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Periodic Vehicle Routing Problem with Driver Consistency and service time optimization

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ABSTRACT

The Periodic Vehicle Routing Problem with Driver Consistency is an extension of the classic Vehicle Routing Problem in which routes for several vehicles have to be determined over a time horizon of several days. Each customer has an associated set of possible visit schedules and must be visited always by the same vehicle. In this paper, we study a variant of the PVRP-DC in which, in addition to routes and visit schedules, service times of customers have to be determined in order to maximize the utility of the service to the company. We call this problem the Periodic Vehicle Routing Problem with Driver Consistency and Service Time Optimization. We present a mixed-integer linear programming formulation for the problem, propose three branch-and-cut methods to solve it, two of which are based on Benders reformulations, and report computational results on benchmark instances with different features.

1. Introduction

The *Periodic Vehicle Routing Problem* (PRVP) asks to determine visit schedules and routes to minimize the total transportation costs for a planning horizon of multiple periods. The single period problem in which every customer must be visited once is the classical vehicle routing problem (VRP). PVRP has been introduced by Beltrami and Bodin (1974) for planning municipal waste collection. Since then, it has been studied in the context of a large variety of applications, such as planning of deliveries of hospital linen to clinics (Banerjea-Brodeur et al., 1998), visits for preventive maintenance or quality assurance (Blakely et al., 2003; Hadjiconstantinou and Baldacci, 1998), delivery of blood products to hospitals (Hemmelmayr et al., 2009b), visits to suppliers or customers in a supply chain Alegre et al. (2007), Claassen and Hendriks (2007), Gaur and Fisher (2004), Golden and Wasil (1987), le Blanc et al. (2006), Ronen and Goodhart (2007), visits to collect recyclable materials and waste (Baptista et al., 2002; Bommisetty et al., 1998; Coene et al., 2010; Nuortio et al., 2006; Shih and Chang, 2001; Shih and Lin, 1999; Teixeira et al., 2004), routes for lottery sales (Jang et al., 2006), or visits to students or patients at home (An et al., 2012; Maya et al., 2012). Many heuristic algorithms have been proposed for PVRP and its variants (Chao et al., 1995; Christofides and Beasley, 1984; Cordeau et al., 1997; Drummond et al., 2001; Gaudioso and Paletta, 1992; Hemmelmayr et al., 2009a; Russell and Gribbin, 1991; Russell and Igo, 1979; Tan and Beasley, 1984). In comparison to heuristics, exact methods are rare and some are developed for simplified variants (see, e.g., Baldacci et al. (2011), Butler et al. (1997) and Mourgaya and Vanderbeck (2007)). For more details on the applications, solution methods and variants of the PVRP, we refer the reader to the surveys by Campbell and Wilson (2014), and by Francis et al. (2008).

More recently new variants of PVRP with a concern of "consistency" have been introduced. Consistency may be in terms of the paths as in Yao et al. (2021) or in terms of visit times and drivers. Time consistency requires different visits to a customer to be

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more or less at the same time of the day. This is especially important, for instance, when administering medication and running tests in home health care services. Driver consistency asks to visit a customer always by the same driver. The aim is to improve the quality of service for customers and to make use of the familiarity of drivers with customers, routes and traffic conditions (see, e.g., Smilowitz et al. (2013)).

The *consistent VRP (ConVRP)*, which takes into account both consistency measures, has been studied by Groër et al. (2009). In this problem, the visit schedules of customers are known in advance and different periods are related to each other with consistency requirements. Exact and heuristic algorithms to solve this problem are proposed by Goeke et al. (2019) where instances with 30 customers and five periods are solved to optimality in reasonable times.

Some studies, such as Braekers and Kovacs (2016), Zhu et al. (2008) and Luo et al. (2015), consider a more general definition of driver consistency by putting an upper bound on the number of drivers visiting a customer. Kovacs et al. (2015a) introduce the generalized ConVRP where each customer is visited by a limited number of drivers and the variation in the arrival times is penalized in the objective function. A multi objective version is studied by Kovacs et al. (2015b) and a variant with multiple daily deliveries and service level agreements is studied by Campelo et al. (2019).

Rodríguez-Martín et al. (2019) present a branch-and-cut algorithm for the *PVRP with driver consistency (PVRP-DC)*, where each customer must be visited by the same vehicle at all visits. Consistency for visit times and travel time limits for vehicles are not considered but visit schedules are not known and are to be determined by the model.

Motivated by the delivery operations for interlibrary loan items, Francis et al. (2006) (see also Francis and Smilowitz (2006) and Lei et al. (2017) for a game theoretic setting) introduce the PVRP with service choice in which the service frequency is a decision and the objective function accounts for the benefits of customers from higher visit frequencies. In this study, we are also interested in the benefits from visits. Different from Francis et al. (2006), we assume that the frequencies of visits are given but the service times are to be optimized. To this end, we introduce a new variant of PVRP-DC in which, in addition to visit schedules and routes, service times of customers at each visit are also determined to maximize the utility of the service for the company. We call this problem the *Periodic Vehicle Routing Problem with Driver Consistency and Service Time Optimization* and abbreviate as PVRP-DC-SO. In PVRP-DC-SO there are upper and lower bounds on service times, and the utility obtained from a visit is a nondecreasing function of the service time. This setting is motivated by a real life application encountered in the context of a project with a major biscuit and chocolate producer in Turkey. The merchandisers of the company visit chain supermarkets (retailers) to refill the empty shelves from the stocks and to better display the products and promotions. A retailer may be visited once, twice, or three times in a week depending on its sales volume. All visits to the same retailer are done by the same merchandiser as the knowledge of the retailer and its customer profile is essential to make the most benefit from the visits. The problem is to determine the schedule of visits and their durations, as well as routes, so that the merchandisers can finish their routes by the end of the working hours and the utility that the company obtains from the visits is maximized. This setting is different from the one of classical vehicle routing problems since the merchandisers travel by public transportation and they do not deliver products. Consequently there are no capacity constraints and the transportation costs do not depend on the routes. However, the working hours are limited.

Even though the motivation comes from this merchandiser routing problem, PVRP-DC-SO may arise in other applications where the vehicles (drivers) provide service instead of delivering products to the points they visit, as it is the case in home health care, education or preventive maintenance services. In these settings, the vehicles used are smaller than the trucks typically considered in the VRP literature; these can even be bikes or the employees may use public transportation as was the case in the motivating application. Hence, the main cost component is the employees' cost rather than the transportation cost. Consequently, making the best use of employees' working days to maximize the utility of the service becomes the major objective. Still the routing decisions play a critical role as the time available for service is determined by the time spent for travel.

Notice that while in the PVRP-DC the objective is to minimize the routes' duration/cost, in the PVPR-DC-SO the objective is to maximize the utility obtained by serving the customers while imposing an upper bound to the time each driver/vehicle invests in making his/her route and visiting his/her customers. Therefore vehicle routes are usually different for these two problems, as illustrated in Figs. 1 and 2. These figures show the routes in the optimal solutions of the PVRP-DC and the PVRP-DC-SO on an instance with ten customers (nodes 1 to 10), depot located at node 0, a time horizon of two days, and two vehicles. Customers 3, 4, 8, 9, and 10 have to be visited both days. Customers 1 and 6 must be visited exclusively the first day, and customer 7 must be visited the second day only. Customers 2 and 5 have to be visited once, either the first or the second day. The service time of each customer can vary between 20 and 30 min for customers 2 and 6, between 30 and 60 min for customers 1, 3, 7, 8, and 9, and between 60 and 120 min for customers 4, 5, and 10. The routes of one vehicle are drawn with solid lines, and the routes of the other with dashed lines. The maximum utility attainable with the PVRP-DC solution is 8839.06, which is less than 9619.06, the optimal value of the PVRP-DC-SO.

The aim of this study is to introduce the PVRP-DC-SO and to propose exact methods to solve it. The remainder of the paper is organized as follows. The problem is formally described and modeled in Section 2. Two alternative formulations based on Benders decomposition are presented in Section 3. In Section 4 we give some implementation details of the three branch-and-cut algorithms used to solve the problem, one based on the initial mathematical formulation of the problem and two others based on the Benders reformulations. In Section 5 we compare the three algorithms through extensive computational experiments. Finally, conclusions are given in Section 6.

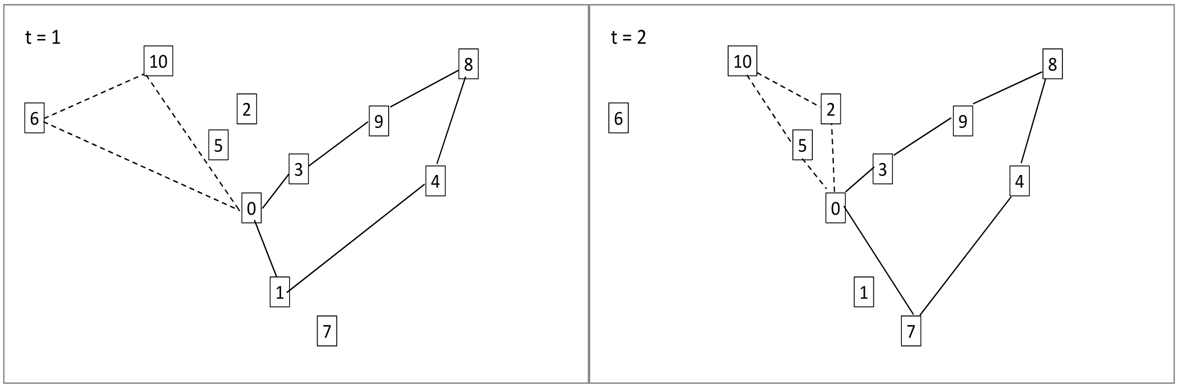

Fig. 1. PVRP-DC optimal solution (routes' duration = 640.65, utility = 8839.06).

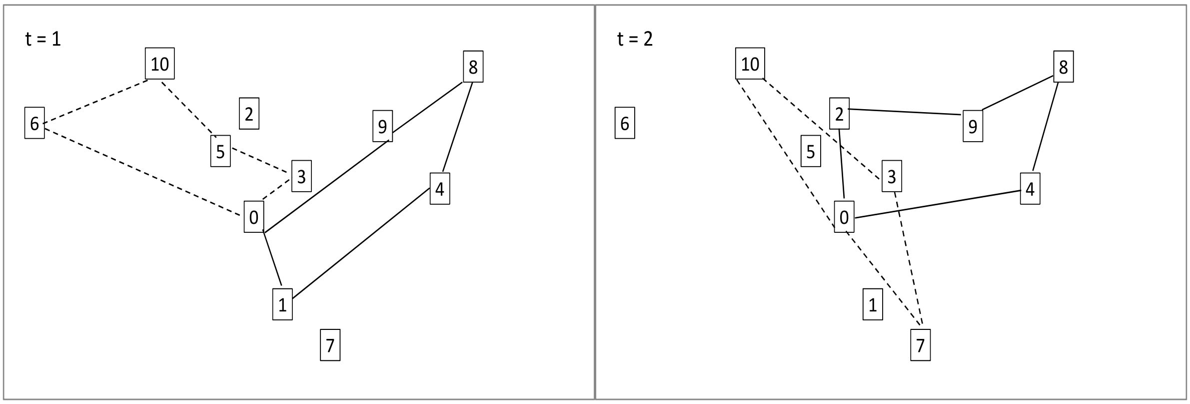

Fig. 2. PVRP-DC-ST optimal solution (routes' duration = 702.68, utility = 9619.06).

2. Problem definition and formulation

Let $V = \{0, 1, \dots, n\}$ be a set of nodes, with node 0 corresponding to the depot and the other nodes corresponding to customers. Let $E = \{e = \{i, j\} : i, j \in V, i < j\}$ be the set of edges and d_e denote the travel time associated with edge $e \in E$. We assume that the travel times are nonnegative, symmetric and satisfy the triangle inequality. We consider a planning horizon $T = \{1, \dots, \tau\}$ of τ periods. Customer $i \in V \setminus \{0\}$ has a positive weight w_i, which can be a measure of its sales volume, and needs to be visited a number of times f_i during the planning horizon, always by the same vehicle. In addition, each visit requires a service time between l_i and u_i units with $0 < l_i \leq u_i$. Let $\rho_{it}(s)$ denote the utility of visiting customer i for s units of service time in period t. We define P_i as the set of possible visit schedules (i.e., allowable visit combinations) for customer i. For instance, if the planning horizon consists of the weekdays (numbered from 1 to 5) and if a customer needs to be visited twice ($f_i = 2$) with at least one day and at most two days between consecutive visits, then Monday–Wednesday, Monday–Thursday, Tuesday–Thursday, Tuesday–Friday, and Wednesday–Friday are the possible visit schedules for this customer, so $P_i = \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}\}$. A homogeneous fleet $K = \{1, \dots, m\}$ of m vehicles is available at the depot. If a vehicle is used in a given period, then it leaves the depot at time 0, visits at least two customers and must be back at the depot by time q. Our aim is to maximize the weighted utility of visiting customers over the planning horizon.

We use the following binary decision variables to formulate the problem. We define z^k_{ip} to be 1 if schedule $p \in P_i$ is chosen to serve customer $i \in V \setminus \{0\}$ and if all visits in that schedule are done by vehicle $k \in K$, and 0 otherwise. We also define x^{tk}_e to be 1 if edge $e \in E$ is traversed by vehicle $k \in K$ in period $t \in T$ and 0 otherwise, and y^{tk}_0 to be 1 if vehicle $k \in K$ is used in period $t \in T$ and 0 otherwise. To simplify the notation, we let $y^{tk}_i = \sum_{p \in P_i : t \in p} z^k_{ip}$ for $i \in V \setminus \{0\}$, $t \in T$ and $k \in K$. This variable is 1 if customer i is visited by vehicle k in period t and is 0 otherwise. The variable s^{tk}_i is the service time for customer $i \in V \setminus \{0\}$ in period $t \in T$ by vehicle $k \in K$ and can be nonzero only if $y^{tk}_i = 1$.

We use some additional notation. For $S \subseteq V$, $\delta(S)$ is the set of edges with one endpoint in S, and $E(S)$ is the set of edges with both endpoints in set S. If $S = \{i\}$, we write $\delta(i)$ instead of $\delta(\{i\})$. In addition, for a given subset of edges $E' \subseteq E$, we define $x^{tk}(E') = \sum_{e \in E'} x^{tk}_e$.

We can model our problem as follows:

$$\max \sum_{t\in T}\sum_{k\in K}\sum_{i\in V\setminus\{0\}} w_i \rho_{it}(s_i^{tk}) \tag{1}$$

$$\text{s.t.} \sum_{p\in P_i}\sum_{k\in K} z_{ip}^k = 1 \qquad i\in V\setminus\{0\}, \tag{2}$$

$$y_i^{tk} = \sum_{p\in P_i: t\in p} z_{ip}^k \qquad i\in V\setminus\{0\}, k\in K, t\in T, \tag{3}$$

$$x^{tk}(\delta(i)) = 2y_i^{tk} \qquad i\in V, k\in K, t\in T, \tag{4}$$

$$x^{tk}(\delta(S)) \geq 2y_i^{tk} \qquad S\subseteq V\setminus\{0\}, i\in S, k\in K, t\in T, \tag{5}$$

$$\sum_{e\in E} d_e x_e^{tk} + \sum_{i\in V\setminus\{0\}} s_i^{tk} \leq q y_0^{tk} \qquad k\in K, t\in T, \tag{6}$$

$$l_i y_i^{tk} \leq s_i^{tk} \leq u_i y_i^{tk} \qquad i\in V\setminus\{0\}, k\in K, t\in T, \tag{7}$$

$$x_e^{tk} \in \{0,1\} \qquad e\in E, k\in K, t\in T, \tag{8}$$

$$y_i^{tk} \in \{0,1\} \qquad i\in V, k\in K, t\in T, \tag{9}$$

$$z_{ip}^k \in \{0,1\} \qquad i\in V\setminus\{0\}, p\in P_i, k\in K. \tag{10}$$

Constraints (2) ensure that a visit schedule and a vehicle is chosen for each customer. Variables y and z are related through constraints (3). Constraints (4) and (5) are the classical degree and connectivity constraints. When $S = V\setminus\{0\}$, constraints (5) can be written as

$$y_i^{tk} \leq y_0^{tk} \qquad i\in V\setminus\{0\}, k\in K, t\in T. \tag{11}$$

These inequalities forbid a vehicle k to visit customers in a period t if k is not used in that period. Constraints (6) ensure that the travel time and the service time on a route does not exceed the available time q. Finally, constraints (7) impose lower and upper bounds on service times and allow service times to take positive values only when there are visits on the corresponding days.

Based on the motivating application, we model $\rho_{it}(s)$ to be a piecewise linear function with two pieces. For each customer $i\in V\setminus\{0\}$, we have a mean service time m_i. The unit utility up to m_i units of service is ρ_i^1. The unit utility after m_i units of time is denoted by ρ_i^2 and is smaller than ρ_i^1. Hence we compute the utility for customer i in period t as $\rho_{it}(s) = \rho_i^1 s$ if $s \leq m_i$ and $\rho_{it}(s) = \rho_i^1 m_i + \rho_i^2(s - m_i)$ if $s > m_i$. Then, the objective function becomes

$$\max \sum_{t\in T}\sum_{k\in K}\sum_{i\in V\setminus\{0\}} w_i(\rho_i^1 a_i^{tk} + \rho_i^2 b_i^{tk}) \tag{12}$$

and we replace constraints (6) and (7) with

$$\sum_{e\in E} d_e x_e^{tk} + \sum_{i\in V\setminus\{0\}} (a_i^{tk} + b_i^{tk}) \leq q y_0^{tk} \qquad k\in K, t\in T, \tag{13}$$

$$l_i y_i^{tk} \leq a_i^{tk} \leq m_i y_i^{tk} \qquad i\in V\setminus\{0\}, k\in K, t\in T, \tag{14}$$

$$0 \leq b_i^{tk} \leq (u_i - m_i) y_i^{tk} \qquad i\in V\setminus\{0\}, k\in K, t\in T. \tag{15}$$

Note that constraints (6) are not the typical capacity constraints present in the Capacitated Vehicle Routing Problem and in the PVRP-DC. Due to this fact, most of the valid inequalities derived by Rodríguez-Martín et al. (2019) for the PVRP-DC are no longer valid for the PVRP-DC-SO. The only family of valid inequalities given in Rodríguez-Martín et al. (2019) for the PVRP-DC that remains valid for the PVRP-DC-SO is the following:

$$x_{\{0,i\}}^{tk} \leq y_i^{tk} \qquad i\in V\setminus\{0\}, k\in K, t\in T. \tag{16}$$

These inequalities ensure that, if an edge adjacent to the depot is traversed by vehicle k in period t, then its other endpoint is visited by vehicle k in the same period t. Note that inequalities $x_{\{i,j\}}^{tk} \leq y_j^{tk}$ for $\{i,j\}\in E\setminus\delta(0)$, $k\in K$ and $t\in T$ are special cases of (5) for $S = \{i,j\}$ and are already part of the model.

Due to the service time variables in constraints (6), these constraints are also different from the distance constraints, for instance, in the Orienteering Problem (see, e.g., Kobeaga et al. (2021)). One can use the lower bounds on the service times to relax constraints (6) in such a way that the left hand side involves only 0–1 variables, however these relaxations are weak.

3. Benders reformulations

In this section, we propose two Benders reformulations and present solution methods for the resulting subproblems.

3.1. Reformulation 1

In the first reformulation, we keep variables y and z in the master problem. In other words, the master problem decides on which vehicles visit which customers on which days. The routes and service times are decided in the subproblem. The master problem is

$$\max \sum_{t\in T}\sum_{k\in K}\eta^{tk}$$

s.t. (2), (3), (9), (10), (11),

$$\sum_{i\in V\setminus\{0\}} y_i^{tk} \geq 2y_0^{tk} \qquad t\in T, k\in K, \tag{17}$$

$$\sum_{i\in \bar{V}} y_i^{tk} \leq (|\bar{V}|-1)y_0^{tk} \qquad t\in T, k\in K, \bar{V}\in C_1, \tag{18}$$

$$\eta^{tk} \leq \eta^*_{\bar{V}} y_0^{tk} + M\left(\sum_{i\in\bar{V}}(y_0^{tk}-y_i^{tk}) + \sum_{i\in V\setminus(\bar{V}\cup\{0\})} y_i^{tk}\right) \qquad t\in T, k\in K, \bar{V}\in C_2, \tag{19}$$

where additional variables $\eta^{tk}\geq 0$ for $k\in K$ and $t\in T$ are defined to compute the utilities, and M is a sufficiently large value. Inequalities (17) impose that if vehicle k is used in period t, i.e., if $y_0^{tk}=1$, then it should visit at least two customers. Observe that (17) are implied in the original formulation by x_e^{tk} being 0–1 and by the degree constraints (4). But we need to impose them explicitly in the above master problem. Set C_1 is the set of subsets of customers that cannot be visited and served in one day with a single vehicle, and constraints (18) are the corresponding feasibility cuts. If a vehicle k is used in period t, i.e., if $y_0^{tk}=1$, then it cannot serve all customers in a set $\bar{V}\in C_1$. Hence, the constraint (18) states that at least one customer in set $\bar{V}$ is not served by vehicle k in period t. These constraints also ensure that if vehicle k is not used in period t, then no customer can be served by this vehicle in period t. Set C_2 is the set of subsets $\bar{V}$ of customers that can be visited and served in one day with a single vehicle, and $\eta^*_{\bar{V}}$ is the resulting maximum utility obtained by visiting the customers in $\bar{V}$ by one vehicle in one day. Constraints (19) ensure that, if the customers visited by vehicle k in period t are those in set $\bar{V}\in C_2$, then η^{tk} is bounded above by $\eta^*_{\bar{V}}$. In other cases, these constraints are redundant because of the big-M.

As the master contains an exponential number of constraints (18) and (19), we work with a relaxation and add these constraints when necessary within a branch-and-cut framework. Because the relaxation is very weak at the earlier iterations, we also add the following valid inequalities to the master problem:

$$\eta^{tk} \leq \sum_{i\in V\setminus\{0\}} \bar{\rho}_i y_i^{tk} \qquad t\in T, k\in K, \tag{20}$$

$$\sum_{i\in V\setminus\{0\}} \bar{d}_i y_i^{tk} \leq (q-\bar{d}_0)y_0^{tk} \qquad t\in T, k\in K, \tag{21}$$

where $\bar{\rho}_i = w_i(\rho_i^1 m_i + \rho_i^2(u_i-m_i))$ is the maximum contribution that customer $i\in V\setminus\{0\}$ can make to the objective function in any period, $\bar{d}_i = \min_{e\in\delta(i)} d_e + l_i$ is the minimum time required to serve i and travel to another node, and $\bar{d}_0 = \min_{e\in\delta(0)} d_e$. Constraints (20) put upper bounds on the η's and constraints (21) aim to eliminate some infeasible visit schedules.

To improve the effectiveness of the approach, we also make use of the linear programming (LP) relaxation of the subproblem to generate cuts. We proceed as follows. At an optimal solution $(\bar{\eta},\bar{y},\bar{z})$ of the current master problem at a node of the branch-and-cut tree with integral $\bar{y}$ and $\bar{z}$, we apply the following separation scheme. For each $t\in T$ and $k\in K$, we first solve the LP

$$\max \sum_{i\in V\setminus\{0\}} w_i(\rho_i^1 a_i + \rho_i^2 b_i) \tag{22}$$

$$\text{s.t.} \; -x(\delta(i)) \leq -2\bar{y}_i^{tk} \qquad i\in V, \tag{23}$$

$$-x(\delta(S)) \leq -2\bar{y}_i^{tk} \qquad S\subseteq V\setminus\{0\}, i\in S, \tag{24}$$

$$\sum_{e\in E} d_e x_e + \sum_{i\in V\setminus\{0\}}(a_i+b_i) \leq q\bar{y}_0^{tk}, \tag{25}$$

$$-a_i \leq -l_i\bar{y}_i^{tk} \qquad i\in V\setminus\{0\}, \tag{26}$$

$$a_i \leq m_i\bar{y}_i^{tk} \qquad i\in V\setminus\{0\}, \tag{27}$$

$$b_i \leq (u_i-m_i)\bar{y}_i^{tk} \qquad i\in V\setminus\{0\}, \tag{28}$$

$$x_e \leq \bar{y}_i^{tk} \qquad e\in E, i\in e, \tag{29}$$

$$a_i, b_i \geq 0 \qquad i\in V\setminus\{0\},$$

$$x_e \geq 0 \qquad e\in E,$$

using a cutting plane algorithm where we separate constraints (24). If this LP turns out to be infeasible, then we introduce the feasibility cut

$$(-2\beta_0 + \gamma q + \sum_{e\in E\cap\delta(0)} \psi_{e0}) y_0^{tk}$$
$$+ \sum_{i\in V\setminus\{0\}} \left(-2\beta_i - 2\sum_{S\subseteq V\setminus\{0\}:i\in S} \delta_{Si} - \sigma_i l_i + \theta_i m_i + \chi_i(u_i - m_i) + \sum_{e\in E\cap\delta(i)} \psi_{ei} \right) y_i^{tk} \geq 0, \tag{30}$$

where β_i, δ_{Si}, γ, σ_i, θ_i, χ_i and ψ_{ei} are the dual variables associated with constraints (23)–(29), respectively. If the primal LP is feasible, then we take the dual optimal solution and check whether the Benders cut

$$\eta^{tk} \leq (-2\beta_0 + \gamma q + \sum_{e\in E\cap\delta(0)} \psi_{e0}) y_0^{tk}$$
$$+ \sum_{i\in V\setminus\{0\}} \left(-2\beta_i - 2\sum_{S\subseteq V\setminus\{0\}:i\in S} \delta_{Si} - \sigma_i l_i + \theta_i m_i + \chi_i(u_i - m_i) + \sum_{e\in E\cap\delta(i)} \psi_{ei} \right) y_i^{tk} \tag{31}$$

is violated by $(\bar{\eta}, \bar{y}, \bar{z})$ or not. If violated, we add the cut.

If solving the LP does not produce any violated cuts, then we solve the integer subproblem. For $t \in T$ and $k \in K$, if $\bar{y}_0^{tk} = 0$, then $\bar{\eta}^{tk} = 0$. If $\bar{y}_0^{tk} = 1$, then the true value of η^{tk} can be computed by solving the following integer program.

$$\hat{\eta}^{tk} = \max \sum_{i\in\bar{V}} w_i(\rho_i^1 a_i + \rho_i^2 b_i)$$
$$\text{s.t. } x(\delta(i)) = 2 \qquad i \in \bar{V} \cup \{0\},$$
$$x(\delta(S) \cap E(\bar{V})) \geq 2 \qquad S \subseteq \bar{V} : 2 \leq |S| \leq |\bar{V}| - 1,$$
$$\sum_{e\in E(\bar{V})} d_e x_e + \sum_{i\in\bar{V}} (a_i + b_i) \leq q,$$
$$l_i \leq a_i \leq m_i \qquad i \in \bar{V},$$
$$0 \leq b_i \leq u_i - m_i \qquad i \in \bar{V},$$
$$x_e \in \{0, 1\} \qquad e \in E(\bar{V} \cup \{0\}),$$

where $\bar{V} = \{i \in V \setminus \{0\} : \bar{y}_i^{tk} = 1\}$. This subproblem can be solved in two parts. First, we solve a travelling salesman problem (TSP) on nodes $\bar{V} \cup \{0\}$ where we minimize the travel time, $\sum_{e\in E(\bar{V})} d_e x_e$. Let $v(\bar{V})$ be the optimal value of this TSP:

- If $v(\bar{V}) > q - \sum_{i\in\bar{V}} l_i$, then the customers in set $\bar{V}$ cannot be served by one vehicle in any period. In this case, we add the feasibility cut (18) for all $k' \in K$ and $t' \in T$.
- If $v(\bar{V}) \leq q - \sum_{i\in\bar{V}} l_i$, to compute the optimal value of the subproblem, denoted by $\hat{\eta}^{tk}$, we maximize the utility of the tour using the remaining time after deducting travel times and minimum service times, $\bar{q} = q - v(\bar{V}) - \sum_{i\in\bar{V}} l_i$, by solving the linear program

$$\hat{\eta}^{tk} = \sum_{i\in\bar{V}} w_i \rho_i^1 l_i + \max \sum_{i\in\bar{V}} w_i(\rho_i^1 a'_i + \rho_i^2 b_i)$$
$$\text{s.t. } \sum_{i\in\bar{V}} (a'_i + b_i) \leq \bar{q},$$
$$0 \leq a'_i \leq m_i - l_i \qquad i \in \bar{V},$$
$$0 \leq b_i \leq u_i - m_i \qquad i \in \bar{V}.$$

This problem can be solved by inspection as follows. Suppose that the nodes in set $\bar{V}$ are renumbered from 1 to $|\bar{V}|$. Let $N = \{1, \ldots, 2|\bar{V}|\}$ and for $j \in N$, let $c_j = w_j \rho_j^1$ and $e_j = m_j - l_j$ if $j \leq |\bar{V}|$, and $c_j = w_i \rho_i^2$ and $e_j = u_i - m_i$ if $j > |\bar{V}|$ and $j - |\bar{V}| = i$. We order the set N in nonincreasing order of c_j values and initialize $\bar{c} = 0$, $\bar{e} = \bar{q}$, and $j = 1$. While $\bar{e} > 0$ and $j \leq 2|\bar{V}|$, let $\bar{c} \leftarrow \bar{c} + c_j \min\{e_j, \bar{e}\}$, $\bar{e} \leftarrow \bar{e} - \min\{e_j, \bar{e}\}$, and $j \leftarrow j + 1$. We check whether $\bar{\eta}^{tk} \leq \hat{\eta}^{tk} = \sum_{i\in\bar{V}} w_i \rho_i^1 l_i + \bar{c}$. If not, we add the violated cut (19) with $\eta_{\bar{V}}^* = \hat{\eta}^{tk}$.

3.2. Reformulation 2

In the second Benders reformulation, in addition to the visit schedules and vehicles, we also decide on the service times in the master problem. The subproblem is then used to check if feasible routes exist for the given schedules and service times. More precisely, the master problem is

$$\max \sum_{t\in T} \sum_{k\in K} \sum_{i\in V\setminus\{0\}} w_i(\rho_i^1 a_i^{tk} + \rho_i^2 b_i^{tk})$$
$$\text{s.t. (2), (3), (9), (10), (14) - (18), (11),}$$
$$v^{tk} + \sum_{i\in V\setminus\{0\}} (a_i^{tk} + b_i^{tk}) \leq q y_0^{tk} \qquad t \in T, k \in K, \tag{32}$$

$$v^{tk} \geq v(\bar{V})y_0^{tk} - \sum_{i \in \bar{V}} M(y_0^{tk} - y_i^{tk}) \qquad t \in T, k \in K, \bar{V} \in C_2, \tag{33}$$

where additional variables $v^{tk} \geq 0$ represent the travel time of the route of vehicle $k \in K$ on day $t \in T$.

Similar to (20) in the first reformulation, we add inequalities

$$v^{tk} \geq \sum_{i \in V} \min_{e \in \delta(i)} d_e y_i^{tk} \qquad t \in T, k \in K \tag{34}$$

to the master problem to impose lower bounds on the v^{tk} variables. We also add the valid inequalities given below. We first have a remark that we use in the validity proofs.

Remark 1. If $\bar{V}$ is the set of all customers visited by vehicle $k \in K$ in period $t \in T$, then for every subset $S \subseteq \bar{V}$, we have $v^{tk} \geq v(S)$ since the travel times are nonnegative and satisfy triangle inequality.

Proposition 1. *For $t \in T$ and $k \in K$, the inequality*

$$v^{tk} \geq \sum_{i \in V \setminus \{0\}} 2d_{\{0,i\}}(\min_{e \in \delta(i)} d_e y_i^{tk} + a_i^{tk} + b_i^{tk})/q \tag{35}$$

is a valid inequality for the second Benders reformulation.

Proof. Let $t \in T$, $k \in K$ and (v, y, z, a, b) be a feasible solution for the second Benders reformulation and $\bar{V} = \{i \in V \setminus \{0\} : y_i^{tk} = 1\}$. As $y_i^{tk} = a_i^{tk} = b_i^{tk} = 0$ for all $i \in V \setminus (\{0\} \cup \bar{V})$, the right hand side of inequality (35) is equal to

$$\sum_{i \in \bar{V}} 2d_{\{0,i\}}(\min_{e \in \delta(i)} d_e + a_i^{tk} + b_i^{tk})/q.$$

By Remark 1, we have $v(\bar{V}) \geq 2d_{\{0,i\}}$ for each $i \in \bar{V}$. Hence,

$$\sum_{i \in \bar{V}} 2d_{\{0,i\}}(\min_{e \in \delta(i)} d_e + a_i^{tk} + b_i^{tk})/q \leq \sum_{i \in \bar{V}} v(\bar{V})(\min_{e \in \delta(i)} d_e + a_i^{tk} + b_i^{tk})/q$$

and since $\sum_{i \in \bar{V}}(\min_{e \in \delta(i)} d_e + a_i^{tk} + b_i^{tk}) \leq q$, we have

$$v(\bar{V}) \sum_{i \in \bar{V}} (\min_{e \in \delta(i)} d_e + a_i^{tk} + b_i^{tk})/q \leq v(\bar{V}).$$

Finally, we have $v^{tk} \geq v(\bar{V})$ and the inequality is satisfied. □

Proposition 2. *For $t \in T$, $k \in K$ and for distinct i, j, l in $V \setminus \{0\}$, the inequality*

$$v^{tk} \geq v(\{i\})y_i^{tk} + (v(\{i,j\}) - v(\{i\}))y_j^{tk} + (v(\{i,j,l\}) - v(\{i,j\}))y_l^{tk} \tag{36}$$

is a valid inequality for the second Benders reformulation.

Proof. Let $t \in T$, $k \in K$, i, j, l be distinct nodes in $V \setminus \{0\}$, (v, y, z, a, b) be a feasible solution for the second Benders reformulation and $S = \{i' \in \{i, j, l\} : y_{i'}^{tk} = 1\}$. Next, we show, case by case, that the right hand side of inequality (36) is less than or equal to $v(S)$. Then by Remark 1, we have that (v, y, z, a, b) satisfies inequality (36).

- If $y_i^{tk} = y_j^{tk} = y_l^{tk} = 1$, then the right hand side becomes $v(\{i,j,l\})$.
- If $y_i^{tk} = y_j^{tk} = 1$ and $y_l^{tk} = 0$, then the right hand side becomes $v(\{i,j\})$.
- If $y_i^{tk} = y_l^{tk} = 1$ and $y_j^{tk} = 0$, then the right hand side is equal to $v(\{i\}) + (v(\{i,j,l\}) - v(\{i,j\}))$ This is less than or equal to $2d_{\{0,i\}} + d_{\{0,j\}} + d_{\{j,i\}} + d_{\{i,l\}} + d_{\{l,0\}} - (d_{\{0,i\}} + d_{\{i,j\}} + d_{\{j,0\}})$, which simplifies to $d_{\{0,i\}} + d_{\{i,l\}} + d_{\{l,0\}} = v(\{i,l\})$.
- If $y_i^{tk} = 0$ and $y_j^{tk} = y_l^{tk} = 1$, then the right hand side becomes $(v(\{i,j\}) - v(\{i\})) + (v(\{i,j,l\}) - v(\{i,j\})) = v(\{i,j,l\}) - v(\{i\})$. This is less than or equal to $d_{\{0,i\}} + d_{\{i,j\}} + d_{\{j,l\}} + d_{\{l,0\}} - 2d_{\{0,i\}}$, which is the same as $-d_{\{0,i\}} + d_{\{i,j\}} + d_{\{j,l\}} + d_{\{l,0\}}$. By triangle inequality, we have $d_{\{i,j\}} \leq d_{\{i,0\}} + d_{\{0,j\}}$. Hence $-d_{\{0,i\}} + d_{\{i,j\}} + d_{\{j,l\}} + d_{\{l,0\}} \leq d_{\{0,j\}} + d_{\{j,l\}} + d_{\{l,0\}} = v(\{j,l\})$.
- If $y_l^{tk} = 1$ and $y_i^{tk} = y_j^{tk} = 0$, then the right hand side is $(v(\{i,j,l\}) - v(\{i,j\})$ and is less than or equal to $d_{\{0,i\}} + d_{\{i,j\}} + d_{\{j,l\}} + d_{\{l,0\}} - (d_{\{0,i\}} + d_{\{i,j\}} + d_{\{j,0\}}) = d_{\{j,l\}} + d_{\{l,0\}} - d_{\{j,0\}}$. Since by triangle inequality, we have $d_{\{j,l\}} \leq d_{\{j,0\}} + d_{\{0,l\}}$, we know that $d_{\{j,l\}} + d_{\{l,0\}} - d_{\{j,0\}} \leq 2d_{\{0,l\}} = v(\{l\})$.
- If $y_j^{tk} = 1$ and $y_i^{tk} = y_l^{tk} = 0$, then the right hand side is $v(\{i,j\}) - v(\{i\}) = d_{\{i,j\}} + d_{\{j,0\}} - d_{\{0,i\}}$ and is not more than $2d_{\{0,j\}} = v(\{j\})$ since $d_{\{i,j\}} \leq d_{\{i,0\}} + d_{\{0,j\}}$.
- If $y_i^{tk} = 1$ and $y_j^{tk} = y_l^{tk} = 0$, then the right hand side is $v(\{i\})$.
- If $y_i^{tk} = y_j^{tk} = y_l^{tk} = 0$, then the right hand side is zero. □

Note that inequalities (36) compute the travel time of tours visiting up to three customers.

Table 1
Results for the instances with 21 nodes.

				B&C0			B&C1						B&C2						
Name	τ	m	q	Time	%-fgap	Slack	Time	%-fgap	(18)	(19)	(30)	(31)	Time	%-fgap	(18)	(33)	(35)	(36)	(41)
A	3	3	420	4.42	0.00	61.32	18.98	0.00	0.04	0.00	1.44	98.52	**7.23**	0.00	0.00	0.18	0.00	3.43	96.39
			450	20.78	0.00	78.50	36.73	0.00	0.02	0.00	0.31	99.66	**18.68**	0.00	0.00	0.20	0.10	1.89	97.81
			500	9.66	0.00	100.00	2.31	0.00	0.00	0.00	1.39	98.61	**0.54**	0.00	0.00	12.31	0.00	12.31	75.38
B	3	3	420	4.58	0.00	30.59	22.83	0.00	0.18	0.33	7.44	92.05	**4.45**	0.00	0.00	0.00	0.39	1.17	98.44
			450	5.69	0.00	38.92	17.78	0.00	0.09	0.00	3.63	96.28	**5.44**	0.00	0.00	0.21	0.21	0.41	99.17
			500	10.17	0.00	75.03	38.28	0.00	0.08	0.50	0.33	99.09	**8.99**	0.00	0.00	0.22	0.11	0.33	99.33
C	3	3	420	7.67	0.00	52.54	19.41	0.00	0.49	0.00	3.55	95.96	**6.44**	0.00	0.15	0.00	1.08	23.03	75.73
			450	9.02	0.00	66.30	30.50	0.00	0.09	0.00	3.24	96.67	**7.63**	0.00	0.00	0.23	0.94	29.47	69.36
			500	735.61	0.00	98.74	40.41	0.00	0.00	0.00	0.35	99.65	**26.44**	0.00	0.00	0.58	0.00	0.58	98.83
A	3	4	420	**0.45**	0.00	100.00	1.55	0.00	0.00	0.00	12.11	87.89	1.00	0.00	0.00	8.26	0.00	7.39	84.35
			450	**0.19**	0.00	100.00	0.52	0.00	1.32	0.00	15.79	82.89	0.28	0.00	0.00	25.00	0.00	12.50	62.50
			500	0.22	0.00	100.00	**0.22**	0.00	4.35	0.00	21.74	73.91	0.26	0.00	0.00	51.61	0.00	32.26	16.13
B	3	4	420	t.l	0.42	88.12	1261.38	0.00	0.00	0.00	0.11	99.88	**171.79**	0.00	0.06	0.39	0.00	0.67	98.88
			450	23.33	0.00	100.00	4.80	0.00	0.10	0.00	2.81	97.09	**1.01**	0.00	0.00	12.05	0.00	11.24	76.71
			500	4.94	0.00	100.00	0.58	0.00	0.00	0.00	12.24	87.76	**0.49**	0.00	0.00	23.40	0.00	5.32	71.28
C	3	4	420	137.14	0.00	100.00	**2.34**	0.00	0.32	0.00	1.62	98.06	2.48	0.00	0.00	2.85	1.14	36.47	59.54
			450	**0.50**	0.00	100.00	2.38	0.00	0.47	0.00	7.33	92.20	0.83	0.00	0.00	7.54	0.29	53.04	39.13
			500	0.39	0.00	100.00	0.50	0.00	0.00	0.00	10.59	89.41	**0.26**	0.00	0.00	7.92	0.83	83.33	7.92
A	5	3	420	623.58	0.00	76.85	1230.78	0.00	0.00	0.00	1.51	98.49	**190.14**	0.00	0.00	0.05	0.00	2.05	97.90
			450	t.l.	0.18	88.45	1447.27	0.00	0.01	0.00	0.40	99.60	**609.06**	0.00	0.00	0.04	0.00	0.00	99.96
			500	113.11	0.00	100.00	18.13	0.00	0.00	0.00	0.06	99.94	**3.47**	0.00	0.00	4.66	0.00	2.02	93.32
B	5	3	420	65.06	0.00	51.53	82.72	0.00	0.08	0.00	0.73	99.19	**28.02**	0.00	0.00	0.00	0.11	7.59	92.30
			450	329.08	0.00	73.42	491.92	0.00	0.01	0.00	0.06	99.92	**72.16**	0.00	0.00	0.06	0.13	3.45	96.37
			500	t.l.	0.13	96.28	51.02	0.00	0.00	0.00	0.00	100.00	**37.14**	0.00	0.00	0.92	0.00	0.06	99.03
C	5	3	420	**99.20**	0.00	25.01	1215.58	0.00	0.04	0.00	4.68	95.27	199.23	0.00	0.00	0.00	0.63	36.87	62.49
			450	**40.58**	0.00	41.41	389.39	0.00	0.06	0.00	2.30	97.64	188.86	0.00	0.00	0.00	0.52	23.81	75.67
			500	1317.90	0.00	67.72	1019.83	0.00	0.00	0.00	0.39	99.61	**334.89**	0.00	0.00	0.00	0.23	16.84	82.93
A	5	4	420	79.42	0.00	100.00	10.39	0.00	0.00	0.00	1.13	98.87	**6.02**	0.00	0.00	6.86	0.00	5.46	87.67
			450	15.48	0.00	100.00	**1.72**	0.00	0.00	0.00	0.44	99.56	2.39	0.00	0.00	18.96	0.00	8.26	72.78
			500	0.77	0.00	100.00	**0.56**	0.00	3.03	0.00	0.00	96.97	2.05	0.00	1.24	41.32	0.00	6.20	51.24
B	5	4	420	53.58	0.00	100.00	12.20	0.00	0.05	0.00	0.47	99.48	**4.34**	0.00	0.00	10.37	0.20	1.42	88.01
			450	3.72	0.00	100.00	**2.05**	0.00	0.00	0.00	5.17	94.83	2.27	0.00	0.00	21.03	0.34	13.45	65.17
			500	3.38	0.00	100.00	**0.58**	0.00	0.00	0.00	3.28	96.72	0.72	0.00	0.00	31.39	0.00	39.42	29.20
C	5	4	420	t.l.	1.94	84.43	t.l.	**1.63**	0.01	0.00	0.08	99.92	t.l.	1.78	0.00	0.00	0.09	2.60	97.31
			450	t.l.	0.19	97.62	t.l.	0.02	0.00	0.00	0.00	100.00	**199.27**	0.00	1.35	2.40	2.55	69.67	24.02
			500	5.73	0.00	100.00	**2.11**	0.00	0.00	0.00	0.00	100.00	2.30	0.00	0.00	3.96	0.66	57.43	37.95

To solve PVRP-DC-SO using this reformulation we do the following. Let $(\bar{v}, \bar{y}, \bar{z}, \bar{a}, \bar{b})$ be an optimal solution of the current master problem. Then, for $t \in T$ and $k \in K$ with $\bar{y}_0^{tk} = 1$, we solve

$$\min \sum_{e \in E} d_e x_e \tag{37}$$

$$\text{s.t. } x(\delta(i)) \geq 2\bar{y}_i^{tk} \qquad i \in V, \tag{38}$$

$$x(\delta(S)) \geq 2\bar{y}_i^{tk} \qquad S \subseteq V \setminus \{0\}, i \in S, \tag{39}$$

$$-x_e \geq -\bar{y}_i^{tk} \qquad e \in E, i \in e, \tag{40}$$

$$x_e \geq 0 \qquad e \in E.$$

If the optimal value is greater than $q\bar{y}_0^{tk} - \sum_{i \in V \setminus \{0\}} (\bar{a}_i^{tk} + \bar{b}_i^{tk})$, then we add the cut

$$\begin{aligned} v^{tk} \geq & (2\beta_0 - \sum_{e \in \delta(0)} \psi_{e0}) y_0^{tk} \\ & + \sum_{i \in V \setminus \{0\}} \left(2\beta_i + 2 \sum_{S \subseteq V \setminus \{0\} : i \in S} \delta_{Si} - \sum_{e \in \delta(i)} \psi_{ei} \right) y_i^{tk}, \end{aligned} \tag{41}$$

where β_i, δ_{Si}, and ψ_{ei} are the dual variables associated with constraints (38)–(40), respectively.

If solving the LP does not produce any cuts and $\bar{y}$ and $\bar{z}$ are integral, we solve a TSP problem on nodes $\bar{V} \cup \{0\}$ where we minimize the travel time, $\sum_{e \in E(\bar{V})} d_e x_e$, to compute $v(\bar{V})$. If $v(\bar{V}) > q - \sum_{i \in \bar{V}} l_i$, then the customers in set $\bar{V}$ cannot be served by one vehicle

Table 2
Results for the instances with 26 nodes.

				B&C0			B&C1						B&C2						
Name	τ	m	q	Time	%-fgap	Slack	Time	%-fgap	(18)	(19)	(30)	(31)	Time	%-fgap	(18)	(33)	(35)	(36)	(41)
A	3	3	420	**8.48**	0.00	17.52	t.l.	1.88	0.05	0.00	14.35	85.61	35.70	0.00	0.00	0.00	0.00	15.38	84.62
			450	**12.70**	0.00	28.31	649.52	0.00	0.07	0.00	6.43	93.50	30.89	0.00	0.00	0.00	0.00	14.29	85.71
			500	**11.08**	0.00	46.85	63.16	0.00	0.11	0.00	5.69	94.20	33.67	0.00	0.00	0.00	0.00	0.00	100.00
B	3	3	420	**5.64**	0.00	25.06	50.88	0.00	0.25	0.29	8.89	90.57	10.80	0.00	0.00	0.00	0.32	21.97	77.71
			450	**5.25**	0.00	34.64	81.33	0.00	0.06	0.00	9.27	90.66	13.28	0.00	0.00	0.00	0.47	36.26	63.27
			500	**6.00**	0.00	55.87	28.92	0.00	0.13	0.39	9.31	90.16	13.38	0.00	0.00	0.00	0.34	29.71	69.95
C	3	3	420	**27.75**	0.00	62.52	233.23	0.00	0.12	0.00	2.81	97.07	34.47	0.00	0.00	0.00	0.38	27.72	71.90
			450	1541.53	0.00	95.54	713.89	0.00	0.01	0.00	0.54	99.45	**260.08**	0.00	0.00	0.24	0.19	26.04	73.53
			500	38.61	0.00	100.00	26.05	0.00	0.00	0.00	0.58	99.42	**2.80**	0.00	0.00	0.75	0.19	68.56	30.50
A	3	4	420	756.99	0.00	71.40	t.l.	1.21	0.03	0.00	0.33	99.64	**693.88**	0.00	0.00	0.12	0.00	1.43	98.45
			450	t.l.	0.73	85.28	t.l.	0.50	0.01	0.00	0.10	99.89	t.l.	**0.19**	0.00	0.09	0.00	0.00	99.91
			500	2.23	0.00	100.00	19.66	0.00	0.00	0.00	1.12	98.88	**2.13**	0.00	0.00	5.08	0.00	0.00	94.92
B	3	4	420	289.83	0.00	66.86	490.59	0.00	0.02	0.09	0.42	99.47	**235.03**	0.00	0.00	0.00	0.25	15.91	83.84
			450	t.l.	**0.89**	87.51	t.l.	1.35	0.01	0.04	0.06	99.89	t.l.	1.01	0.00	0.00	0.08	1.04	98.88
			500	852.92	0.00	100.00	14.13	0.00	0.00	0.00	1.09	98.91	**12.81**	0.00	0.00	1.18	0.26	7.15	91.41
C	3	4	420	18.06	0.00	100.00	3.05	0.00	0.00	0.00	5.41	94.59	**2.05**	0.00	0.00	1.47	0.29	68.10	30.14
			450	16.53	0.00	100.00	1.45	0.00	0.00	0.00	18.58	81.42	**1.18**	0.00	0.00	2.54	0.09	88.66	8.71
			500	0.47	0.00	100.00	1.72	0.00	0.00	8.82	16.91	74.26	**0.29**	0.00	0.00	2.42	0.12	95.15	2.30
A	5	3	420	1042.06	0.00	33.19	t.l.	7.64	0.02	0.00	5.94	94.04	**705.81**	0.00	0.00	0.00	0.00	0.35	99.65
			450	1183.28	0.00	49.27	t.l.	0.64	0.03	0.00	2.43	97.53	**1008.70**	0.00	0.00	0.05	0.00	4.28	95.66
			500	t.l.	**1.27**	72.65	t.l.	2.73	0.00	0.05	0.26	99.69	t.l.	2.37	0.00	0.05	0.00	0.15	99.80
B	5	3	420	**520.06**	0.00	17.66							1559.03	0.00	0.00	0.00	0.28	14.51	85.20
			450	**377.75**	0.00	28.61	t.l.	11.13	0.03	0.06	5.34	94.57	836.06	0.00	0.00	0.00	0.36	22.42	77.22
			500	**298.58**	0.00	47.14	t.l.	0.75	0.07	0.14	1.21	98.57	890.86	0.00	0.00	0.00	0.36	0.00	99.64
C	5	3	420	t.l.	2.23	62.97	t.l.	2.89	0.04	0.10	0.85	99.01	t.l.	**1.82**	0.00	0.01	0.11	0.00	99.88
			450	t.l.	0.59	85.63	t.l.	1.38	0.00	0.04	0.17	99.79	t.l.	**0.06**	0.00	0.05	0.08	0.00	99.87
			500	65.42	0.00	100.00	6.78	0.00	0.14	0.00	0.29	99.57	**6.51**	0.00	0.00	2.75	0.39	49.65	47.21
A	5	4	420	t.l.	2.06	85.57	t.l.	1.38	0.01	0.00	0.10	99.90	t.l.	**0.83**	0.00	0.04	0.01	0.04	99.91
			450	1187.85	0.00	100.00	524.75	0.00	0.01	0.00	0.04	99.95	**30.58**	0.00	0.00	1.96	0.00	1.66	96.38
			500	19.69	0.00	100.00	6.84	0.00	0.16	0.00	0.31	99.53	**5.58**	0.00	0.00	11.30	0.00	14.49	74.20
B	5	4	420	t.l.	**0.42**	65.71	t.l.	2.39	0.01	0.00	0.39	99.60	t.l.	1.04	0.00	0.00	0.25	12.82	86.93
			450	t.l.	3.31	76.67	t.l.	**1.86**	0.00	0.00	0.19	99.81	t.l.	2.03	0.00	0.11	0.19	10.17	89.54
			500	t.l.	0.30	97.24	340.38	0.00	0.00	0.00	0.00	100.00	**24.48**	0.00	0.00	1.25	0.42	35.51	62.83
C	5	4	420	22.09	0.00	100.00	**5.98**	0.00	0.20	0.00	0.00	99.80	7.92	0.00	0.00	6.86	1.62	0.00	91.52
			450	6.98	0.00	100.00	2.63	0.00	1.46	0.00	1.46	97.07	**1.92**	0.00	0.00	32.58	5.06	0.00	62.36
			500	3.17	0.00	100.00	1.44	0.00	0.00	0.00	1.33	98.67	**0.69**	0.00	0.00	1.61	0.10	94.47	3.82

in any period. In this case, we add the feasibility cut (18) for all $k' \in K$ and $t' \in T$. Otherwise, we compare $\bar{v}^{tk}$ with $v(\bar{V})$ and if the latter is larger, we add the violated cut (33).

4. Implementation details

We have implemented three branch-and-cut algorithms to solve the PVRP-DC-SO. The first one, B&C0, is based on the mathematical model presented in Section 2. The other two algorithms, B&C1 and B&C2, are based on the Benders reformulations 1 and 2 given in Section 3, respectively. In this section we give some implementation details for these algorithms.

4.1. Improvements

A preprocessing procedure proposed by Rodríguez-Martín et al. (2019) is implemented since it significantly reduces the computation times. It works as follows. First, for each customer $i \in V \setminus \{0\}$ and period $t \in T$, if t does not appear in any of the allowed visits schedules for i, we set $y_i^{tk} = 0$ for all vehicles $k \in K$. Second, we apply a symmetry breaking strategy. This is very important because, as the PVRP-DC-SO is a vehicle routing problem with identical vehicles, for any given solution we can permute the vehicles assigned to the routes, taking into account the driver consistency, and obtain an equivalent solution with the same objective function value. The method we use to eliminate symmetries is the following. For each vehicle $k \in \{1, \ldots, |K|-1\}$, we select a customer i' and we introduce a set of constraints that prevent i' from being visited by a vehicle with an index higher than k. The constraints that avoid that customer i' is visited by a vehicle with index higher than k are: $\sum_{p \in P_{i'}} z_{i'p}^{k'} = 0$ for all $k' \in K$ such that $k' > k$. For each vehicle $k \in \{1, \ldots, |K|-1\}$, we choose the customer i' as the non previously selected customer with the largest visit frequency.

Table 3
Results for the instances with 31 nodes.

				B&C0			B&C1						B&C2						
Name	τ	m	q	Time	%-fgap	Slack	Time	%-fgap	(18)	(19)	(30)	(31)	Time	%-fgap	(18)	(33)	(35)	(36)	(41)
A	3	3	420	**597.98**	0.00	5.06							t.l.	5.53	0.00	0.00	0.00	0.30	99.70
			450	**47.25**	0.00	15.45	t.l.	42.62	0.02	0.12	19.42	80.44	164.98	0.00	0.06	0.00	0.57	0.00	99.37
			500	**23.27**	0.00	32.74	1143.63	0.00	0.02	0.13	9.28	90.57	130.88	0.00	0.00	0.00	0.37	37.00	62.63
B	3	3	420	**101.22**	0.00	9.59							842.87	0.00	0.00	0.00	0.00	0.63	99.37
			450	**65.34**	0.00	20.32	t.l.	41.67	0.04	0.03	19.35	80.58	461.13	0.00	0.00	0.00	0.00	0.00	100.00
			500	**35.48**	0.00	40.07	t.l.	11.11	0.04	0.00	5.29	94.67	460.73	0.00	0.00	0.00	0.00	0.00	100.00
C	3	3	420	t.l.	1.25	3.32							t.l.	**0.84**	0.00	0.00	0.00	0.36	99.64
			450	**10.28**	0.00	13.16							129.08	0.00	0.04	0.00	0.00	1.10	98.86
			500	**14.92**	0.00	31.60	t.l.	25.20	0.10	0.24	10.25	89.41	124.83	0.00	0.00	0.00	0.00	0.27	99.73
A	3	4	420	**1305.69**	0.00	49.28	t.l.	9.83	0.06	0.04	7.73	92.17	t.l.	1.21	0.00	0.00	0.20	15.09	84.70
			450	t.l.	2.03	71.17	t.l.	2.50	0.04	0.00	1.74	98.22	t.l.	**1.98**	0.00	0.00	0.10	10.30	89.60
			500	t.l.	0.36	88.58	174.61	0.00	0.01	0.10	0.73	99.16	**68.25**	0.00	0.00	0.12	0.20	18.53	81.15
B	3	4	420	**860.47**	0.00	61.58	t.l.	5.81	0.02	0.00	3.01	96.97	1035.44	0.00	0.00	0.00	0.00	0.64	99.36
			450	t.l.	1.93	79.15	t.l.	3.85	0.04	0.04	1.01	98.91	t.l.	**1.76**	0.00	0.03	0.00	0.00	99.97
			500	726.20	0.00	100.00	t.l.	0.86	0.01	0.03	0.15	99.82	**20.81**	0.00	0.00	0.78	0.00	1.75	97.47
C	3	4	420	**745.77**	0.00	43.85	t.l.	12.98	0.02	0.10	4.21	95.67	t.l.	2.26	0.00	0.00	0.00	0.20	99.80
			450	t.l.	**0.76**	57.48	t.l.	6.64	0.02	0.00	1.90	98.08	t.l.	0.87	0.00	0.01	0.00	0.07	99.92
			500	t.l.	0.75	89.04	t.l.	1.28	0.00	0.03	0.12	99.84	t.l.	**0.52**	0.00	0.08	0.00	0.00	99.92
A	5	3	420	t.l.	**1.39**	24.70	t.l.	23.45	0.00	0.00	13.20	86.80	t.l.	2.08	0.00	0.00	0.18	0.00	99.82
			450	t.l.	**1.25**	34.93	t.l.	12.96	0.03	0.13	9.15	90.69	t.l.	1.27	0.00	0.00	0.19	11.93	87.88
			500	t.l.	**1.15**	64.84	t.l.	5.12	0.02	0.06	1.28	98.64	t.l.	2.68	0.00	0.01	0.16	8.76	91.06
B	5	3	420																
			450	**1063.17**	0.00	10.47							t.l.	0.64	0.00	0.00	0.34	8.84	90.82
			500	**347.55**	0.00	30.84	t.l.	14.39	0.01	0.13	11.35	88.50	1198.78	0.00	0.00	0.00	0.18	16.55	83.27
C	5	3	420	t.l.	**0.35**	28.12	t.l.	10.74	0.01	0.00	5.60	94.39	t.l.	1.78	0.00	0.00	0.00	0.04	99.96
			450	t.l.	**0.25**	42.33	t.l.	21.36	0.01	0.00	4.06	95.93	t.l.	1.75	0.00	0.00	0.00	0.12	99.88
			500	t.l.	**0.31**	62.84	t.l.	6.21	0.03	0.06	0.53	99.38	t.l.	1.19	0.00	0.10	0.00	0.18	99.72
A	5	4	420	t.l.	2.22	63.61	t.l.	1.63	0.01	0.06	1.32	98.60	t.l.	**1.20**	0.00	0.00	0.17	0.00	99.83
			450	t.l.	0.41	92.66	t.l.	0.12	0.00	0.00	0.28	99.72	**76.88**	0.00	0.00	0.15	0.17	6.92	92.77
			500	66.58	0.00	100.00	34.48	0.00	0.02	0.45	0.42	99.11	**4.41**	0.00	0.64	10.54	3.51	0.00	85.30
B	5	4	420	t.l.	4.01	36.37	t.l.	18.78	0.01	0.07	6.95	92.98	t.l.	**2.84**	0.00	0.00	0.00	0.00	100.00
			450	t.l.	2.95	59.73	t.l.	8.30	0.00	0.07	2.49	97.43	t.l.	**2.64**	0.00	0.05	0.00	0.60	99.35
			500	t.l.	**0.40**	88.09	t.l.	1.41	0.01	0.12	0.24	99.63	t.l.	0.84	0.00	0.12	0.00	0.61	99.27
C	5	4	420	t.l.	2.14	80.67	t.l.	4.51	0.04	0.07	0.53	99.36	t.l.	**1.89**	0.00	0.04	0.00	0.02	99.94
			450	t.l.	1.20	86.04	t.l.	1.28	0.00	0.05	0.05	99.90	**89.35**	0.00	0.00	0.65	0.03	26.30	73.02
			500	14.11	0.00	100.00	17.64	0.00	0.21	0.00	0.28	99.50	**13.28**	0.00	0.07	2.94	0.00	0.20	96.80

4.2. Separation procedures

The general connectivity constraints (5), as well as constraints (24) and (39) can be separated exactly in polynomial time by solving min-cut/max-flow problems on appropriately defined support graphs. For example, the separation procedure of (5) is fully described in Rodríguez-Martín et al. (2019). In B&C0, inequalities (11) and (16) are separated by complete enumeration and before (5).

The separation procedures for the constraints (18), (19), (30), and (31) used in B&C1 are described in Section 3.1. For integral solutions, the violation of (18), (19) is checked only if no violated constraints (30) and (31) have been found. For fractional solutions, only (18) and (19) are separated, and only each 50 branch-and-cut nodes.

Similarly, the separation procedures for the (18), (33), and (41) used in B&C2 are described in Section 3.2. The violation of constraints (41) is checked each 100 branch-and-cut nodes, and for all integral solutions. For integral solutions, besides, constraints (18) and (33) are separated when no constraint (41) has been violated. Constraints (35) and (36) are separated exactly by simple enumeration, and only at the root node of the branch-and-cut algorithm.

5. Computational experiments

The three branch-and-cut algorithms described in the previous section were coded in C++ and ran on a personal computer with a processor Intel Core i5-10400 CPU at 2.90 GHz and 8 GB of RAM. We used CPLEX 22.1.0.0 as mixed integer linear programming solver, with the default settings. We used a routine from *Concorde TSP* to solve the min-cut problems.

Table 4
Effect of preprocessing and valid inequalities in B&C0.

					B&C0-0		B&C0-1		B&C0	
n	Name	τ	m	q	Time	%-fgap	Time	%-fgap	Time	%-fgap
21	A	3	3	420	23.91	0.00	4.63	0.00	4.42	0.00
				450	353.14	0.00	21.90	0.00	20.78	0.00
				500	6.27	0.00	14.87	0.00	9.66	0.00
	B	3	3	420	44.20	0.00	6.66	0.00	4.58	0.00
				450	260.50	0.00	6.32	0.00	5.69	0.00
				500	183.53	0.00	13.57	0.00	10.17	0.00
	C	3	3	420	46.78	0.00	8.37	0.00	7.67	0.00
				450	314.33	0.00	12.84	0.00	9.02	0.00
				500	1800.00	0.13	755.38	0.00	735.61	0.00
Aver					336.96	0.01	93.84	0.00	89.73	0.00
	A	3	4	420	0.70	0.00	2.78	0.00	0.45	0.00
				450	0.16	0.00	0.33	0.00	0.19	0.00
				500	0.11	0.00	0.42	0.00	0.22	0.00
	B	3	4	420	t.l.	0.42	t.l.	0.42	t.l.	0.42
				450	1086.66	0.00	40.35	0.00	23.33	0.00
				500	5.92	0.00	1.28	0.00	4.94	0.00
	C	3	4	420	1314.73	0.00	110.91	0.00	137.14	0.00
				450	7.23	0.00	37.79	0.00	0.50	0.00
				500	0.25	0.00	0.59	0.00	0.39	0.00
Aver					468.42	0.05	221.60	0.05	218.57	0.05

Table 5
Effect of preprocessing and valid inequalities in B&C1.

					B&C1-0		B&C1-1		B&C1	
n	Name	τ	m	q	Time	%-fgap	Time	%-fgap	Time	%-fgap
21	A	3	3	420	131.75	0.00	18.47	0.00	18.98	0.00
				450	216.55	0.00	34.98	0.00	36.73	0.00
				500	1.25	0.00	2.30	0.00	2.31	0.00
	B	3	3	420	159.14	0.00	22.75	0.00	22.83	0.00
				450	104.48	0.00	38.25	0.00	17.78	0.00
				500	287.97	0.00	37.41	0.00	38.28	0.00
	C	3	3	420	79.95	0.00	42.58	0.00	19.41	0.00
				450	94.50	0.00	26.42	0.00	30.50	0.00
				500	370.05	0.00	41.09	0.00	40.41	0.00
	Aver				160.63	0.00	29.36	0.00	25.25	0.00
	A	3	4	420	0.95	0.00	0.44	0.00	1.55	0.00
				450	0.48	0.00	0.52	0.00	0.52	0.00
				500	0.48	0.00	0.38	0.00	0.22	0.00
	B	3	4	420	t.l.	0.40	1523.86	0.00	1261.38	0.00
				450	3.59	0.00	6.61	0.00	4.80	0.00
				500	1.02	0.00	0.91	0.00	0.58	0.00
	C	3	4	420	8.97	0.00	9.81	0.00	2.34	0.00
				450	1.55	0.00	1.75	0.00	2.38	0.00
				500	0.20	0.00	0.44	0.00	0.50	0.00
	Aver				201.92	0.04	171.63	0.00	141.58	0.00

5.1. Instances

To evaluate the algorithms we conducted computational experiments on a subset of the benchmark dataset used in Rodríguez-Martín et al. (2019). These are randomly generated instances with a number of customers n in $\{20, 25, 30\}$. Customers' coordinates are in $[0, 100] \times [0, 100]$, and the depot 0 is placed at (50, 50). The weight w_i of each customer varies between 1 and 15. The number of days τ in the time horizon is 3 or 5, and the number m of vehicles available at the depot is 3 or 4. Each customer has an associated visit frequency f_i between 1 and τ, and a set of allowed visit schedules P_i with that frequency.

Note that in this dataset the visit frequency and number of visits schedules of the customers have been generated randomly among all possibilities. The spacial distribution of customers is also random. So, for the same given n, τ, and m, some instance may be more complex than others. For example, if $\tau = 3$, f_i can take the values 1, 2 or 3. Then, if a customer i has $f_i = 3$, there is only one possible visit schedule for that customer, but if $f_i = 1$ or $f_i = 2$, the set P_i may have cardinality 1, 2 or 3, and this is decided also randomly. When $\tau = 5$ the number of possibilities for each customer increases significantly.

Table 6
Effect of preprocessing and valid inequalities in B&C2.

					B&C2-0		B&C2-1		B&C2-2		B&C2	
n	Name	τ	m	q	Time	%-fgap	Time	%-fgap	Time	%-fgap	Time	%-fgap
21	A	3	3	420	74.50	0.00	11.06	0.00	10.69	0.00	7.23	0.00
				450	182.88	0.00	32.70	0.00	25.09	0.00	18.68	0.00
				500	1.31	0.00	1.55	0.00	1.03	0.00	0.54	0.00
	B	3	3	420	28.53	0.00	5.22	0.00	6.31	0.00	4.45	0.00
				450	50.28	0.00	6.61	0.00	7.78	0.00	5.44	0.00
				500	116.73	0.00	11.98	0.00	13.14	0.00	8.99	0.00
	C	3	3	420	32.17	0.00	8.63	0.00	9.16	0.00	6.44	0.00
				450	29.09	0.00	10.08	0.00	10.47	0.00	7.63	0.00
				500	86.42	0.00	25.13	0.00	22.17	0.00	26.44	0.00
Average					66.88	0.00	12.55	0.00	11.76	0.00	9.53	0.00
	A	3	4	420	1.45	0.00	1.05	0.00	1.20	0.00	1.00	0.00
				450	0.63	0.00	0.53	0.00	0.91	0.00	0.28	0.00
				500	0.73	0.00	0.38	0.00	0.39	0.00	0.26	0.00
	B	3	4	420	t.l.	0.38	259.91	0.00	193.34	0.00	171.79	0.00
				450	5.61	0.00	2.23	0.00	4.03	0.00	1.01	0.00
				500	1.22	0.00	0.92	0.00	0.48	0.00	0.49	0.00
	C	3	4	420	3.64	0.00	4.73	0.00	4.41	0.00	2.48	0.00
				450	1.56	0.00	1.30	0.00	0.98	0.00	0.83	0.00
				500	1.19	0.00	0.83	0.00	0.23	0.00	0.26	0.00
Average					201.78	0.04	30.21	0.00	22.89	0.00	19.82	0.00

For simplicity, we set the duration d_e equal to the Euclidean distance between the extremes of edge e. The maximum time limit q of a route takes values in $\{420, 450, 500\}$. We define the lower and upper bounds l_i and u_i for the service time of each customer in function of its weight, since we consider that more time should be dedicated to more important customers. Specifically, we do the following: if $w_i \leq 5$, then $l_i = 20$ and $u_i = 30$; if $5 < w_i \leq 10$, then $l_i = 30$ and $u_i = 60$; if $10 < w_i \leq 15$, then $l_i = 60$ and $u_i = 120$. There are three instances for each combination of n, τ and m, named A, B and C, which makes a total of 36 instances. As parameter q takes three values, the total number of test cases in our computational experiments is 108. The whole set of instances is available at http://dx.doi.org/10.17632/jybm8hkp9y.1.

5.2. Comparison of the algorithms

For the computation, we set $m_i = \frac{u_i+l_i}{2}$, $\rho_i^1 = 1$ and $\rho_i^2 = 0.5$ for all $i \in V \setminus \{0\}$. So $\rho_{it}(s) = s$ if $s \leq m_i$, and $\rho_{it}(s) = m_i + 0.5(s - m_i)$ if $s > m_i$. We set the big-M to $\max_{t\in\tau} \sum_{i\in V\setminus\{0\}:\exists p\in P_i:t\in p} \bar{\rho}_i$ in constraints (19) and to $v(\bar{V})$ in constraints (33). The value $\max_{t\in\tau} \sum_{i\in V\setminus\{0\}:\exists p\in P_i:t\in p} \bar{\rho}_i$ is an upper bound for the revenue that can be collected by a vehicle in a day, and $v(\bar{V})$ is the value of the TSP on the vertex set $\bar{V}$. We ran the branch-and-cut algorithms on each instance with each q value, with a time limit of 1800 s. In order not to overload the article, the full results are presented in Appendix A. Tables 1, 2 and 3 summarize the results for the instances with 21, 26 and 31 nodes respectively. For each instance (characterized by its name, number of periods τ, and number of vehicles m) and q, and for each branch-and-cut algorithm, the following data is displayed:

- *time*: Total computation time, in seconds. When the time limit is reached, we write t.l. in this column.
- *%-fgap*: Percentage gap between the objective function value of the best solution found and the lower bound at the end of the computation. It is 0 when an optimal solution is found before the time limit is reached.
- *slack*: This number is computed as the average over all customers i of $(\sum_{t\in T,k\in K}(a_i^{tk} + b_i^{tk})/f_i - l_i)/(u_i - l_i) * 100$. Therefore if the *slack* is close to zero, then q is a very tight bound. On the contrary, if it is close to 100 then q is loose. We only report it once, for B&C0, since it does not depend on the algorithm, though it is accurate only when an optimal solution has been found.
- (18), (19), (30), and (31) for B&C1: Percentage of the number of violated cuts of each type over the total number of cuts added.
- (18), (33), (35), (36), and (41) for B&C2: Percentage of the number of violated cuts of each type over the total number of cuts added.

Blank cells indicate that the algorithm was unable to find even a feasible solution within the time limit. Bold figures are used to highlight the winning algorithm, either in terms of computing time or in terms of final gap if the time limit has been reached.

We can see in Table 1 all the outputs for the instances with 20 customers (or 21 nodes). In this case, the algorithm B&C2, based on the second Benders reformulation, clearly outperforms the other two methods. In fact, B&C2 finds the optimal solution with the shortest time in 23 out of the 36 cases, while B&C0 and B&C1 do so in five and seven cases, respectively. Only one test case in this table is not solved to optimality within the time limit by any of the algorithms, $n20 - \tau5 - m4$-C with $q = 420$, though B&C1 finishes with the smallest gap.

Table 7
Results for instances with 21 nodes when $\rho_i^1 = 1$ and $\rho_i^2 = 0.2$ in the utility function.

				B&C0			B&C1			B&C2		
Name	τ	m	q	Sol	Time	%-fgap	Sol	Time	%-fgap	Sol	Time	%-fgap
A	3	3	420	22733.35	**3.88**	0.00	22733.35	17.92	0.00	22733.35	7.52	0.00
			450	23165.86	27.03	0.00	23165.86	52.59	0.00	23165.86	**23.88**	0.00
			500	23294.00	1.45	0.00	23294.00	3.36	0.00	23294.00	**1.19**	0.00
B	3	3	420	25916.98	8.19	0.00	25916.98	41.77	0.00	25916.98	**6.03**	0.00
			450	27135.56	6.73	0.00	27135.56	11.48	0.00	27135.56	**5.53**	0.00
			500	28163.27	**9.45**	0.00	28163.27	37.39	0.00	28163.27	11.05	0.00
C	3	3	420	25523.52	8.80	0.00	25523.52	24.05	0.00	25523.52	**7.63**	0.00
			450	26171.35	11.08	0.00	26171.35	23.80	0.00	26171.35	**7.50**	0.00
			500	26762.53	596.81	0.00	26762.53	23.64	0.00	26762.53	**21.56**	0.00
A	3	4	420	23294.00	16.41	0.00	23294.00	**1.27**	0.00	23294.00	1.47	0.00
			450	23294.00	0.56	0.00	23294.00	**0.53**	0.00	23294.00	1.09	0.00
			500	23294.00	**0.13**	0.00	23294.00	0.44	0.00	23294.00	0.42	0.00
B	3	4	420	28361.82	t.l.	0.23	28372.57	t.l.	0.15	28377.72	**198.70**	0.00
			450	28428.00	28.11	0.00	28428.00	5.03	0.00	28428.00	**2.11**	0.00
			500	28428.00	1.48	0.00	28428.00	**0.72**	0.00	28428.00	0.77	0.00
C	3	4	420	26778.00	134.78	0.00	26778.00	14.47	0.00	26778.00	**3.27**	0.00
			450	26778.00	**0.95**	0.00	26778.00	1.69	0.00	26778.00	1.16	0.00
			500	26778.00	4.72	0.00	26778.00	**0.70**	0.00	26778.00	0.94	0.00
A	5	3	420	38737.99	1528.19	0.00	38737.99	1700.03	0.00	38737.99	**202.75**	0.00
			450	39239.89	t.l.	0.12	39241.31	t.l.	0.06	39241.31	**380.36**	0.00
			500	39286.00	147.09	0.00	39286.00	9.47	0.00	39286.00	**4.81**	0.00
B	5	3	420	42293.77	166.81	0.00	42293.77	139.98	0.00	42293.77	**50.58**	0.00
			450	43116.13	157.91	0.00	43116.13	204.41	0.00	43116.13	**89.28**	0.00
			500	43628.87	t.l.	0.02	43638.00	293.81	0.00	43638.00	**52.19**	0.00
C	5	3	420	46713.17	576.96	0.00	46713.17	t.l.	1.96	46713.17	**226.41**	0.00
			450	49795.01	794.78	0.00	49795.01	1430.25	0.00	49795.01	**338.66**	0.00
			500	51858.74	607.55	0.00	51858.74	754.25	0.00	51858.74	**354.78**	0.00
A	5	4	420	39286.00	**2.75**	0.00	39286.00	8.48	0.00	39286.00	3.84	0.00
			450	39286.00	14.31	0.00	39286.00	1.97	0.00	39286.00	**1.88**	0.00
			500	39286.00	0.73	0.00	39286.00	**0.48**	0.00	39286.00	0.61	0.00
B	5	4	420	43638.00	458.56	0.00	43638.00	**1.89**	0.00	43638.00	5.67	0.00
			450	43638.00	20.42	0.00	43638.00	**0.83**	0.00	43638.00	1.55	0.00
			500	43638.00	0.91	0.00	43638.00	**0.52**	0.00	43638.00	1.38	0.00
C	5	4	420	52571.04	t.l.	0.85	52585.81	t.l.	0.82	52646.35	t.l.	**0.69**
			450	52986.11	t.l.	0.06	53012.39	t.l.	0.01	53018.00	**320.28**	0.00
			500	53018.00	10.97	0.00	53018.00	**0.84**	0.00	53018.00	1.80	0.00

If we look only at B&C0 and B&C1, the comparison is slightly favorable to the latter. B&C1 outperforms B&C0 in terms of computing times in 17 cases, while the contrary happens in 16 cases, and B&C0 fails to solve to optimality three more cases than B&C1. Therefore, we could conclude that for the instances with 20 customers, the two methods based on Benders reformulations are better options than the branch-and-cut based on the initial mathematical model.

Regarding the effect of the parameter q, we observe that when already $q = 420$ allows to dedicate to each customer the maximum service time (i.e., the slack is 100), then the problem gets generally easier as q increases (see for example what happens with $n20$-$\tau3$-$m4$-C). When q forces a restriction on the service times, and the subsequent slack increases with q without reaching the value 100, then, generally, the hardest case for an instance is when the slack is between 70 and 80 percent (see for example the results for instance $n20$-$\tau3$-$m3$-A).

Table 2 shows the results for the instances with 25 customers. We observe that B&C0 outperforms the other two methods in 13 out to the 36 cases, B&C2 does so in 21 cases, and B&C1 in two cases. On the other hand, B&C2 fails to find the optimum within the time limit in eight cases, and B&C0 fails in nine cases; B&C1 does so in 14 cases and moreover it is unable to find even a feasible solution in one case. In this table we can see that the instances with a temporal horizon of three periods (first half of the table) are easier than those with 5 periods. For a given time horizon, the hardest instances are those with four vehicles, and B&C2 outperforms B&C0 in those cases.

The results for the larger instances with 30 customers are displayed in Table 3. In these cases, B&C1 has the worst performance once more, finding the optimal solution within the time limit in only four out of 36 cases, and being unable to find a solution in six cases. If we look at the whole table, we see that B&C0 gives the best result in 21 cases while B&C2 wins in 14 cases, they reach the time limit without proving optimality in 19 and 20 cases, respectively, and fail to find even a feasible solution in one case. Again, as in Table 2, there is a difference between the instances with three and five periods. When $\tau = 3$, B&C0 performs clearly better than B&C2 but, when $\tau = 5$, B&C0 is able to find the optimum in one less case than B&C2 and the two methods have similar

Table 8
Results for instances with 21 nodes when $\rho_i^1 = 1$ and $\rho_i^2 = 0.8$ in the utility function.

				B&C0			B&C1			B&C2		
Name	τ	m	q	Sol	Time	%-fgap	Sol	Time	%-fgap	Sol	Time	%-fgap
A	3	3	420	25559.88	**5.28**	0.00	25559.88	26.34	0.00	25559.88	14.77	0.00
			450	27122.81	34.27	0.00	27122.81	40.36	0.00	27122.81	**33.55**	0.00
			500	27581.00	2.30	0.00	27581.00	2.56	0.00	27581.00	**1.27**	0.00
B	3	3	420	26673.59	**5.28**	0.00	26673.59	17.27	0.00	26673.59	5.84	0.00
			450	29385.57	**4.98**	0.00	29385.57	12.64	0.00	29385.57	7.05	0.00
			500	32784.58	12.72	0.00	32784.58	51.48	0.00	32784.58	**12.08**	0.00
C	3	3	420	27533.69	**6.20**	0.00	27533.69	29.47	0.00	27533.69	13.78	0.00
			450	29655.53	**8.33**	0.00	29655.53	31.94	0.00	29655.53	12.30	0.00
			500	31675.11	248.16	0.00	31675.11	32.91	0.00	31675.11	**27.91**	0.00
A	3	4	420	27581.00	6.27	0.00	27581.00	**1.06**	0.00	27581.00	1.50	0.00
			450	27581.00	**0.19**	0.00	27581.00	0.44	0.00	27581.00	0.88	0.00
			500	27581.00	**0.13**	0.00	27581.00	0.28	0.00	27581.00	0.20	0.00
B	3	4	420	33404.78	t.l.	0.76	33490.91	1415.36	0.00	33490.91	**208.67**	0.00
			450	33657.00	50.52	0.00	33657.00	9.94	0.00	33653.87	**3.06**	0.00
			500	33657.00	8.64	0.00	33657.00	**0.38**	0.00	33657.00	1.36	0.00
C	3	4	420	31737.00	95.63	0.00	31737.00	16.97	0.00	31737.00	**4.36**	0.00
			450	31737.00	25.02	0.00	31737.00	1.91	0.00	31737.00	**0.78**	0.00
			500	31737.00	0.91	0.00	31737.00	0.88	0.00	31737.00	**0.63**	0.00
A	5	3	420	44691.28	439.63	0.00	44691.28	1610.53	0.00	44691.28	**287.41**	0.00
			450	46335.66	t.l.	0.30	46335.66	1344.58	0.00	46335.66	**619.41**	0.00
			500	46474.00	192.27	0.00	46474.00	**7.39**	0.00	46474.00	10.95	0.00
B	5	3	420	47174.26	126.52	0.00	47174.26	124.13	0.00	47174.26	**37.88**	0.00
			450	50062.00	203.98	0.00	50062.00	386.69	0.00	50062.00	**64.69**	0.00
			500	51714.58	t.l.	0.06	51747.00	**40.84**	0.00	51747.00	116.83	0.00
C	5	3	420	47571.65	**27.98**	0.00	46162.20	t.l.	9.58	47571.65	162.94	0.00
			450	52419.18	**37.00**	0.00	52419.18	274.97	0.00	52419.18	146.05	0.00
			500	58685.84	t.l.	0.23	58685.84	t.l.	0.17	58685.84	**791.55**	0.00
A	5	4	420	46474.00	44.28	0.00	46474.00	13.20	0.00	46474.00	**3.63**	0.00
			450	46474.00	**0.92**	0.00	46474.00	2.77	0.00	46474.00	1.66	0.00
			500	46474.00	1.22	0.00	46474.00	**0.39**	0.00	46474.00	0.91	0.00
B	5	4	420	51747.00	69.45	0.00	51747.00	12.33	0.00	51747.00	**6.81**	0.00
			450	51747.00	53.59	0.00	51747.00	4.05	0.00	51747.00	**1.42**	0.00
			500	51747.00	1.38	0.00	51747.00	**0.91**	0.00	51747.00	0.98	0.00
C	5	4	420	61449.63	t.l.	**2.31**	61434.05	t.l.	2.33	61442.16	t.l.	2.32
			450	62705.21	t.l.	0.26	62844.56	t.l.	0.04	62867.00	**1172.59**	0.00
			500	62867.00	90.17	0.00	62867.00	**1.56**	0.00	62867.00	2.80	0.00

performances. Note also that for one test case in the table, $n30$-$\tau5$-$m3$-B with $q = 420$, none of the methods is able to find even a feasible solution.

Finally, note that the Benders subproblems decompose for each vehicle and each period. Consequently, the branch-and-cut algorithms based on the Benders reformulations (B&C1 and B&C2) perform better when the number of vehicles and the number of periods are larger, and in those cases they outperform B&C0. The hardest instances in our benchmark are those with a time horizon of five periods ($\tau = 5$), and four vehicles ($m = 4$). There are nine of such instances for each problem size (see the bottom nine lines of the tables). We can see that B&C2 is the method that provides the best results, either in terms of computing time or final gap, in three out of nine cases when $n = 20$ (see Table 1), in six out of nine cases when $n = 25$ (see Table 2), and in eight out of nine cases when $n = 30$ (see Table 3). When looking at the performance of B&C0 on these instances, the winning cases are zero out of nine for $n = 20$, and one out of nine for $n = 25$ and $n = 30$. That is, one of the Benders based branch-and-cut algorithms is better than B&C0 in 25 out of the 27 hardest instances in our benchmark set.

5.3. Effect of preprocessing and valid inequalities

In this section we evaluate the impact of the preprocessing procedure described in Section 4.1, and the different valid inequalities proposed, on the performance of the three branch-and-cut methods presented. To this end we conducted several computational experiments on the subset of instances with 21 nodes, three days, and three and four vehicles. Tables 4 to 6 show the obtained results. In the lines *aver* we report average results for the nine instances with the same number of vehicles.

The algorithms compared are:

In Table 4:

- B&C0-0: B&C0 without preprocessing and without valid inequalities (16).

Table 9
Results for instances with 21 nodes and larger service time intervals.

Name	τ	m	q	B&C0 Sol	B&C0 Time	B&C0 %-fgap	B&C1 Sol	B&C1 Time	B&C1 %-fgap	B&C2 Sol	B&C2 Time	B&C2 %-fgap
A	3	3	420	27039.49	**2.47**	0.00	27039.49	24.31	0.00	27039.49	7.89	0.00
			450	29224.72	**2.88**	0.00	29224.72	10.91	0.00	29224.72	7.48	0.00
			500	32175.70	**3.09**	0.00	32175.70	12.20	0.00	32175.70	9.00	0.00
B	3	3	420	27295.12	6.88	0.00	27295.12	89.88	0.00	27295.12	**5.52**	0.00
			450	30463.60	5.67	0.00	30463.60	17.94	0.00	30463.60	**5.23**	0.00
			500	34831.72	6.22	0.00	34831.72	19.97	0.00	34831.72	**5.39**	0.00
C	3	3	420	28588.30	**7.94**	0.00	28588.30	56.34	0.00	28588.30	13.25	0.00
			450	31109.37	**11.53**	0.00	31109.37	68.91	0.00	31109.37	18.13	0.00
			500	34507.01	**8.23**	0.00	34507.01	24.64	0.00	34507.01	15.88	0.00
A	3	4	420	34285.26	14.44	0.00	34285.26	15.22	0.00	34285.26	**12.25**	0.00
			450	35600.42	73.55	0.00	35600.42	37.22	0.00	35600.42	**14.48**	0.00
			500	36155.00	129.03	0.00	36155.00	**0.94**	0.00	36155.00	1.48	0.00
B	3	4	420	37797.45	7.06	0.00	37797.45	30.66	0.00	37797.45	**6.34**	0.00
			450	39917.56	9.92	0.00	39917.56	23.72	0.00	39917.56	**8.13**	0.00
			500	42754.99	34.97	0.00	42754.99	77.66	0.00	42754.99	**31.50**	0.00
C	3	4	420	36701.47	**82.84**	0.00	36701.47	200.59	0.00	36701.47	93.64	0.00
			450	38537.65	182.88	0.00	38537.65	568.92	0.00	38537.65	**143.16**	0.00
			500	40704.18	t.l.	0.84	40704.18	t.l.	0.23	40704.18	**205.83**	0.00
A	5	3	420	47689.42	**24.34**	0.00	47689.42	59.66	0.00	47689.42	34.77	0.00
			450	51145.06	**18.98**	0.00	51145.06	135.88	0.00	51145.06	32.91	0.00
			500	55775.39	**32.45**	0.00	55775.39	75.77	0.00	55775.39	56.75	0.00
B	5	3	420	49533.69	33.02	0.00	49533.69	49.34	0.00	49533.69	**20.06**	0.00
			450	53373.71	40.19	0.00	53373.71	76.19	0.00	53373.71	**27.94**	0.00
			500	58906.89	55.63	0.00	58906.89	56.69	0.00	58906.89	**20.30**	0.00
C	5	3	420	48421.84	**39.86**	0.00	48421.84	492.50	0.00	48421.84	213.61	0.00
			450	54130.77	**58.89**	0.00	54130.77	245.03	0.00	54130.77	127.41	0.00
			500	61648.22	t.l.	0.49	61648.22	t.l.	0.55	61648.22	**365.59**	0.00
A	5	4	420	57283.82	t.l.	2.08	57268.47	t.l.	1.56	57303.61	**1465.47**	0.00
			450	59598.20	t.l.	**1.21**	59346.05	t.l.	2.10	59366.83	t.l.	1.36
			500	60843.57	t.l.	0.01	60850.00	302.86	0.00	60850.00	**299.58**	0.00
B	5	4	420	62501.99	100.22	0.00	62501.99	222.66	0.00	62501.99	**41.75**	0.00
			450	65281.36	594.61	0.00	65281.36	250.42	0.00	65281.36	**54.47**	0.00
			500	67874.40	t.l.	0.13	67876.22	730.36	0.00	67876.22	**130.14**	0.00
C	5	4	420	66676.85	t.l.	0.23	66676.85	t.l.	0.41	66676.85	**979.19**	0.00
			450	70595.89	t.l.	0.81	70595.89	t.l.	0.77	70595.89	t.l.	**0.42**
			500	76250.53	t.l.	1.93	76030.15	t.l.	2.19	76250.53	t.l.	**1.44**

- B&C0-1: B&C0 with preprocessing and without valid inequalities (16).
- B&C0: Complete branch-and-cut algorithm.

In Table 5:

- B&C1-0: B&C1 without preprocessing and without valid inequalities (20)–(21).
- B&C1-1: B&C1 with preprocessing and without valid inequalities (20)–(21).
- B&C1: Complete branch-and-cut algorithm.

In Table 6:

- B&C2-0: B&C2 without preprocessing and without valid inequalities (34)–(36).
- B&C2-1: B&C2 with preprocessing and without valid inequalities (34)–(36).
- B&C2-2: B&C2 with preprocessing, with (34), and without (35)–(36).
- B&C2: Complete branch-and-cut algorithm.

We can see that the use of the preprocessing, which combines variable fixing and symmetry breaking, produces a significant reduction of the average computing times in the three cases. The results also show that all the valid inequalities proposed contribute to further improve the performance of the algorithms. To be precise, and looking at Tables 4 to 6, in the case of B&C0, using the preprocessing results in a percentage reduction of 60.83% in the average computing time, and a further 2.26% reduction is obtained with the valid inequalities. In the case of B&C1, the preprocessing produces an average percentage reduction of 44.56% in the time, and the valid inequalities give an additional 16.99% reduction. Finally, in the case of B&C2, the average computing time is reduced by 84.08% with the preprocessing, another 18.96% when separating (34), and a further 15.29% when separating also (35) and (36).

5.4. Impact of the utility function and the service times bounds

In the PVRP-DC-SO, the objective is to maximize the total weighted utility obtained by serving the customers. The service time s devoted to each customer i in each visit must lie in the interval $[l_i, u_i]$, being m_i the mean service time, and the utility of visiting a customer i at period t is given by the piecewise function $\rho_{it}(s)$, defined as $\rho_{it}(s) = \rho_i^1 s$ if $s \leq m_i$ and $\rho_{it}(s) = \rho_i^1 m_i + \rho_i^2(s - m_i)$ if $s > m_i$. In this section we perform some further computational experiments to get some insight on the impact of the utility function and the service time bounds on the performance of the algorithms.

We first evaluate the impact of the utility function definition. To this end, we run the three branch-and-cut algorithms on the instances with 21 nodes, setting the parameters ρ_i^2 to 0.2 and 0.8. We keep $\rho_i^1 = 1$. Note that, with respect to the results given in Section 5.2 and Appendix A, where $\rho_i^2 = 0.5$, these new settings imply to reduce the utility due to the part to the service time that exceeds m_i (when $\rho_i^2 = 0.2$), or to augment it (when $\rho_i^2 = 0.8$). The results are displayed in Tables 7 and 8. The first thing to notice is that when $\rho_i^2 = 0.2$ there is an average reduction of around 7% in the optimal value with respect to the results with $\rho_i^2 = 0.5$, and when $\rho_i^2 = 0.8$ there is an average increment of around 7% in the optimal value. The computing times are similar for the three parameter settings. Regarding the comparison among the three branch-and-cut algorithms in Tables 7 and 8, it is still favorable to B&C2. In fact, this algorithm gives the best results in 22 out of the 36 cases reported in Table 7, and in 18 out of the 36 cases in Table 8. For the hardest instances in the tables, those with $\tau = 5$ and $m = 4$, the two algorithms based on the Benders reformulations clearly outperform B&C0.

To measure the impact of the service times, we made an experiment enlarging the gap between their lower and upper bounds. To be precise, for each customer i, we set $l_i = 20$ and $u_i = 40$ if $w_i \leq 5$, $l_i = 30$ and $u_i = 90$ if $5 < w_i \leq 10$, and $l_i = 60$ and $u_i = 180$ if $10 < w_i \leq 15$. This implies to double the length of the service time intervals, with respect to the settings in Section 5.2. The new results are shown in Table 9. Comparing these results with the previous ones (see Tables 1, 10, 13 and 16), we see that the larger service times make the problem harder to solve, especially when $\tau = 5$ and $m = 4$. In that case, the number of instances not solved to optimality within the time limit, increases from two to seven (out of nine) for B&C0, from two to five for B&C1, and from one to three for B&C2. Still, B&C2 performs better than the other two methods.

6. Conclusions

In this paper we addressed a new variant of the Periodic VRP where, besides imposing driver consistency, the times used to serve the customers have to be optimized in order to maximize the benefit/utility the carrier gets. This complex problem has not been studied before, and we presented three mathematical models for it, two of them based on Benders reformulation, and implemented the corresponding branch-and-cut exact algorithms. We evaluated the performance of the three methods on benchmark instances from the literature with up to 31 nodes, five periods, four vehicles, and different time limits on the routes. Our computational study indicates that the algorithms based on the initial mathematical formulation and on the second Benders reformulation are more effective, the latter being the best option for the instances with $n = 20$ and also for the hardest instances, i.e., those with larger number of customers, five periods, and four vehicles.

The current work can be extended in several directions. First, different utility functions may be considered. Second, depending on the application setting, additional consistency measures may be incorporated. Third, skill sets of drivers may be considered in assigning them to customers. Fourth, time dependency of travel times may be taken into account. We believe these and other extensions give rise to very interesting and challenging practical problems.

Data availability

The data used is available at http://dx.doi.org/10.17632/jybm8hkp9y.1.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.trb.2022.11.004.

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