_{j=0}^{h-1} \Gamma_{t+h-j}) W_t. \tag{9}$$

By multiplying e_1' on both sides of Equation 9 we have

$$\epsilon_{t+h} = \left(e_1' \left(\Pi_{j=0}^{h-1} \Gamma_{t+h-j} \right) W_t \right)^{1/2}. \tag{10}$$

For $s = 1$ we have the following matrix representation:

$$\begin{pmatrix} \epsilon^2_{t+h} \\ \epsilon^2_{t+h-1} \\ . \\ \epsilon^2_{t+h-r+1} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \nu^2_{t+h-1} & \alpha_2 & \dots & \alpha_{r-1} & \alpha_r & \omega \\ 1 & 0 & \dots & 0 & 0 & 0 \\ . & . & & . & & . \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon^2_{t+h-1} \\ \epsilon^2_{t+h-2} \\ . \\ \epsilon^2_{t+h-r} \\ 1 \end{pmatrix},$$

or, more compactly,

$$V_{t+h} = G_{t+h} V_{t+h-1}.$$

Therefore, by a similar method as above, we have

$$\epsilon_{t+h} = \left(e_1' \left(\Pi_{j=0}^{h-1} G_{t+h-j} \right) V_t \right)^{1/2}, \tag{11}$$

for $h \geqslant 2$. Note that we have $E_t(\epsilon_{t+1}) = \epsilon_{t+1}$. Now, based on the observed values $Z_1, \dots, Z_t$ we estimate W_t (for $s > 1$) or V_t (for $s = 1$). Therefore, by simulating i.i.d. sequence $\{\nu_t\}$ we simulate values of ϵ_{t+h}. Now, by calculating a sample mean of the simulated values we have an estimation for $E_t(\epsilon_{t+h})$. Using the GEVStableGarch package we can estimate parameters of ν_0, and therefore we can estimate $E(|\nu_0|)$. This enables us to predict future absolute values of α-stable GARCH models by

$$E_t(|Z_{t+h}|) = E_t(\epsilon_{t+h}) E(|\nu_0|). \tag{12}$$

In the literature, there is a closed formula for $E(|\nu_0|)$ (see Samorodnitsky and Taqqu, (1994), p. 18, or Mittnik, Paolella, and Rachev, 2002). On the other hand, using the estimated parameters we can estimate $E(|\nu_0|)$ by simulation, and this estimation is different from calculating expectation using its formula. Therefore, in this paper, we calculate $E(|\nu_0|)$ by simulating the values of $|\nu_0|$ and obtaining the mean value.

In the following remark, we extend the prediction method for predicting α-stable power-GARCH models.

Remark 1. *The process $\{Z_t\}$ is called α-stable power-GARCH(r, s, ξ) if it satisfies the following equations:*

$$Z_t = \epsilon_t \nu_t \tag{13}$$

and

$$\epsilon_t^{\xi} = \omega + \sum_{i=1}^{r} \alpha_i \epsilon_{t-i}^{\xi} + \sum_{i=1}^{s} \beta_i |Z_{t-i}|^{\xi}, \tag{14}$$

where $\{\nu_t\}$ is an i.i.d. sequence of α-stable random variables, $\{\alpha_i\}$ and $\{\beta_i\}$ are as in Equation 2, $1 < \alpha < 2$ and $0 < \xi < \alpha$. The stationarity of $\alpha-$stable power-GARCH models is considered in Mittnik et al. (2002). Using the same procedure for obtaining predictors of Z_{t+h} and ϵ_{t+h} from the GARCH model in Equations 1 and 2, we can find the conditional expectations of Z_{t+h} and ϵ_{t+h} satisfying Equations 13 and 14. For $s > 1$ we can write

$$\epsilon_{t+h} = \left(e_1'(\Pi_{j=0}^{h-1}\Gamma^*_{t+h-j})W_t^*\right)^{1/\xi}, \tag{15}$$

where

$$\Gamma_t^* = \begin{pmatrix} \alpha_1+\beta_1|\nu_{t-1}|^\xi & \alpha_2 & \dots & \alpha_r & \beta_2 & \dots & \beta_s & \omega \\ 1 & 0 & \dots\ 0 & 0 & 0 & \dots & 0 & 0 \\ . & & 0 & 0 & & & & \\ 0 & & \dots\ 1 & 0 & 0 & \dots & 0 & 0 \\ |\nu_{t-1}|^\xi & 0 & \dots & & & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots\ 0 & 0 & 0 \\ . & & \dots & & & \dots & & \\ 0 & 0 & \dots\ 0 & 0 & 0 & \dots\ 1 & 0 & 0 \\ 0 & 0 & \dots\ 0 & 0 & 0 & \dots\ 0 & 0 & 1 \end{pmatrix},$$

$$W_t^* = \begin{pmatrix} \epsilon_t^\xi \\ \epsilon_{t-1}^\xi \\ . \\ \epsilon_{t-r+1}^\xi \\ |Z_{t-1}|^\xi \\ |Z_{t-2}|^\xi \\ . \\ |Z_{t-s}|^\xi \\ 1 \end{pmatrix}.$$

The method for $s = 1$ is similar to obtaining Equation 11. Now, by the same method for obtaining Equation 12, we can calculate $E_t(|Z_{t+h}|)$ for α-stable power-GARCH models.

3 | PREDICTION OF ARMA-GARCH-M MODELS

In the following theorem, we express a representation for the conditional expectation $E_t(X_{t+h})$, for $h > 0$, where $\{X_t\}$ follows an ARMA-GARCH-M model satisfying Equation 3 or 4 with finite variance innovations $\{\nu_t\}$. First, we present a nonparametric method for estimating $E_t(\epsilon_{t+h})$, for $h > 0$.

Let $\{z_t, t = 1, \dots, N\}$ be an observed sample from Equations 1 and 2. We assume that r and s are known, $E(\nu_1) = 0$ and $E(\nu_1^2) = 1$. In the following, we give a nonparametric algorithm based on a sieve bootstrap procedure for approximating $E_N(\epsilon_{N+h})$ (see Chen, Gel, Balakrishna, & Abraham, 2011). Let $\kappa_t = z_t^2 - \epsilon_t^2$ and $m = \max(r, s)$. The sequence $\{\kappa_t, t \in \mathbb{Z}\}$ is an uncorrelated sequence and it satisfies the following system:

$$z_t^2 = \omega + \sum_{i=1}^m (\alpha_i+\beta_i)z_{t-i}^2 + \kappa_t - \sum_{i=1}^s \alpha_i\kappa_{t-j}. \tag{16}$$

- Step 1. Using the ARMA representation in Equation 16 and the least square method, estimate the coefficients $\hat{\omega}_0$, $\widehat{\alpha_1+\beta_1}, \dots, \widehat{\alpha_m+\beta_m}, \hat{\alpha}_1, \dots, \hat{\alpha}_s$, and then compute $\hat{\beta}_i = \widehat{\alpha_i+\beta_i} - \hat{\alpha}_i, i = 1, \dots, s$, where $\hat{\beta}_i = 0$, for $i = s+1, \dots, m$.
- Step 2. Extract the squared volatilities $\{\epsilon_t^{2*}, t = 1, \dots, N\}$ using

$$\epsilon_t^{2*} = \hat{\omega} + \sum_{i=1}^s \hat{\beta}_i z_{t-i}^2 + \sum_{i=1}^r \hat{\alpha}_i \epsilon_{t-i}^{2*}, \quad t = m+1 \dots, N,$$

  where $\epsilon_t^{2*} = \hat{\omega}/(1 - \sum_{i=1}^m \widehat{(\alpha_i+\beta_i)})$ for $t = 1, \dots, m$. We use the estimation of $E(\epsilon_t^2)$ for setting initial values ϵ_t^{2*}, $t = 1, \dots, m$.
- Step 3. Construct $\nu_t^{2*} = z_t^2/\epsilon_t^{2*}$, $t = m+1, \dots, N$. The empirical distribution of the centered squared residuals

$$\tilde{\nu}_t^{2*} = \left(\hat{\nu}_t^{2*} - \frac{1}{N-m}\sum_{t=m+1}^N \hat{\nu}_t^{2*}\right), \ t = m+1 \dots, N,$$

  is

$$F_{\tilde{\nu}_t^{2*}}(y) = \sum_{t=m+1}^N I(\tilde{\nu}_t^{2*} \leqslant y),$$

  where $I(.)$ denotes to the indicator function.
- Step 4. Sample with replacement from $F_{\tilde{\nu}_t^{2*}}(y)$ to obtain the bootstrap process $\{\nu_{N+s}^{2*}, s = 1, \dots, h\}$. Then calculate Equation 10 or 11.
- Step 5. Repeat step 4 B times.

In the following theorem, we present the conditional expectation of ARMA-GARCH-M models.

Theorem 2. *Let $\{X_t\}$ be an ARMA-GARCH-M(p, q, r, s) model satisfying Equation 3 or 4. Let*

$$F_t = \begin{pmatrix} X_t \\ X_{t-1} \\ . \\ X_{t-p+1} \\ Z_t \\ Z_{t-1} \\ . \\ Z_{t-q+1} \\ 1 \end{pmatrix}, \quad and \quad \Gamma = \begin{pmatrix} \phi_1 & \dots & \phi_p & \theta_1 & \dots & \theta_q & \varphi \\ 1 & \dots\ 0 & 0 & 0 & \dots & 0 & 0 \\ . & 0 & 0 & & & & 0 \\ 0 & \dots\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & & & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 & \dots\ 0 & 0 & 0 \\ . & \dots & & & \dots & & 0 \\ 0 & \dots\ 0 & 0 & 0 & \dots\ 1 & 0 & 0 \\ 0 & \dots\ 0 & 0 & 0 & \dots\ 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

Then

$$E_t(X_{t+h}) = \begin{cases} e_1'\Gamma^h F_t + \delta\sum_{i=0}^{h-1} e_1'\Gamma^i e_1 E_t(\epsilon_{t+h-i}), \text{under model } (3) \\ e_1'\Gamma^h F_t + \delta\sum_{i=0}^{h-1} e_1'\Gamma^i e_1 E_t(\epsilon_{t+h-i}^2), \text{under model } (4). \end{cases} \tag{18}$$

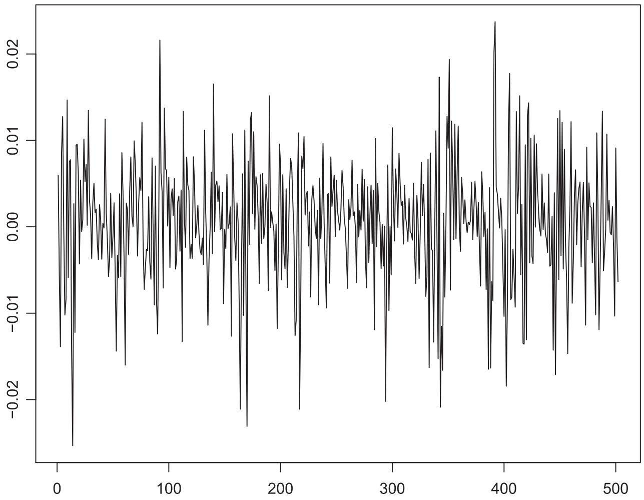

FIGURE 1 S&P 500 stock index daily log-return in the period May 31, 2013 to May 29, 2015

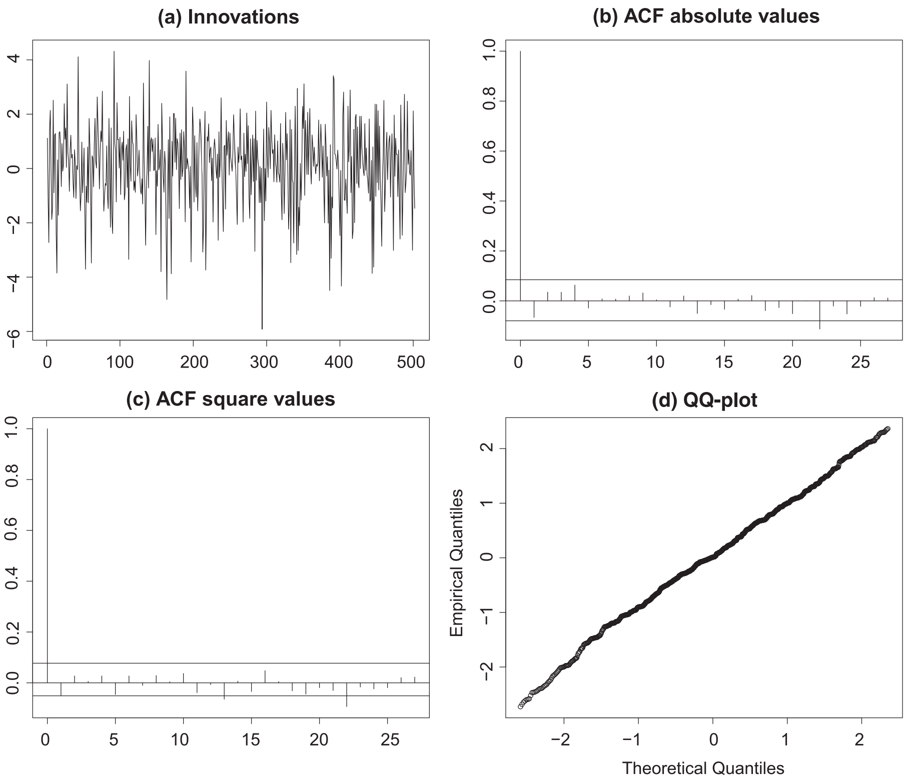

FIGURE 2 (a) Residuals $\{\hat{v}_t\}$, (b) sample ACF for the absolute values of mean-corrected $\{\hat{v}_t\}$, (c) sample ACF for the squares of mean-corrected $\{\hat{v}_t\}$, and (d) stable QQ-plot for $\{\hat{v}_t\}$

Proof. *For the model in Equation 3 we have the following matrix representation:*

$$F_{t+h} = \Gamma F_{t+h-1} + (e_1 + e_{p+1})Z_{t+h} + e_1 \delta \epsilon_{t+h}. \tag{19}$$

By repeated substitution in Equation 19 we have

$$\begin{aligned} F_{t+h} = &\sum_{i=0}^{h-1} \Gamma^i((e_1 + e_{p+1})Z_{t+h-i}) + \Gamma^h F_t \\ &+ \delta \sum_{i=0}^{h-1} \Gamma^i e_1 \epsilon_{t+h-i}, \end{aligned} \tag{20}$$

where Γ^0 denotes a $(p+q+1) \times (p+q+1)$ identity matrix. We have $Z_{t+h} = e'_{p+1} F_{t+h}$ and $E_t(Z_{t+h}) = 0$ for $h > 0$. Therefore, multiplying e'_1 on both sides of Equation 20 and taking the conditional expectation on both sides imply Equation 18 under model 3. Using a similar procedure, we can find the conditional expectation under model 4. □

In practice, based on the observed values $Z_1, \dots, Z_t$, we estimate F_t and also, using the stated algorithm in this section, we approximate $E_t(\epsilon_{t+h-i})$, for $i = 0, \dots, h-1$ in Equation 20.

Under model 4 we need to estimate $E_t(\epsilon^2_{t+h-i})$, $i = 1, \dots, h-1$. Baillie and Bollerslev (1992) present an expression for the conditional mean of square returns, which is stated in the following.

Theorem 3. *(Baillie & Bollerslev, 1992) Let $\{Z_t\}$ be a GARCH(r,s) process satisfying Equations 1 and 2. Then*

$$E_t(Z_{t+h}^2) = \kappa_h + \sum_{i=0}^{r-1} \zeta_{i,h}\epsilon_{t-i}^2 + \sum_{i=0}^{m-1} \eta_{i,h} Z_{t-i}^2, \quad for \ h \geqslant 1,$$

where $m = \max(r, s)$,

$$\Pi = \begin{pmatrix} \alpha_1+\beta_1 & \dots & & \alpha_m+\beta_m & -\alpha_1 & \dots & & -\alpha_r \\ 1 & \dots & 0 & 0 & 0 & \dots & & 0 \\ . & \dots & & & & \dots & & \\ 0 & \dots & 1 & 0 & 0 & \dots & & 0 \\ 0 & \dots & & & & \dots & & 0 \\ 0 & \dots & & 0 & 1 & \dots & 0 & 0 \\ . & \dots & & & & \dots & & \\ 0 & \dots & & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

and

$$\begin{aligned} \kappa_h &= e_1'(I + \Pi + \cdots + \Pi^{h-1})e_1\omega, \\ \zeta_{i,h} &= -e_1'\Pi^h e_{m+i+1}, & i &= 0, \dots, r-1, \\ \eta_{i,h} &= e_1'\Pi^h(e_{i+1} + e_{m+i+1}), & i &= 0, \dots, r-1, \\ \eta_{i,h} &= e_1'\Pi^h e_{i+1}, & i &= r, \dots, m-1. \end{aligned}$$

4 | EMPIRICAL STUDY

In this section, we consider two empirical datasets. First, we use an α-stable power-GARCH model for the S&P 500 stock market. Then, using the introduced method in Section 2, we predict the future values of returns and volatilities. Second, we use the presented method in Section 3 for predicting wind speed data, which can be modeled by an ARMA-GARCH-M process.

4.1 | Financial data

In this section, we demonstrate h-step-ahead prediction method using S&P 500 stock index daily log-returns in the period May 31, 2013 to May 29, 2015. The data length is 502 and its time plot is shown in Figure 1.

A popular model for log-returns of stock markets is a GARCH model. Because of high fluctuations for data in Figure 1, we guess that a model with α-stable innovations would be more appropriate compared to Gaussian innovations. We show that an α-stable power-GARCH model is a satisfactory model for data, and it is more appropriate than the GARCH model with Gaussian innovations.

First, by using the GEVStableGarch package in R (see Sousa, Otiniano, Lopes, & Wuertz, 2015), we fit an α-stable power-GARCH(1,1,1) model to the data and extract its mean-corrected residual process $\{\hat{v}_t\}$.

Before calculating prediction values, we check the appropriateness for assuming an i.i.d. α-stable innovation process. Figure 2(b,c) shows the correlograms of $|\hat{v}_t|$ and $\hat{v}_t^2$, respectively. Now, assuming the independence of $\{\hat{v}_t\}$, we obtain the

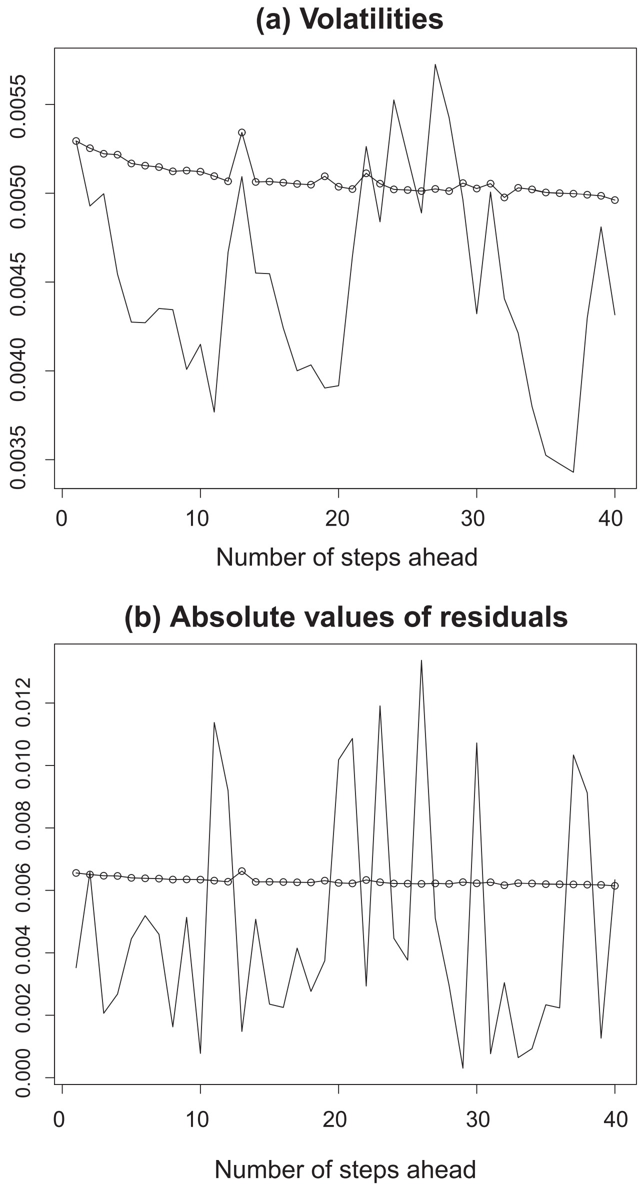

FIGURE 3 (a) Smooth line displays $\{\epsilon_t, t = 463, \dots, 502\}$ and dotted line displays the corresponding predicted values. (b) Smooth line displays $\{|Z_t|, t = 463, \dots, 502\}$ and dotted line displays the corresponding predicted values

estimates of α-stable parameters:

$$(1.8559, -0.9899,\ 1,\ 0), \tag{21}$$

for the tail index and skewness intensity, respectively. Note that we consider standard α-stable distributions and, therefore, scale and location equal 1 and 0, respectively. In general, if $\{X_t\}$ is an i.i.d. sequence from a regularly varying tail distribution with tail index α less than 2, then

$$(n/\log(n))^{1/\alpha} \sum_{t=1}^{n-h} X_t X_{t+h} \Big/ \sum_{t=1}^{n} X_t^2,\ \ h > 0,$$

has an α-stable distribution for enough large n (see Davis & Resnick, 1985). Based on this fact, smooth lines in Figure 2(b, c) show 95% confidence intervals for the autocorrelation function (ACF) of $|\hat{v}_t|$ and $\hat{v}_t^2$, respectively. We obtained these bands by simulating 100,000 independent sample ACFs for the absolute values and squares of 502 mean-corrected i.i.d. stable random variables with parameters in Equation 21. Since all values of ACF are in the confidence intervals except lag 22 for absolute values and lags 13 and 22 for square values, which are not far from the bands, the residuals appear approximately i.i.d., and so we conclude that an α-stable power-GARCH(1,1,1) model is satisfactorily fitted for the series Figure 2(b,c). A QQ-plot, with empirical quantiles for the residuals plotted against theoretical quantiles of an α-stable distribution with parameters in Equation 21, is given in Figure 2(d). Because the QQ-plot is remarkably linear, it appears reasonable to model the i.i.d. noise $\{v_t\}$ as an α-stable sequence with parameters in Equation 21. Moreover, we test the following hypothesis using a nonparametric test:

H_0: $\{v_t\}$ comes from an α-stable distribution with parameters in Equation 21.

The Kolmogorov–Smirnov test is nonparametric and shows the p-value of 0.6357. Therefore, we accept H_0 and conclude that an α-stable power-GARCH(1,1,1) model is an appropriate model for the data. If we consider the GARCH(1,1) model in Equations 1 and 2 with Gaussian innovations, then we obtain the p-value of 0.1557.

For evaluating the appropriateness of the prediction method we consider the last 40 values of data. Figure 3 shows the values of $\{\epsilon_t, t = 463, \dots, 502\}$ and absolute values of log-returns $\{Z_t, t = 463, \dots, 502\}$ with corresponding h-step-ahead prediction values. We obtain the predicted values using Section 2. The values of mean square errors (MSE) for volatilities and absolute values of returns are obtained 6.42×10^{-7} and 1.31×10^{-5}, respectively.

4.2 | Wind speed data

Liu, Erden, and Shi (2011) show the appropriateness of ARMA-GARCH-M to model the mean and volatility of wind speed data. In this section, we use the hourly wind speed data in Colorado, USA, from February 1, 2015 to February 28, 2015. Data are available at https://www.ncdc.noaa.gov/societal-impacts/wind. Figure 4 displays data with length 1,084. We use ARMA-GARCH-M-II for data. As in Liu, Erden, and Shi (2011), we can use GARCH(1,1) for volatility, and a Gaussian distribution for the innovations $\{v_t\}$. For obtaining orders of mean equation we use p and q in $\{1, \dots, 10\}$ such that they produce the highest p-value for the following hypothesis:

$$H_0 : \{v_t\} \text{ comes from a Gaussian distribution.}$$

The Kolmogorov–Smirnov test with $p = q = 10$ gives a p-value of 0.8809. Similar to the discussion in Section 4.1, we find that an ARMA-GARCH-M-II(1,1,10,10) model with Gaussian innovations is appropriate for wind speed data. Now, using Equation 18 we predict the last 44 observations in terms of the past values $\{X_1, \dots, X_{1040}\}$. Figure 5 shows the values $\{X_{1041}, \dots, X_{1084}\}$ with the corresponding predicted values, and the MSE value is 14.7808.

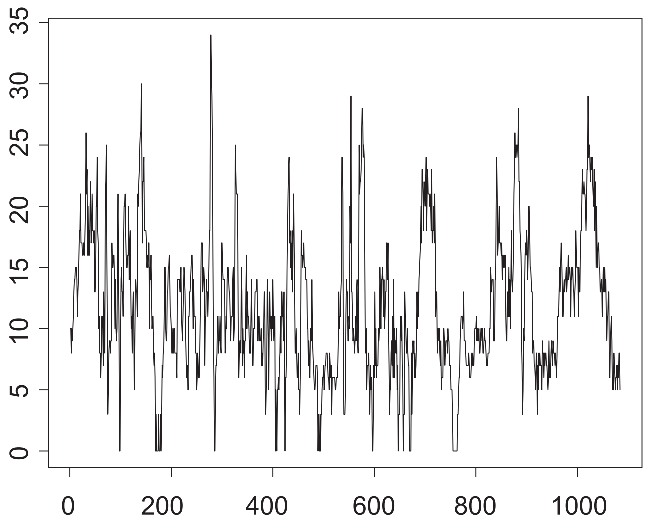

FIGURE 4 Hourly wind speed data in the period 1 February 2015 to 28 February 2015

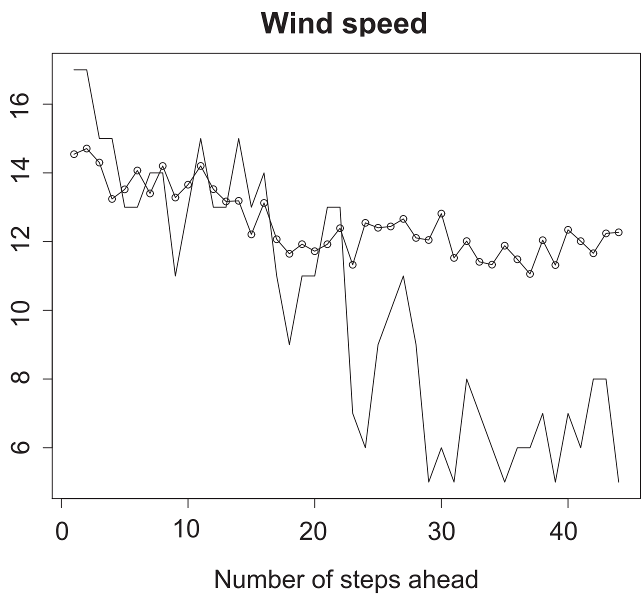

FIGURE 5 Smooth line displays $\{X_t, t = 1041, \dots, 1084\}$ and dotted line displays the corresponding predicted values

5 | CONCLUSIONS

In this paper, we consider the best prediction of three modifications of GARCH models: α-stable GARCH, α-stable power-GARCH, and ARMA-GARCH-M. These models are especially applicable to financial data and wind energy data. In this paper, we present the best predictions based on the conditional expectations for these models in terms of infinite past values. Moreover, a sufficient condition for stationarity of α-stable GARCH models is presented. To the best of our knowledge, the best predictions for α-stable GARCH and α-stable power-GARCH models have not yet been considered, and the presented prediction method for ARMA-GARCH-M models is easy to calculate. We show the appropriateness of the methods by applying the models to financial market and wind speed data.

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