Kalman filtering

Linearised position-velocity model

A dynamic process can be generally described by the following matrix differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t), \tag{1}$$

where $\mathbf{x}$ is the state vector, $\mathbf{F}$ is the system dynamic matrix, $\mathbf{G}$ is input matrix, t is time and $\mathbf{u}$ is a vector forcing function, whose elements are white noise.
In the case of the position-velocity model we assume that a vehicle is moving with a constant velocity and that the velocity vector is changing randomly. The state vector contains position vector (coordinates) and velocity vector:

$$\mathbf{x} = \begin{bmatrix} e & n & v_e & v_n \end{bmatrix}^T = \begin{bmatrix} \mathbf{p}^T & \mathbf{v}^T \end{bmatrix}^T \tag{2}$$

The kinematic model:

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{0} + \boldsymbol{\omega}_a \end{aligned} \tag{3}$$

The matrixes from equation (1) become:

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \omega_{ae} \\ \omega_{an} \end{bmatrix} \tag{4}$$

The discrete solution of the differential equation (1) can be generally written as:

$$\mathbf{x}_{k+1} = \mathbf{T}_{k+1,k}\mathbf{x}_k + \mathbf{w}_k \tag{5}$$

where subscript k is a short notation of t_k, which is time of a discrete epoch k and

$$\mathbf{T}_{k+1,k} = e^{\mathbf{F}\cdot\Delta t} \approx \mathbf{I} + \mathbf{F}_k\Delta t + \frac{1}{2}\mathbf{F}_k^2\Delta t^2 + \frac{1}{3!}\mathbf{F}_k^3\Delta t^3 + \frac{1}{4!}\mathbf{F}_k^4\Delta t^4 + \dots \tag{6}$$

is the transition matrix between epochs k and k+1, where $\Delta t = t_{k+1} - t_k$ and

$$\mathbf{w}_k = \int_k^{k+1} \mathbf{T}_{k+1,\tau}\mathbf{G}_\tau \mathbf{u}_\tau d\tau \tag{7}$$

is the driven response at epoch k+1 due to the presence of the white noise input during interval Δt. The covariance of $\mathbf{w}_k$, can be expressed as:

$$\begin{aligned}
&\mathbf{Q}_k = \mathrm{E}\left[\mathbf{w}_k\mathbf{w}_k^T\right] = \\
&\mathrm{E}\left[\int_k^{k+1}\int_k^{k+1} \mathbf{T}(k+1,s)\mathbf{G}(s)\mathbf{u}(s)\mathbf{u}^T(t)\mathbf{G}^T(t)\mathbf{T}^T(k+1,t)\,dt\,ds\right] = \\
&\int_k^{k+1}\int_k^{k+1} \mathbf{T}(k+1,s)\mathbf{G}(s)\mathbf{E}\left[\mathbf{u}(s)\mathbf{u}^T(t)\right]\mathbf{G}^T(t)\mathbf{T}^T(k+1,t)\,dt\,ds
\end{aligned} \tag{8}$$

$$\mathrm{E}\left[\mathbf{u}(s)\mathbf{u}^T(t)\right] = \mathbf{Q} = \begin{bmatrix} q_{ae} & 0 \\ 0 & q_{an} \end{bmatrix} \tag{9}$$

q_{ae} and q_{an} are power spectral density (PSD) of the random acceleration. The unit of q_{ae} and q_{an} is $\left[\left(\frac{m}{s^2\sqrt{Hz}}\right)^2\right]$. Using Equation (6), the solution of the integral (8) can be approximated by following expansion:

$$\begin{aligned}
\mathbf{Q}_k &= \mathrm{E}\left[\mathbf{w}_k\mathbf{w}_k^T\right] \approx \mathbf{Q}_G\Delta t + \left(\mathbf{F}\mathbf{Q}_G + \mathbf{Q}_G\mathbf{F}^T\right)\frac{\Delta t^2}{2} + \\
&+\left[\mathbf{F}^2\mathbf{Q}_G + 2\mathbf{F}\mathbf{Q}_G\mathbf{F}^T + \mathbf{Q}_G\left(\mathbf{F}^T\right)^2\right]\frac{\Delta t^3}{6} + \\
&+\left[\mathbf{F}^3\mathbf{Q}_G + 3\mathbf{F}\mathbf{Q}_G\left(\mathbf{F}^T\right)^2 + 3\mathbf{F}^2\mathbf{Q}_G\mathbf{F}^T + \mathbf{Q}_G\left(\mathbf{F}^T\right)^3\right]\frac{\Delta t^4}{24} + \dots
\end{aligned} \tag{10}$$

where

$$\mathbf{Q}_G = \mathbf{G}\mathbf{Q}\mathbf{G}^T \tag{11}$$

Since, in the case of our PV model $\mathbf{F}^n = \mathbf{0}, \quad n \ge 2$, the process noise covariance matrix will become exactly:

$$\mathbf{Q}_k = \mathbf{Q}_G\Delta t + \left(\mathbf{F}\mathbf{Q}_G + \mathbf{Q}_G\mathbf{F}^T\right)\frac{\Delta t^2}{2} + \mathbf{F}\mathbf{Q}_G\mathbf{F}^T\frac{\Delta t^3}{3} \tag{12}$$

Taking into account (4):

$$
\mathbf{Q}_k = \begin{bmatrix} \frac{q_e \Delta t^3}{3} & 0 & \frac{q_e \Delta t^2}{2} & 0 \\ 0 & \frac{q_n \Delta t^3}{3} & 0 & \frac{q_n \Delta t^2}{2} \\ \frac{q_e \Delta t^2}{2} & 0 & q_e \Delta t & 0 \\ 0 & \frac{q_n \Delta t^2}{2} & 0 & q_n \Delta t \end{bmatrix} \tag{13}
$$

The transition matrix **T** can be computed by equation (6) as:

$$
\mathbf{T}_{k+1,k} = \mathbf{I} + \mathbf{F}_k \Delta t = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}
$$

Kalman filtering

The discrete Kalman filter algorithm has the following steps:

1. Initialisation:

$$
\mathbf{x}_0, \quad \mathbf{Q}_{x0} = \mathrm{var}[\mathbf{x}_0] \tag{15}
$$

2. Time propagation

$$
\mathbf{x}_k^- = \mathbf{T}_{k,k-1}\mathbf{x}_{k-1}, \quad \mathbf{Q}_{x,k}^- = \mathbf{T}_{k,k-1}\mathbf{Q}_{x,k-1}\mathbf{T}_{k,k-1}^T + \mathbf{Q}_k \tag{16}
$$

3. Gain calculation:

$$
\mathbf{K}_k = \mathbf{Q}_{x,k}^- \mathbf{H}_k^T \left[\mathbf{R}_k + \mathbf{H}_k \mathbf{Q}_{x,k}^- \mathbf{H}_k^T \right]^{-1} \tag{17}
$$

4. Measurement update

$$
\mathbf{x}_k = \mathbf{x}_k^- + \mathbf{K}_k \left[\tilde{\mathbf{L}}_k - \mathbf{h}_k(\mathbf{x}_k^-) \right] \tag{18}
$$

5. Covariance update

$$
\mathbf{Q}_{x,k} = \left[\mathbf{I} - \mathbf{K}_k \mathbf{H}_k \right] \mathbf{Q}_{x,k}^- \tag{19}
$$

The tilde symbol (~) denotes measured and minus in superscript predicted quantity. $\tilde{\mathbf{L}}_k$ is the vector of observations with covariance matrix **R**. The observation equations (generally non-linear) can be written as:

$$
\mathbf{L} = \mathbf{h}(\mathbf{x}) \tag{20}
$$

H is the design matrix:

$$\mathbf{H} = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_k^-} \tag{21}$$

If we measure position and absolute value of the velocity vector, then the observation equations are as follows:

$$\mathbf{L}_k = \mathbf{h}_k(\mathbf{x}_k) = \begin{bmatrix} e_k & n_k & \sqrt{v_{ek}^2 + v_{nk}^2} \end{bmatrix}^T \tag{22}$$

or introducing measured values and their random errors $\boldsymbol{\varepsilon}$:

$$\tilde{\mathbf{L}} + \boldsymbol{\varepsilon} = \begin{bmatrix} \tilde{e} \\ \tilde{n} \\ \tilde{v} \end{bmatrix} + \boldsymbol{\varepsilon} = \begin{bmatrix} e \\ n \\ \sqrt{v_e^2 + v_n^2} \end{bmatrix} \tag{23}$$

The last equation in (23)

$$v = \tilde{v} + \varepsilon_v = \sqrt{v_e^2 + v_n^2} \tag{24}$$

is not linear, therefore it is linearised around the predicted values v_e^- and v_n^-

$$\begin{aligned} v &= v^- + \frac{\partial v^-}{\partial v_e^-}\Delta v_e + \frac{\partial v^-}{\partial v_n^-}\Delta v_n + \cdots \\ v^- &= \sqrt{(v_e^-)^2 + (v_n^-)^2} \end{aligned} \tag{25}$$

Design matrix (21) contains partial derivatives of $\mathbf{h}(\mathbf{x})$ (Equation (20)) with respect to all variables in the state vector:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{v_e^-}{v^-} & \dfrac{v_n^-}{v^-} \end{bmatrix} \tag{26}$$

$$\mathbf{h}_k(\mathbf{x}_k^-) = \begin{bmatrix} e_k^- & n_k^- & \sqrt{(v_{ek}^-)^2 + (v_{nk}^-)^2} \end{bmatrix}^T \tag{27}$$

Kalman filter loop