On Fibering 3-manifolds

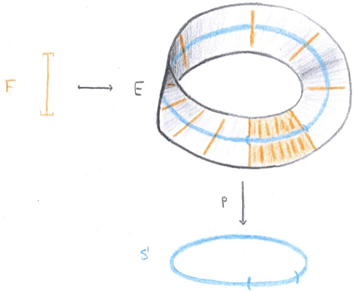
Isaac M. Craig

Bryn Mawr College

MAA EPaDel Section Meeting
24 March 2018

Background

What is a fiber bundle?



- E total space
- B base space
- F fiber space
- $p\colon E \to F$ continuous and *locally trivial*, i.e. locally E looks like a product of some neighborhood $U \subset B$ with F

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

What 3-manifolds can be fibered over S^1?

Suppose E is a compact 3-manifold that fibers over S^1 with connected fiber F.

$$\cdots \to \pi_2(S^1) \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \underset{\varphi}{\longrightarrow} \pi_1(S^1) \longrightarrow \pi_0(F) \to \cdots$$

Exactness implies $\ker(\varphi) = \pi_1(F)$ and $\pi_1(E)/\pi_1(F) \cong \mathbb{Z}$, so

$\pi_1(E)$ *has a finitely-generated, normal subgroup with quotient* $\mathbb{Z}$

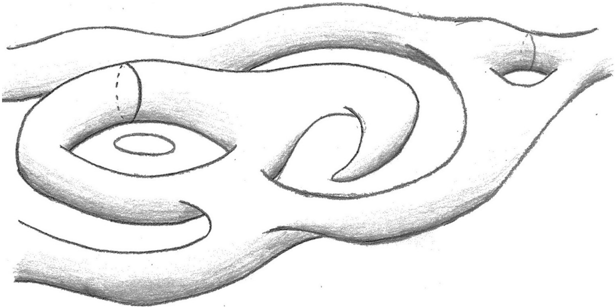
Fiber Lemma (Stallings '62)

Let E be a compact 3-manifold such that $\pi_1(E)$ has a finitely-generated, normal subgroup G with quotient $\mathbb{Z}$. Then G is the fundamental group of a surface T embedded in E.

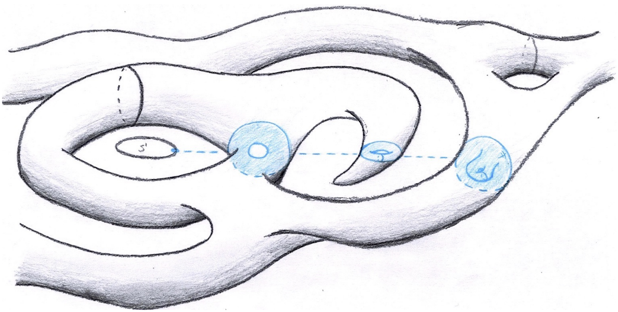
Fiber Theorem (Stallings '62)

If in addition to the Fiber Lemma, E is irreducible and $G \neq \mathbb{Z}_2$, then E is the total space of a fiber bundle over the circle with fiber T.

Fiber Lemma (1. Induced Map)



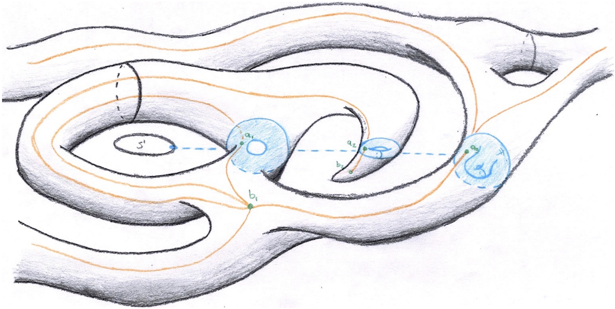
- Recall a fiber bundle is a map $E \to S^1$ with algebraic information
- By assumption, there is a map $\varphi\colon \pi_1(E) \to \pi_1(E)/G$
- Using obstruction theory, there exists a map $f\colon E \to S^1$ with $f_* = \varphi$ (note that $G \cong \ker(f_*)$)



- Taking a regular value $p \in S^1$, the preimage $f^{-1}(p) = T$ is a properly embedded submanifold

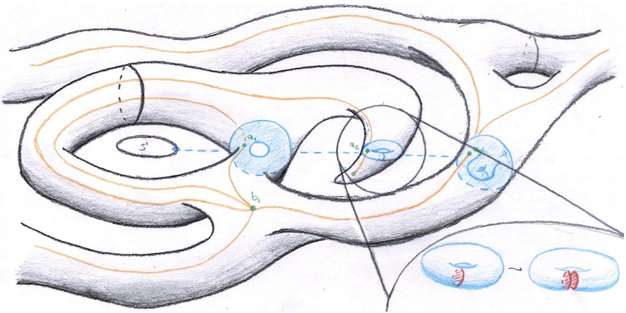
- Wish list: we want T to be connected, incompressible (i.e. π_1-injective), and have $\pi_1(T)$ embed onto $\ker(f_*)$

Fiber Lemma (3. Graph)



- Construct a graph Γ onto which E retracts
- The algebra forces $\pi_1(\Gamma) \cong \pi_1(S^1)$ so that $\Gamma \simeq S^1$
- Homotopy of f pushes away all but one T_i, so $f^{-1}(p) = T$ is connected

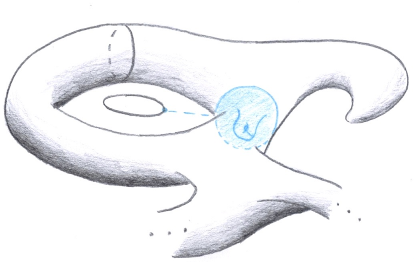
Fiber Lemma (4. Incompressible)



- Incompressible: no essential curve in T that bounds a disk in E

- Idea: homotop away any compressing disks

Fiber Lemma (5. Huh?)



Summary:

- T is a connected, properly embedded surface in E
- T is incompressible, so inclusion induced map $\pi_1(T) \to \pi_1(E)$ is injective
- $\pi_1(T) \subseteq \ker(f_*)$, and in fact is all of $\ker(f_*)$

Fiber Lemma (Stallings '62)

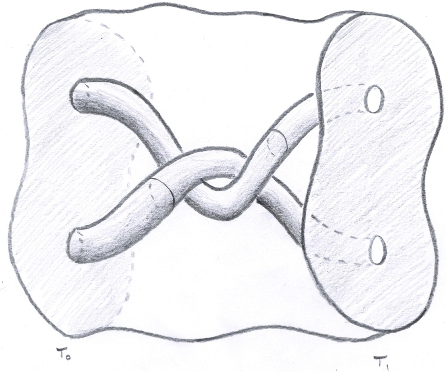
Let E be a compact 3-manifold such that $\pi_1(E)$ has a finitely-generated, normal subgroup G with quotient $\mathbb{Z}$. Then G is the fundamental group of a surface T embedded in E.

Fiber Theorem (Stallings '62)

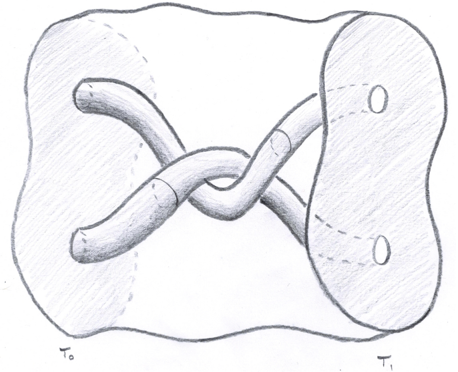
If in addition to the Fiber Lemma, E is irreducible and $G \neq \mathbb{Z}_2$, then E is the total space of a fiber bundle over the circle with fiber T.

- Proof of Fiber Theorem involves two cases, depending on boundary of T
- We will prove case where T has boundary, and reduce the closed case
- Idea is to consider the *split* M of the 3-manifold E along T, and show it is homeomorphic to $T \times I$

Fiber Theorem (1. Boundary of Split)

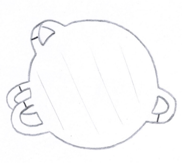


- Boundary of M consists of T_0 to T_1, connected by annuli
- As with a product, the split M retracts onto T_0 by some $r\colon M \to T_0$
- The induced map r_* is an isomorphism $\pi_1(M) \cong \pi_1(T_0)$

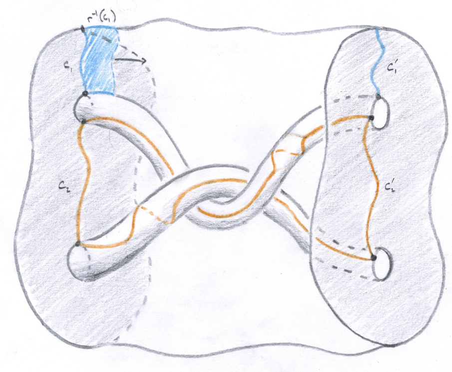


- Every surface with boundary has representation with handles, cross-caps, and holes

- Cut each "band" by some arc C_i producing a disk

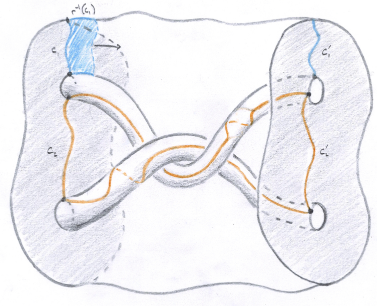


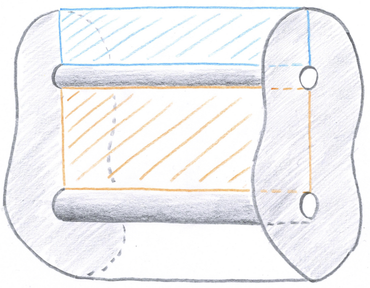
Fiber Theorem (3. Cut up M



- Making the retract $r\colon M \to T_0$ transverse to each C_i, the preimage $D_i = r^{-1}(C_i)$ is a properly embedded, incompressible surface in M
- The algebra forces each D_i to be a disk with $T_1 \setminus (\cup D_i) \cap T_1$ a disk

Fiber Theorem (4. Homeomorphism on Boundary)





- Define homeomorphism on $\partial M = T_0 \cup T_1 \cup \{\text{ annuli }\}$
- Extend across each disk D_i in the obvious way
- An Euler characteristic argument shows that what is left in M is bounded by a 2-sphere, so there is an extension

Thank you!

Thank you!