The Period Vehicle Routing Problem and its Extensions

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Summary. This chapter presents an overview of the Period Vehicle Routing Problem, a generalization of the classic vehicle routing problem in which driver routes are constructed over a period of time. We survey the evolution of the PVRP and present a synopsis of modeling and solution methods, including classical heuristics, metaheuristics, and mathematical programming based methods. We review three important variants of the problem: the PVRP with Time Windows, the Multi-Depot PVRP, and the PVRP with Service Choice. We present case studies and highlight related implementation issues, including metrics that quantify the operational complexity of implementing periodic delivery routes. Finally, we discuss potential directions for future work in the area.

Key words: Vehicle routing; periodic distribution problems.

1 Introduction

With rising fuel costs and increased competitiveness among supply chains, vehicle delivery operations have come under greater scrutiny. Particularly, in periodic delivery operations, where deliveries are made to a set of customers over multiple days, optimizing these repetitive operations can add up to significant cost savings. Periodic deliveries occur in a wide range of applications, including courier services, elevator maintenance and repair, vending machine replenishment, the collection of waste and the delivery of interlibrary loan material. These problems can be modeled as Period Vehicle Routing Problems

B. Golden et al. (eds.), *The Vehicle Routing Problem*,
doi: 10.1007/978-0-387-77778-8_4,

(PVRP). The PVRP was introduced in the seminal paper by Beltrami and Bodin in 1974 and has evolved into a significant body of work with several exciting variants and applications arising in recent years.

The PVRP is a generalization of the classic vehicle routing problem in which vehicle routes must be constructed over multiple days (we use "day" as a general unit of time throughout this chapter). During each day within the planning period, a fleet of capacitated vehicles travels along routes that begin and end at a single depot. The underlying graph G=(N,A) is assumed to be a complete network with known travel costs along the set of arcs, A. The set of nodes, N, includes the depot and customers that are visited with predetermined frequency over the planning period. The objective of the PVRP is to find a set of tours for each vehicle that minimizes total travel cost while satisfying operational constraints (vehicle capacity and visit requirements).

Let D be the set of days, indexed $d \in D$, that constitute the planning period. We define the following:

A *schedule* is a collection of days within the planning period in which nodes receive service. Allocating a node to a schedule implies that the node will receive service in every day of that schedule.

Denote the set of all schedules by S and index this set by $s \in S$. Each schedule in S is fully described by a vector a_{sd} such that:

$$a_{sd} = \begin{cases} 1 & \text{If day } d \in D \text{ belongs in schedule } s \in S \\ 0 & \text{Otherwise} \end{cases} \tag{1}$$

The PVRP is defined as follows:

Given: A complete network graph G=(N,A) with known arc costs $c_{ij}, \forall (i,j) \in A$; a planning period of $|D|$ days indexed by d; a depot node indexed $i = 0$; a set of customer nodes $N_c = N \backslash \{0\}$ with each node $i \in N_c$ having a total demand of W_i over the planning period, and requiring a fixed number of visits f_i; a set of vehicles K each with capacity C; a set of schedules S.

Find: An allocation of customer nodes to schedules such that each node is visited the required number of times; a routing of vehicles for each day to visit the selected nodes during that day; with

Objective: Minimum cost of visiting the nodes.

From the above definition, it can be seen that PVRP involves three simultaneous decisions:

- Select a schedule from a candidate set of schedules for each node
- Assign a set of nodes to be visited by each vehicle on each day
- Route the vehicles for each day of the planning period

Note that in the classic VRP, only the last two decisions need to be made, and over a single day only. In the PVRP, each node requires a number of visits

f_i during the planning period. Hence, for each node $i \in N_c$, the PVRP must choose a schedule from a non-empty subset of candidate schedules $S_i \subseteq S$ such that:

$$S_i = \{s \in S \ : \ \sum_{d \in D} a_{sd} = f_i\}. \tag{2}$$

Note that if $|S_i| = 0$ for any $i \in N_c$, there is no feasible solution to the problem, as no schedule can satisfy the visit requirements of node i. Further, if $|S_i| = 1, \ \forall i \in N_c$, each node has only one possible schedule that can satisfy its visit requirement. In this case, the exact allocation of nodes to schedules is known and the problem decomposes into $|D|$ separate VRP problems.

Sometimes additional constraints may be imposed. Some formulations constrain the maximum length (distance or time) to a maximum length, L. In the PVRP literature, it is assumed that a fraction $1/f_i$ of the total demand has to be delivered to customer i in each visit. Hence at each visit, a demand of $w_i = W_i/f_i$ is delivered.

The following decision variables are used in various formulations of the PVRP:

$$x_{ijk}^d = \begin{cases} 1 & \text{If vehicle } k \in K \text{ traverses arc } (i,j) \in A \text{ on day } d \in D \\ 0 & \text{Otherwise} \end{cases} \tag{3a}$$

$$y_{ik}^s = \begin{cases} 1 & \text{If vehicle } k \in K \text{ visits node } i \in N_c \text{ on schedule } s \in S \\ 0 & \text{Otherwise} \end{cases} \tag{3b}$$

Some formulations use aggregated versions of the above variables as follows:

$$\tilde{x}_{ik}^d = \sum_{j \in N} x_{ijk}^d = \begin{cases} 1 & \text{If vehicle } k \in K \text{ visits node } i \in N_c \text{ on day } d \in D \\ 0 & \text{Otherwise} \end{cases} \tag{3c}$$

$$z_i^s = \sum_{k \in K} y_{ik}^s = \begin{cases} 1 & \text{If node } i \in N_c \text{ is visited on schedule } s \in S \\ 0 & \text{Otherwise} \end{cases} \tag{3d}$$

Section 2 traces the evolution of the PVRP in the literature and the various solution methods proposed for this problem. The above notation is used to present some PVRP formulations in Section 2.2. Section 3 presents important variants of the PVRP. Section 4 discusses some issues that arise in the implementation of the PVRP. We review papers that describe the implementation of the PVRP in Section 4.1 and describe metrics that quantify operational complexity in Section 4.2. Finally, Section 5 examines possible future research directions for the PVRP.

2 Evolution of PVRP Models and Solution Methods

Figure 1 presents an overview of this section, in which we survey the development of the PVRP from identification to definition to select literature

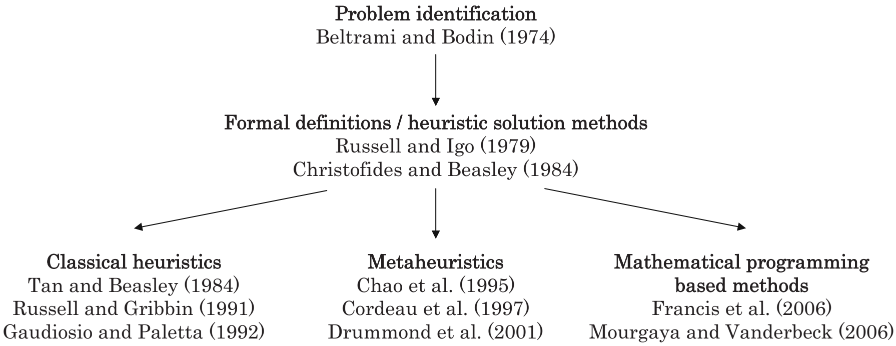

Fig. 1. Evolution of models and solution methods for the PVRP.

regarding solution methods. In particular, we focus on two papers that define the PVRP from different viewpoints – the Assignment Routing Problem that emphasizes the difficulty of the assignment decision (as in Russell and Igo [52]), and the Period Routing Problem that emphasizes the difficulty of the related routing problem (as in Christofides and Beasley [16]). Two different formulations of the PVRP are presented. Finally, we survey the literature that led to the development of current solution methods for the PVRP.

2.1 Motivating Problems

The first problem motivating the PVRP is introduced by Beltrami and Bodin [8] for assigning hoist compactor trucks in municipal waste collection. The authors describe a situation in which garbage sites need to be visited with different frequencies. They propose heuristics to solve the problem, but do not formulate or define the problem formally; however, they do include a good exposition of the difficulty of the PVRP, as compared to the standard VRP.

The example in Beltrami and Bodin [8] has nodes of two types: those demanding service three days a week and those demanding service six days a week. They allow two schedules MWF (Monday-Wednesday-Friday visits) and TRS (Tuesday-Thursday-Saturday) that visit nodes at the same frequency of three visits per week. Nodes with desired frequency of $f_i = 6$ are duplicated; each node and its copy are assigned to different three-day schedules. Thus three different service options are created while operating only two schedules.

In the paper, Beltrami and Bodin adopt a cluster-first, route-second approach since the agency operating the vehicles "decided *a priori* the day assignment for each site". After such an *a priori* assignment, the nodes to be visited on each day of the week are known and independent VRPs are solved for each day of the week. In fact, given this assignment for the problem in

Beltrami and Bodin [8], only two VRPs need to be solved – one for the MWF days and another for the TRS days.

In Foster and Ryan [27], the authors discuss a periodic variant of the Vehicle Scheduling Problem (VSP) that does not address the issues of schedules and visit frequency directly. This periodic VSP is found as the first stage of many cluster-first, route-second approaches to the PVRP. Foster and Ryan envision the VSP as a vehicle routing problem with some additional constraints such as: visiting customers on specified days; evenly spaced deliveries during the planning period; balancing the load between multiple visits to the same customer; routes capacitated by a maximum route duration; and preventing extreme variations in the fleet size. The authors handle these constraints by designing suitably restricted “feasible routes”, designed using a modified shortest path algorithm.

2.2 Defining the PVRP

Russell and Igo [52] provide a formal definition of the PVRP as the “Assignment Routing Problem”. Here the authors specifically examine the difficulties of choosing a schedule for each node together with solving the routing problem, which is not directly addressed in the first two papers. The authors consider the Assignment Routing Problem as a mixed integer problem, imposing constraints on the vehicle capacity as well as the maximum duration of any route. Additionally, each node has a set of permissible “day assignments” which is similar in spirit to the set S_i defined in eq. (2). The authors do not formulate the problem or attempt to solve it optimally, but propose three heuristics instead. The viewpoint presented in Russell and Igo [52] is that the problem is one of picking a valid day combination for a specified service frequency.

Christofides and Beasley [16] present the first formulation of the PVRP. They define the PVRP as the problem of designing a set of routes for each day of a given $|D|-$day planning period to meet the required customer visit frequency. They present an integer programming formulation of the PVRP using two sets of decision variables – one for the assignment of customers to schedules, and another for the routing of a given vehicle on a given day. The formulation follows the VRP formulation of Golden et al. [35]. Their formulation is presented below with modified notation. They use three decision variables: x^d_{ijk} defined in eq. (3a), z^s_i defined in eq. (3d), and a binary aggregate decision variable v^d_i which takes value 1 if node $i \in N_c$ is visited on day $d \in D$, and value 0 otherwise.

The formulation for the PVRP by Christofides and Beasley [16] is:

$$\min \sum_{d \in D} \sum_{(i,j) \in A} \sum_{k \in K} c_{ij} x^d_{ijk} \tag{4a}$$

subject to

$$\sum_{s \in S_i} z_i^s = 1 \qquad \forall i \in N_c \tag{4b}$$

$$v_i^d = \sum_{s \in S_i} z_i^s a_{sd} \qquad \forall d \in D; i \in N_c \tag{4c}$$

$$\sum_{k \in K} x_{ijk}^d \leq \frac{v_i^d + v_j^d}{2} \qquad \forall d \in D; i, j \in N_c (i \neq j) \tag{4d}$$

$$\sum_{j \in N_c} x_{ijk}^d = \sum_{j \in N_c} x_{jik}^d \qquad \forall i \in N; k \in K; d \in D \tag{4e}$$

$$\sum_{k \in K} \sum_{i \in N} x_{ijk}^d = \begin{cases} v_j^d & \forall j \in N_c \\ |K| & j = 0 \end{cases} \qquad \forall d \in D \tag{4f}$$

$$\sum_{i,j \in Q} x_{ijk}^d \leq |Q| - 1 \qquad \forall Q \subseteq N_c; k \in K; d \in D \tag{4g}$$

$$\sum_{j \in N_c} x_{0jk}^d \leq 1 \qquad k \in K; d \in D \tag{4h}$$

$$\sum_{i \in N_c} w_i \sum_{j \in N} x_{ijk}^d \leq C \qquad \forall k \in K; d \in D \tag{4i}$$

$$\sum_{(i,j) \in A} c_{ij} x_{ijk}^d \leq L \qquad \forall k \in K; d \in D \tag{4j}$$

$$z_i^s \in \{0, 1\} \qquad \forall i \in N_c; s \in S_i \tag{4k}$$

$$x_{ijk}^d \in \{0, 1\} \qquad \forall (i, j) \in A; k \in K; d \in D \tag{4l}$$

The objective function (4a) minimizes the arc travel costs. Constraints (4b) ensure that a feasible schedule is chosen for each node, while constraints (4c) define v_i^d on the days within the assigned schedule. Constraints (4d) allow arcs only between customers assigned for delivery on day $d \in D$. Constraints (4e) are the flow conservation constraints. Constraints (4f) ensure that nodes are included on routes for days within their assigned schedule. Constraints (4g) are the subtour elimination constraints. Constraints (4h) ensure that a vehicle is used no more than once a day. Constraints (4i) and (4j) are the physical capacity constraints and route length constraints, respectively. Finally, constraints (4k) and (4l) define the sets of variables.

Christofides and Beasley [16] do not attempt to solve the PVRP to optimality given the complexity of the problem. They propose the use of a median relaxation to approximate the PVRP cost as the sum of the radial distances between nodes and the depot; however, Christofides and Eilon [17] show that such a metric is a good estimator of the total distance for problem instances which have $|N| \gg \kappa^2$, where κ is the average value of the maximum number of customers on a route.

Tan and Beasley [56] summarize the results of Beltrami and Bodin [8], Russell and Igo [52], and Christofides and Beasley [16] and propose a problem that can be solved more simply than the PVRP itself. Their formulation is of interest because it attempts to circumvent the computationally difficult routing constraints. The authors use a simplified representation of the Fisher and Jaikumar [26] VRP formulation that does not explicitly specify the routing constraints.

Tan and Beasley define a cost measure θ_{ikd} that indicates the distance or cost of visiting node $i \in N_c$ with vehicle $k \in K$ on day $d \in D$. In their formulation, they use the decision variables $\tilde{x}_{ik}^d$ defined in Equation (3c) and z_i^s defined in Equation (3d). They formulate the PVRP as:

$$min \sum_{i \in N_c} \sum_{k \in K} \sum_{d \in D} \theta_{ikd} \tilde{x}_{ik}^d \tag{5a}$$

subject to

$$\sum_{s \in S_i} z_i^s = 1 \qquad \forall i \in N_c \tag{5b}$$

$$\sum_{k \in K} \tilde{x}_{ik}^d = \sum_{s \in S_i} a_{sd} z_i^s \qquad \forall i \in N_c, d \in D \tag{5c}$$

$$\sum_{i \in N_c} w_i \tilde{x}_{ik}^d \leq C \qquad \forall k \in K, d \in D \tag{5d}$$

$$\tilde{x}_{ik}^d \in \{0, 1\} \qquad \forall i \in N_c, k \in K, d \in D \tag{5e}$$

$$z_i^s \in \{0, 1\} \qquad \forall i \in N_c, s \in S_i \tag{5f}$$

The objective function (5a) minimizes the cost of service as specified by θ_{ikd}. Constraints (5b) assign each node to one schedule. Constraints (5c) ensure that vehicles are routed on the appropriate day to visit the corresponding schedule. Constraints (5d) ensure that assignments to vehicles do not violate capacity restrictions. Constraints (5e) and (5f) define the binary assignment variables. The above model describes a problem that is clearly more complex than a multi-day VRP assignment, as constraints (5b) and (5c) do not allow the problem to be decomposed by days.

Given the difficulty of solving this problem, Tan and Beasley suggest that the assignment of nodes to vehicles be neglected to reduce the size of the problem. They make the decision of allocating nodes to days in the first phase and the routing decision for each day in the second phase. They propose an aggregated cost measure, Θ_i^d, which represents the distance cost of visiting node $i \in N_c$ by *any* vehicle route on day $d \in D$. They solve an assignment problem, assigning nodes to days and ensuring that the total demand in each day does not exceed the total vehicle capacity ($C \cdot |K|$). The objective of the assignment problem is to minimize the distance cost of traveling between the nodes. Their modified formulation is as follows:

$$min \sum_{i \in N_c} \sum_{d \in D} \sum_{s \in S_i} \Theta_i^d a_{sd} z_i^s \tag{6a}$$

subject to

$$\sum_{s \in S_i} z_i^s = 1 \qquad \forall i \in N_c \tag{6b}$$

$$\sum_{i \in N_c} \sum_{s \in S_i} w_i a_{sd} z_i^s \leq C \cdot |K| \qquad \forall d \in D \tag{6c}$$

$$z_i^s \in \{0, 1\} \qquad \forall i \in N_c, s \in S_i \tag{6d}$$

The objective function (6a) minimizes the cost of routing as measured by Θ_i^d. Constraints (6b) assign each node to one schedule. Constraints (6c) ensure that the assignment does not exceed the total capacity available on any given day. Constraints (6d) define the binary assignment variables. After this stage of the problem, they solve $|D|$ independent VRPs for each day $d \in D$. This method depends heavily on evaluating the contribution matrix Θ_i^d correctly. Unfortunately, evaluating Θ_i^d accurately requires solving VRPs for all possible route and day combinations for each node $i \in N_c$.

In summary, two viewpoints have emerged in defining the PVRP: Russell and Igo [52] and Tan and Beasley [56] approach the problem as an extension of the assignment problem with a routing component; Christofides and Beasley [16] formulate the PVRP as a routing problem with a selection decision involved. In the following sections, we review solution methods based on these two viewpoints.

2.3 Solution Methods

Two-phase solution methods similar to that of Beltrami and Bodin [8] are commonly found in early heuristics for the PVRP. Recent PVRP literature has focused on metaheuristic methods of solving the problem that can escape the trap of local optimality that plagues conventional heuristics. In this section, we review the classical heuristics, the metaheuristics, as well as recent mathematical programming based approaches to solving the PVRP.

Classical Heuristics

Russell and Igo [52] present three heuristics to solve the PVRP: an improvement heuristic, and two construction heuristics. The first heuristic involves creating route clusters for all days using nodes whose day assignments are fixed, i.e. all nodes $i \in N_c$ such that $|S_i| = 1$. In the refuse-collection application described in their paper, these are nodes that require daily service and nodes that require service on fixed days. Then, the remaining unallocated nodes are assigned in descending order of required visit frequency. This allocation of nodes to days is made according to metrics that relate average distance of the node to route clusters. After initial construction, an improvement phase attempts to reassign nodes to other schedules. The authors note that the heuristic does not perform well as it is dependent on problem-specific metrics to construct routes and suggest its use only to obtain feasible starting solutions for other heuristics.

Their second heuristic is an improvement heuristic that reoptimizes the allocation and routing of nodes. It is a modified version of the MTOUR heuristic for the VRP (Russell [50]) which is itself based on a similar heuristic for the TSP (Lin and Kernighan [43]).

The third heuristic is an implementation of the Clarke-Wright savings method, similar to that of Beltrami and Bodin [8], with additional conditions to ensure that any proposed savings move results in a feasible allocation of nodes to days. For large problem instances ($|N| > 700$ nodes), savings moves are only considered within restricted neighborhoods of nodes, thus reducing computational effort.

Christofides and Beasley note that even their proposed relaxations of the PVRP (into a median problem and a periodic TSP) are hard. They do not attempt to solve either the relaxations or the PVRP optimally. They propose a two-stage heuristic method: first, they allocate nodes to days; second, they attempt node exchanges with the aim of minimizing the vehicle routing costs. An interesting point in their approach is that they have a merit order of nodes according to which they make initial allocations. For instance, nodes with fixed delivery combinations are allocated first and the remaining nodes are allocated in descending order of demand per visit. The idea is to reduce the possibility of infeasible solutions. This general heuristic is modified to provide solutions to the PVRP, the median problem and the periodic TSP. Routes developed from the relaxations tend to be inferior to the PVRP solutions given directly by their heuristic. Between the two relaxations, the routes developed from the periodic TSP relaxation tend to be superior to the routes developed from the median relaxation.

As discussed in Section 2.2, Tan and Beasley [56] also solve the PVRP in two stages: First, they determine the allocation of nodes to schedules using an assignment problem; Second, they solve independent VRPs for each of the days. It is clear that the performance of the method depends crucially on the cost measure Θ_i^d. While it is not desirable to find the cost measure by evaluating routes, the measure must still be a reasonable representation of the actual cost of serving routes on days associated with the schedule. The authors propose to do this by finding $|K| \cdot |D|$ seed points and associating each seed point with a day. They create a measure to determine the desirability of associating each seed point with a given day and solve an assignment problem to find the best possible allocation so that there are $|K|$ seed points in each day.

Russell and Gribbin [51] propose a solution method that consists of an initial route design using a network approximation, followed by three improvement phases. Their network flow model is similar to the formulation of Tan and Beasley, except that it has only one seed point for each day rather than $|K|$ seed points. As before, the cost metric is difficult to calculate;

however, Russell and Gribbin make a simple approximation of this cost metric as a sum of insertion costs into the round-trip tour between the depot and the appropriate seed points. This approximation considerably speeds up the construction phase. The first improvement heuristic uses the interchange method of Christofides and Beasley [16] to make improvements in individual tours. The second heuristic applies this interchange idea at the vehicle routing level. Finally, the authors propose a binary integer program to further refine the proposed solution. This phase assesses the possible reassignment of nodes between delivery combinations; however, the authors state that only slight improvements are observed in this phase.

Gaudioso and Paletta [31] suggest an alternative heuristic for the tactical problem of minimizing fleet size, rather than the operational problem of reducing distance. They impose constraints on the maximum route duration as well as the vehicle capacity. Gaudioso and Paletta do not impose a schedule set from which to choose day combinations, but instead place restrictions on the minimum and maximum number of days between visits for each node. This tactical version of the PVRP is shown to be $\mathcal{NP}-$hard by reduction to the bin packing problem. Their heuristic adopts a construction phase that allocates nodes to delivery combinations one at a time. The authors propose a number of improvement schemes to post-process the routes constructed by their method. They note that the distance cost of their solution is usually greater than other PVRP solution methods for two possible reasons: one, their objective is to minimize fleet size and not distance; and, two, they use a simple algorithm to solve the embedded TSP to optimize the routes after nodes have been allocated to delivery combinations.

According to the classification scheme of Cordeau et al. [20], all of these heuristics are construction-improvement, single-thread (no parallel processing).

Metaheuristics

Chao et al. [14] develop a special-purpose, metaheuristic method to solve the PVRP. This method generates an initial feasible solution to the PVRP and then iteratively uses improvement steps to progress towards the optimal solution to the problem. The initial feasible solution is obtained using the formulation of Christofides and Beasley [16]. They solve a linear relaxation of the assignment problem of allocating nodes to delivery days, while minimizing the maximum load carried in any given day. While the resulting solution may not be capacity feasible, it is still useful as an initial starting point.

In the next stage of the metaheuristic, the authors use the concept of feasible schedules (defining node-specific feasible sets S_i) like most of the PVRP literature. Attempts are made to improve the solution by moving a node from one schedule to another. If the proposed movement is valid for that node (i.e. moving node i to a schedule s such that $s \in S_i$) and if the move reduces the total distance, then the movement is immediately accepted. If there is an increase in the total distance, then the move is accepted if it is less than a certain threshold value. Otherwise, the node is not moved from its initial allocation. This process is continued and the threshold gradually reduced. The iterations terminate when it is no longer possible to make movements with positive cost savings. Chao et al. [14] also describe methods to improve this heuristic through capacity relaxation and post processing.

In terms of the metaheuristic classification scheme of Cordeau et al. [20], this metaheuristic is construction-improvement, single thread (no parallel processing). With respect to the Blum-Roli classification (see Blum and Roli [10]), it is single-point search (i.e. from a single trial solution and not a population of solutions), static objective function, single neighborhood, memoryless method.

Cordeau et al. [21] present a Tabu search method for solving several different routing problems, including the PVRP. Their Tabu search method has been modified to use specific insertion and route improvement techniques developed by the authors; however, there is no significant change to the core of the Tabu search technique that is specific to the PVRP.

The objective function is a weighted combination of travel cost and overcapacity penalties (thus intermediate solutions may violate capacity and time constraints). The method begins at a feasible solution and iterates by experimenting with moves (insertion of a node into a different schedule). The insertion process used is the GENI procedure (least-cost insertion) of Gendreau et al. [33]. The GENI procedure can also be applied to choose a node for removal from its current route. No additional reoptimization of the tour is required as $4-$opt route modifications are incorporated into the GENI procedure. The neighborhood of possible moves is the set of all solutions obtained by moving any customer i to another route and to any other schedule in its feasible set S_i. Initially, the method permits some infeasible movements that may cause violations of the capacity and route duration constraints.

In order to diversify the exploration of the solution space, the authors utilize a diversification stage in the Tabu search. This stage adds an additional penalty to the objective value of solutions that contain frequently added movements. The authors do not employ the final intensification stage that performs detailed exploration of the neighborhood of the best-known solutions. The overall method is seen to produce good solutions for the PVRP, improving on or producing comparable results to those of Chao et al. [14].

This metaheuristic can be classified as a single-construction, multiple improvement thread heuristic (Cordeau et al. [20]). The Blum-Roli classification is single-point search, dynamic objective function, single neighborhood, memory-usage method.

Drummond et al. [23] propose a metaheuristic based on a combination of genetic algorithm concepts and local search heuristics. This metaheuristic is a parallel-thread population mechanism heuristic (Cordeau et al. [20]). The Blum-Roli classification is population-based search, static objective function, single neighborhood, memory-usage method. Their method is an implementation of genetic algorithms (see Baker and Ayechew [5] and Reeves and Rowe

[48] for an overview) on a parallel computing framework together with modified local search methods.

The chromosome used specifies the days on which each customer is visited and the corresponding demand accumulation at each customer location on each day. The fitness level of each chromosome corresponds to the PVRP travel cost, and is calculated by solving a set of VRPs for each day using a Clarke and Wright [19] savings method. Subsequently standard crossover and mutation operations are used to create a diverse, evolving population of solutions.

Although their method is computationally intensive, the proposed parallel computing version converges rapidly. Further, the authors claim that their method is fairly robust for a broad range of heuristic parameters and may not require exhaustive tuning. They present a numerical study, comparing their solutions to those of Cordeau et al. [21], providing improved solutions to some problem instances. The authors discuss the implications of simultaneously evolving different groups of populations due to the parallel computational framework and the possibility of "migrating" solutions between different populations.

All three papers show numerical studies on a set of 20 test cases that are commonly used in the PVRP literature, shown in Table 1. Instances 1-10, from Christofides and Beasley [16], are based on VRP instances in Eilon et al. [24], with visit frequencies determined by demand at the nodes. Instance 11, from Russell and Igo [52], represents aggregated data from a industrial refuse application. Instances 12 and 13 are generated by Russell and Gribbin [51], with instance 12 drawn from a refuse collection application and instance 13 from a fast food application. Instances 14 - 20 are created by Chao et al. [14] according to two location distribution models: windmill and Star of David. Tables 1 and 2 summarize the relative performance of these metaheuristics in terms of solution time and relative solution quality.

In Table 1, the first column is the PVRP instance number. Columns 2-5 are characteristics of the test instance: the number of nodes; the number of vehicles; the number of days; and the vehicle capacity, respectively. Column 6, labeled CGW, shows the solution time for the Chao et al. [14] heuristic. Column 7 and 8, labeled CGL(T*) and CGL(T), shows the solution time for the Cordeau et al. [21] heuristic to find the best solution, and the overall solution time, respectively. Column 9, labeled DOV, shows the solution time for the parallel computing version of the Drummond et al. [23] heuristic. It is difficult to make direct comparisons of computing time as the authors use different computing software and hardware. Blank cells indicate that no solution is reported for that instance and metaheuristic.

Table 2 compares the objective value of the best solution obtained from the three metaheuristics with the values obtained from the classical heuristics. Column 1 refers to the same instance numbers as Table 1. Columns 2-4, labeled CB, TB, and RG, list the objective values obtained using Christofides and Beasley [16], Tan and Beasley [56], and Russell and Gribbin [51] respec-

Table 1. Comparison of solution times (in minutes) for PVRP metaheuristics.

Instance	$\|N\|$	$\|K\|$	$\|D\|$	C	CGW	CGL(T*)	CGL(T)	DOV
1	50	3	2	160	1.1	0.91	3.39	
2	50	3	5	160	6.8	0.6	4.06	0.61
3	50	1	5	160	0.6	3.18	3.73	0.09
4	75	6	5	140	7.7	4.74	5.19	1.76
5	75	1	10	140	5.4	7.46	7.48	2.77
6	75	1	10	140	3	6.87	7.84	0.27
7	100	4	2	200	5.5	6.6	7.63	6.04
8	100	5	5	200	13.5	8.97	10.7	10.11
9	100	1	8	200	4.6	6.7	10.03	3.39
10	100	4	5	200	14.7	5.57	9.68	15.84
11	126	4	5	235	205.4	9.72	14.17	23.59
12	163	3	5	140	11.9	11.51	18.37	97.41
13	417	9	7	2000	33.7	57.74	59.98	2.82
14	20	2	4	20	0.2	0.01	1.15	0.07
15	38	2	4	30	0.5	0.04	2.58	0.34
16	56	2	4	40	0.3	0.34	4.28	1.45
17	40	4	4	20	5.3	0.07	3.01	0.57
18	76	4	4	30	11.1	4.84	6.46	3.59
19	112	4	4	40	60.6	9.08	11.9	20.09
20	184	4	4	60	150.5	6.34	23.44	90.73

tively. Columns 5-7, labeled CGW, CGL, and DOV, list the objective values obtained using Chao et al. [14], Cordeau et al. [21], and Drummond et al. [23] respectively. As before, a blank cell indicates that no data is available for that instance-heuristic pair. Solution methods based on the metaheuristic approach produce the best known results for the PVRP. This is expected, as metaheuristics have been proven to be particularly effective for VRP-like problems (see Gendreau et al. [34] for an overview of metaheuristics for the VRP).

Mathematical Programming Based Approaches

Francis et al. [29] develop an exact solution method based on Lagrangian relaxation of an integer programming formulation of the PVRP. We present this formulation in Section 3.3. Although this is a formulation for an extended problem, it contains the PVRP as a special case. Their formulation is a multi-dimensional extension of Fisher and Jaikumar [26] and contains two sets of decisions variables: x_{ijk}^d defined in eq. (3a) and y_{ik}^s defined in eq. (3b).

The authors show that the dimensions of the problem can be reduced when the set of schedules S is such that one schedule contains all days in the planning period (a "daily" schedule) and all other schedules are disjoint from each other such that no day occurs in more than one schedule (besides the daily schedule). Under these conditions, there are at most $|S| - 1$ different

Table 2. Relative performance of heuristic techniques on the PVRP test set.

Instance	CB	TB	RG	CGW	CGL	DOV
1	547.4		537.3	524.6	524.6	
2	1,443.1	1,481.3	1,355.4	1,337.2	1,330.1	1,291.1
3	546.7			524.6	524.6	533.9
4	843.9		867.8	860.9	837.9	871.7
5	2,187.3	2,192.5	2,141.3	2,089.0	2,061.4	2,089.3
6	938.2			881.1	840.3	770.8
7	839.2		833.6	832.0	829.4	844.7
8	2,151.3	2,281.8	2,108.3	2,075.1	2,054.9	2,113.0
9	875.0			829.9	829.5	836.7
10	1,674.0	1,833.7	1,638.5	1,633.2	1,630.0	1,660.9
11	847.3	878.5	820.3		817.6	775.9
12			1,312.0	1,237.4	1,239.6	1,215.4
13			3,638.1	3,629.8	3,602.8	4,604.7
14				954.8	954.8	864.1
15				1,862.6	1,862.6	1,792.1
16				2,875.2	2,875.2	2,749.7
17				1,614.4	1,597.8	1,613.7
18				3,217.7	3,159.2	3,143.2
19				4,846.5	4,902.6	4,792.2
20				8,367.4	8,367.4	8,299.7

routes for each vehicle. Further, the dimension on the routing variables can be reduced from the full set of days $d \in D$ to a reduced set of representative days, one for each schedule in the set S. Their Lagrangian relaxation phase removes the constraints that link the two sets of decision variables, and the problem decomposes into a capacitated assignment subproblem and a number of prize-collecting traveling salesman subproblems. Any remaining gaps are closed using a branch-and-bound phase. A heuristic variation of this approach truncates the branch-and-bound phase within $\delta\%$ of the optimal. Using this variation, problem instances with up to 50 nodes are solved to within $\delta = 2\%$ of optimality. This provides the first known exact solution method, a heuristic method with a bounded gap, and a lower bound for the PVRP class of problems.

Mourgaya and Vanderbeck [45] solve a tactical version of the PVRP in which visit schedules and customer assignments to vehicles are solved simultaneously. The sequencing of customers within vehicle routes is determined in an operational problem. The authors consider two objectives: a "workload balancing" objective that ensures an equal distribution of customers among vehicles and a "regionalization" objective that clusters customers geographically as a proxy for tour length (similar in spirit to the tour length estimation approach taken by Tan and Beasley [56]). Focusing solely on the tactical problem facilitates the solution of larger problem instances.

The authors use the term "scenarios" to denote sets of schedule options. Customers are assigned to a scenario that is feasible given their visit requirements. Their formulation of the tactical PVRP uses a binary decision variable, $\hat{X}_{ij}^d$ which equals 1 if nodes $i \in N_c$ and $j \in N_c$ are visited by the same vehicle on day $d \in D$. The problem can then be written as follows:

$$\min \sum_{d \in D} \sum_{i \in N_c} \sum_{j \in N_c} c_{ij} \hat{X}_{ij}^d \tag{7a}$$

subject to

$$\sum_{s \in S_i} z_i^s \geq 1 \qquad \forall i \in N_c \tag{7b}$$

$$\sum_{k \in K} \tilde{x}_{ik}^d - \sum_{s \in S_i} a_{sd} z_i^s = 0 \qquad \forall i \in N_c; d \in D \tag{7c}$$

$$\tilde{x}_{ik}^d + \tilde{x}_{jk}^d - \hat{X}_{ij}^d \geq 1 \qquad \forall i \in N_c; j \in N_c; d \in D; k \in K \tag{7d}$$

$$\sum_{i \in N_c} w_i \tilde{x}_{ik}^d \geq C \qquad \forall k \in K; d \in D \tag{7e}$$

$$\tilde{x}_{ik}^d \in \{0,1\} \qquad \forall i \in N_c; k \in K; d \in D \tag{7f}$$

$$\hat{X}_{ij}^d \in \{0,1\} \qquad \forall i \in N_c; j \in N_c; d \in D \tag{7g}$$

$$z_i^s \in \{0,1\} \qquad \forall i \in N_c; s \in S \tag{7h}$$

The objective function (7a) is an estimate of the travel cost (the authors do not specify how they compute the measure c_{ij}). Constraints (7b) ensure that each node is assigned to a feasible scenario. Constraints (7c) link the scenario assignment and vehicle visit variables. Constraints (7d) check if two nodes are visited by the same vehicle. Constraints (7e) are vehicle capacity constraints. Finally, constraints (7f)-(7h) are variable definition constraints.

The authors note that this formulation suffers from a weak linear programming relaxation. Also, as has been noted by Tan and Beasley [56], the value of such a formulation depends greatly on being able to estimate the cost measure accurately. To circumvent both these problems, the authors present a Dantzig-Wolfe reformulation of the problem where the decisions are assigning scenarios to customers and customers to "clusters" (sets of customers to be visited together on a given day), thus ignoring assignments of customers to specific vehicles.

This reformulated problem can be solved using a column generation method. The objective is to minimize the cost of serving chosen clusters, where the cost of serving a particular cluster is determined by a pricing subproblem. The pricing subproblem, used to identify suitable clusters, includes constraints (7d) and (7e) and is hence hard to solve precisely. Simple heuristics are used to generate columns at each iteration, and the subproblems are

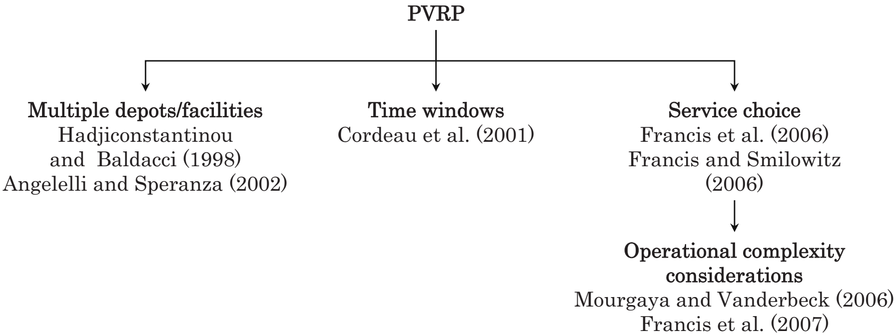

Fig. 2. Variants of the PVRP.

solved either exactly or through heuristics. The heuristics used for creating these subproblems can thus be tailored to achieve balanced workloads for all clusters. On the other hand, the minimum cost objective function tends to favor denser clusters, hence satisfying the balancing and regionalization objectives at the same time.

The authors report gaps of 14%-30% on average, depending on the heuristics used. More importantly, they present a first look at the difficulty of modeling and optimizing measures other than travel cost in the PVRP setting. The authors also comment on the poor performance of Lagrangian bounds for the tactical PVRP, echoing similar statements by Francis et al. [29] on the poor performance of the Lagrangian lower bound for the PVRP in general.

To summarize, the PVRP literature relating to solution methods recognizes that the problem is computationally hard. Research in this area has focused on heuristics for the PVRP. Of the heuristics reviewed, the classical heuristics tend to solve the assignment and routing decisions sequentially. More recent work has focused on metaheuristics and mathematical programming based approaches, recognizing the need to take an integrated approach to the PVRP problems.

3 PVRP Variants

Figure 2 outlines the literature relating to three important PVRP variants. We review these variants in this section. First, Section 3.1 reviews the multi-depot version of the PVRP (MDPVRP), in which periodic deliveries are made using a fleet of vehicles that are based across a number of depots. Also notable is the existence of a similar problem, the PVRP with Intermediate Facilities (PVRPIF), where the vehicles are based in a single depot, but capacity replenishment is possible at points along the routes. Second, Section 3.2 examines the PVRP with Time Window constraints (PVRPTW), in which customers may be visited only at certain times during the planning period. Finally, Section 3.3 shows the PVRP with Service Choice (PVRP-SC) which extends the PVRP to make visit frequency an endogenous decision of the problem.

All these variants have analogs in single-visit VRP models. The MDPVRP has an analog in the Multi-Depot VRP (MDVRP). Incidentally, Cordeau et al. [21] show that the MDVRP is a special case of the PVRP. The PVRPTW has an analog in the well-studied VRPTW (Solomon [55]). Finally, the PVRP-SC is similar to the class of problems called Vehicle Routing Problem with Profits (which includes the Team Orienteering Problem of Chao et al. [15]) in which the service frequency of a node can be 0 (not visited) or 1 (visited).

Note that there is currently no known dynamic variant of the PVRP in the literature. The closest work is the Dynamic Multi-Period Routing Problem (DMPRP) of Angelelli et al. [1]. The authors consider a distribution system in which customer orders arise dynamically, and orders arising in a certain day can be served either in the same day or the next. This model differs from the PVRP as there is no periodicity in customer demand in the DMPRP model – customers need to be visited only once during the planning period and a single order triggers only a single visit.

3.1 MDPVRP

Cordeau et al. [21] present a formulation of the PVRP and show that the Multi-Depot Vehicle Routing Problem (MDVRP) is a special case by associating depots with days. In their paper, Hadjiconstantinou and Baldacci [37] combine the ideas of periodicity and multiple-depots, extending the PVRP to include multiple depots. This greatly increases the difficulty of the resulting problem as it involves the additional decisions of assigning vehicles to depots as well as customer nodes to depots. Their Multi-Depot Period Vehicle Routing Problem (MDPVRP) is the problem of designing a set of routes for each day of a given $|D|-$day planning period. Each route of day $d \in D$ must be executed by one of a homogenous fleet of $|K|$ vehicles (service teams visiting customers) based at a certain depot (i.e., it must start and finish at its assigned depot). Their heuristic is as follows:

1. Assign each node to its nearest depot.
2. At each depot:
   a) Arrange the nodes according to some chosen order (say, decreasing visit frequency).
   b) Using this ordered list, use least-cost insertion to add nodes to routes such that their visit frequency f_i is satisfied.
   c) When all nodes have been assigned to feasible route combinations, solve VRPs for each day of the planning period using Tabu search.
3. Attempt to improve the overall solution by interchanging customer routes and/or assignments.

The heuristic is repeated several times by changing the ordering rule in Step 2(a). This idea is similar to the ordering rule of Christofides and Beasley [16]. Infeasible vehicle assignments (that violate the capacity constraints or fleet size) are not permitted. Note that in solving the MDPVRP, the heuristic assigns customers to be visited by particular depots, effectively defining geographical "boundaries" or "service territories" for each depot.

The authors introduce the notion of changing the customer service levels as defined by average frequency of visits (as in the PVRP), but do not incorporate this decision into their model. Instead, they use this idea to develop a strategic decision-making tool, solving the MDPVRP with different service frequency combinations, and constructing cost-benefit tradeoff curves. While this limits the number of service combinations that can be considered, this approach yields good results for their problem instance. The authors present illustrative tradeoff curves between increasing service level and rising travel costs and fleet size.

The PVRP with Intermediate Facilities (PVRPIF) is similar to the MDPVRP. While Angelelli and Speranza [2] do not allow multiple vehicle depots, they do use the idea of "drop-off points", or intermediate facilities, at which vehicle can stop along their vehicle routes, allowing them to replenish their capacities. Vehicles start and end their routes at their own depots, but visit these intermediate facilities along the way. Such problems arise in applications like waste collection with recycling facilities or goods collection with warehouse facilities. The authors solve the resulting extended PVRP problem using a Tabu search method.

3.2 PVRPTW

Cordeau et al. [22] extend the earlier work by Cordeau et al. [21] including time-windows. Their method provides a Tabu search method for the PVRPTW, which can be used to solve the VRPTW and MDVRPTW as special cases (recall that Cordeau et al. [21] show that the VRP and the MDVRP are special cases of the PVRP). The PVRPTW is the problem of designing $|K|$ different vehicle routes such that all customers are visited with their desired service frequency over the planning period, and each visit lies within a specified time interval. In order to solve this complex problem, the authors modify the Tabu search heuristic presented in Cordeau et al. [21]. The change to the heuristic is minor, principally requiring an additional penalty term to be added to the objective function for violations of time window constraints. The authors also create a set of new instances for the PVRPTW and MDVRPTW, and present numerical solutions, although the quality of the solutions cannot be specifically gauged in the absence of optimal solutions or lower bounds. The authors do provide a comparison of the performance of their heuristic on the Solomon VRPTW test instances (Solomon [55]), where it performs favorably when compared to the best known solution.

3.3 PVRP-SC

Francis et al. [29] extend the PVRP to make visit frequency a decision of the problem. The extended problem is called the PVRP with Service Choice (PVRP-SC). This increases the difficulty of solving the problem in two ways: first, there is the added complexity of determining the service frequency; second, the vehicle capacity requirement when visiting a node also becomes a decision of the model. We review the formulation of the PVRP-SC from Francis et al. [29]. Each schedule has an monetary benefit α^s. A weight $\beta \geq 0$ converts vehicle travel and stopping time into comparable costs in the objective function. Note that the demand accumulated between visits, w_i^s, depends on the demand of the node $i \in N_c$ and the frequency of schedule $s \in S$, but is approximated by the maximum accumulation between visits. The stopping time has a variable component τ_i^s. The decision variables used in this formulation are x_{ijk}^d and y_{ik}^s (eqns. 3a and 3b).

The formulation for PVRP-SC by Francis et al. [29] is:

$$\min \sum_{k \in K} \left[\sum_{d \in D} \sum_{(i,j) \in A} c_{ij} x_{ijk}^d + \sum_{s \in S} \sum_{i \in N_c} \gamma^s \tau_i^s y_{ik}^s - \beta \sum_{s \in S} \sum_{i \in N_c} W_i \alpha^s y_{ik}^s \right] \quad (8a)$$

subject to

$$\sum_{s \in S} \sum_{k \in K} \gamma^s y_{ik}^s \geq f_i \qquad \forall i \in N_c \quad (8b)$$

$$\sum_{s \in S} \sum_{k \in K} y_{ik}^s \leq 1 \qquad \forall i \in N_c \quad (8c)$$

$$\sum_{s \in S} \sum_{i \in N_c} w_i^s a_{sd} y_{ik}^s \leq C \qquad \forall k \in K; d \in D \quad (8d)$$

$$\sum_{j \in N} x_{ijk}^d = \sum_{s \in S} a_{sd} y_{ik}^s \qquad \forall i \in N_c; k \in K; d \in D \quad (8e)$$

$$\sum_{j \in N} x_{ijk}^d = \sum_{j \in N} x_{jik}^d \qquad \forall i \in N; k \in K; d \in D \quad (8f)$$

$$\sum_{i,j \in Q} x_{ijk}^d \leq |Q| - 1 \qquad \forall Q \subseteq N_c; k \in K; d \in D \quad (8g)$$

$$y_{ik}^s \in \{0,1\} \qquad \forall i \in N_c; k \in K; s \in S \quad (8h)$$

$$x_{ijk}^d \in \{0,1\} \qquad \forall (i,j) \in A; k \in K; d \in D \quad (8i)$$

The objective function (8a) is a weighted combination of travel and stopping costs net of service benefit. (see Francis and Smilowitz [28] for an analysis of the impact of the value of α on the resulting solution). Constraints (8b) ensure that the visit requirements for each node are satisfied, while constraints (8c) ensure that each node is assigned to a single schedule and vehicle. Constraints (8d) are capacity constraints, and constraints (8e) link the two sets of variables. Constraints (8f) and (8g) are flow conservation and subtour elimination constraints. Note that the PVRP-SC includes the PVRP as a special case by fixing constraints (8b) to equality and hence this is an IP formulation for the PVRP as well. The constraints (8b) may be alternatively expressed as $\sum_{k \in K} \sum_{s \in S_i} y_{ik}^s = 1, \ \forall i \in N_c$ with the redefinition of S_i as $S_i = \{s \in S \ : \ \sum_{d \in D} a_{sd} \geq f_i\}$.

Francis et al. [29] solve this problem using the Lagrangian relaxation method, combined with branch and bound procedure, described in Section 2.3. The authors present solutions for their motivating problems and for a standard test case from Christofides and Beasley [16]. Instances of up to 50 nodes can be solved within 2% of optimality. Results from Francis et al. [29] indicate that the magnitude of the savings obtained by introducing service choice in the PVRP for a given instance depends on geographic distribution of nodes (in particular, nodes of highest visit requirements).

4 Implementation Issues

As with the VRP, implementations of the PVRP pose particular challenges. Given the difficulty of the problems, it is practically impossible to incorporate every real-world constraint into the model. As a result, the solution obtained from such models does not always capture the needs of the system designers. Significant post-processing may be required to convert the solution into a practical routing implementation. In Section 4.1, we present case studies in which real-world constraints are combined with PVRP models to produce routing solutions appropriate to the problem instance. In Section 4.2, we present an approach that seeks to quantify the operational complexity of PVRP solutions by defining metrics suitable for periodic delivery operations.

4.1 Case Studies

Banerjea-Brodeur et al. [6] describe a situation in which the PVRP is used to plan the deliveries of linen to 58 different clinics within a hospital. Deliveries are required at preset service frequencies to each clinic, using a fixed fleet of carts. Although the number of carts is small, the carts can make multiple trips during the course of the day, thus effectively increasing the fleet size. The problem is solved using the Tabu search method of Cordeau et al. [21].

This implementation highlights several practical features relating to PVRP modeling. First, the authors report that they do not explore the entire solution space due to the layout of the hospital – the underlying graph is not a complete network. Second, allowances are made for elevator transfers in computing the distance matrix to account for waiting time and elevator travel time, in addition to the physical walking distance. Third, the hospital places a high premium on the stability of the solution to allow hospital agencies to plan operations around the deliveries. Finally, producing a balanced work schedule is considered to be important.

Blakely et al. [9] describe an application of the PVRP to control the routing and scheduling of service teams for preventive maintenance of elevators at customer locations. Each customer has a pre-specified service frequency depending on the type of elevator present at their location. The authors use a weighted multi-objective function which includes total travel costs, overtime costs, penalties for violating time windows, as well as a penalty for imbalanced loads.

The authors formulate the problem as a mathematical model, but do not solve it optimally. Rather, they solve the problem using a multi-stage heuristic. First, an initial solution is built by clustering together nodes that are geographically close, and assigning each cluster to one service team (recall the Tan and Beasley [56] method of using seed points). Second, the solution is improved by considering service times at each node, and real travel times between the nodes. Attempts are made to improve the solution by iteratively changing the cluster assignments between nodes. These improving moves are guided by a Tabu search metaheuristic. Third, when no further improvements are possible through such assignment changes, the routes are formed for each service team (vehicle) for each day of the week. At this stage, the heuristic takes into account customer visit frequency, time windows, service duration, and individual characteristics of the service team.

The authors highlight several important modeling considerations. First, they impose a fixed stopping time at nodes to account for service "set-ups" such as parking, regardless of service duration and in addition to travel time. Second, the street network data used did not use inter-modal links such as ferry crossings; in effect the underlying network had to be augmented by adding phantom road links that represented ferry service. Third, the fleet of service teams is not homogenous as service technicians had different equipment repair skill sets. Fourth, in some cases, it is required to pre-assign a service team to a particular customer. Finally, as with the linen delivery case, it is important to balance the work schedule between teams.

Hemmelmayr et al. [38] investigate the periodic delivery of blood products to hospitals by the Austrian Red Cross. In this case, the regularity of deliveries is of paramount importance. The authors model the problem as a detailed integer program with specific constraints and a fixed set of routes, and also as a PVRP with tour length constraints. They solve the integer program using a commercial IP solver, and the PVRP with a Variable Neighborhood Search heuristic. The authors show an average improvement in operating cost of about 30% with either method. They note that the vehicles are not capacity-constrained due to the small size of the deliveries, but only time-constrained by the perishability of the product.

An interesting note in their implementation is that using their periodic delivery solution requires negotiation with the hospitals involved to get them to accept the delivery schedule. Such considerations often arise in the implementation of periodic deliveries where the service provider is transitioning to a scheme in which they, rather than the customer, decide the service frequency (periodic delivery to Vendor-Managed Inventory, for instance). This underscores the importance of recognizing and appropriately modeling customer requirements that might make them averse to such changes.

Many other case studies have appeared in the literature, reflecting the growing popularity of the PVRP model for solving real-world problems. In the rest of this section, we briefly review some recent applications and implementations.

le Blanc et al. [42] use the PVRP as the basis for modeling periodic deliveries. The authors examine a special supply chain network called Factory Gate Pricing in which a number of suppliers are periodically visited by transport vehicles. They use a classical construction-improvement PVRP heuristic to solve their problem, similar to the third heuristic of Russell and Igo [52]. The PVRP may also be contained as a subproblem in more complex problems. Parthanadee and Logendran [46] consider a multi-product, multi-depot periodic distribution problem that contains the PVRP as a subproblem. They solve their problem using Tabu search heuristics.

Periodic delivery problems are also found in the general logistics literature. Golden and Wasil [36] survey vehicle routing problems arising in the soft drink industry, particularly the problem of *driver-sell* in which the driver visits retail locations periodically to replenish customer demand. The authors describe desirable features of routing systems for such an application such as the ability to accept predefined driver territories or generate new territorial boundaries, the ability to determine service frequency (as in the PVRP-SC), and to balance driver workloads. An existing routing software for a soft drink distributor is also described. Carter et al. [12] consider a complex periodic distribution problem arising in grocery delivery. The problem is solved using a Lagrangian heuristic. Gaur and Fisher [32] describe the development of a system to solve a vehicle routing and delivery scheduling problem for a European supermarket chain that includes the PVRP as a subproblem. They solve their problem using a combination of clustering heuristics to solve the scheduling problem and Lagrangian relaxation for truck routing. Claassen and Hendriks [18] model a milk collection problem arising in the dairy industry with emphasis on generating a stable collection schedule. The authors ignore the routing decision (following the idea of Tan and Beasley [56] and Mourgaya and Vanderbeck [45]) and focus on assigning schedules to customers and clusters of customers to vehicles. This problem is then further reduced by using some special properties of their application, and can be solved exactly.

Recycling and waste collection applications have been modeled using the PVRP since the seminal paper by Beltrami and Bodin [8]. Recently, a number of case studies have appeared in this area that solve problems of larger size, and incorporate more complex constraints. Shih and Lin [54], Shih and Chang [53], and Pontin et al. [47] consider the problem of collecting infectious waste from multiple medical facilities to a single disposal location. Shih and Lin [54] and Shih and Chang [53] both use a two-phase heuristic in which sets of possible routes are generated in the first phase and visit periods are determined in the second phase based on the identified routes. Pontin et al. [47] propose a genetic algorithm approach similar to Drummond et al. [23]. Bommisetty et al. [11] considers a similar problem of collecting recyclable material in a college campus. Baptista et al. [7] examines the problem of recycling paper containers in a Portuguese city. Their solution method is a simple extension of Christofides and Beasley [16]. Teixeira et al. [57] also address a problem arising in waste collection. This paper is notable among PVRP case studies as it deals with multiple products (three different types of waste). The authors use a cluster-first route-second heuristic to solve their problem, such that the delivery regions and product mix are already decided when the route sequencing decisions have to be made for each day.

Finally, in the service operations literature, Jang et al. [39] examine the problem of routing lottery sales representatives to visit lottery retail locations for the Missouri Lottery system. They also solve their problem using a cluster-first route-second heuristic. Workload balance is an important criterion in their models, besides the conventional objective of routing efficiency. Hence, they use the idea of seeding their construction phase in a manner similar to Fisher and Jaikumar [26], with the seed points being chosen from historical routes.

In the next section, we examine metrics that can quantify operational complexity, from the perspective of the service provider as well as the customer.

4.2 Operational Complexity

Francis et al. [30] explicitly consider the trade-offs between operational flexibility and operational complexity in a PVRP setting. Operational flexibility is defined as the ability to make changes to operating conditions. For example, the ability to determine the frequency of service provided to customers (as in the PVRP-SC), or to have different drivers visit customers rather than committing to a single driver (as is the case in Blakely et al. [9]). The authors show that having such flexibility leads to gains in terms of vehicle routing costs and improved customer service, but at the expense of increases in operational complexity.

The authors define operational complexity as the difficulty of a implementing a solution, either from the perspective of the service provider, or its customers. Complex solutions are characteristically those that are hard to convey to drivers without complex mapping, involve a high degree of driver learning, and tend to cause dissatisfaction among drivers and customers.

Francis et al. [30] introduce a set of metrics to quantify the operational complexity in a periodic distribution solution resulting from a PVRP/PVRP-SC. They define three measures:

1. Driver coverage expresses the service region visited by a driver over the planning period as a percentage of the total area.

2. Arrival span is the variability in the time of day when customers are visited over the planning period.
3. Crewsize is the number of different drivers seen by a customer over the planning period.

The driver coverage metric is defined for each driver, and is averaged over all drivers. The authors motivate this metric with the learning/forgetting model of Zhong et al. [58] that shows improved driver performance due to increased geographic familiarity resulting from consistent dispatches. A small driver coverage area is also consistent with geographical clustering approaches currently in practice (see Blakely et al. [9], for example). The arrival span and crewsize are defined for each customer, and averaged across all customers in the region. These measures are motivated by the need to provide better customer service by establishing consistent arrival times (small arrival span) and building relationships between customers and drivers (small crewsize). Further, the authors describe situations in which improvements in these measures help customers reduce staffing requirements, address security/access concerns, or obtain time-sensitive material (such as the blood products in Hemmelmayr et al. [38]).

The authors develop a set of randomized test cases in four different configurations representing different demand patterns, such as Traditional City (TC) configuration where the high demand nodes are located in the city center, to the Vanishing City (VC) configuration where the high demand is located in the outlying areas. They solve the PVRP and PVRP-SC for these test cases with a range of flexibility options, including different schedule choice, constraints on service choice and crewsize. They use a Tabu search heuristic based on that of Cordeau et al. [21] with changes to incorporate the operational flexibility options. After the solution is obtained, they calculate the value of the three complexity metrics. To correctly calculate the metrics, the authors develop an integer program to assign drivers to vehicle routes each day. The objective of this IP is to minimize the total driver coverage, and is similar in spirit to industry practice in which dispatchers assign drivers to familiar areas.

The authors show that introducing flexibility tends to increase the operational complexity of the resulting solutions, although the increase in complexity depends on the type of operational flexibility being introduced (for instance, restricting crewsize has little impact on the objective function, but reduces operational complexity significantly). The authors also note the importance of geographic distribution on the value of introducing operational flexibility, with greater returns obtained when the high frequency nodes are located closer to the depot.

Although the authors do not incorporate these operational metrics into their objective function, this can be accomplished by the use of penalties. Such an approach is clearly desirable in real applications of periodic distributions, although the weights to be applied on various terms of the objective function are not easily determined. Blakely et al. [9] present one way to work around such a situation. Their routing software optimizes a weighted multi-objective function. Rather than impose a fixed set of weights in the objective function, the software allows local supervisors to vary the weights on travel cost, workload balance, and other factors. The supervisors can thus obtain routing solutions that have appropriate operational complexity for their local business needs.

5 Future Research Directions

Future research into newer variants and solution methods for the PVRP is a rich and fertile area. With increase in computational capabilities, models for periodic delivery operations have become more complex, and capable of modeling problem instances in greater detail. The improvement in computational complexity could be exploited to create a dynamic variant of the PVRP, building on the work done by Angelelli et al. [1]. Older modeling techniques are being refreshed with the use of modern heuristics and solution methods. For instance, the vehicle scheduling problem of Foster and Ryan [27] has been revisited by Kang et al. [40], who provide an exact algorithm for the periodic version of the problem. Work on mathematical programming based approaches to these problems, such as the ongoing work of Mingozzi and Valletta [44] for solving the PVRP and MDVRP, provides faster and better quality solutions for the use of researchers and industry planners.

One possible area of interest is the pricing of periodic delivery services. Francis et al. [29] show that if customers are flexible in the frequency of service that they receive, then the cost savings can be achieved by giving customers a level of service appropriate to their geographic location as well as their willingness to pay. Francis and Smilowitz [28] model a tactical version of the PVRP-SC to allow managers of such services to analyze the effect of price and service schedule changes.

Another area of future research is the incorporation of multiple objectives in the optimization of periodic deliveries as discussed in Section 4. Mourgaya and Vanderbeck [45] present multiple objectives besides simple travel cost minimization. Other potential objectives that might be considered could be: monetary, such as the minimization of long term fleet acquisition and depreciation costs; partly-monetary, such as union-mandated balancing of driver workloads; or non-monetary, such as maximization of customer service aspects.

Francis et al. [30] introduce operational complexity measures that can be considered either endogenously through variable and parameter definitions or exogenously in post-processing. Future work could focus on adding complexity measures into the objective function of the PVRP and its variants, thereby allowing the solution method to choose the appropriate balance between complexity and flexibility. Cordeau et al. [22] extend the notion of time windows into periodic deliveries through the PVRPTW. In the routing literature, time windows for node visits have been incorporated with soft penalties for violations, which could form the basis for adding soft penalties for variations in visit times for nodes across days in the PVRP. Here too, we encounter the issue of pricing time window violations relative to the travel costs. Multi-stage solution methods could be conceived in which tactical versions of the PVRP could be used for high-level planning and pricing purposes (such as the approximation model of Francis and Smilowitz [28]), and the results of which could be used in detailed routing models. Research in this area would involve parametric analysis of the relative weighting of complexity costs to operational benefits and determining a frontier of efficient solutions for different levels of complexity.

The PVRP-SC is also closely related to another class of periodic delivery problems, the Inventory Routing Problem (IRP), which also determines visit frequency, route configuration, and delivery quantity. We refer the reader to Anily and Federgruen [4], Chan et al. [13], Federgruen and Simchi-Levi [25], Anily and Bramel [3], and Kleywegt et al. [41]. Some modeling differences exist such as service-related costs (PVRP-SC) versus unit holding costs (IRP) and delivery amount determined by schedule choice and delivery policies (PVRP-SC) versus being modeled as direct decision variable (IRP). Rusdiansyah and Tsao [49] model the IRP as an integrated IRP/PVRP with Time Windows. Francis et al. [30] explore connections between the PVRP-SC and the IRP, and suggests modeling the PVRP-SC as an IRP with deterministic demand. Exploration of the relationship between these two distribution problems is an important area of new research.

As discussed in this chapter, the PVRP is being used successfully to plan and operate periodic delivery operations in a wide range of applications. Future research in this area could focus on the incorporation of real-world constraints and operational complexity measures into PVRP models, as well as the development of models and solution methods for the dynamic version of the PVRP. With increasing price-competitiveness between supply chains, the ability to run “last-mile” delivery operations efficiently is likely to be a significant operational advantage.

Acknowledgement

This research has been supported by the RATS grant from The University of Chicago GSB, grant DMI-0348622 from the National Science Foundation, and the Alfred P. Sloan Foundation.

References

1. Angelelli, E., M. W. P. Savelsbergh, and M. G. Speranza (2006). Competitive analysis of a dispatch policy for

a dynamic multi-period routing problem. Working paper; http://www2.isye.gatech.edu/~mwps/publications/.

2. Angelelli, E. and M. G. Speranza (2002). The periodic vehicle routing problem with intermediate facilities. *European Journal of Operational Research 137*(2), 233–247.
3. Anily, S. and J. Bramel (1998). Vehicle routing and the supply chain. In S. Tayur, R. Ganeshan, and M. Magazine (Eds.), *Quantitative Models for Supply Chain Management*. Kluwer Academic Publishers.
4. Anily, S. and A. Federgruen (1990). One warehouse multiple retailer systems with vehicle routing costs. *Management Science 36*(1), 92–114.
5. Baker, B. M. and M. A. Ayechew (2003). A genetic algorithm for the vehicle routing problem. *Computers and Operations Research 30*(5), 787–800.
6. Banerjea-Brodeur, M., J.-F. Cordeau, G. Laporte, and A. Lasry (1998). Scheduling linen deliveries in a large hospital. *Journal of the Operational Research Society 49*(8), 777–780.
7. Baptista, S., R. C. Oliveira, and E. Zúquete (2002). A period vehicle routing case study. *European Journal of Operational Research 139*(2), 220–229.
8. Beltrami, E. J. and L. D. Bodin (1974). Networks and vehicle routing for municipal waste collection. *Networks 4*(1), 65–94.
9. Blakely, F., B. Bozkaya, B. Cao, W. Hall, and J. Knolmajer (2003). Optimizing periodic maintenance operations for Schindler Elevator Corporation. *Interfaces 33*(1), 67–79.
10. Blum, C. and A. Roli (2003). Metaheuristics in combinatorial optimization: Overview and conceptual comparison. *ACM Computing Surveys 35*(3), 268–308.
11. Bommisetty, D., M. Dessouky, and L. Jacobs (1998). Scheduling collection of recyclable material at Northern Illinois University campus using a two-phase algorithm. *Computers and Industrial Engineering 35*(3–4), 435–438.
12. Carter, M. W., J. M.Farvolden, G. Laporte, and J. Xu (1996). Solving an integrated logistics problem arising in grocery distribution. *INFOR 34*(4), 290–306.
13. Chan, L., A. Federgruen, and D. Simchi-Levi (1998). Probabilistic analysis and practical algorithms for inventory-routing models. *Operations Research 46*(1), 96–106.
14. Chao, I.-M., B. L. Golden, and E. A. Wasil (1995). An improved heuristic for the period vehicle routing problem. *Networks 26*(1), 25–44.
15. Chao, I.-M., B. L. Golden, and E. A. Wasil (1996). The team orienteering problem. *European Journal of Operational Research 88*(3), 464–474.
16. Christofides, N. and J. E. Beasley (1984). The period routing problem. *Networks 14*(2), 237–256.
17. Christofides, N. and S. Eilon (1969). Expected distances in distribution problems. *Operational Research Quarterly 20*(4), 437–443.

18. Claassen, G. D. H. and T. H. B. Hendriks (2007). An application of Special Ordered Sets to a periodic milk collection problem. *European Journal of Operational Research 180*(2), 754–769.
19. Clarke, G. and J. Wright (1964). Scheduling of vehicles from a central depot to a number of delivery points. *Operations Research 12*(4), 568–581.
20. Cordeau, J.-F., M. Gendreau, A. Hertz, G. Laporte, and J.-S. Sormany (2005). New heuristics for the vehicle routing problem. In A. Langevin and D. Riopel (Eds.), *Logistics Systems: Design and Optimization*, pp. 279–297. Springer US.
21. Cordeau, J.-F., M. Gendreau, and G. Laporte (1997). A tabu search heuristic for periodic and multi-depot vehicle routing problems. *Networks 30*(2), 105–119.
22. Cordeau, J.-F., G. Laporte, and A. Mercier (2001). A unified tabu search heuristic for vehicle routing problems with time windows. *Journal of the Operational Research Society 52*(8), 928–936.
23. Drummond, L. M. A., L. S. Ochi, and D. S. Vianna (2001). An asynchronous parallel metaheuristic for the period vehicle routing problem. *Future Generation Computer Systems 17*(4), 379–386.
24. Eilon, S., C. T. Watson-Gandy, and N. Christofides (1971). *Distribution Management: Mathematical modeling and practical analysis*. London: Griffin.
25. Federgruen, A. and D. Simchi-Levi (1995). Analysis of vehicle routing and inventory-routing problems. In M. O. Ball, T. L. Magnanti, C. L. Monma, and G. L. Nemhauser (Eds.), *Handbooks in Operations Research and Management Science*, Volume 8, pp. 297–373. Elsevier Science B.V.
26. Fisher, M. L. and R. Jaikumar (1981). A generalized assignment heuristic for vehicle routing problems. *Networks 11*(2), 109–124.
27. Foster, B. A. and D. M. Ryan (1976). An integer programming approach to the vehicle scheduling problem. *Operational Research Quarterly 27*(2), 367–384.
28. Francis, P. M. and K. R. Smilowitz (2006). Modeling techniques for periodic vehicle routing problems. *Transportation Research: Part B 40*(10), 872–884.
29. Francis, P. M., K. R. Smilowitz, and M. Tzur (2006). The period vehicle routing problem with service choice. *Transportation Science 40*(4), 439–454.
30. Francis, P. M., K. R. Smilowitz, and M. Tzur (2007). Flexibility and complexity in periodic distribution problems. *Naval Research Logistics (Forthcoming).*
31. Gaudioso, M. and G. Paletta (1992). A heuristic for the period vehicle routing problem. *Transportation Science 26*(2), 86–92.
32. Gaur, V. and M. L. Fisher (2004). A periodic inventory routing problem at a supermarket chain. *Operations Research 52*(6), 813–822.
33. Gendreau, M., A. Hertz, and G. Laporte (1994). A tabu search heuristic for the vehicle routing problem. *Management Science 40*(10), 1276–1290.

34. Gendreau, M., G. Laporte, and J.-Y. Potvin (2002). Metaheuristics for the capacitated VRP. In P. Toth and D. Vigo (Eds.), *The vehicle routing problem*, pp. 129–154. Philadelphia: SIAM.
35. Golden, B. L., T. L. Magnanti, and H. Q. Nguyen (1977). Implementing vehicle routing algorithms. *Networks 7*(2), 113–148.
36. Golden, B. L. and E. A. Wasil (1987). Computerized vehicle routing in the soft drink industry. *Operations Research 35*(1), 6–17.
37. Hadjiconstantinou, E. and R. Baldacci (1998). A multi-depot period vehicle routing problem arising in the utilities sector. *The Journal of the Operational Research Society 49*(12), 1239–1248.
38. Hemmelmayr, V., K. F. Doerner, R. F. Hartl, and M. W. P. Savelsbergh (2006). Delivery strategies for blood products supplies. Working paper; http://www2.isye.gatech.edu/∼mwps/publications/.
39. Jang, W., H. H. Lim, T. J. Crowe, G. Raskin, and T. E. Perkins (2006). The Missouri Lottery optimizes its scheduling and routing to improve efficiency and balance. *Interfaces 36*(4), 302–313.
40. Kang, K. H., Y. H. Lee, and B. K. Lee (2005). An exact algorithm for multi depot and multi period vehicle scheduling problem. In O. Gervasi, M. L. Gavrilova, V. Kumar, A. Laganà, H. P. Lee, Y. Mun, D. Taniar, and C. J. K. Tan (Eds.), *Computational Science and Its Applications ICCSA 2005*, pp. 350–359. Springer Berlin.
41. Kleywegt, A. J., V. S. Nori, and M. W. P. Savelsbergh (2002). The stochastic inventory routing problem with direct deliveries. *Transportation Science 36*(1), 94–118.
42. le Blanc, H. M., F. Cruijssen, H. A. Fleuren, and M. B. M. de Koster (2004). Factory gate pricing: An analysis of the Dutch retail distribution. Working paper; http://econpapers.repec.org/paper/dgreureri/30001575.htm.
43. Lin, S. and B. W. Kernighan (1973). An effective heuristic for the traveling salesman problem. *Operations Research 21*(2), 498–516.
44. Mingozzi, A. and A. Valletta (2005). The multi-depot and periodic vehicle routing problem. Presented at the 2005 Symposium on Abstraction, Reformulation, and Approximation, Edinburgh.
45. Mourgaya, M. and F. Vanderbeck (2006). Column generation based heuristic for tactical planning in multi-period vehicle routing. *European Journal of Operational Research (In press).*
46. Parthanadee, P. and R. Logendran (2006). Periodic product distribution from multi-depots under limited supplies. *IIE Transactions 38*(11), 1009–1026.
47. Pontin, M. I., M. N. Ziletti, and A. Mendez (2004). Optimal routing for infectious waste collection in the City of Rio Cuarto. Presented at the V Pan American Workshop on Applied and Computational Mathematics, Honduras.
48. Reeves, C. R. and J. E. Rowe (2003). *Genetic Algorithms - Principles and Perspectives.* Boston: Kluwer Academic Publishers.

49. Rusdiansyah, A. and D.-B. Tsao (2005). An integrated model of the periodic delivery problems for vending-machine supply chains. *Journal of Food Engineering 70*(3), 421–434.
50. Russell, R. A. (1977). An effective heuristic for the M-tour traveling salesman problem with some side conditions. *Operations Research 25*(3), 517–524.
51. Russell, R. A. and D. Gribbin (1991). A multiphase approach to the period routing problem. *Networks 21*(7), 747–765.
52. Russell, R. A. and W. Igo (1979). An assignment routing problem. *Networks 9*(1), 1–17.
53. Shih, L.-H. and H.-C. Chang (2001). A routing and scheduling system for infectious waste collection. *Environmental Modeling and Assessment 6*(4), 261–269.
54. Shih, L.-H. and Y.-T. Lin (1999). Optimal routing for infectious waste collection. *Journal of Environmental Engineering 125*(5), 479–484.
55. Solomon, M. M. (1987). Algorithms for the vehicle routing and scheduling problems with time window constraints. *Operations Research 35*(2), 254–265.
56. Tan, C. C. R. and J. E. Beasley (1984). A heuristic algorithm for the period vehicle routing problem. *Omega 12*(5), 497–504.
57. Teixeira, J., A. P. Antunes, and J. P. de Sousa (2004). Recyclable waste collection planning - a case study. *European Journal of Operational Research 158*(3), 543–554.
58. Zhong, H., R. W. Hall, and M. Dessouky (2007). Territory planning and vehicle dispatching with driver learning. *Transportation Science 41*(1), 74–89.