to the shop at the locations designated by circles a, b, c, and d. The objective is to assign the new centers to the proposed locations in a manner that will minimize the total materials handling traffic between the existing centers and the proposed ones. Table 5.43 summarizes the frequency of trips between the new centers and the old ones. Materials handling equipment travels along the rectangular aisles intersecting at the locations of the centers. For example, the one-way travel distance (in meters) between center 1 and location b is $30 + 20 = 50$ m.

TABLE 5.43

		New center			
		I	II	III	IV
Existing center	1	10	2	4	3
	2	7	1	9	5
	3	0	8	6	2
	4	11	4	0	7

5.4.2 Simplex Explanation of the Hungarian Method

The assignment problem in which n workers are assigned to n jobs can be represented as an LP model in the following manner: Let c_{ij} be the cost of assigning worker i to job j, and define

$$x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is assigned to job } j \\ 0, & \text{otherwise} \end{cases}$$

Then the LP model is given as

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1, \ i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \ j = 1, 2, \ldots, n$$

$$x_{ij} = 0 \text{ or } 1$$

The optimal solution of the preceding LP model remains unchanged if a constant is added to or subtracted from any row or column of the cost matrix (c_{ij}). To prove this point, let p_i and q_j be constants subtracted from row i and column j. Thus, the cost element c_{ij} is changed to

$$c'_{ij} = c_{ij} - p_i - q_j$$

Now

$$\sum_i \sum_j c'_{ij} x_{ij} = \sum_i \sum_j (c_{ij} - p_i - q_j) x_{ij} = \sum_i \sum_j c_{ij} x_{ij} - \sum_i p_i\left(\sum_j x_{ij}\right) - \sum_j q_j\left(\sum_i x_{ij}\right)$$
$$= \sum_i \sum_j c_{ij} x_{ij} - \sum_i p_i(1) - \sum_j q_j(1)$$
$$= \sum_i \sum_j c_{ij} x_{ij} - \text{constant}$$

Because the new objective function differs from the original by a constant, the optimum values of x_{ij} must be the same in both cases. The development thus shows that steps 1 and 2 of the Hungarian method, which call for subtracting p_i from row i and then subtracting q_j from column j, produce an equivalent assignment model. If a feasible solution can be found among the created zero entries of the cost matrix created by steps 1 and 2, then it must be optimum because the cost in the modified matrix cannot be less than zero.

If the created zero entries cannot yield a feasible solution (as Example 5.4-2 demonstrates), then step 2a (dealing with the covering of the zero entries) must be applied. The validity of this procedure is again rooted in the simplex method of linear programming and can be explained by duality theory (Chapter 4) and the complementary slackness theorem (Chapter 7). We will not present the details of the proof here because they are somewhat involved.

The reason $(p_1 + p_2 + \dots + p_n) + (q_1 + q_2 + \dots + q_n)$ gives the optimal objective value is that it represents the dual objective function of the assignment model. This result can be seen through comparison with the dual objective function of the transportation model given in Section 5.3.4. (See Bazaraa et al. [1990], pp. 499–508 for the details.)

5.5 THE TRANSSHIPMENT MODEL

The transshipment model recognizes that it may be cheaper to ship through intermediate or ***transient*** nodes before reaching the final destination. This concept is more general than that of the regular transportation model, where direct shipments only are allowed between a source and a destination.

This section shows how a transshipment model can be converted to (and solved as) a regular transportation model using the idea of a **buffer**.

Example 5.5-1

Two automobile plants, $P1$ and $P2$, are linked to three dealers—$D1$, $D2$, and $D3$—by way of two distribution centers, $T1$ and $T2$, according to the network shown in Figure 5.9. The supply amounts at plants $P1$ and $P2$ are 1000 and 1200 cars, and the demand amounts at dealers $D1$, $D2$, and $D3$, are 800, 900, and 500 cars. The shipping cost per car (in hundreds of dollars) between pairs of nodes is shown on the connecting links (or arcs) of the network.

Transshipment occurs in the network in Figure 5.9 because the entire supply amount of 2200 (=1000 + 1200) cars from nodes $P1$ and $P2$ could potentially pass

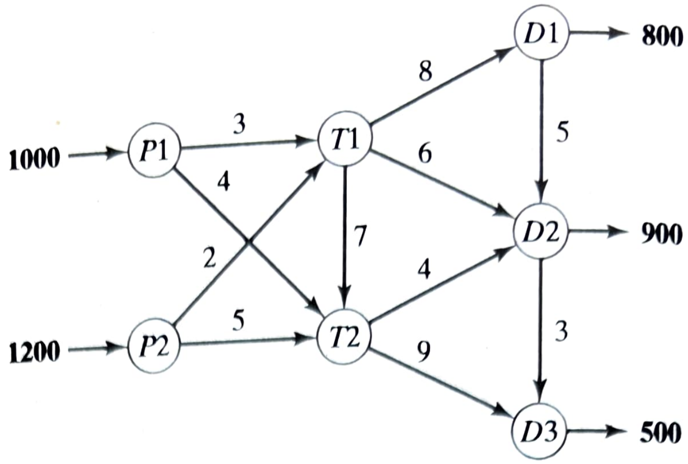

FIGURE 5.9
Transshipment network between plants and dealers

through any node of the network before ultimately reaching their destinations at nodes $D1$, $D2$, and $D3$. In this regard, the nodes of the network with both input and output arcs ($T1$, $T2$, $D1$, and $D2$) act as both sources and destinations, and are referred to as **transshipment nodes**. The remaining nodes are either **pure supply nodes** ($P1$) or **pure demand nodes** ($D3$). The transshipment model can be converted into a regular transportation model with 6 sources ($P1$, $P2$, $T1$, $T2$, $D1$, and $D2$) and five destinations ($T1$, $T2$, $D1$, $D2$, and $D3$). The amounts of supply and demand at the different nodes are computed as

Supply at a *pure supply node* = Original supply

Demand at a *pure demand node* = Original demand

Supply at a *transshipment node* = Original supply + Buffer amount

Demand at a *transshipment node* = Original demand + Buffer amount

The buffer amount should be sufficiently large to allow the entire *original* supply (or demand) units to pass through any of the *transshipment* nodes. Let B be the desired buffer amount, then

$$\begin{aligned} B &= \text{Total supply (or demand)} \\ &= 1000 + 1200 \text{ (or } 800 + 900 + 500) \\ &= 2200 \text{ cars} \end{aligned}$$

Using the buffer B and the unit shipping costs given in the network, the equivalent regular transportation model is constructed as in Table 5.44.

TORA solution of the transportation model (file ch5ToraTransshipEx5-5-1.txt) is shown in Figure 5.10. Note the effect of transshipment: Dealer $D2$ receives 1400 cars, keeps 900 cars to satisfy its demand, and sends the remaining 500 cars to dealer $D3$.

PROBLEM SET 5.5A[8]

1. The network in Figure 5.11 gives the shipping routes from nodes 1 and 2 to nodes 5 and 6 by way of nodes 3 and 4. The unit shipping costs are shown on the respective arcs.

[8]You are encouraged to use TORA, Excel Solver, or LINGO to solve the problems in this set.

TABLE 5.44

	T1	T2	D1	D2	D3	
P1	3	4	M	M	M	1000
P2	2	5	M	M	M	1200
T1	0	7	8	6	M	B
T2	M	0	M	4	9	B
D1	M	M	0	5	M	B
D2	M	M	M	0	3	B
	B	**B**	**800 + B**	**900 + B**	**500**	

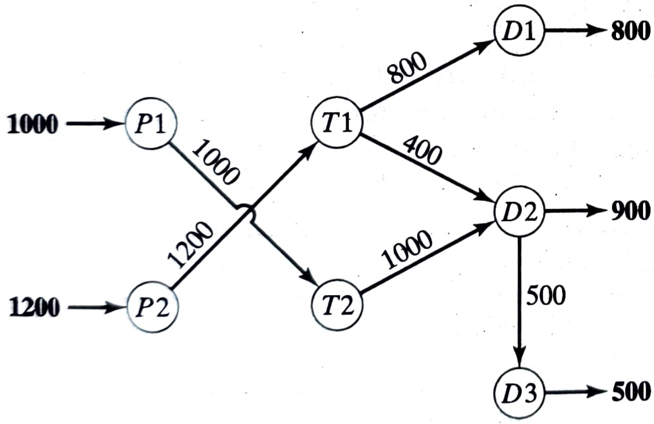

FIGURE 5.10
Solution of the transshipment model

(a) Develop the corresponding transshipment model.

(b) Solve the problem, and show how the shipments are routed from the sources to the destinations.

2. In Problem 1, suppose that source node 1 can be linked to source node 2 with a unit shipping cost of \$1. The unit shipping cost from node 1 to node 3 is increased to \$5. Formulate the problem as a transshipment model, and find the optimum shipping schedule.

3. The network in Figure 5.12 shows the routes for shipping cars from three plants (nodes 1, 2, and 3) to three dealers (nodes 6 to 8) by way of two distribution centers (nodes 4 and 5). The shipping costs per car (in \$100) are shown on the arcs.

(a) Solve the problem as a transshipment model.

(b) Suppose that distribution center 4 can sell 240 cars directly to customers. Find the new optimum solution.

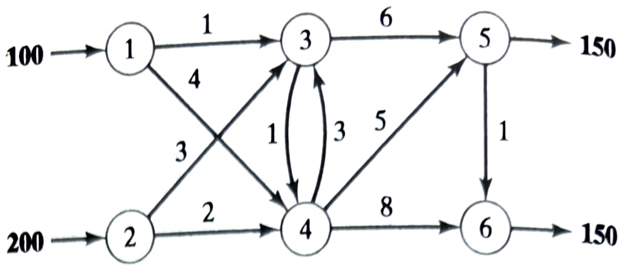

FIGURE 5.11
Network for Problem 1, Set 5.5a

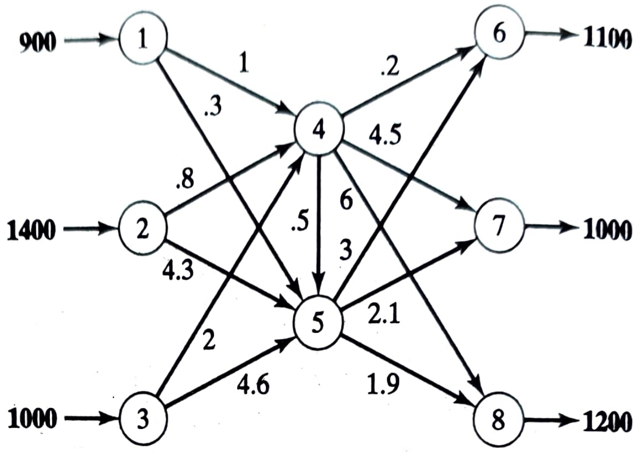

FIGURE 5.12
Network for Problem 3, Set 5.5a

4. Consider the transportation problem in which two factories supply three stores with a commodity. The number of supply units available at sources 1 and 2 is 200 and 300; those demanded at stores 1, 2, and 3 are 100, 200, and 50, respectively. Units may be transshipped among the factories and the stores before reaching their final destination. Find the optimal shipping schedule based on the unit costs in Table 5.45.

TABLE 5.45

		Factory 1	Factory 2	Store 1	Store 2	Store 3
Factory	1	\$0	\$6	\$7	\$8	\$9
	2	\$6	\$0	\$5	\$4	\$3
Store	1	\$7	\$2	\$0	\$5	\$1
	2	\$1	\$5	\$1	\$0	\$4
	3	\$8	\$9	\$7	\$6	\$0

5. Consider the oil pipeline network shown in Figure 5.13. The different nodes represent pumping and receiving stations. Distances in miles between the stations are shown on the network. The transportation cost per gallon between two nodes is directly proportional to the length of the pipeline. Develop the associated transshipment model, and find the optimum solution using TORA.

6. *Shortest-Route Problem*. Find the shortest route between nodes 1 and 7 of the network in Figure 5.14 by formulating the problem as a transshipment model. The distance between the different nodes in shown on the network. (*Hint:* Assume that node 1 has a net supply of 1 unit, and node 7 has a net demand also of 1 unit.)

3.10 THE TRANS-SHIPMENT PROBLEM

The transportation problem assumes that direct routes exist from each source to each destination. However, there are situations in which units may be shipped from one source to another or to other destinations before reaching their final destination. This is called a trans-shipment problem. For example, movement of material involving two different modes of transport—road and railways or between stations connected by broad gauge and metre gauge lines will necessarily require trans-shipment. For the purpose of trans-shipment the distinction between a source and destination is dropped so that a transportation problem with m sources and n destinations gives rise to a trans-shipment problem with $m + n$ sources and $m + n$ destinations. The basic feasible solution to such a problem will involve $[(m + n) + (m + n) - 1]$ or $2m + 2n - 1$ basic variables and if we omit the variables appearing in the $(m + n)$ diagonal cells, we are left with $m + n - 1$ basic variables.

In the trans-shipment problem, as each source or destination is a potential point of supply as well as demand, the total supply, say of N units, is added to the actual supply of each source, as well as to the actual demand at each destination. Also the 'demand' at each source and 'supply' at each destination are set equal to N.

Therefore, we may assume the supply and demand of each location to be fictitious one. These quantities (N) may be regarded as *buffer stocks* and each of these buffer stocks should at least be equal to the total supply/demand in the given problem.

The given trans-shipment problem can, therefore, be regarded as the extended transportation problem and can hence be solved by the transportation technique. In the final solution, units transported from a point to itself *i.e.*, in diagonal cells are ignored as they do not have any physical meaning since there is no transportation involved.

EXAMPLE 3.10-1

A firm has two factories X and Y and three retail stores A, B and C. The number of units of a product available at factories X and Y are 200 and 300 respectively, while demanded at retail stores are 100, 150 and 250 respectively. Rather than shipping directly from sources to destinations, it is decided to investigate the possibility of trans-shipment. Find the optimal shipping schedule. The transportation costs in rupees per unit are given below.

Table 3.217

		Factory		Retail store		
		X	Y	A	B	C
Factory	X	0	6	7	8	9
	Y	6	0	5	4	3
Retail store	A	7	2	0	5	1
	B	1	5	1	0	4
	C	8	9	7	6	0

Solution. For this trans-shipment problem, buffer stock = total supply = total demand = 500 units. Adding 500 units to each supply/demand point, we get table 3.218. Initial b.f.s. obtained by the Vogel's approximation method is also shown.

Table 3.218

To

From	X	Y	A	B	C	Supply
X	0 (500)	6	7 (200)	8	9	700
Y	6	0 (500)	5	4 (50)	3 (250)	800
A	7	2	0 (400)	5 (100)	1	500
B	1	5	1	0 (500)	4	500
C	9	9	7	6	0 (500)	500
Demand	500	500	600	650	750	

The optimal solution obtained after one iteration is given in table 3.219. The diagonal allocations in this table may be ignored since they have no physical meaning. The remaining allocations may be interpreted as follows:

Table 3.219

To

From	X	Y	A	B	C
X	0 (500)	6	7 (100)	8 (100)	9
Y	6	0 (500)	5	4 (50)	3 (250)
A	7	2	0 (500)	5	1
B	1	5	1	0 (500)	4
C	8	9	7	6	0 (500)

Factory X supplies 100 units each to retail stores A and B. Factory Y supplies 50 units to retail store B and 250 units to C.

Exs 70, 71

EXAMPLE 3.10-2

Table 3.220 represents the supply from the plants, the requirement at the distribution centres and the unit transportation costs.

Table 3.220

Distribution centres

Plants	*A*	*B*	*C*	*Supply*
1	*11*	*13*	*25*	*150*
2	*13*	*15*	*35*	*300*
Requirement	*150*	*150*	*150*	

When each plant is also considered a destination and each distribution centre is also considered as origin, some additional cost data are necessary, which are given in the tables below:

Table 3.221

To Plant

From plant	1	2
1	*0*	*75*
2	*11*	*0*

Table 3.222

To dist. centres

From dist. centre	*A*	*B*	*C*
A	*0*	*33*	*11*
B	*11*	*0*	*13*
C	*75*	*13*	*0*

Table 3.223

To Plant

From dist. centre	*1*	*2*
A	*13*	*25*
B	*35*	*13*
C	*55*	*65*

Find the optimal shipping schedule for the trans-shipment problem.

Solution. From the given four tables we get the following transportation formulation of the trans-shipment problem:

Table 3.224

To

From	1	2	A	B	C	*Supply*
1	0 (150)	75	11 (300)	13 (150)	25	600
2	11 (300)	0 (450)	13	15	35	750
A	13	25	0 (300)	33	11 (150)	450
B	35	13	11	0 (450)	13	450
C	55	65	75	13	0 (450)	450
Requirement	450	450	600	600	600	

A buffer stock of 450 units, which is the total supply as well as total requirement in the original tranportation problem, is added to each row and column of the trans-shipment problem. The optimal solution is also given in table 3.224. The diagonal allocations in the table may be igonred since they have no physical meaning. The remaining allocations may be interpreted as follows:

(*i*) Plant 2 supplies 300 units to plant 1. This increases the supply capacity of plant 1 to 450 units including the 150 units originally available in it.

(*ii*) Plant 1 transports 300 units to distribution centre A and 150 units to B.

(*iii*) Distribution centre A sends 150 units to C out of 300 units available in it.

The total cost of trans-shipment $= 11 \times 300 + 13 \times 150 + 11 \times 300 + 11 \times 150 = 10{,}200.$

3.11 DUAL OF THE TRANSPORTATION PROBLEM

We know any linear programming problem has its dual. Since the transportation problem is a special type of L.P. problem, it also has its dual with usual interpretations and applications. To illustrate let us consider example 3.5-1 along with the transportation cost table 3.3. The mathematical model (Primal) for this problem is re-written below:

Minimize $Z = 2x_{11} + 3x_{12} + 11x_{13} + 7x_{14} + x_{21} + 0x_{22} + 6x_{23} + x_{24} + 5x_{31} + 8x_{32} + 15x_{33} + 9x_{34},$

subject to constraints

$$\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &= 6,\\
x_{21} + x_{22} + x_{23} + x_{24} &= 1,\\
x_{31} + x_{32} + x_{33} + x_{34} &= 10,\\
x_{11} + x_{21} + x_{31} &= 7,\\
x_{12} + x_{22} + x_{32} &= 5,\\
x_{13} + x_{23} + x_{33} &= 3,\\
x_{14} + x_{24} + x_{34} &= 2,
\end{aligned}$$

where $x_{ij} \geq 0;\ i = 1, 2, 3;\ j = 1, 2, 3, 4.$

Now the dual of this L.P. problem can be written as

maximize $Z' = 6u_1 + u_2 + 10u_3 + 7v_1 + 5v_2 + 3v_3 + 2v_4,$

subject to

$$\begin{aligned}
u_1 + v_1 &\leq 2,\\
u_1 + v_2 &\leq 3,\\
u_1 + v_3 &\leq 11,\\
u_1 + v_4 &\leq 7,\\
u_2 + v_1 &\leq 1,
\end{aligned}$$