

Kalman filtering

Linearised position-velocity model

A dynamic process can be generally described by the following matrix differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t), \quad (1)$$

where \mathbf{x} is the state vector, \mathbf{F} is the system dynamic matrix, \mathbf{G} is distribution matrix, t is time and \mathbf{u} is a vector forcing function, whose elements are white noise.

In the case of the position-velocity model we assume that a vehicle is moving with a constant velocity and that the velocity vector is changing randomly. The state vector contains position vector (coordinates) and velocity vector:

$$\mathbf{x} = [\mathbf{e} \quad \mathbf{n} \quad \mathbf{v}_e \quad \mathbf{v}_n]^T = [\mathbf{p}^T \quad \mathbf{v}^T]^T \quad (2)$$

The kinematic model:

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{0} + \boldsymbol{\omega}_a \end{aligned} \quad (3)$$

The matrixes from equation (1) become:

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \boldsymbol{\omega}_{ae} \\ \boldsymbol{\omega}_{an} \end{bmatrix} \quad (4)$$

The discrete solution of the differential equation (1) can be generally written as:

$$\mathbf{x}_k = \mathbf{T}_{k-1,k} \mathbf{x}_{k-1} + \mathbf{w}_{k-1,k} \quad (5)$$

where subscript k is a short notation of t_k , which is time of a discrete epoch k and

$$\mathbf{T}_{k-1,k} = e^{\mathbf{F} \cdot \Delta t} \approx \mathbf{I} + \mathbf{F} \Delta t + \frac{1}{2} \mathbf{F}^2 \Delta t^2 + \frac{1}{3!} \mathbf{F}^3 \Delta t^3 + \frac{1}{4!} \mathbf{F}^4 \Delta t^4 + \dots \quad (6)$$

is the transition matrix between epochs k and $k+1$, where $\Delta t = t_k - t_{k-1}$ and

$$\mathbf{w}_k = \int_{k-1}^k \mathbf{T}_{k-1,\tau} \mathbf{G}_\tau \mathbf{u}_\tau d\tau \quad (7)$$

is the driven response at epoch k due to the presence of the white noise input during interval Δt . The covariance of \mathbf{w}_k , can be expressed as:

$$\begin{aligned} \mathbf{Q}_k &= E[\mathbf{w}_{k-1,k} \mathbf{w}_{k-1,k}^T] = \\ &E\left[\int_{k-1}^k \int_{k-1}^k \mathbf{T}(k-1,s) \mathbf{G}(s) \mathbf{u}(s) \mathbf{u}^T(t) \mathbf{G}^T(t) \mathbf{T}^T(k-1,t) dt ds\right] = \\ &\int_{k-1}^k \int_{k-1}^k \mathbf{T}(k-1,s) \mathbf{G}(s) E[\mathbf{u}(s) \mathbf{u}^T(t)] \mathbf{G}^T(t) \mathbf{T}^T(k-1,t) dt ds \end{aligned} \quad (8)$$

$$E[\mathbf{u}(s) \mathbf{u}^T(t)] = \mathbf{Q} = \begin{bmatrix} q_{ae} & 0 \\ 0 & q_{an} \end{bmatrix} \quad (9)$$

q_{ae} and q_{an} are power spectral density (PSD) of the random acceleration. The unit of q_{ae} and q_{an} is $\left[\left(\frac{m}{s^2 \sqrt{Hz}}\right)^2\right]$. Using Equation (6), the solution of the integral (8) can be approximated by following expansion:

$$\begin{aligned} \mathbf{Q}_k &= E[\mathbf{w}_{k-1,k} \mathbf{w}_{k-1,k}^T] \approx \mathbf{Q}_G \Delta t + (\mathbf{F} \mathbf{Q}_G + \mathbf{Q}_G \mathbf{F}^T) \frac{\Delta t^2}{2} + \\ &+ \left[\mathbf{F}^2 \mathbf{Q}_G + 2 \mathbf{F} \mathbf{Q}_G \mathbf{F}^T + \mathbf{Q}_G (\mathbf{F}^T)^2 \right] \frac{\Delta t^3}{6} + \\ &+ \left[\mathbf{F}^3 \mathbf{Q}_G + 3 \mathbf{F} \mathbf{Q}_G (\mathbf{F}^T)^2 + 3 \mathbf{F}^2 \mathbf{Q}_G \mathbf{F}^T + \mathbf{Q}_G (\mathbf{F}^T)^3 \right] \frac{\Delta t^4}{24} + \dots \end{aligned} \quad (10)$$

where

$$\mathbf{Q}_G = \mathbf{G} \mathbf{Q} \mathbf{G}^T \quad (11)$$

Since, in the case of our PV model $\mathbf{F}^n = \mathbf{0}$, $n \geq 2$, the process noise covariance matrix will become exactly:

$$\mathbf{Q}_k = \mathbf{Q}_G \Delta t + (\mathbf{F} \mathbf{Q}_G + \mathbf{Q}_G \mathbf{F}^T) \frac{\Delta t^2}{2} + \mathbf{F} \mathbf{Q}_G \mathbf{F}^T \frac{\Delta t^3}{3} \quad (12)$$

Taking into account (4):

$$\mathbf{Q}_k = \begin{bmatrix} \frac{q_e \Delta t^3}{3} & 0 & \frac{q_e \Delta t^2}{2} & 0 \\ 0 & \frac{q_n \Delta t^3}{3} & 0 & \frac{q_n \Delta t^2}{2} \\ \frac{q_e \Delta t^2}{2} & 0 & q_e \Delta t & 0 \\ 0 & \frac{q_n \Delta t^2}{2} & 0 & q_n \Delta t \end{bmatrix} \quad (13)$$

The transition matrix \mathbf{T} can be computed by equation (6) as:

$$\mathbf{T}_{k-1,k} = \mathbf{I} + \mathbf{F}_k \Delta t = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (14)$$

Kalman filtering

The discrete Kalman filter algorithm has the following steps:

1. Initialisation:

$$\mathbf{x}_0, \quad \mathbf{Q}_{x0} = \text{var}[\mathbf{x}_0] \quad (15)$$

2. Time propagation

$$\mathbf{x}_k^- = \mathbf{T}_{k-1,k} \mathbf{x}_{k-1}, \quad \mathbf{Q}_{x,k}^- = \mathbf{T}_{k-1,k} \mathbf{Q}_{x,k-1} \mathbf{T}_{k-1,k}^T + \mathbf{Q}_k \quad (16)$$

3. Gain calculation:

$$\mathbf{K}_k = \mathbf{Q}_{x,k}^- \mathbf{H}_k^T [\mathbf{R}_k + \mathbf{H}_k \mathbf{Q}_{x,k}^- \mathbf{H}_k^T]^{-1} \quad (17)$$

4. Measurement update

$$\mathbf{x}_k = \mathbf{x}_k^- + \mathbf{K}_k [\tilde{\mathbf{L}}_k - \mathbf{h}_k(\mathbf{x}_k^-)] \quad (18)$$

5. Covariance update

$$\mathbf{Q}_{x,k} = [\mathbf{I} - \mathbf{K}_k \mathbf{H}_k] \mathbf{Q}_{x,k}^- \quad (19)$$

The tilde symbol (\sim) denotes measured and minus in superscript predicted quantity. $\tilde{\mathbf{L}}_k$ is the vector of observations with covariance matrix \mathbf{R} . The observation equations (generally non-linear) can be written as:

$$\mathbf{L} = \mathbf{h}(\mathbf{x}) \quad (20)$$

\mathbf{H} is the design matrix:

$$\mathbf{H} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \bigg|_{\mathbf{x}=\mathbf{x}_k^-} \quad (21)$$

If we measure position and absolute value of the velocity vector, then the observation equations are as follows:

$$\mathbf{L}_k = \mathbf{h}_k(\mathbf{x}_k) = \begin{bmatrix} \mathbf{e}_k & \mathbf{n}_k & \sqrt{v_{ek}^2 + v_{nk}^2} \end{bmatrix}^T \quad (22)$$

or introducing measured values and their random errors $\boldsymbol{\varepsilon}$:

$$\tilde{\mathbf{L}} + \boldsymbol{\varepsilon} = \begin{bmatrix} \tilde{\mathbf{e}} \\ \tilde{\mathbf{n}} \\ \tilde{\mathbf{v}} \end{bmatrix} + \boldsymbol{\varepsilon} = \begin{bmatrix} \mathbf{e} \\ \mathbf{n} \\ \sqrt{v_e^2 + v_n^2} \end{bmatrix} \quad (23)$$

The last equation in (23)

$$v = \tilde{v} + \varepsilon_v = \sqrt{v_e^2 + v_n^2} \quad (24)$$

is not linear, therefore it is linearised around the predicted values v_e^- and v_n^-

$$\begin{aligned} v &= v^- + \frac{\partial v^-}{\partial v_e^-} \Delta v_e + \frac{\partial v^-}{\partial v_n^-} \Delta v_n + \dots \\ v^- &= \sqrt{(v_e^-)^2 + (v_n^-)^2} \end{aligned} \quad (25)$$

Design matrix (21) contains partial derivatives of $\mathbf{h}(\mathbf{x})$ (Equation (20)) with respect to all variables in the state vector:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{v_e^-}{v^-} & \frac{v_n^-}{v^-} \end{bmatrix} \quad (26)$$

$$\mathbf{h}_k(\mathbf{x}_k^-) = \begin{bmatrix} e_k^- & n_k^- & \sqrt{(v_{ek}^-)^2 + (v_{nk}^-)^2} \end{bmatrix}^T \quad (27)$$

Smoothing

After the last observation we get an estimation of the state vector denoted as \mathbf{x}_N , and covariance matrix $\mathbf{Q}_{x,N}$. The smoothed estimation for the previous epochs is given by

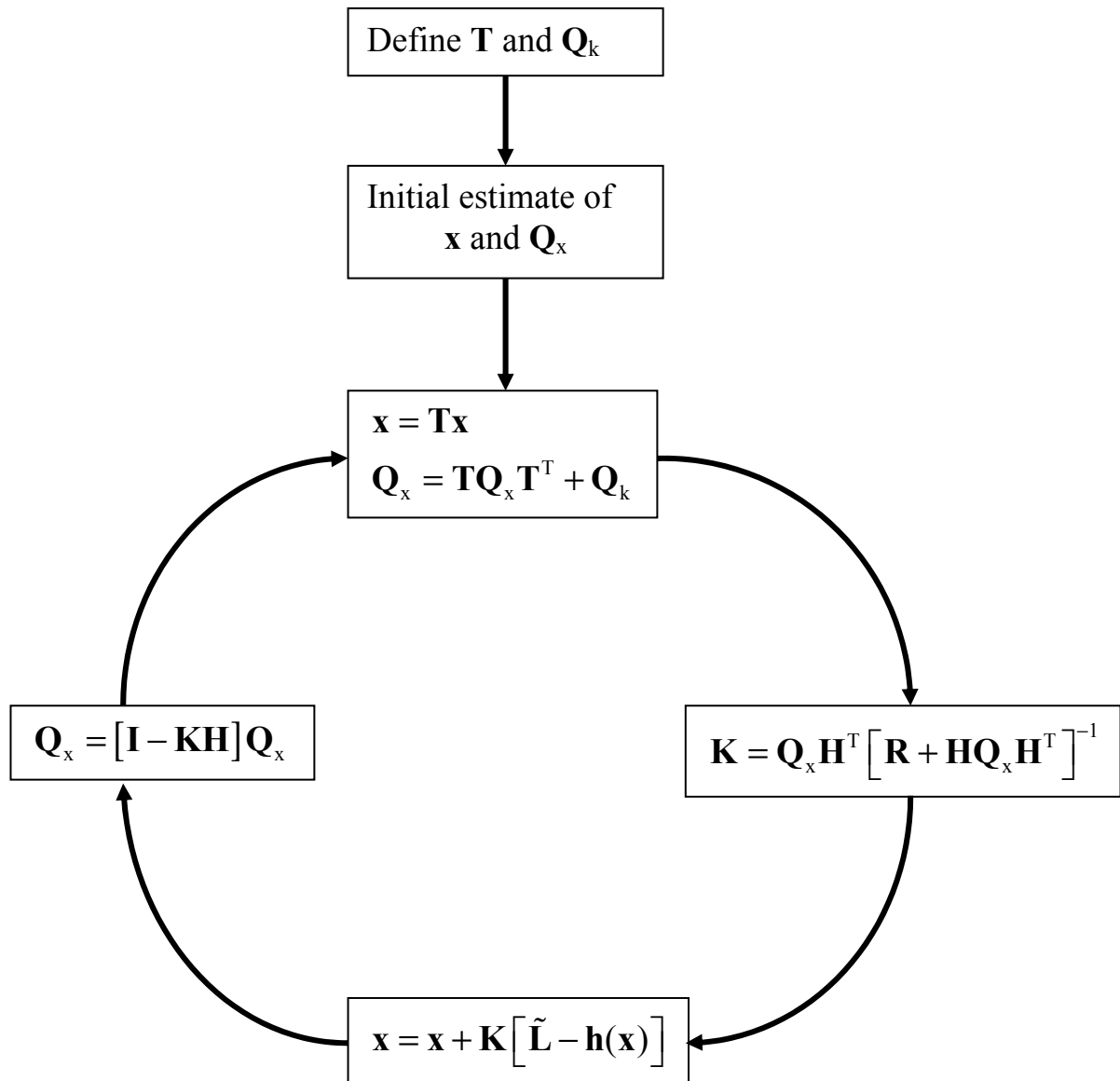
$$\hat{\mathbf{x}}_k = \mathbf{x}_k + \mathbf{D}_k \left[\hat{\mathbf{x}}_{k+1} - \mathbf{x}_{k+1}^- \right] \quad (28)$$

$$\mathbf{D}_k = \mathbf{Q}_{x,k} \mathbf{T}_{k-1,k}^T \left(\mathbf{Q}_{x,k+1}^- \right)^{-1} \quad (29)$$

and the covariance matrix of the smoothed state vector:

$$\hat{\mathbf{Q}}_{x,k} = \mathbf{Q}_{x,k} + \mathbf{D}_k \left[\hat{\mathbf{Q}}_{x,k+1} - \mathbf{Q}_{x,k+1}^- \right] \mathbf{D}_k^T \quad (30)$$

Kalman filter loop



References

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