S-Parameters for Three and Four Cascaded Two-Ports

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Abstract

This paper presents explicit formulas for the resultant S-parameters for cascaded three and four two-ports in terms of the S-parameters of the individual two-ports. The formulas are derived in two ways (multiplication of the T-parameters and signal flow graph) giving the same result. They are also confirmed by simulation.

1. Introduction

The scattering parameters are fundamental in the characterization of electrical devices at high frequencies. They are particularly useful for analyzing and designing multiport high frequency and microwave networks. However, the S-parameters are not directly suitable for the analysis of a cascaded network of two or more two-ports. Such cascaded networks are usually analyzed by multiplying the individual matrices using the ABDC or T-parameters. Since many microwave networks consists of cascaded connections of both passive and active elements, it is desirable to have explicit formulas directly involving the S-parameters. Such formulas are useful for analyzing waveguide and microstrip discontinuities. As far as the authors are aware, such formulas are only available for two cascaded two-ports[1,2]. In this paper, we provide the formulas for three and four cascaded two-ports.

The problem of determining the S-parameters for cascaded two-ports can be solved in three ways: (1) multiplying the ABCD matrices of the three or four two-ports, (2) multiplying the transfer scattering matrices, (3) using the signal flow graph (SFG) to derive the overall S-parameters. Using the ABCD matrices will involve knowing in advance the reference characteristic impedance Z_o. Therefore, we choose to use the last two approaches.

2. Background

For a single two-port network, the incident voltage waves a1 and a2 are related to the reflected waves b1 and b2 as:

$$b_1 = S_{11}a_1 + S_{12}a_2 \qquad (1a)$$

$$b_2 = S_{21}a_1 + S_{22}a_2 \qquad (1b)$$

We now relate the S-parameters to the chain transfer parameters (also known as the transfer scattering parameters) or simply T-parameters, which are suitable for the analysis of cascaded two-ports. It should be noted that there are different definitions of the T-parameter in the literature [3-5]. Here we follow the type described by Hewlett Parkard [4] since it is the most common.

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \qquad (2)$$

By manipulating the matrices in eqs. (1) and (2), we can readily show that

$$T = \begin{pmatrix} -\dfrac{\Delta S}{S_{21}} & \dfrac{S_{11}}{S_{21}} \\ -\dfrac{S_{22}}{S_{21}} & \dfrac{1}{S_{21}} \end{pmatrix} \qquad S = \begin{pmatrix} \dfrac{T_{12}}{T_{22}} & \dfrac{\Delta T}{T_{22}} \\ \dfrac{1}{T_{22}} & -\dfrac{T_{21}}{T_{22}} \end{pmatrix} \qquad (3)$$

where $\Delta S = S_{11}S_{22} - S_{12}S_{21}$, the determinant of the S-matrix and $\Delta T = T_{11}T_{22} - T_{12}T_{21}$, the determinant of the T-matrix.

3. Derivation

With that preliminary background, we now consider three cascaded two-ports as shown in Fig. 1. The chain matrix of the overall cascade connection can be written as

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = T \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \qquad (4)$$

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where

$$T = T_a T_b T_c \qquad (5a)$$

or

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} T_{11}^a & T_{12}^a \\ T_{21}^a & T_{22}^a \end{pmatrix} \begin{pmatrix} T_{11}^b & T_{12}^b \\ T_{21}^b & T_{22}^b \end{pmatrix} \begin{pmatrix} T_{11}^c & T_{12}^c \\ T_{21}^c & T_{22}^c \end{pmatrix} \qquad (5b)$$

For example,

$$T_{22} = T_{21}^a T_{11}^b T_{12}^c + T_{21}^a T_{12}^b T_{22}^c + T_{22}^a T_{21}^b T_{12}^c + T_{22}^a T_{22}^b T_{22}^c \qquad (6)$$

Substituting for the individual T-matrices using eq. (3) yields

$$\begin{aligned} S_{21} &= \frac{1}{T_{22}} = \frac{1}{T_{21}^a T_{11}^b T_{12}^c + T_{21}^a T_{12}^b T_{22}^c + T_{22}^a T_{21}^b T_{12}^c + T_{22}^a T_{22}^b T_{22}^c} \\ &= \frac{1}{-\frac{S_{22}^a}{S_{21}^a}\left(-\frac{\Delta S^b}{S_{21}^b}\right)\frac{S_{11}^c}{S_{21}^c} - \frac{S_{22}^a}{S_{21}^a}\frac{S_{11}^b}{S_{21}^b}\frac{1}{S_{21}^c} - \frac{1}{S_{21}^a}\frac{S_{22}^b}{S_{21}^b}\frac{S_{11}^c}{S_{21}^c} + \frac{1}{S_{21}^a}\frac{1}{S_{21}^b}\frac{1}{S_{21}^c}} \qquad (7a) \\ &= \frac{S_{21}^a S_{21}^b S_{21}^c}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c + S_{22}^a \Delta S^b S_{11}^c\right]} \end{aligned}$$

where $\Delta S^b = S_{11}^b S_{22}^b - S_{12}^b S_{21}^b$, the determinant of the S^b-matrix. By taking a similar approach for other elements of the overall S-matrix, we obtain

$$S_{11} = \quad S_{11}^a + \frac{S_{12}^a S_{21}^a (S_{11}^b - \Delta S^b S_{11}^c)}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c + S_{22}^a \Delta S^b S_{11}^c\right]} \qquad (7b)$$

$$S_{12} = \quad \frac{S_{12}^a S_{12}^b S_{12}^c}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c + S_{22}^a \Delta S^b S_{11}^c\right]} \qquad (7c)$$

$$S_{22} = \quad S_{22}^c + \frac{(S_{22}^b - S_{22}^a \Delta S^b) S_{12}^c S_{21}^c}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c + S_{22}^a \Delta S^b S_{11}^c\right]} \qquad (7d)$$

As mentioned earlier, the signal flow graph (SFG) can be used to derive the same formulas. SFG is a method of writing a set of equations, whereby the related variables are represented by points and the interrelations by directed lines giving a direct picture of signal flow. The main advantage of such a graphical technique in solving cascaded networks are the convenient pictorial representation and the painless method of proceeding directly to the solution from the graph. The cascaded connection of three two-ports in Fig. 1 is represented by the flow graph of Fig. 2. With Fig. 2, we can readily derive eq. (7).

Similar steps can be taken to obtain the S-parameters for four cascaded two-ports. The results are

$$S_{11} = \quad S_{11}^a + \frac{S_{12}^a S_{21}^a \left(S_{11}^b - \Delta S^b S_{11}^c + \Delta S^b \Delta S^c S_{11}^d - S_{11}^b S_{22}^c S_{11}^d\right)}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c - S_{22}^c S_{11}^d + S_{22}^a \Delta S^b S_{11}^c - S_{22}^a \Delta S^b \Delta S^c S_{11}^d + S_{22}^b \Delta S^c S_{11}^d + S_{22}^a S_{11}^b S_{22}^c S_{11}^d\right]} \qquad (8a)$$

$$S_{12} = \quad \frac{S_{12}^a S_{12}^b S_{12}^c S_{12}^d}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c - S_{22}^c S_{11}^d + S_{22}^a \Delta S^b S_{11}^c - S_{22}^a \Delta S^b \Delta S^c S_{11}^d + S_{22}^b \Delta S^c S_{11}^d + S_{22}^a S_{11}^b S_{22}^c S_{11}^d\right]} \qquad (8b)$$

$$S_{21} = \quad \frac{S_{21}^a S_{21}^b S_{21}^c S_{21}^d}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c - S_{22}^c S_{11}^d + S_{22}^a \Delta S^b S_{11}^c - S_{22}^a \Delta S^b \Delta S^c S_{11}^d + S_{22}^b \Delta S^c S_{11}^d + S_{22}^a S_{11}^b S_{22}^c S_{11}^d\right]} \qquad (8c)$$

$$S_{22} = \quad S_{22}^d + \frac{S_{12}^d S_{21}^d \left(S_{22}^c - S_{22}^b \Delta S^c + S_{22}^a \Delta S^b \Delta S^c - S_{22}^a S_{11}^b S_{22}^c\right)}{\left[1 - S_{22}^a S_{11}^b - S_{22}^b S_{11}^c - S_{22}^c S_{11}^d + S_{22}^a \Delta S^b S_{11}^c - S_{22}^a \Delta S^b \Delta S^c S_{11}^d + S_{22}^b \Delta S^c S_{11}^d + S_{22}^a S_{11}^b S_{22}^c S_{11}^d\right]} \qquad (8d)$$

where $\Delta S^c = S^c_{11}S^c_{22} - S^c_{12}S^c_{21}$, the determinant of the S^c-matrix

4. Conclusion

Explicit formulas for finding the composite S-parameters for three and four cascaded two-ports have been derived in this letter. The formulas are derived in two ways using direct multiplication of the T-parameters and signal flow graph and verified by simulation.

5. References

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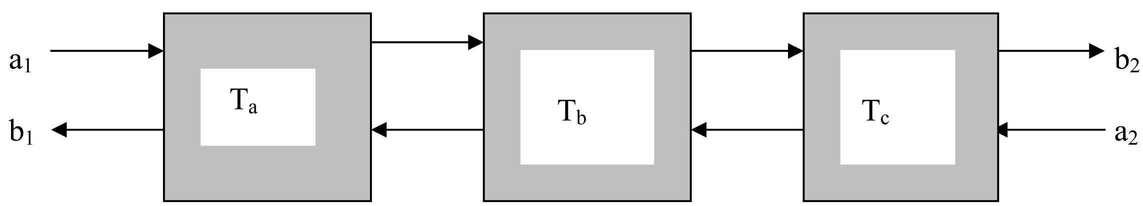

Fig. 1 Cascaded connection of three two-ports.

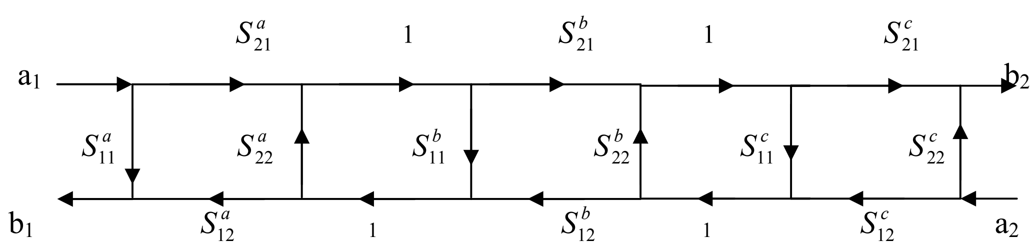

Fig. 2 Signal flow graph for three cascaded two-ports.