On Fibering 3-Manifolds

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Abstract

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Master's Thesis

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In 1962, John Stallings showed that a 3-manifold E fibers over the circle if and only if its fundamental group fits into a short exact sequence $G \to \pi_1(E) \to \mathbb{Z}$ with G finitely generated. This exposition provides an expository account of this result and its proof. To motivate the importance of fiber bundles in topology, the first chapter discusses the basics of fiber bundles. The following two chapters prove Stallings' Fibering Theorem in two stages. First, we prove the Fiber Lemma, which claims that the group G is represented by the fundamental group of a 2-sided, embedded, incompressible surface $T \subset E$. This lemma prepares us for the Fibering Theorem, which claims E is a fiber bundle over the circle with fiber T. The fourth chapter concludes the exposition with a discussion of supplementary topics. We assume familiarity with many topics in 3-manifold topology and algebraic topology; for convenience, these are discussed in an appendix.

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Contents

1 Introduction

Fiber bundles were first introduced in 1932 by H. Seifert, and eventually popularized by H. Whitney in 1935. The importance of these types of spaces was quickly recognized, and the period 1935-1950 saw the development of many distinct theories. The slow, disjointed development of this theory was likely due to the second world war, which separated many mathematicians at that time. It was not until the 1950's Brussels colloquium on fiber bundles that a generally accepted theory was established (cf. [7], [12]). Today, these spaces remain an important topic of study.

Fiber bundles are perfect examples of spaces that are both simple and interesting. The definition of a fiber bundle is straightforward and intuitive, yet flexible enough to include elaborate spaces. We will discuss these spaces and their generalizations in the first section. In particular, we highlight their relationship with algebraic topology.

The importance of fiber bundles in 3-manifold topology comes from this algebraic relationship. In particular, fiber bundles work well with homotopy groups of any dimension. As we will see, a fiber bundle E locally decomposes into a local product of a base space B and a fiber space F, giving rise to a long exact sequence of homotopy groups:

$$\begin{tikzcd}
\pi_*(F) \arrow[rr] & & \pi_*(E) \arrow[dl] \\
& \pi_*(B) \arrow[ul, "-1"] &
\end{tikzcd}$$

If E is a compact 3-manifold with base space the circle, the fiber F will be a compact surface. Excluding the sphere and projective plane, surfaces have well-understood homotopy groups, meaning we can often use this long exact sequence to understand the homotopy groups of E. What 3-manifolds are fiber bundles over the circle?

In 1962, John Stallings (1935 - 2008) gave necessary and sufficient conditions on the fundamental group of a 3-manifold to determine if it fibers over the circle [11]. Not surprisingly, these conditions were motivated by the above long exact sequence, ending with

$$0 = \pi_2(S^1) \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \xrightarrow[\varphi]{} \pi_1(S^1) \xrightarrow[0]{} \pi_0(F) = 0.$$

By exactness, the homomorphism φ induces an isomorphism $\pi_1(E)/\pi_1(F) \cong \mathbb{Z}$. In particular, this tells us the fiber F is a connected, incompressible[1], embedded surface whose fundamental group represents a finitely generated, normal subgroup of $\pi_1(E)$ with quotient $\mathbb{Z}$. Stallings observed that this condition on $\pi_1(E)$ is in fact sufficient for E to fiber over the circle. The proof of this theorem, which we refer to as Stallings' Fibering Theorem, is achieved in two parts: the Fiber Lemma and the Fibering Theorem.

Fiber Lemma Let E be a compact 3-manifold such that $\pi_1(E)$ has a finitely generated, normal subgroup G with quotient $\mathbb{Z}$. Then G is the fundamental group of a surface T, properly embedded in E; that is, the inclusion induced map $i_*\colon \pi_1(T) \to \pi_1(E)$ maps isomorphically onto G.

Fibering Theorem If E is an irreducible 3-manifold satisfying the hypotheses of the Fiber Lemma, then E is the total space of a fiber bundle over the circle with fiber T, the surface produced in the Fiber Lemma.

We prove these results in the second and third chapters, respectively. The fourth chapter concludes the exposition with a discussion of supplementary topics. In particular, we define virtual fiberings and discuss Thurston's Virtual Fibering Conjecture, now a theorem of Agol and Wise.

Virtual Fibering Conjecture Every closed, irreducible, atoroidal 3-manifold with infinite fundamental group has a finite cover which is a surface bundle over the circle.

[1] Occasionally referred to as π_1-injective; this property will be discussed in the third section.

2 Fibrations

It is generally difficult to compute the homotopy groups of a given topological space, and these computations become increasingly difficult as the dimension of the space is exceeded by the dimension from the groups. Thus, it is very fortunate to find classes of spaces in which these computations are comparatively simple.

One class of spaces that works particularly well with homotopy groups are *fibrations*. This partnership comes in the form of an induced long exact sequences of homotopy groups, which significantly reduces the burden of computing homotopy groups of any dimension.

2.1 Fibration Fundamentals

This exposition is mainly concerned with fiber bundles – maps that locally express a space as a local product between two relatively simpler spaces. As product structures (local or otherwise) are a rigid property to impose, it is often useful to relax this condition. One generalization of fiber bundles are *fibrations*. Intuitively, these are maps that parameterize one space by another. In this section, we discuss their properties.

The general theory behind fibrations will not be of great importance in the upcoming chapters. Instead, we focus on building an intuition for these spaces. For this reason, we have deferred all proofs to the appendix, which includes a thorough discussion of lifts, extensions, and fibrations.

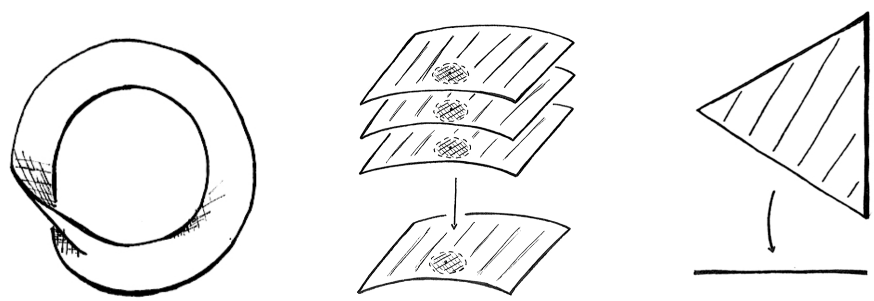
Definition 1. A map $p\colon E \to B$ has the ***homotopy lifting property*** (HLP) with respect to X if for any homotopy $g_t\colon X \to B$ and any lift h_0 of g_0 there exists a homotopy $h_t\colon X \to E$ of h_0 lifting g_t.

Definition 2. A map $p\colon E \to B$ which satisfies the homotopy lifting property with respect to all spaces is called a ***Hurewicz fibration***, or simply a ***fibration***. More generally, a map which has the homotopy lifting property with respect to all CW-complexes is called a ***Serre fibration***.

The above definition is far from intuitive, so we begin by considering some examples.

(a) Projections $\pi_E \colon E \times F \to E$ are perhaps the most straightforward example. Given maps as in Definition 1, the lift $h_0 \colon X \to E \times F$ induces a map $f \colon E \to F$ in the second component, i.e. $h_0(x) = (g_0(x), f(x))$. A lift of each g_t can be defined by $h_t(x) = (g_t(x), f(x))$.

(b) Similar to product spaces, we will later see that the projections of fiber bundles have the HLP. For example, the Möbius band M is a local product over the circle S^1 with an interval $I = [0, 1]$ and projects onto its core circle as a fibration. Due to examples like this, fiber bundles are often referred to as *twisted products*. We will later see that any fiber bundle is a Serre fibration.



(b) (c) (d)

(c) Recall that covering spaces have the homotopy lifting property, and in particular, the lift is unique (see for example [3] p. 60).

(d) The cone on an interval projects onto its I factor to form a fibration. However, the projection of the comb space onto its base interval is not a fibration (exercise).

The spaces E and B in a fibration $p \colon E \to B$ are commonly referred to as the ***total space*** and ***base space***, respectively. In general, the base space is assumed to be path-connected. For each $b \in B$, the ***fiber at b*** is the subspace $F_b = p^{-1}(b)$. The following proposition relates the fibers of fibrations.

Proposition 2.1. The fibers of a Hurewicz fibration are homotopy equivalent.

We are not often concerned with the point $b \in B$ associated with a fiber F_b. Instead, we may simply consider the ***fiber F*** of a fibration.

The triumph of fibrations comes in the form of their induced long exact sequence of homotopy groups, a beneficial tool in computing higher homotopy groups of a fibration.

Theorem 2.2. Suppose $p\colon E \to B$ is a Serre fibration with basepoints $b \in B$ and $x \in F_b$. If B is path-connected, there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(F,x) \longrightarrow \pi_n(E,x) \xrightarrow{p_*} \pi_n(B,b) \longrightarrow \pi_{n-1}(F,x) \longrightarrow \cdots \longrightarrow \pi_0(E,x) \to 0.$$

2.2 Fibre Bundles

Fiber bundles are maps that express a space as a local product. This local structure need not be mirrored globally, as with the Möbius band, above. For example, each of the above fibrations (a - c) is a fiber bundle. A hasty reader might be tempted to conjecture that all fibrations are fiber bundles, however, we will soon find a property which distinguishes the two (namely, we will see that (d) is not a bundle). First, the definition.

Definition 3. A ***fiber bundle*** with fiber F is a map $p\colon E \to B$ such that B is a ***local product***; that is, each point $b \in B$ belongs to a neighborhood $U \subset B$ such that $h\colon p^{-1}(U) \to U \times F$ is a homeomorphism making the following diagram commute:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

Such a map h is referred to as a ***local trivialization*** of the bundle.

For simplicity, a fiber bundle is often given in the condensed notation $F \hookrightarrow E \xrightarrow{p} B$. Note that we are *given* a space F which forms a local product with B. The local trivialization then preserves the structure of this space in each of the fibers, as the following theorem shows.

Theorem 2.3. The fibers of a fiber bundle are homeomorphic.

At this point, the reader should be able to come up with multiple examples of fibrations which are not fiber bundles. As we have previously stated, the other implication is correct; fiber bundles are a special case of fibrations.

Theorem 2.4. A fiber bundle is a Serre fibration.

Specializing further, a covering space is a special type of fiber bundle[2] , in which the fiber is a discrete space. Thus, we have the following containments:

$$\text{Covering Spaces} \subset \text{Fiber Bundles} \subset \text{Serre fibrations} \subset \text{Hurewicz fibrations.}$$

We may then apply the above results on fibrations and fiber bundles to covering spaces. In particular, the long exact sequence of a fibration, gives the following result.

COROLLARY 2.5. For a covering space $p\colon \widetilde{X} \to X$, the map $p_*\colon \pi_n(\widetilde{X}) \to \pi_n(X)$ is an isomorphism for $n \geqslant 2$ and a monomorphism for $n = 1$. Moreover, the cosets of $p_*(\pi_1(\widetilde{X})) < \pi_1(X)$ are in bijection with the fiber.

[2] We must restrict ourselves to covering spaces whose fibers have the same cardinality, otherwise we contradict Theorem 2.3.

3 Fiber Lemma

We now begin the proof of Stallings' Fibering Theorem. This chapter proves the Fiber Lemma, which characterizes the subgroups of the fundamental group of a 3-manifold that are realizable as the fundamental group of a connected, irreducible, properly embedded surface in the 3-manifold.

The original proof of Stallings' Theorem uses a classical piecewise-linear approach, and although these methods have been largely replaced by their contemporary counterparts[3], we will occasionally follow this traditional approach. In such cases, we make note of a modern approach.

THEOREM 3.1. (Fiber Lemma) Let E be a compact 3-manifold such that $\pi_1(E)$ has a finitely generated, normal subgroup G with quotient $\mathbb{Z}$. Then G is the fundamental group of a surface T, properly embedded in E; that is, the inclusion induced map $i_*\colon \pi_1(T) \to \pi_1(E)$ maps isomorphically onto G.

3.1 Candidate Fibration

Recall that a fiber bundle is given as a map from a total space to a base space. As we inevitably wish to show that E is a fiber bundle over the circle, it is reasonable to believe that a map $E \to S^1$ will be rich with information, provided that it reflects the assumed algebra. Note that by hypothesis, there is a surjective map $\varphi\colon \pi_1(E) \to \mathbb{Z}$ whose kernel is G. The following lemma[4] shows this map is induced by a map $f\colon E \to S^1$.

LEMMA 3.2. Let X be a simplicial-complex and Y an aspherical space. Then any homomorphism $\varphi\colon \pi_1(X) \to \pi_1(Y)$ is induced by a map $h\colon X \to Y$.

Proof. We define h by extending across each skeleton of X. As Y is aspherical, it suffices to extend across the 2-skeleton X^2 (see Appendix A2 Corollary 3).

We begin with some notation. Choose a maximal tree Γ in the 1-skeleton of X, and

[3]Topological, smooth, and piecewise-linear categories coincide for 3-manifolds, so many of the combinatorial piecewise-linear arguments can be circumnavigated by smooth arguments.

[4]This argument heavily utilizes simplicial-complexes, so it is very combinatorial in nature. Some of the notational headache can be reduced by considering CW-complexes (see for example [3] page 90).

choose a basepoint $b \in \Gamma$. For each $v \in \Gamma$, let e_v denote the unique simple path in Γ from b to v. For adjacent vertices u and v, let e_{uv} denote the directed edge connecting u to v. Each such edge induces a loop $\alpha_{uv} = e_u e_{uv} e_v^{-1}$ based at b.

Define h as follows. Map the tree Γ to a basepoint $y \in Y$, and each remaining edge e_{uv} to any y-based loop representing the element $\varphi([\alpha_{uv}])$. This defines h on the 1-skeleton of X, and by construction, we have $h_*([\alpha_{uv}]) = \varphi([\alpha_{uv}])$ for any loop α_{uv} To extend across the 2-skeleton, consider a 2-simplex Δ with vertices u, v, and w. The path

$$\alpha = e_u e_{uv} e_{vw} e_{wu} e_u^{-1}$$

defines a loop based at $b \in X$. It is clear that $\alpha_{uv}\alpha_{vw}\alpha_{wu} \simeq \alpha$ and that α is nullhomotopic across Δ. It follows that

$$1 = \varphi([\alpha]) = \varphi([\alpha_{uv}])\varphi([\alpha_{vw}])\varphi([\alpha_{wu}]) = h_*([\alpha_{uv}])h_*([\alpha_{vw}])h_*([\alpha_{wu}]) = h_*(\alpha).$$

Thus, $h(\alpha)$ is nullhomotopic in Y, whereby we can extend across Δ by a nullhomotopy. Repeating this process for each 2-simplex in X, we have defined h on the 2-skeleton X^2, as desired. In fact, $\pi_1(X)$ is generated by the $[\alpha_{uv}]$, over which we have $h_*([\alpha_{uv}]) = \varphi([\alpha_{uv}])$, so this map induces φ. □

Returning to the proof of the Fiber Lemma, there now exists a map $f\colon E \to S^1$ inducing the given quotient map $\varphi\colon \pi_1(E) \to \mathbb{Z}$. Noting that $\ker(f_*) = \ker(\varphi) = G$, the remainder of the Fiber Lemma will prove the existence of a surface $T \subset E$ whose fundamental group maps isomorphically onto $\ker(f_*)$ by the inclusion induced homomorphism.

Note that we are free to change f by a homotopy, since the induced map will be unchanged. We will repeatedly make use of this freedom, and it should be understood that any homotopy of f is relabeled again as f, for convenience.

3.2 Candidate Fiber

If the map produced in the previous section is a candidate fiber bundle, it should possess properties reminiscent of this type of map. In particular, we will show it produces a candidate fiber: a 2-sided, embedded surface $T \subset E$, a first approximation to the fiber. In subsequent sections, we use homotopies of f to reshape T to be connected and incompressible, so that $\pi_1(T)$ maps isomorphically onto $\ker(\varphi)$, as required for the Fiber Lemma.

We give two approaches to obtaining this surface, one smooth and one piecewise-linear.

For a smooth approach[5], let T be the pull-back of a regular value $p \in S^1$ by a smooth map homotopic to f. Then T is a properly embedded surface in E. Pulling back the normal bundle of the point guarantees it is 2-sided.

For a more hands-on argument, choose a simplicial map f (with respect to suitable triangulations of E and S^1) homotopic to the map constructed in the first section[6]. Consider the possible preimages of a non-vertex point $p \in S^1$. Figure 2 shows that the four vertices of a 3-simplex must be mapped to adjacent vertices in S^1. The leftmost drawing cannot occur because $f_* = \varphi$ and each 2-simplex in contractible; the rightmost cannot occur because f is simplicial and no additional vertices are valid.

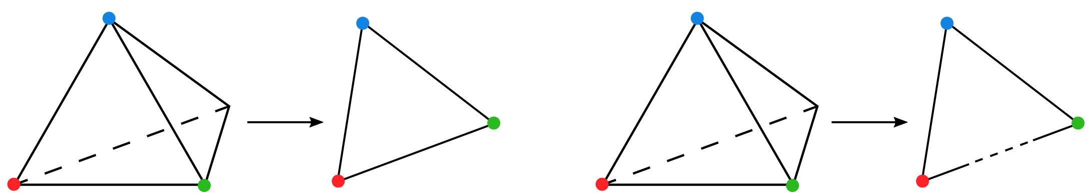


Figure 1. Disallowed simplicial mappings of a 3-simplex to S^1.

Moreover, for any point $p \in S^1$ that is not a vertex, the orientation of S^1 pulls back to a normal direction on $f^{-1}(p)$. Locally, the result is a properly embedded, 2-sided surface in each 3-simplex, as can be seen in Figure 3.

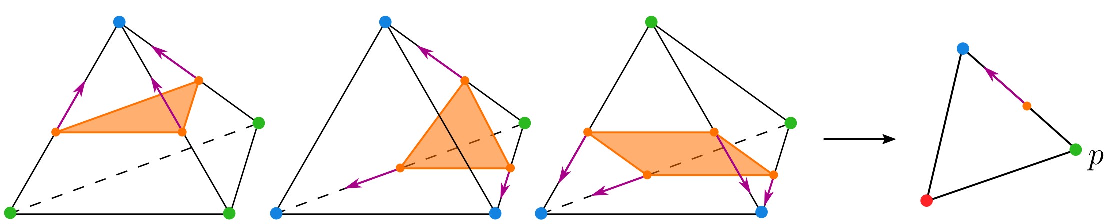


Figure 2. The possible preimages of a non-vertex p in a 3-simplex of E.

In fact, the entire preimage $f^{-1}(p)$ is a 2-sided, properly embedded surface $T \subset E$; we need only show it is locally 2-dimensional. Consider an arbitrary $x \in T$ not in ∂E; there are three cases depending on which skeleton of E contains x, shown in Figure 4. The

[5]This requires the Whitney Approximation Theorem, Sard's Theorem, and the Regular Value Theorem
[6]This requires the Simplicial Approximation Theorem.

leftmost drawing shows $x \in E^3 \setminus E^2$, in which case it belongs to some 3-simplex of the form from Figure 3. Similarly, the middle shows $x \in E^2 \setminus E^1$, where we use the fact that each 2-simplex is the face of exactly two 3-simplices, to see that x lies in the interior of a disk formed by gluing together two triangles along a edge containing x. In the rightmost drawing, $x \in E^1 \setminus E^0$ is contained in an edge of finitely many 3-simplices, whose union is homeomorphic to a 3-ball.

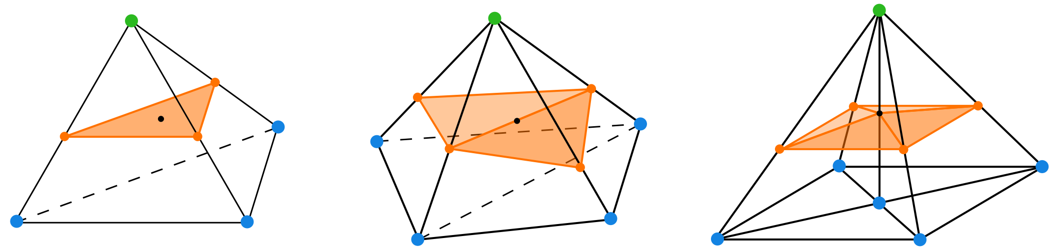


Figure 3. Each point in T belongs to one of three skeletons in E, around which T is 2-dimensional.

The case where $x \in \partial E$ follows mutatis mutandis, where simplices on the boundary of E yield boundary components of T. Thus T is a properly embedded surface in E. The normal direction from Figure 3 shows this surface is 2-sided in M. We should note that there is no guarantee of connectedness or orientability. The former will be dealt with in the following section. For now, we conclude that $T = T_1 \cup \cdots \cup T_n$, where each T_i is a connected, properly embedded, 2-sided compact surface in E.

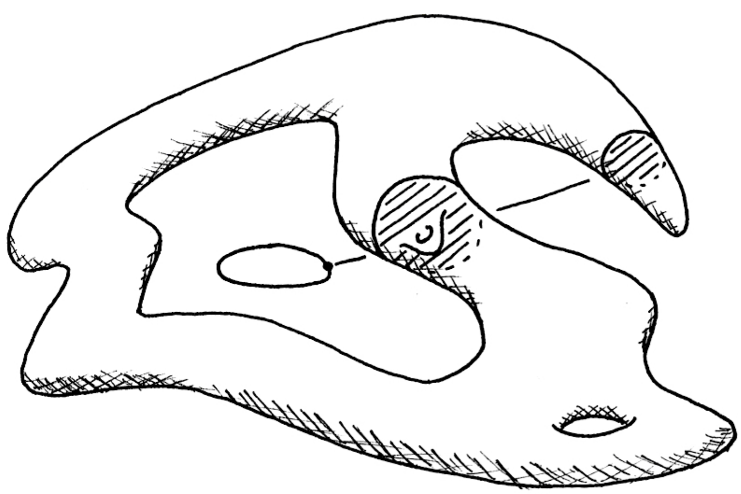


Figure 4. The preimage of p under the map $f\colon E \to S^1$.

3.3 Connectedness

In the previous section, we found a suitable $p \in S^1$ so that $f^{-1}(p)$ is as a collection $T_1 \cup \cdots \cup T_n$ of 2-sided, embedded submanifolds of E. This section produces a homotopy of f having a single component in the preimage $f^{-1}(p)$. To do this, we zoom out two dimensions, producing a graph whose vertices and edges are representative of the surfaces and 3-manifold, respectively. This graph shows how the surfaces are positioned in E. We will then conclude the graph has the homotopy type of a circle.

Choose a point a_i on each surface T_i and a point b_j in each component E_j of $E \setminus T$. Whenever T_i is a boundary component of E_j, choose a path in E_j connecting a_i to b_j in this component. This can be done so that no two paths intersect away from their endpoints. As each surface is 2-sided, each a_i corresponds to exactly two paths, which connect to one or two b_j. Let $\Gamma \subset E$ be the embedded graph whose vertices are the a_i and b_j and whose edges are paths in $E \setminus T$.

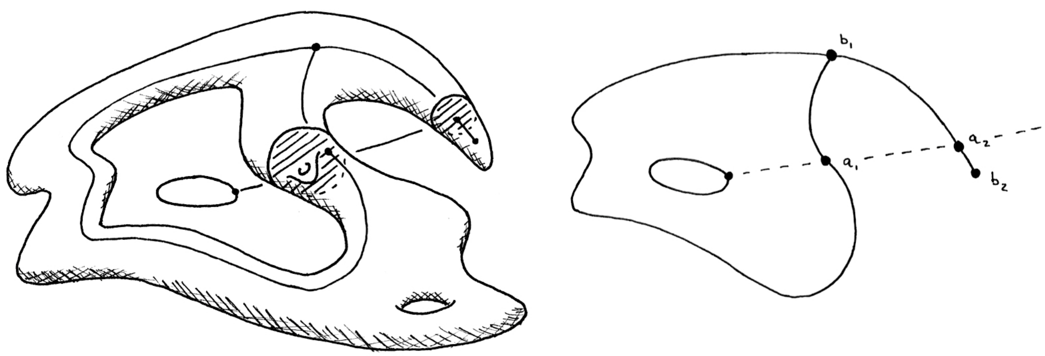


Figure 5. The graph Γ onto which E retracts.

We now determine the homotopy type of Γ. To begin, we define a retract $\rho\colon E \to \Gamma$. Each T_i has a product neighborhood $T_i \times I$ in E, and each a_i has a compatible product neighborhood $a_i \times I$ in Γ. We send $T_i \times I$ to $a_i \times I$ preserving the second coordinate, so that T_i is sent to a_i and $a_i \times I$ remains fixed. It remains to define ρ on each component E_j. Map $E_j \cap \Gamma$ identically to itself. Each $E_j \cap \Gamma$ is a tree, and there is no obstruction to defining an extension on this contractible space. Thus, we have defined a map $\rho\colon E \to \Gamma$ which restricts to the identity on Γ. Notice that by construction, the maps $(f|_\Gamma \circ \rho)$ and f both have preimage T at the point $p \in S^1$. By the Brown Representability Theorem[7] we

[7]This is a very deep theorem, and we encourage the curious reader to consult Appendix A4 for details.

have $f \simeq f|_\Gamma \circ \rho$, which produces the following commutative diagram:

$$\begin{tikzcd}
\pi_1(E) \arrow[rr, "f_*"] \arrow[dr, "\rho_*"'] & & \pi_1(S^1) \\
& \pi_1(\Gamma) \arrow[ur, "(f|_\Gamma)_*"'] &
\end{tikzcd}$$

We claim that $(f|_\Gamma)_*$ is an isomorphism. Both f_* and ρ_* are surjective, so $(f|_\Gamma)_*$ must be surjective. By commutativity, the kernel of $(f|_\Gamma)_*$ is given by $\rho_*(\ker(f_*))$, a finitely generated, normal subgroup of a free group having infinite index. Any such group is trivial by the following lemma, concluding that $(f|_\Gamma)_*$ is an isomorphism.

<u>Lemma 3.3</u>. If N is a finitely generated, normal subgroup of a free group F, then N has finite index in F.

<u>Proof</u>. View F as the fundamental group of a wedge of circles X, and consider the covering space $\widetilde{X} \to X$ corresponding to the finitely generated, normal subgroup $N \triangleleft F$. By normality of N, the group of covering automorphisms $\pi_1(X)/\pi_1(\widetilde{X}) \cong F/N$ acts transitively on the fibers of the cover. In particular, if N has infinite index in F, a loop in $\widetilde{X}$ will be distinctly repeated infinitely many times in $\widetilde{X}$. Conversely, the space $\widetilde{X}$ is a 1-complex with fundamental group a finitely generated free group. Hence the retract of a maximal tree in $\widetilde{X}$ produces a wedge of finitely many circles. It follows that the index cannot be infinite, as desired. $\square$

By Whitehead's Theorem, we conclude that Γ has the homotopy type of the circle. There is a clear homotopy $g \simeq f|_\Gamma$ having $g^{-1}(p) = \{a_i\}$ for some $a_i \in \Gamma$ (Figure 6).

Figure 6. A homotopy of the graph which removes a surface from the preimage.

It follows that $g\rho\colon E \to S^1$ satisfies $g\rho \simeq (f|_\Gamma)\rho \simeq f$ and $(g\rho)^{-1}(p) = T_i$, as desired.

In conclusion, we have produced a map $f\colon E \to S^1$ having $f^{-1}(p) = T$ a 2-sided, properly embedded, connected surface in E.

3.4 Incompressibility

The previous two sections established the geometric properties of the map $f\colon E \to S^1$, namely, that a candidate fiber T can be produced. The algebra of this surface has been entirely neglected, and the remainder of the Fiber Lemma focuses on correcting this offense. In particular, we show the inclusion induced map $\pi_1(T) \to \pi_1(E)$ embeds onto $\ker(f_*)$. This section proves injectivity of this map by showing T is incompressible. We begin by defining incompressibility and proving a historic lemma.

A properly embedded surface S in a 3-manifold W is ***compressible in W*** if there exists an embedded disk with essential boundary in S. Such a disk is referred to as a ***compressing disk***, and if no such disk exists, we say S is ***incompressible in W***. These types of surfaces are important in the study of 3-manifolds, as their fundamental groups accurately reflects the geometry of the ambient 3-manifold.

Incompressibility is commonly phrased algebraically. A properly embedded surface S in a 3-manifold W is ***π_1-injective*** if the inclusion induced map $\pi_1(S) \to \pi_1(W)$ is injective. It is clear that every π_1-injective surface is incompressible, as any compressing disk would make $\ker(\pi_1(S) \to \pi_1(W))$ nontrivial. The following lemma provides conditions in which the converse holds.

Lemma 3.4. (cf. [1]) Let S be a 2-sided, properly embedded surface in a 3-manifold X. The S is incompressible if and only if it is π_1-injective.

Proof. The sufficiency holds in general and was proven above; for the necessity we show that if S is not π_1-injective, there exists a compressing disk of S in X. Suppose there is a map $g\colon D^2 \to X$ having $g|_{\partial D^2}$ an essential loop in S. We will find a compressing disk by applying the Loop Theorem[8], however, we must first work to meet the conditions of this theorem. In particular, we need g to be a proper map. Making g transverse to S, its

[8] The Loop Theorem is a classic result in the theory of 3-manifold topology. We encourage the reader to explore Appendix A5 for details.

image intersects S in finitely many disjoint, closed curves. Consider an innermost such curve. If this curve is inessential, the intersection can be removed by replacing the disk it bounds on S with one pushed off from the surface (see surgery done in Appendix A5). Otherwise, this curve is essential in S, and restricting g to the disk bound by this curve produces a proper map $g' \colon (D^2, S^1) \to (X, S)$. In any case, such a map can be found.

Now, since S is 2-sided in X, we can split X along S, producing a 3-manifold X_S whose boundary contains two copies of S. Let $\iota \colon X_S \to X$ be the map that reglues the copies of S in X_S to reproduce X. The map g' is proper, so its image belongs entirely on one side of S, as shown in Figure 7.

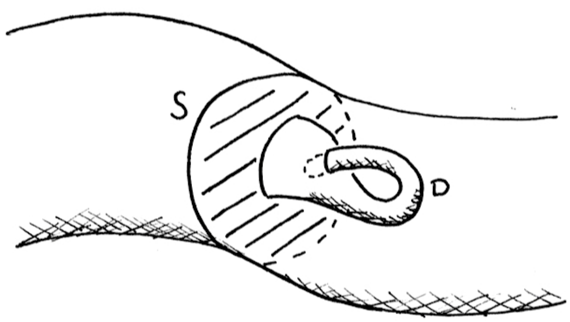


Figure 7. The properly embedded disk $(D^2, S^1) \to (X, S)$ lies on one side of S.

This induces a proper map into (X_S, S), which by Dehn's Lemma, produces a proper embedding $h \colon (D, S^1) \to (X_S, S)$. Composing with ι produces the desired compressing disk of S in X. □

Returning to the proof of the Fiber Lemma, recall that T is a 2-sided surface in E. By this lemma, it suffices to show that any compressing disk induces a map homotopic to f with one fewer compressing disks.

Suppose there is a compressing disk $g \colon (D^2, S^1) \to (E, T)$. The normal bundle of $g(D^2)$ is trivial[9], and by a homotopy we can make this product neighborhood intersect T in an annulus. Taking a regular neighborhood of the normal bundle gives a 3-ball D having $T \cap D$ an annulus and $\partial D \cap T$ a pair of disjoint circles, shown in Figure 8.

[9]The normal bundle of any contractible space is trivial.

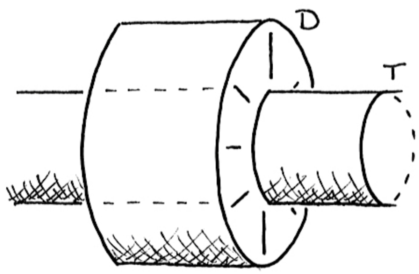


Figure 8. A regular neighborhood D of the compressing disk.

Our goal is to find a homotopy of $f|_D$ to a map in which these circles bound a pair of disjoint disks in D. Noting that $D \times I \cong B^4$, a homotopy of $f|_D$ is simply a map $B^4 \to S^1$. Since the circle is aspherical, there is no obstruction to extending a map defined on

$$\partial B^4 \cong \partial(D \times I) = (\partial D \times I) \cup (D \times \partial I).$$

It follows that any extension f' of $f|_{\partial D}$ is homotopic to $f|_D$ via the homotopy obtained by extending the map on $(\partial D \times I) \cup (D \times \partial I)$ that is identically $f|_{\partial D}$ on $(\partial D \times I)$, $f|_D$ on $D \times \{0\}$, and f' on $D \times \{1\}$.

We define an extension of $f|_{\partial D}$ as follows. Map a pair of disjoint disks bounding the aforementioned pair of circles in ∂D to the point $p \in S^1$. These disks separate D into three regions, each of which is bounded by S^2, as shown in Figure 9.

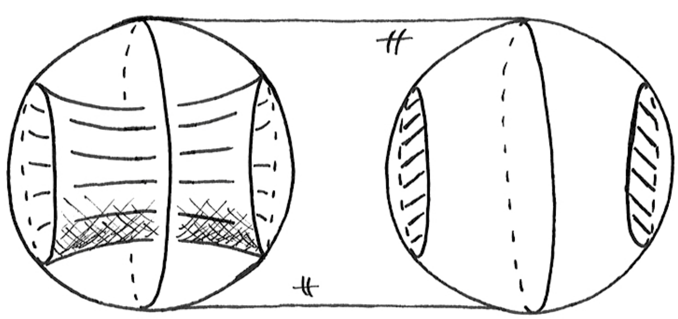


Figure 9. A homotopy of $f|_D$ expressed as a map on ∂B^4..

Since $\pi_2(S^1)$ is trivial, the map defined on each of these spheres extends to their interior. Moreover, because the point p is 2-sided in S^1, this can be done so that $D \cap f^{-1}(p)$ is simply the aforementioned pair of disks. Thus f is homotopic to a map having one fewer compressing disk in $f^{-1}(p)$.

The above homotopy might disconnect the surface, as in Figure 10, in which case we repeat the second section of this proof. We claim that only a finite number of compressions are necessary. Indeed, each compression raises the Euler characteristic of T by two, and if necessary, applying the previous section lowers the Euler characteristic by at most one (by discarding a surface that is not a sphere). In total, the Euler characteristic increases, and because the Euler characteristic is bounded above, only finitely many compressions are needed to guarantee $\ker(\pi_1(T) \to \pi_1(E))$ trivial.

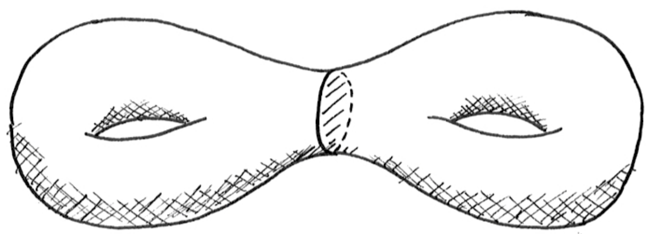


Figure 10. A disconnecting compressing disk.

It is trivial to generalize this section to include a wider range of codomains. Indeed, we only relied on the the fact that the codomain is an aspherical space containing a 2-sided, hypersurface[10].

LEMMA 3.5. Let X be a compact 3-manifold, and let Y be a connected, 2-sided, embedded, piecewise-linear hypersurface in an n-manifold Z having $\pi_2(Z)$ and $\pi_3(Z)$ trivial. Then any map $k\colon X \to Z$ is homotopic to a map $h\colon X \to Z$ having $h^{-1}(Y)$ is an incompressible, 2-sided surface[11] in X.

3.5 Conclusion

The previous sections drastically changed T to ensure incompressibility and connectedness. We now let the algebra take over, claiming that no further modifications are necessary. That is, we claim $\pi_1(T) \hookrightarrow \pi_1(E)$ is an embedding with image $G = \ker(f_*)$. Certainly $i_*(\pi_1(T)) \subseteq \ker(f_*)$ since f maps T identically to a point. The remainder of this section is dedicated to showing it is all of the kernel.

[10]A ***hypersurface*** of a manifold is an embedded, codimension 1 submanifold of the ambient manifold.
[11]Note that without further assumptions, we are not able to guarantee $h^{-1}(Y)$ is connected.

Before we begin, recall the construction from Lemma 3.4 in which we split a 3-manifold along a properly embedded 2-sided surface to produce a 3-manifold with boundary. Let M be the resulting 3-manifold formed by splitting E along T, let T_0 and T_1 be the two copies of T in ∂M, and let $\iota\colon M \to E$ be the corresponding map described in that lemma.

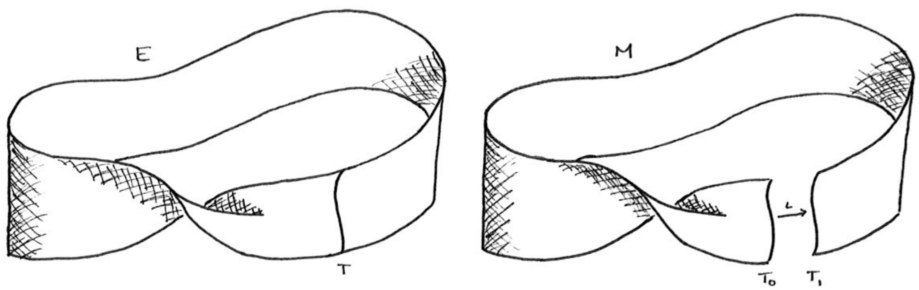


Figure 11. Viewed down one dimension, the split M of the 3-manifold E along the surface T.

Now, consider the cover space $p\colon \widetilde{E} \to E$ corresponding to the finitely generated, normal subgroup $\ker(f_*)$. The group of deck transformations for this cover space is given by $\pi_1(E)/\pi_1(\widetilde{E}) \cong \mathbb{Z}$. Since the image of $f \circ \iota$ is contractible in S^1, there is a lift of ι into $\widetilde{E}$. We claim that this lift is an embedding. Note that ι is an embedding away from T_0 and T_1, so because $\iota = p \circ \widetilde{\iota}$ the lift must also be an embedding away from these surfaces. If $\widetilde{\iota}$ is not an embedding on $T_0 \cup T_1 \subset M$, consider a path γ in M connecting any two points $x_0 \in T_0$ and $x_1 \in T_1$ with $\widetilde{\iota}(x_0) = \widetilde{\iota}(x_1)$. This path lifts to a loop in $\widetilde{E}$, and consequently, it represents an element of $\ker(f_*)$. However, it is also mapped to a nontrivial element of $\pi_1(\Gamma)$ by $\rho \circ \iota$. This contradicts the fact that $\ker(\rho_*) \cong \ker(f_*)$, as seen in section 3.3. We conclude that $\widetilde{\iota}$ is an embedding. Since ι is also onto, this implies that $\widetilde{E}$ is a union of copies of M, glued together along the surfaces T_0 and T_1. More concretely, we have shown that $\widetilde{E} = \bigcup_{\tau \in \mathbb{Z}} \tau(\widetilde{\iota}(M))$.

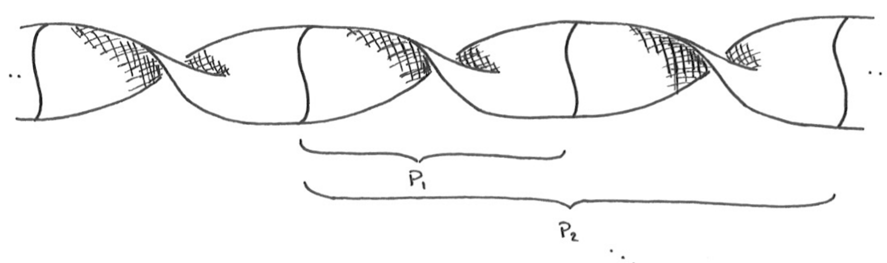


Figure 12. A covering space of E expressed as infinitely many copies of M attached along T.

Now, assume that $\pi_1(T)$ is not all of $\ker(f_*)$ and consider some $[\alpha] \in \ker(f_*) \setminus \pi_1(T)$. We use this element to contradict the hypothesis that $\ker(f_*)$ is finitely generated. Choose a generator τ_1 of the group $\mathbb{Z}$ of deck transformations, and let $\tau_n = (\tau_1)^n$. Choose a lift P_1 of M in $\widetilde{E}$, and in general, let $P_{n+1} = P_n \cup \tau_n(P_1)$. By van Kampen's Theorem,

$$\pi_1(P_{n+1}) = \pi_1(P_n) *_{\pi_1(T)} \pi_1(\tau_n P_1)$$

The inclusion induced maps for P_n and $\tau_n(P_1)$ are both injective for free products with amalgamation (see for example [2], pp. 30). However, we claim that they are not surjective, leading to a strictly increasing sequence of groups $\{\pi_1(P_n)\}$. Let $\widetilde{\alpha}$ be the lift of α into P_1. We will show $\widetilde{\alpha}$ and $\tau_1(\widetilde{\alpha})$ are not homotopic. Consider the copy of T separating P_1 and $\tau_1(P_1)$. Any homotopy between these loops can be made transverse to this surface, intersecting it in finitely many curves. Because T is 2-sided, at least one curve is nontrivial, and an innermost curve argument (as in Lemma 3.4) removes any trivial intersections. This results in a homotopy between the lift $\tau_1(\widetilde{\alpha})$ and a loop in T. No such homotopy can exist by assumption, so the inclusion induced map $\pi_1(P_{n-1}) \to \pi_1(P_n)$ is not onto. We have produced a strictly increasing sequence of groups satisfying $\pi_1(\widetilde{E}) = \varinjlim \pi_1(P_n)$, contradicting that $\pi_1(\widetilde{E}) = \ker(f_*)$ is finitely generated.

This concludes the Fiber Lemma, showing $T \subset E$ is a connected, properly embedded surface in E whose fundamental group embeds into $\pi_1(E)$ onto G.

As a final note, the supposed loop in $\ker(f_*) \setminus \pi_1(T)$ can be replaced by any loop from $\pi_1(M) \setminus \pi_1(T_0)$ to yield the same conclusion. That is, the inclusion induced map $\pi_1(T_0) \to \pi_1(M)$ is a surjection. By section 3.3 we have proved the following.

Corollary 3.6. The inclusion induced map $\pi_1(T_0) \to \pi_1(M)$ is an isomorphism.

4 Fibering Theorem

THEOREM 4.1. (Fibering Theorem) If E is an irreducible 3-manifold satisfying the hypotheses of the Fiber Lemma, then E is the total space of a fiber bundle over the circle with fiber T, the surface produced in the Fiber Lemma.

We first note that irreducibility cannot be removed from the hypothesis. Suppose we are given a fiber bundle E over the circle with fiber T. Then either the infinite cyclic cover $\widetilde{E}$ of E or the orientable double cover of $\widetilde{E}$ is irreducible. Indeed, this implies that E is irreducible. We conclude that a reducible 3-manifold does not fiber over the circle. This excludes the case $T = S^2$ because, for example, $S^2 \times S^1$ is reducible.

We exclude one other case. If the group G from the Fiber Lemma is $\mathbb{Z}_2$, then we have a split short exact sequence $\mathbb{Z}_2 \to \pi_1(E) \to \mathbb{Z}$. So by the Splitting Lemma, $\pi_1(E) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$. Then by Thurston's Geometrization Conjecture, now a theorem of Perelman, it follows that E is diffeomorphic to $\mathbb{R}\mathrm{P}^2 \times S^1$, a fiber bundle. Henceforth, we may assume $G \neq \mathbb{Z}_2$.

Recall that a fiber bundle comes equipped with local trivializations, which locally express the space as a product. By splitting a fiber bundle along some fiber, these maps show the resulting space is homeomorphic to a product. We will prove the Fibering Theorem in this manner. Namely, we construct a homeomorphism $M \to T_0 \times I$, where M is the 3-manifold constructed in 3.5 by splitting E along the candidate fiber T. Moreover, the boundary of M contains two copies of this surface, T_0 and T_1, having fundamental groups isomorphic to $\pi_1(M)$ (Corollary 3.6). To summarize, the Fibering Theorem follows from the following Theorem.

THEOREM 4.2. Let M be a compact, irreducible 3-manifold, and let T_0 be an embedded, connected surface (not S^2 or $\mathbb{R}\mathrm{P}^2$) in ∂M whose induced map $\pi_1(T_0) \to \pi_1(M)$ is an isomorphism. Then M is homeomorphic to $T_0 \times I$.

To prove Theorem 4.2, we use a similar strategy as the Fiber Lemma. The proof of this result began by constructing an algebraically relevant map $f\colon E \to S^1$ that served as a candidate fiber bundle of E over S^1. This map allowed us to produce a candidate fiber T in E satisfying the conditions of the Fiber Lemma. Our approach here is very similar. Given that our objective is to show $M \cong T_0 \times I$, there should exist a retract $r\colon M \to T_0$. We construct such a map, allowing us to produce a collection of disks which separate M into a collection of balls. The same can be done for $T_0 \times I$, leading to an obvious homeomorphism.

4.1 Retract

In this section, we construct a retract $r\colon M \to T_0$ inducing the isomorphism produced in Corollary 3.6. As a retract onto T_0 is an extension of the identity map on T_0, this construction uses a relative version of Lemma 3.2.

<u>Lemma 4.3</u>. Let K be a connected simplicial-complex with connected subcomplex L, and let Y be aspherical. Given a map $g\colon L \to Y$ and a homomorphism $\phi\colon \pi_1(K) \to \pi_1(Y)$, there exists a map $G\colon K \to Y$ extending f and inducing ϕ. That is, the following diagrams are commutative, and the leftmost diagram induces the rightmost diagram.

$$\begin{tikzcd} K \arrow[dr, dashed, "G"] & \\ L \arrow[u, hook] \arrow[r, "g"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \pi_1(K) \arrow[dr, "\phi"] & \\ \pi_1(L) \arrow[u, hook] \arrow[r, "g_*"'] & \pi_1(Y) \end{tikzcd}$$

<u>Proof</u>. As usual, we define the map $G\colon K \to Y$ by extending across each skeleton. Since Y is aspherical, it suffices to define G on the 2-skeleton.

We begin by introducing some notation. Choose a maximal tree Γ_L in the 1-skeleton of L, and extend this to a maximal tree Γ in the 1-skeleton of K, as in Figure 1. Choose a basepoint $b \in \Gamma_L$. For each vertex $v \in \Gamma_L$, let e_v denote the unique, simple path in Γ_L connecting b to v. For each vertex $v \in \Gamma_L$, let Λ_v denote the unique subtree of Γ growing from v into K (possibly $\Lambda_v = \{v\}$). For each vertex $v \in \Gamma \setminus \Gamma_L$, let $\widetilde{v} \in \Gamma_L$ denote the unique vertex whose induced tree $\Lambda_{\widetilde{v}}$ contains v. For each vertex $v \in \Gamma \setminus \Gamma_L$, let $\lambda_{\widetilde{v}v}$ denote the unique simple path in $\Lambda_{\widetilde{v}}$ from $\widetilde{v}$ to v. For each pair of adjacent vertices u and v, let e_{uv} denote the directed edge connecting u to v. Each such edge induces a loop based at b,

$$\gamma_{uv} = e_{\widetilde{u}} \lambda_{\widetilde{u}u} e_{uv} \lambda^{-1}_{\widetilde{v}v} e^{-1}_{v'}.$$

Define G as follows. Map L identically to itself, and send each tree Λ_u to $g(u)$. Map each remaining edge e_{uv} to any based loop representing the element $\phi([\gamma_{uv}])$. The proof continues identically to Lemma 3.2. □

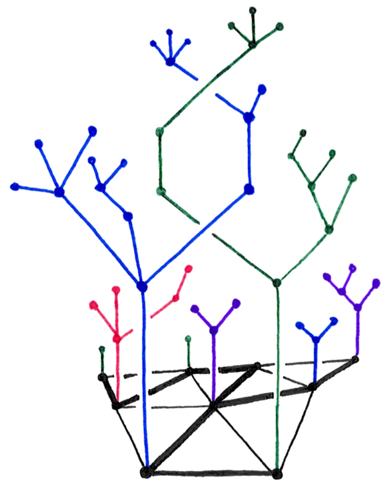


Figure 1. Extending the maximal tree in L to a maximal tree in K.

Returning to the proof of the Fibering Theorem, we apply this lemma to the diagrams

$$\begin{tikzcd} M \arrow[dr, dashed, "r"] & \\ T_0 \arrow[u, hook] \arrow[r, "\mathrm{id}"'] & T_0 \end{tikzcd} \qquad \begin{tikzcd} \pi_1(M) \arrow[dr, "\cong"] & \\ \pi_1(T_0) \arrow[u] \arrow[r, "\mathrm{id}_*"'] & \pi_1(T_0) \end{tikzcd}$$

We need only check that T_0 is aspherical. The following Lemma shows this is true for any surface except S^2 and $\mathbb{RP}^2$, both of which have been excluded.

LEMMA 4.4. A compact surface $S \neq S^2, \mathbb{RP}^2$ is aspherical.

Proof. If $S \neq S^2$ is closed and orientable, it has universal cover the plane, and if $S \neq \mathbb{RP}^2$ is closed and nonorientable, it is covered by some surface with nonzero genus. Otherwise, S has boundary and is homotopy equivalent to a wedge of circles, covered by some contractible 1-complex. In any case, S is aspherical by Corollary 2.5. $\square$

We conclude that there exists a retract $r\colon M \to T_0$ inducing the given isomorphism. As in the Fiber Lemma, we are not concerned by homotopies of this map.

4.2 Splitting Homeomorphism

In this section, we split M into simpler pieces, over which we can define a homeomorphism $M \to T_0 \times I$. This is done by finding arcs in T_0 whose complement is a disk, extending these arcs to disks in M, and removing product neighborhoods of these disks. We will see that the resulting space is a 3-ball. A similar deconstruction of $T_0 \times I$ produces identical subspaces, whereby we define an obvious homeomorphism.

The reader may have noticed that the described curves on T_0 only exist when T_0 is a surface with boundary. For this reason, the proof has two cases. We begin with the case where the boundary of T_0 is nonempty. Section 4.4 will reduce to this section from the closed case. Note that the retract $r\colon M \to T_0$ exists in both cases.

Any connected, compact surface with boundary can be constructed from a disk by attaching handles (bands) with orientation preserving/reversing maps, as in Figure 2.

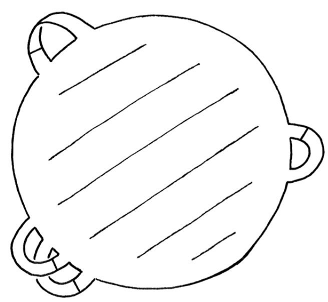


Figure 2. Every surface with boundary is obtained by attaching handles.

Representing T_0 in this way, we find finitely many arcs Q_i (namely, the cocores of the handles) whose complement in T_0 is a disk. Each arc is a contractible, 2-sided, hypersurface in T_0, so by a relative version of Lemma 3.5, there is a map homotopic to the retract $r\colon M \to T_0$ having each $r^{-1}(Q_i)$ an incompressible, properly embedded surface in M. Consider the component D_i of $r^{-1}(Q_i)$ containing Q_i. We have the following commutative diagram, where unlabeled maps are inclusion induced.

$$\begin{tikzcd} & \pi_1(M) \arrow[dr, "r_*"] & \\ \pi_1(D_i) \arrow[ur] \arrow[dr, "(r|_{Di})_*"'] & & \pi_1(T_0) \\ & \pi_1(Q_i) \arrow[ur] & \end{tikzcd}$$

Since the top two maps are injective and Q_i is contractible, it follows that D_i is simply connected. As D_i has boundary, it is a disk.

Just as the Q_i reduced T_0 to a 2-ball, we show that the disks D_i reduce M to a 3-ball. Each disk D_i has a product neighborhood $D_i \times I \subset M$, and each arc Q_i has a compatible product neighborhood $Q_i \times I \subset T_0$. Let $D = \bigcup(D_i \times I)$ and $Q \cong \bigcup(Q_i \times I)$. We will show that what remains, $B = M \setminus D$, is a 3-ball.

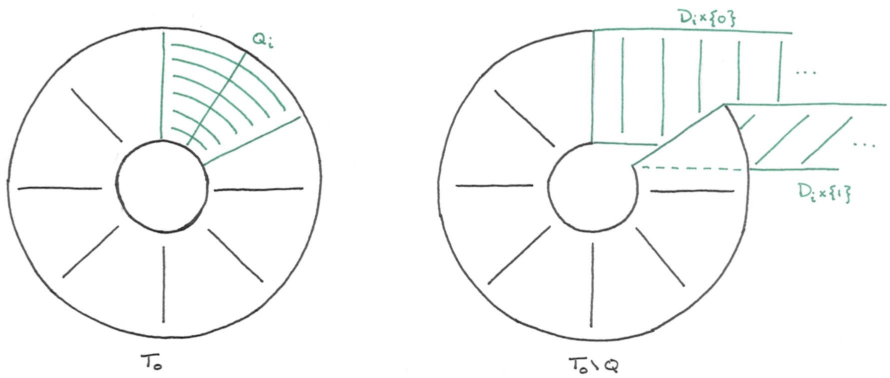


Figure 3. The set S (right) in the in the case where T_0 is an annulus.

First observe that it suffices to show that $F = \partial M \setminus \mathrm{int}(T_0)$ is incompressible in M. For in this case, the disk $C = \bigcup(D_i \times \partial I) \cup (T_0 \setminus Q)$, shown in Figure 3, lies in ∂B with $\partial C \subset F$. We have $C \subset \partial B$ and $\partial C \subset F$. Then ∂C bounds an embedded disk $C' \subset F$ by incompressibility, since otherwise, C is a compressing disk of F in M, a contradiction. Then $\partial B = C \cup C'$ is a pair of embedded disks attached on their boundary, namely, a 2-sphere. By irreducibility of M, B is homeomorphic to a 3-ball.

It remains to prove that F is incompressible in M.

<u>Lemma 4.5</u>. The surface $F = \partial M \setminus \mathrm{int}(T_0)$ is incompressible in M.

<u>Proof</u>. Assume to the contrary that we have a compressing disk D of M. We show that D allows us to write $\pi_1(M)$ as a free product of $\pi_1(T_0)$, which using Corollary 3.6, will produce a contradiction. By Dehn's Lemma, we can make D embedded; there are two cases depending on whether D separates M.

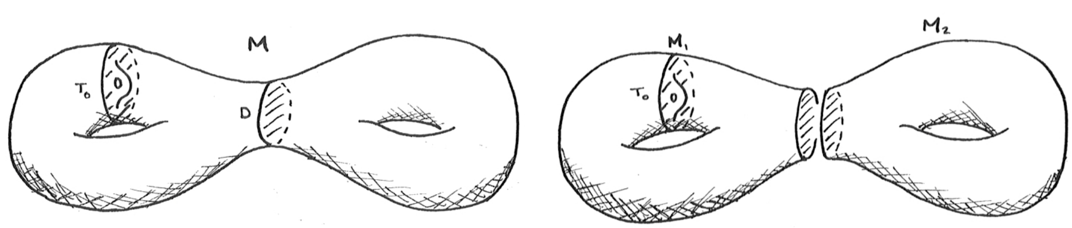


Figure 4. If D is separating, we obtain two 3-manifolds M_1 and M_2.

If D is separating, let M_1 and M_2 denote the components of $M \setminus D$, as in Figure 4. Then $\pi_1(M)$ can be presented by van Kampen's Theorem as

$$\pi_1(M) \cong \pi_1(M_1) *_{\pi_1(D)} \pi_1(M_2) = \pi_1(M_1) * \pi_1(M_2) \cong \pi_1(T_0) * \pi_1(M_2).$$

Corollary 3.6 implies $\pi_1(M_2)$ is trivial. Every simply-connected manifold is orientable[12], and the boundary of an orientable manifold is also orientable. Since ∂D is essential in M, we cannot have $\partial M_2 = S^2$. It follows from the Half-Life Lemma[13] that $H_1(M_2)$ is nontrivial, contradicting M_2 simply-connected.

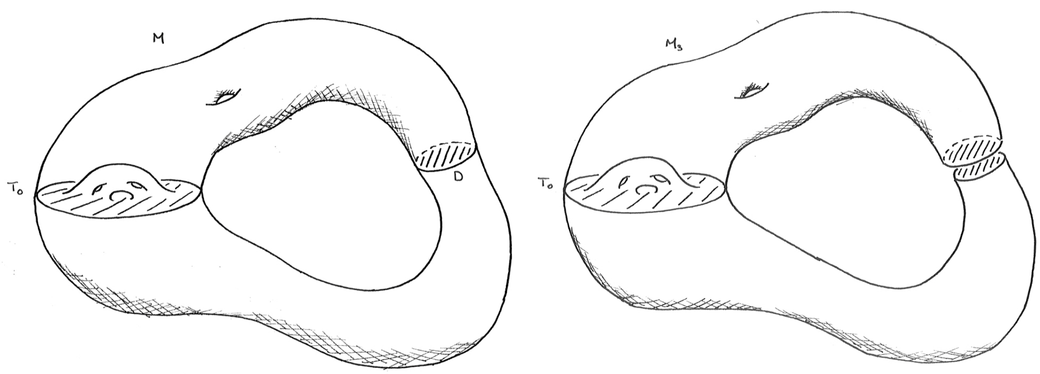


Figure 5. If D is nonseparating, we consider M_3, the split of M along D.

On the other hand, if D is nonseparating, the split of M along D forms a connected 3-manifold M_3 containing two copies D_0 and D_1 of the disk, as in Figure 5. For any homeomorphism $\alpha\colon D_0 \to D_1$, we can present $\pi_1(M)$ by an HNN Extension[14] of M_3 with α. That is, any presentation $\pi_1(M_3) = \langle S|R\rangle$ gives

$$\pi_1(M) \cong \pi_1(M_3)*_{\alpha_*} = \langle S, t|R, tht^{-1} = \alpha_*(h) \text{ for all } h \in \pi_1(D)\rangle.$$

Noting that $\pi_1(D)$ is trivial, and $\pi_1(M_3)$ is generated by $\pi_1(T_0)$, we have

$$\pi_1(M) \cong \langle S, t|R\rangle \cong \pi_1(T_0) * \mathbb{Z}.$$

This contradicts Corollary 3.6. $\square$

From the above comments, B is a 3-ball. We have decomposed M as $\bigcup(D_i \times I)$ and B, a collection of 3-balls. Note that we can similarly decompose $T_0 \times I$ as a collection of 3-balls: the product neighborhoods of $Q_i \times I$ and their complement.

[12]See for example [3] page 234.
[13]See for example [4] page 58
[14]See for example [5] page 173 or [2] page 35.

4.3 Homeomorphism

The decompositions of M and $T_0 \times I$ in the previous section have nearly completed the proof; we define the obvious homeomorphism $h\colon M \to T_0 \times I$ as follows. Send T_0 identically to $T_0 \times \{0\}$. Then each product neighborhood $(Q_i \times I) \subset T_0$ is sent identically to $(Q_i \times I) \times \{0\}$. Extend the definition h on $Q \times \{0\}$ so that $D \subset M$ is mapped to $Q \times I$. What remains of the domain and codomain are, respectively, $M \setminus D$ and $(T_0 \times I) \setminus (Q \times I)$, both of which are 3-balls. Thus, we can extend to a homeomorphism $h\colon M \to T_0 \times I$.

4.4 Reduction

The previous two sections proved that Theorem 4.2 holds for a surface T_0 with boundary. Suppose now that T_0 is a closed surface; we will reduce to the case with boundary. To do this, we split M along an annulus whose boundary contains a nonseparating curve on T_0. This resulting space will then satisfy the properties of Theorem 4.2.

Since T_0 is a surface with boundary that is not S^2 or $\mathbb{RP}^2$, there is a 2-sided, nonseparating curve $C \subset T_0$. We repeat the beginning of section 4.2 with the curve C in place of the arcs Q_i. The curve C is an aspherical, 2-sided, hypersurface in T_0, so by Lemma 3.5, there is a map homotopic to the retract $r\colon M \to T_0$ having $r^{-1}(C)$ an incompressible, properly embedded surface in M. Consider the component A or $r^{-1}(C)$ containing C. We have the following commutative diagram, where unlabeled maps are inclusion induced.

$$\begin{tikzcd}
 & \pi_1(M) \arrow[dr, "r_*"] & \\
\pi_1(A) \arrow[ur] \arrow[dr, "(r|_A)_*"'] & & \pi_1(T_0) \\
 & \pi_1(C) \arrow[ur] &
\end{tikzcd}$$

The top two maps are injective and $(r|_A)_*$ is onto. It follows that $\pi_1(C) = \mathbb{Z}$, so C is either an annulus or Möbius band. The latter cannot occur, as the map r retracts A onto its boundary. Since A is 2-sided in M, we can consider the split M along A. Let M^* be the resulting space, $\iota\colon M^* \to M$ the usual map (see Lemma 3.4), and let $T_0^* = T_0 \cap M^*$. To apply Theorem 4.2, we need only ensure $\pi_1(T_0^*) \to \pi_1(M^*)$ is an isomorphism.

To see this map is injective, consider the following commutative diagram, where unlabeled maps are inclusion induced.

$$\begin{tikzcd}
& \pi_1(M^*) \arrow[dr, "\iota_*"] & \\
\pi_1(T_0^*) \arrow[ur] \arrow[dr, "(\iota|_{T_0^*})_*"'] & & \pi_1(M) \\
& \pi_1(T_0) \arrow[ur] &
\end{tikzcd}$$

We claim that the bottom two maps are injective, implying the upper left map is injective. We need only check injectivity of the lower left map. Consider a loop α representing an element of $\ker(\pi_1(T_0^*) \to \pi_1(T_0))$. Because C is 2-sided, α can be viewed as a nullhomotopic loop in $T_0 \setminus (C \times I)$. This homotopy can be easily pushed away from the curve C, implying that α is nullhomotopic in T_0^*, as desired.

To see this map is onto, we use a similar argument as injectivity. Consider a loop β in M^* based on T_0^*. Because A is 2-sided, β can be viewed as a loop in $M \setminus (A \times I)$. By Corollary 3.6, there is a homotopy of β to a loop in T_0. Making this homotopy transverse to $A \times I$, it intersects in finitely many curves and arcs. At this point, the reader is well versed in removing these intersections. We eventually produce a homotopy in $M \setminus (A \times I)$. Hence, we view this homotopy in M^* to see that β is homotopic to a loop in T_0^*.

We conclude that $\pi_1(T_0^*) \to \pi_1(M^*)$ is an isomorphism. Since T_0^* has boundary, sections 4.2 and 4.3 and Theorem 4.2 show that $M^* \cong T_0^* \times I$. Gluing two annuli on ∂M^* and two annuli on $T_0^* \times I$ we have $M \cong T_0 \times I$, as desired.

Supplementary Topics

Examples

It would be a crime if we did not discuss examples of 3-manifolds which fiber over the circle. We will give two classes of examples in this section. The first examples are certain knot complements. The ***knot complement*** of a knot $K \subset S^3$ is the space $X_K = S^3 - K$ obtained by removing a tubular neighborhood of K from S^3. If X_K is a fiber bundle over the circle, we say K is a ***fibered knot***.

We can oftentimes apply Stallings Fibering Theorem to understand when a knot K is a fibered knot. A standard computation in algebraic topology shows that $H_1(X_K) \cong \mathbb{Z}$. We then have the short exact sequence

$$0 \to \pi_1(X_K)' \to \pi_1(X_K) \to H_1(X_K) \to 0.$$

Stallings' Fibering Theorem applies so long as the commutator subgroup is finitely-generated. A theorem of Crowell and Rapaport (cf. [8]) says this group is finitely generated if the Alexander polynomial of the knot Δ_K satisfies $|\Delta_K(0)| = 1$. It follows that the unknot and trefoil are fibered knots.

Taking a different approach, we can construct fiber bundles over the circle. Consider any surface T, and any self-diffeomorphism $f : T \to T$. The space $(T \times I)/\sim$ obtained by identifying $x \sim f(x)$ for each $x \in T$ is a fiber bundle over the circle. In general, the set of self-diffeomorphisms are very complicated, so we make no further claims about the fiber bundles they produce.

Circle Bundles

This exposition has focused on fiber bundles of a 3-manifold over a circle with fiber a surface, or *surface bundles*. A different approach is to reverse the roles of the fiber and base space. A 3-manifold that fiber over a surface with fiber a circle is called a *circle bundle*. Stallings Fibering Theorem can be generalized to include circle bundles. The following theorem does this.

THEOREM 4.6. (cf. [5]) Let M be a 3-manifold fitting into the short exact sequence

$$1 \to N \to \pi_1(M) \to Q \to 1,$$

where N is finitely generated and Q is a free group.

(i) If $Q \cong \mathbb{Z}$, then M is either $\mathbb{R}\mathrm{P}^2 \times S^1$ or surface bundle with fiber a connected 2-manifold whose fundamental group embeds isomorphically onto N.

(ii) If rank$Q > 1$, then M is a circle bundle with fiber a circle whose fundamental group embeds isomorphically onto N.

Virtual Fibering Conjecture

We saw in the beginning of Chapter 4 that reducible 3-manifolds do not fiber over the circle, but how atypical is it for a 3-manifolds not to fiber? In some senses, this is very atypical. A property P of a 3-manifold M is ***virtual*** if it is true for M in some finite covering space. Virtual properties became popular in the early twenty-first century, and two important conjectures emerged about virtual properties: the Virtual Fibering Conjecture and Virtual Haken Conjecture (cf. [1]). The former addresses the question posed at the beginning of this section.

VIRTUAL FIBERING CONJECTURE Every closed, irreducible, atoroidal 3-manifold with infinite fundamental group has a finite cover which is a surface bundle over the circle.

This theorem was proven in 2009 by combined works of Agol and Wise.

Appendix

A1 Homotopy Theory

Homotopy theory appears frequently when studying fibrations. We give a brief overview of higher homotopy groups and their relative generalizations. As with the fundamental group, there is a dependence on the choice of basepoint. We characterize spaces in which this dependence can be completely eliminated. This will be particularly important when we tackle Obstruction Theory (Appendix A2).

1.1 Higher Homotopy Groups

Recall the fundamental group of a pointed space (X, x) is defined as the set of homotopy classes of based loops into X, written $[(S^1, *), (X, x)]$, under concatenation. Higher homotopy groups simply raise the dimension of the spheres which we map; the n-th homotopy group of a pointed space (X, x) is the set $[(S^n, *), (X, x)]$ under a similar operation.

Although this intuitive approach has a clear geometric advantage, it has its theoretic limitations. In particular, the core of homotopy theory is supported by the ability to produce homotopies. Not surprisingly, we favor a definition involving cubes. Let I^n denote the n-cube and ∂I^n its boundary. For a pointed space (X, x), the set

$$\pi_n(X, x) := [(I^n, \partial I^n), (X, x)]$$

forms the ***n-th homotopy group of (X, x)*** under the operation $+$ defined on representatives $f, g\colon (I^n, \partial I^n) \to (X, x)$ as

$$(f+g)(s_1, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & \text{if } 0 \leqslant s_1 \leqslant 1/2 \\ g(2s_1 - 1, s_2, \ldots, s_n) & \text{if } 1/2 \leqslant s_1 \leqslant 1 \end{cases}$$

This operation is well-defined with respect to homotopy classes (i.e. $[f] + [g] = [f + g]$), endowing $\pi_n(X, x)$ with a group structure. It is clear this definition jives with our beloved fundamental group. Moreover, many properties from $\pi_1(X, x)$ are easily replicated in higher dimensions. For example, every map between pointed spaces $h\colon (X, x) \to (Y, y)$ induces a homomorphism $h_*\colon \pi_n(X, x) \to \pi_n(Y, h(x))$ by mapping $h_*([f]) = [f \circ h]$.

We list a few additional results here. Let (X, x) and (Y, y) be pointed spaces.

Theorem 1.1. Homotopy groups at distinct points are isomorphic in path-connected spaces.

Theorem 1.2. For all $n > 0$, we have $\pi_n(X \times Y, (x, y)) \cong \pi_n(X, x) \times \pi_n(Y, y)$.

Theorem 1.3. The group $\pi_n(X, x)$ is abelian for $n \geqslant 2$.

The reader is encouraged to prove these results independently, and might reference [3] and [6]. Now that we have the basics established, we develop the relative homotopy groups.

1.2 Relative Homotopy Groups

A homotopy group relative to a subspace considers maps whose image represents an n-sphere once we quotient by a subspace. Again there are multiple ways to define these groups. Let (X, x) be a pointed space with subspace $A \subset X$ containing the basepoint. For an intuitive approach, we define

$$\pi_n(X, A, x) = [(D^n, S^{n-1}, s), (X, A, x)],$$

for $n \geqslant 1$, So the image of each D^n is an n-sphere once we quotient out by A. As before, a practical definition is obtained with cubes. For the n-cube I^n, let J^{n-1} be the complement in ∂I^n of the bottom face I^{n-1} (i.e. the subspace of I^n having last coordinate 0). We define

$$\pi_n(X, A) = [(I^n, \partial I^n, J^{n-1}), (X, A, x)].$$

So ∂I^n is mapped to x, except on the bottom face, which is mapped into A. By quotienting by J^{n-1} we recover the pair (D^n, S^{n-1}), so these definitions are equivalent. For $n \geqslant 2$, this forms a group (see [3]) under the usual operations.

Of particular importance is knowing when an element represents the trivial element; as we are quotienting away the portion of any map lying in A, the following proposition should be intuitively clear.

Proposition 1.4. An element of $\pi_n(X, A, x)$ is trivial if and only if it is represented by a map whose image lies in A.

As with absolute homotopy groups, maps of pairs induce homomorphisms between the n-th relative homotopy groups (when $n \geqslant 2$). In addition, a boundary map can be obtained by simply restricting a map to the bottom face; that is, for any given map

$f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x)$, define $\partial f\colon (I^{n-1}, \partial I^{n-1}) \to (A, x)$ as the restriction to the bottom face. Then $\partial\colon \pi_n(X, A, x) \to \pi_{n-1}(A, x)$ defined by $\partial[f] = [\partial f]$ is a well-defined homomorphism (when $n \geqslant 2$). This will be useful when producing a long exact sequence of homotopy groups (Appendix A3 Theorem 7).

1.3 Basepoint Free Homotopy Groups

The objective of this section is to determine which spaces Y have the property that an arbitrary map $S^n \to Y$ determines an element in $\pi_n(Y, y)$. One might call such spaces *basepoint free*. If Y is not path-connected, there is no hope to guarantee that the sphere lands in the same path-component as the basepoint, so at the very least we let Y be path-connected. Note that any given map $f\colon S^n \to Y$ induces an element of $\pi_n(Y, y)$ by pinching a neighborhood on S^n to a path which connects the sphere to the basepoint (Figure A1.1). This construction is hindered by a dependence on which path you choose to the basepoint, so one characterization of basepoint free spaces might be: any space in which paths to the basepoint can be chosen arbitrarily.

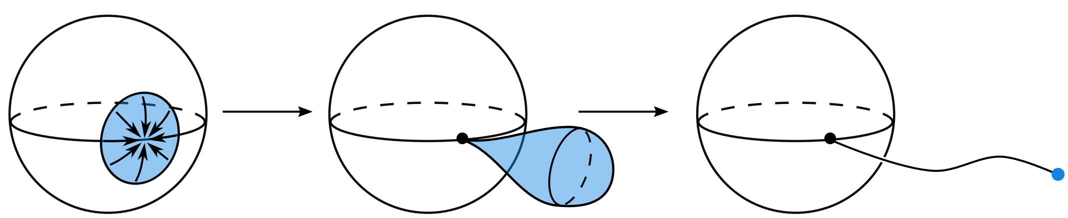


Figure A1.1: Each map $S^n \to Y$ is homotopic to a map based at any point in Y.

To rigorize this characterization, we begin by defining an action of $\pi_1(Y)$ on $\pi_n(Y)$. Let $f_0\colon (S^n, x) \to (Y, y)$ be a map, and let $\alpha\colon I \to Y$ a loop based at y. Thinking of α as a homotopy of a subspace of S^n, we can extend to a homotopy $f_t\colon S^n \to Y$ such that $f_t(x) = \alpha(t)$ for all $t \in I$ (see Corollary A3.4). Thus, we define a right-action on $\pi_n(Y)$ by $[f_0] \cdot [\alpha] = [f_1]$. At first glance, this action may seem trivial because $f_0 \simeq f_1$, however, the homotopy f_t is not relative to the basepoint x. The following proposition shows that the orbits of this action are in correspondence with the homotopy classes of maps $S^n \to Y$.

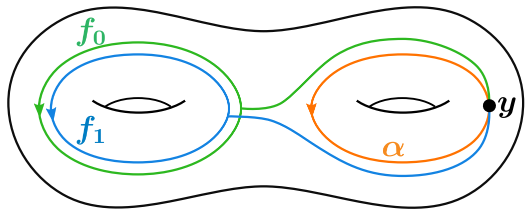


Figure A1.2: The loop α acts on $[f_0] \in \pi_1(Y, y)$ to produce the element $[f_1]$.

PROPOSITION 1.5. For each point $y \in Y$, there is a bijection between $[S^n, Y]$ and the set of orbits in $\pi_n(Y, y)$ under the action of $\pi_1(Y, y)$, described above.

Proof. Fix a $y \in Y$. As discussed above, a map $f\colon S^n \to Y$ is homotopic to some $f_\alpha\colon (S^n, x) \to (Y, y)$, where α is a path connecting $f(x)$ to y. We claim that the assignment $[f] \mapsto [f_\alpha]$ is the desired bijection. It must first be verified that this map is well-defined. Suppose that $f \simeq g$ with α and β arbitrary paths connecting $f(x)$ and $g(x)$ to y, respectively. There is a homotopy-induced path γ from $f(x)$ to $g(x)$, giving rise to an element $[\alpha.\gamma.\beta^{-1}]$ in $\pi_1(Y, y)$. Since $f \simeq g$, we have

$$[f_\alpha] = [g_\beta] \cdot [\alpha.\gamma.\beta^{-1}].$$

Thus, two representatives from $[f]$ are mapped to the same orbit. Now, suppose α and β are two arbitrary paths from $f(x)$ to y. These paths give rise to an element $[\alpha.\beta^{-1}]$ in $\pi_1(Y, y)$, yielding

$$[f_\alpha] = [f_\beta] \cdot [\alpha.\beta^{-1}].$$

We conclude that the map is well-defined. Now, as any $[f] \in \pi_n(Y, y)$ has $[f] \mapsto [f]$ this map is onto. Moreover, Y is path-connected, so the argument from well-definedness together with the homotopy extension property (Corollary A3.4) gives one-to-one. We conclude that this map is a bijection. □

We will say that Y is ***n*-simple** if for each $y \in Y$, the group $\pi_1(Y, y)$ acts trivially on $\pi_n(Y, y)$. Thus, this proposition shows that an arbitrary map $S^n \to Y$ determines an element of $\pi_n(Y, y)$ whenever Y is n-simple.

COROLLARY 1.6. Any simply-connected space is n-simple.

COROLLARY 1.7. Any aspherical space with abelian fundamental group is n-simple.

A2 Obstruction Theory

In the study of 3-manifolds, it is often necessary to construct a map on an arbitrary 3-manifold. Since every 3-manifold admits a simplicial structure, one way to do this is through iterative extensions across skeleta. Unfortunately, we are not always guaranteed an extension. More generally, we want to know when a simplicial-complex X and a map on its n-skeleton X^n has an extension over X^{n+1}. This question has a very computable, concise answer, which we establish almost immediately. The simplicity of this answer, however, is not the beauty of obstruction theory. This comes in the form of the Eilenberg Extension Theorem, which shows that certain maps on X^n are twice-extendable when restricted to X^{n-1}. Throughout this section, we assume Y is n-simple[15]. Further reading on this topic can be found in [6].

2.1 The Obstruction Cocycle

Let X be a CW complex with n-skeleton X^n. In order to extend a map $f\colon X^n \to Y$ to a map on X^{n+1}, it suffices to extend f across each individual $(n+1)$-simplex in X. The boundary of such a simplex σ is homeomorphic to an n-sphere, so there is an induced element $[f|_{\partial\sigma}]$ belonging to $\pi_n(Y)$. The following lemma rephrases the extension problem.

Lemma 2.1. For an $(n+1)$-simplex σ, a map $f\colon \partial\sigma \to Y$ extends across σ if and only if $[f|_{\partial\sigma}]$ is trivial in $\pi_n(Y)$.

This lemma is the motivation behind obstruction theory. Using $\pi_n(Y)$ as the coefficient group, we are able to consolidate the extension problem into simplicial cohomology. We define the ***obstruction of f*** to be the cochain

$$c_f^{n+1}\colon C_{n+1}(X^{n+1}) \to \pi_n(Y),$$

which maps each $(n+1)$-cell σ to the element $[f|_{\partial\sigma}]$. The following proposition is a first answer to the extension problem.

Proposition 2.2. A map $g\colon X^n \to Y$ has an extension over X^{n+1} if and only if the

[15] This assumption is for simplicity, and it can oftentimes be removed. Note that in Stallings Fibering Theorem, our codomain is usually n-simple by Corollary A1.7

obstruction of g is trivial.

Although uncommon, there are many spaces in which nearly every map extends. Recall that an ***aspherical space*** has trivial n-th homotopy groups for $n > 1$. Equivalently, an aspherical space is a $K(G, 1)$ space (see Appendix A4).

Corollary 2.3. For a simplicial-complex X and aspherical space Y, any $f\colon X^2 \to Y$ extends to a map on X.

On the surface, the obstruction cochain merely consolidates the extension problem. The depth it provides to the theory comes from the fact that it represents an element of cohomology, called the ***obstruction class***.

Proposition 2.4. The obstruction of a map $g : X^n \to Y$ is a cocycle.

Proof. It suffices to show that $\delta c_g^{n+1}(\sigma) = 0$ for every $(n+2)$-simplex σ in X. Fixing such a simplex σ, note that its boundary can be thought of as both a subcomplex $B = \partial\sigma$ and an $(n+1)$-cycle in $C_{n+1}(B)$. We show the obstruction is a coboundary on $C_{n+1}(B)$, since then there is a map d such that for every $\gamma \in C_{n+1}(B)$ we have

$$\delta c_g^{n+1}(\gamma) = c_g^{n+1}(\partial\gamma) = \delta d(\partial\gamma) = \delta\delta d(\gamma) = 0.$$

In particular, we are done by setting $\gamma = \partial\sigma$. It remains to define $d\colon C_{n+1}(B) \to \pi_n(Y)$ satisfying $\delta d \equiv c_g^{n+1}$. Free abelian groups have the useful quality that a map defined on one of its direct summands extends to a map on the whole group. With this in mind, consider the free abelian group $Z_n(B)$, and note that because $Z_{n-1}(B) = B_{n-1}(B)$ is also free abelian, we have the following split short exact sequence

$$0 \longrightarrow Z_n(B) \hookrightarrow C_n(B) \xrightarrow{\partial_n} B_{n-1}(B) \longrightarrow 0.$$

By the Splitting Lemma, $C_n(B) \cong Z_n(B) \oplus B_{n-1}(B)$, so we need only define d on $Z_n(B)$.

The trick in connecting $Z_n(B)$ to $\pi_n(Y)$ will be using the Hurewicz map on B^n. Let $k = g|_{B^n}$ be a restriction of the given map, with induced map $k_*\colon \pi_n(B^n) \to \pi_n(Y)$. Since B^n is $(n-1)$-connected, the Hurewicz map $h\colon \pi_n(B^n) \to H_n(B^n)$ is either an isomorphism (when $n > 1$) or an epimorphism with $\ker(h) \subseteq \ker(k_*)$. In either case, we have a well-defined map $H_n(B^n) \to \pi_n(Y)$. Note that B^n contains no $(n+1)$-cells, so $Z_n(B) \cong H_n(B^n)$. We define $d\colon C_n(B) \to \pi_n(Y)$ to be the extension of the resulting map $Z_n(B) \to \pi_n(Y)$. The following figure illustrates this definition in the case $n > 1$.

$$\begin{tikzcd} C_{n+1}(B) \arrow[r, "\partial"] \arrow[dr, "c_g^{n+1}"'] & Z_n(B) \arrow[r, "\cong"] \arrow[d, dashed, "d"] & H_n(B^n) \arrow[d, "h^{-1}"] \\ & \pi_n(Y) & \pi_n(B^n) \arrow[l, "k_*"'] \end{tikzcd}$$

It is straightforward to check that this definition gives $c_g^{n+1} \equiv \delta d$, as desired. $\square$

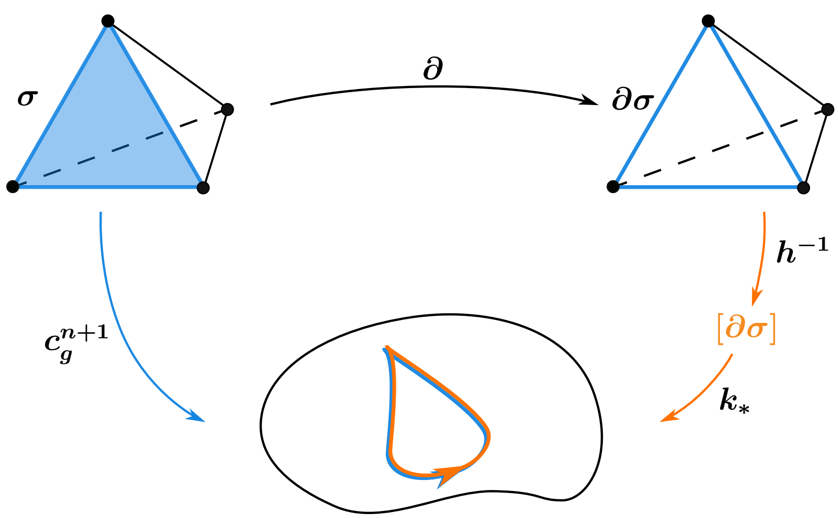


Figure A2.1: For each $(n+1)$-simplex σ, we have $c_g^{n+1}(\sigma) = \delta d(\sigma)$.

2.2 Eilenberg Extension Theorem

The above proof was successful because we were able to show the obstruction was a coboundary on a certain subset of the domain. What if the obstruction is itself a coboundary, or equivalently, the obstruction class is trivial in cohomology? The following theorem of Samuel Eilenberg (1913-1998) says that in such a case, we are able to produce an extension of a restricted map. Of course the obstruction class is not always trivial, presenting a much more hopeless case. For this reason, it a nontrivial obstruction class is referred to as a ***primary obstruction***.

<u>Theorem 2.5</u>. (Eilenberg Extension Theorem) A map $f\colon X^n \to Y$ with trivial obstruction class agrees with some map $h\colon X^{n+1} \to Y$ on the subcomplex X^{n-1}.

Proof. The obstruction to f is a coboundary by hypothesis, so there is a homomorphism $d\colon C_n(X) \to \pi_n(Y)$ satisfying $c_f^{n+1} = \delta d$. In particular, for every $(n+1)$-simplex σ,

$$[f|_{\partial\sigma}] = c_f^{n+1}(\sigma) = \delta d(\sigma) = \sum_{i=0}^{n+1}(-1)^i d(\sigma_i). \tag{1}$$

Our goal is to modify the map on each face σ_i by cutting out a neighborhood where the map is constant and replacing with a map representing the inverse of $d(\sigma_i)$.

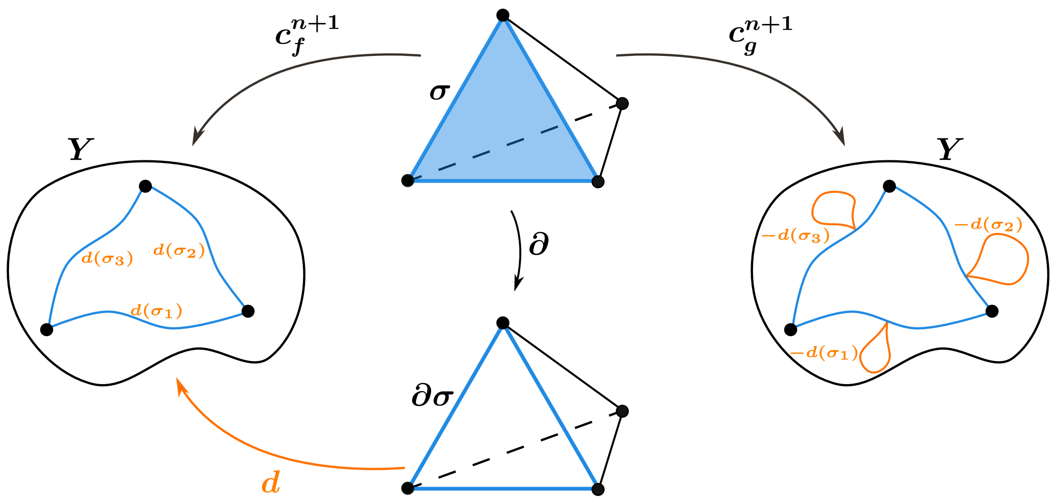


Figure A2.2: For each face of an n-simplex σ,

we redefine f so that its obstruction cancels with d.

Let γ be an arbitrary n-simplex in X. We can find a homotopy relative to $\partial\gamma$ from f to a map which is constant on a neighborhood $B_\gamma \approx B^n$ of the barycenter of γ. Since the restriction of f to the boundary of this neighborhood is constant, by changing f over the interior of B_γ, we can present $f|_{B_\gamma}$ as any element of $\pi_n(Y)$. Specifically, redefine $f|_{B_\gamma}$ to represent the element $\pm d(\gamma)$, where the sign differs with the orientation of the simplex. Doing so on each n-simplex in X, we produce a map $g : X^n \to Y$ agreeing with f on the subcomplex X^{n-1}. Moreover, on each $(n+1)$-simplex σ in X this map satisfies

$$c_g^{n+1}(\sigma) = [g|_{\partial\sigma}] = [f|_{\partial\sigma}] + \sum_{i=0}^{n}[g|_{B_{\sigma_i}}] = 0.$$

Hence, the obstruction of g is trivial, so g extends to a map $h\colon X^{n+1} \to Y$. This map agrees with f on X^{n-1}. □

A3 Fibrations

Although Appendix A1 gives an approachable introduction to higher homotopy groups, the truth is that they are generally difficult to compute. One difficulty arises when computing homotopy groups using higher dimensional spheres than the dimension of the given space[16]. In this section, we will give a large class of spaces, called fibrations, which work well with homotopy groups of all dimensions.

3.1 The Extension & Lifting Problems

To set up the definition of a *fibration*, we first discuss a general problem in mathematics. Let (X, A) be a pair of spaces, and let $p\colon E \to B$ be a map. Given a map $f\colon A \to E$, a map $h\colon X \to E$ is called an ***extension*** of f if we have $h|_A = f$. Similarly, given a map $g\colon X \to B$, a map $h\colon X \to E$ is called a ***lift*** of f if we have $ph = g$. Either question can be visualized as a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & E \\ X \arrow[ur, dashed, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

A slightly more restricted definition is formed by combining these diagrams: given f and g as above, a map $h\colon X \to E$ is called a ***lift-extension*** of f and g if it simultaneously extends f and lifts g.

Definition 1. Given a map $p\colon E \to B$, a pair of spaces (X, A) has the ***lift-extension property*** (LEP) with respect to p if every pair of maps $A \to E$ and $X \to B$ has a lift extension $X \to E$.

A particular case of the lift-extension problem arises in homotopy theory, and is the motivation behind fibrations.

Definition 2. A map $p\colon E \to B$ has the ***homotopy lifting property*** (HLP) with respect to X if for any homotopy $g_t\colon X \to B$ and any lift h_0 of g_0 there exists a homotopy $h_t\colon X \to E$ of h_0 lifting g_t.

[16] For example, it was famously shown that $\pi_3(S^2) \cong \mathbb{Z}$ using the Hopf fibration

Definition 3. A map has the ***relative homotopy lifting property*** (RHLP) with respect to a pair (X, A) if whenever we are given a solution H_A (Figure A3.1 right), there exists a lift H_X agreeing with H_A on $A \times I$.

$$
\begin{tikzcd}
X \times \{0\} \arrow[r, "h_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "G"'] \arrow[ur, dashed, "H_X"] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \cup (A \times I) \arrow[r, "h_0 \cup H_A"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "G"'] \arrow[ur, dashed, "H_X"] & B
\end{tikzcd}
$$

Figure A3.1: Commutative diagrams for the HLP and RHLP (left and right, respectively). We set $G(x, t) = g_t(x)$, and similarly define $H_A\colon A \times I \to E$ or $H_X\colon X \times I \to E$.

Each of the HLP and RHLP can be rephrased using the lift-extension property. The proof is essentially by-definition, or by noting that the existence questions posed by their commutative diagrams are the same.

Lemma 3.1. A map $p\colon E \to B$ has the HLP with respect to X if and only if the pair $(X \times I, X \times \{0\})$ has the lift-extension property with respect to p, and it has the RHLP with respect to (X, A) if and only if the pair $(X \times I, (X \times \{0\}) \cup (A \times I))$ has the lift-extension property with respect to p.

3.2 The Basics of Fibrations

At face value, the homotopy lifting property might not seem like much, however, its generality gives rise to many classes of spaces. Our main concern will be maps having the HLP with respect to a certain collections of spaces. These types of maps are called *fibrations*.

Definition 4. A map $p\colon E \to B$ which satisfies the homotopy lifting property with respect to all spaces is called a ***Hurewicz fibration***, or simply a ***fibration***. More generally, a map which has the homotopy lifting property with respect to all CW-complexes is called a ***Serre fibration***.

The spaces E and B in the above definition are commonly referred to as the ***total space*** and ***base space***, respectively. Throughout the rest of this section, we assume that the B is path-connected. For each $b \in B$, the ***fiber at b*** is the subspace $F_b = p^{-1}(b) \subset E$.

The upcoming proposition shows these fibers are all homotopy equivalent, so we often consider ***the fiber F*** of a fibration.

PROPOSITION 3.2. For a Hurewicz fibration $p\colon E \to B$, the fibers are homotopy equivalent.

Proof. A beautiful proof can be found in [3]. To sketch the proof, any path α in the base space can be easily lifted to a homotopy of the fibers. The end of this homotopy produces a map $H_\alpha\colon F_{\alpha(0)} \to F_{\alpha(1)}$. It can be shown that this map is independent of the homotopy type of α; for any $\alpha \simeq \beta$ relative to boundary, we have $H_\alpha = H_\beta$. Furthermore, the lifted homotopy plays nicely with respect to composition; $H_\alpha H_\beta = H_{\alpha\beta}$. We then conclude that H_α is a homotopy equivalence $F_{\alpha(0)} \simeq F_{\alpha(1)}$, with homotopy inverse $H_{\overline{\alpha}}$. □

Since the distinction between Serre and Hurewicz fibrations are somewhat pathological, we will focus on properties of Serre fibrations. The following four results highlight some of the basics for Serre fibrations. We begin by providing some equivalent definitions; in particular, it suffices to satisfy the HLP with respect to disks.

THEOREM 3.3. Let $p\colon E \to B$ be a map. The following are equivalent:

(a) p is a Serre fibration;

(b) p has the HLP with respect to all D^n;

(c) p has the RHLP with respect to all (D^n, S^{n-1});

(d) p has the RHLP with respect to all CW-pairs (X, A).

Proof. (a ⇒ b). Every n-disk is a CW-complex, so every Serre fibration has the HLP with respect to all n-disks. (b ⇒ c). By Lemma A3.1, this follows immediately from the fact that

$$(D^n \times I, D^n \times \{0\}) \cong (D^n \times I, (D^n \times \{0\}) \cup (S^{n-1} \times I))$$

(c ⇒ d). It suffices to show this for each skeleton (X^n, A). Inducting on n, there is a given homotopy on the pair of spaces $(X^{n-1} \times I) \cup (A \times I)$ by hypothesis, and we can extend this homotopy across each n-cell using the assumption. (d ⇒ a). Setting $A = \varnothing$, we are done by Lemma A3.1. □

COROLLARY 3.4. Every CW-pair (X, A) has the homotopy extension property.

Proof. Set B to be a point and consider at the commutative diagram in Figure A3.1. □

<u>Theorem 3.5</u>. Let $p\colon E \to B$ be a Serre fibration and (X, A) a CW-pair. For any maps $f\colon A \to E$ and $g\colon X \to B$ the following hold:

(a) if there exists a lift-extension of f and some map g' that is homotopic to g relative to A, then there exists a lift-extension for f and g;

(b) if there exists a lift-extension of g and some map f' that is fiberwise-homotopic to f, then there exists a lift-extension for f and g.

<u>Proof</u>. We sketch (a). There is a homotopy $X \times I \to B$ from g' to g, which is constant on $A \times I$. Since $ih' = f$, we can lift this to a homotopy $A \times I \to E$ such that $(a, t) \mapsto f(a)$. Corollary A3.4 then extends to a homotopy on X, and the end of this homotopy is the desired map. □

<u>Corollary 3.6</u>. A map lifts (extends) if it is homotopic to a map that lifts (extends).

<u>Proof</u>. For lifting, set $A = \varnothing$ in (a). For extending, set B to be a point in (b). □

3.3 Long Exact Sequence of a Serre Fibration

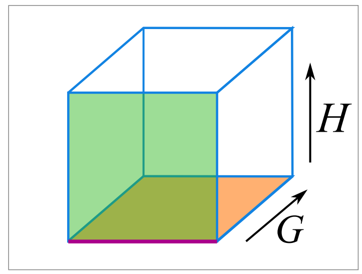
<u>Theorem 3.7</u>. A pointed pair (X, A, x) induces a long exact sequence of homotopy groups
$$\cdots \xrightarrow{\partial} \pi_n(A, x) \xrightarrow{i_*} \pi_n(X, x) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x) \longrightarrow \cdots \longrightarrow \pi_0(X, x).$$

<u>Proof</u>. There are three places to check exactness. We begin with exactness at
$$\pi_{n+1}(X, A, x) \xrightarrow{\partial} \pi_n(A, x) \xrightarrow{i_*} \pi_n(X, x).$$
It is straightforward to check that an element $[f]$ belongs to $\mathrm{im}(\partial)$ precisely when there exists a map $g\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x)$ such that $g|_{I^n} \simeq f$. Since most of this map is collapsed, we can easily simplify this to a map of pairs $(I^n, \partial I^n) \to (A, x)$ whose inclusion into X is nullhomotopic relative to the basepoint. Since g is homotopic to f on the bottom face, this map of pairs can be taken to be f. On the other hand, an element of $\ker(i_*)$ has a representative $h\colon (I^n, \partial I^n) \to (A, x)$ such that ih is nullhomotopic in X relative to the basepoint. It is easy now to see that these sets agree. Next, we consider exactness at
$$\pi_n(A, x) \xrightarrow{i_*} \pi_n(X, x) \xrightarrow{j_*} \pi_n(X, A, x).$$

The composition ji sends a any map $(I^n, \partial I^n) \to (A, x)$ to a map $(I^n, \partial I^n, x) \to (A, x, x)$. Since the image lies entirely in A, by Proposition A1.?? this map represents the trivial element in $\pi_n(X, A, x)$. Thus $\mathrm{im}(i_*) \subset \ker(j_*)$. Conversely, if f represents an element of $\ker(j_*)$, then by the same proposition, jf is homotopic to a map whose image lies entirely in A. The opposite inclusion follows, as desired. Finally, we show exactness at

$$\pi_n(X, x) \xrightarrow{j_*} \pi_n(X, A, x) \xrightarrow{\partial} \pi_{n-1}(A, x).$$

The composition ∂j_* is trivial by definition, so we have $\mathrm{im}(j_*) \subset \ker(\partial)$. For the reverse inclusion, consider some map $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x)$ such that ∂f is nullhomotopic relative to ∂I^{n-1}. It suffices to construct a homotopy of f to a map $(I^n, \partial I^n) \to (X, x)$. Visually, we do this on the cube $I^n \times I$, where I^n is the bottom face of the cube (orange), having I^{n-1} as its bottom face (purple). By assumption, we have a homotopy $F\colon I^{n-1} \times I \to A$ between ∂f and a constant map to x relative to ∂I^{n-1}. View the homotopy moving across the front face (green). Similarly, we can view f as a homotopy of ∂f relative to ∂I^{n-1} moving across the bottom face (orange). Since (I^n, I^{n-1}) is a CW-pair, by Corollary A3.4 there is a homotopy $G\colon (I^{n-1} \times I) \times I \to X$ of F. Changing the direction in which we view I, we have a homotopy H between f and some map H_1 relative to J^{n-1}. Following the definition of each of these maps, each edge in the cube (except I^{n-1}) is mapped to x. Therefore, the map H_1 is a map $(I^n, \partial I^n) \to (X, x)$ that is homotopic to f, as desired.



□

THEOREM 3.8. Suppose $p\colon E \to B$ is a Serre fibration with basepoints $b \in B$ and $x \in F_b$. If B is path-connected, there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(F, x) \longrightarrow \pi_n(E, x) \xrightarrow{p_*} \pi_n(B, b) \longrightarrow \pi_{n-1}(F, x) \longrightarrow \cdots \longrightarrow \pi_0(E, x) \to 0.$$

Proof. The majority of the work was done in the previous theorem, since inputing the pair (E, F) nearly gives the desired long exact sequence. We need only show that $\pi_n(E, F, x) \cong \pi_n(B, b)$. We clam that the map $p\colon (E, F, x) \to (B, b, b)$ induces this isomorphism. Note that this induced map is not the map p_* in the long exact sequence, which is induced by the fibration; which map we refer to can be deduced by context.

To see p_* is onto, consider some $f\colon (I^n, \partial I^n) \to (B, b)$. The constant map of ∂I^n to x lifts $f|_{J^{n-1}}$. By Corollary A3.4 on $(I^{n-1}, \partial I^{n-1})$, the map ∂f has a lift. Pairing these lifts, we have have a lift of $f|_{I^{n-1}}$ into E. The same Corollary on (I^n, I^{n-1}) produces a lift $\widetilde{f}\colon (E, F, x) \to (B, b, b)$ of the map f. By construction we have $p_*([\widetilde{f}]) = [p\widetilde{f}] = [f]$, as desired. For injectivity, the same lifting process lifts a homotopy of two maps in the image to a homotopy of the maps. $\square$

Corollary 3.9. For a covering space $p\colon \widetilde{X} \to X$, the map $p_*\colon \pi_n(\widetilde{X}) \to \pi_n(X)$ is an isomorphism for $n \geqslant 2$ and a monomorphism for $n = 1$. Moreover, the cosets of $p_*(\pi_1(\widetilde{X})) < \pi_1(X)$ are in bijection with the fiber.

Proof. Use the long exact sequence of a Serre fibration, accessible by the previous theorem. The first claim follows immediately. The tail of this sequence gives a short exact sequence, giving the second claim. The final claim uses the first isomorphism theorem on this short exact sequence. $\square$

3.4 Fibre Bundles

One particularly fruitful class of fibrations are *fiber bundles*, whose total spaces can be thought of as *twisted product spaces*. That is, they are spaces with a local product structure that is not necessarily mirrored globally.

Definition 5. A ***fiber bundle*** with fiber F is a map $p\colon E \to B$ such that B is a ***local product***; that is, each point $b \in B$ belongs to a neighborhood $U \subset B$ such that $h\colon p^{-1}(U) \to U \times F$ is a homeomorphism making the following diagram commute:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

Such a map h is referred to as a ***local trivialization*** of the bundle.

Note that we are *given* a space F which forms a local product with B. The local trivialization then preserves the structure of this space in each of the fibers, as the following theorem shows. So the fibers of a bundle are built in to their definition. For simplicity, a fiber bundle is often given in the condensed notation $F \hookrightarrow E \xrightarrow{p} B$.

<u>THEOREM 3.10</u>. The fibers of a fiber bundle are homeomorphic.

<u>Proof</u>. Each $b \in B$ belongs to some neighborhood U, determined by a local trivialization $h\colon p^{-1}(U) \to U \times F$. Note that the map $h|_{F_b}\colon F_b \to \{b\} \times F$ is continuous and injective; moreover, by commutativity of the above diagram, it is onto. The codomain is clearly just the fiber F, as desired. $\square$

Fiber bundles are a special case of fibrations. However, this fact is not immediate from the definitions, and the type of fibration (Serre or Hurewicz) admitted by a fiber bundle depends on properties of the base space. A fiber bundle over a paracompact space is a Hurewicz fibration. More specific to our needs is the fact that:

<u>THEOREM 3.11</u>. A fiber bundle is a Serre fibration.

The proof of this theorem is rather dull (a curious reader might reference [3]). A glance at the definition of a covering space should convince the skeptical reader that most covering spaces are fiber bundles. So to bring the above discussion together:

Covering Spaces[17] $\subset$ Fiber Bundles $\subset$ Serre fibrations $\subset$ Hurewicz fibrations

A4 Brown & Eilenberg

In this section, we discuss *Eilenberg-MacLane spaces*, which can be thought of as the building blocks for topological spaces (see Postnikov towers). These spaces have some fantastic properties that highlight the previous two sections. We end with a deep theorem of Brown.

4.1 Eilenberg-MacLane Spaces

To motivate Eilenberg-MacLane spaces, suppose we are given a sequence of groups $\{G_n\}$. Can we construct a space having each G_n as its nth homotopy group? Firstly, this is impossible if for any $n > 1$ the group G_n is nonabelian (Theorem A1.3). Restricting to this case, suppose there exists a space X_n having exactly one nontrivial homotopy group $\pi_n(X_n) = G_n$. Then by Theorem A1.2, a solution is obtained by the space $\Pi_n X_n$.

[17] We must restrict ourselves to covering spaces whose fibers have the same cardinality, otherwise we contradict Theorem A3.10.

<u>Definition 6</u>. Given a group G and a positive integer n, a space X having one nontrivial homotopy group $\pi_n(X) \cong G$ is called an ***Eilenberg-MacLane space of type $K(G,n)$.***

The above question is now: given a positive integer n and a group G (abelian for $n > 1$), does there exist a $K(G,n)$ space? This question has an affirmative answer (see e.g. [3]). Although they can be constructed, there are few examples in the topologists tool belt. We provide some of these common examples here:

- Certainly S^1 is a $K(\mathbb{Z},1)$, having fundamental group $\mathbb{Z}$ and contractible universal cover.
- The CW-complex $\mathbb{R}\mathrm{P}^2$ is a $K(\mathbb{Z}_2,1)$.
- The CW-complex $\mathbb{C}\mathrm{P}^\infty$ is a $K(\mathbb{Z},2)$. To see this, let $n > 0$ and consider the usual quotient map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$. This is a fiber bundle with fiber S^1. The long exact sequence of this fibration implies
$$\pi_k(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & k = 2 \\ \pi_k(S^{2n+1}) & k \neq 2 \end{cases}$$
So as n increases, it is easy to show the higher homotopy groups vanish. Recalling $\mathbb{C}\mathrm{P}^\infty$ is constructed using a limiting process on $\mathbb{C}\mathrm{P}^n$, it will follow $\mathbb{C}\mathrm{P}^\infty$ is a $K(\mathbb{Z},2)$ space.
- Note that any of the above $K(G,n)$ can be generalized to a $K(G^k,n)$ for any $k > 0$ by using Theorem A1.2.

Eilenberg-MacLane spaces are unique up to homotopy equivalence. In particular, the only $K(\mathbb{Z},1)$ space is the circle. A space is ***aspherical*** if it is a $K(G,1)$.

4.2 Brown Representability Theorem

Eilenberg-MacLane spaces are deeply related to cohomology. The Brown Representability Theorem is a deep theorem in category theory that determines which objects in the category of pointed, connected CW-complexes have a representation in the category of sets. Applied to cohomology, this result yields the following theorem.

<u>Theorem 4.1</u>. Let X be a CW complex, and let Y be a $K(G,n)$ for some abelian group G. Then $H^n(X;G) \cong [X,Y]$.

The proof of this theorem is out of the scope of this exposition, however, we apply the theorem to the following scenario. Let X be a compact 3-manifold, and let Y be aspherical with abelian fundamental group. Applying duality (Poincaré or Lefschetz) to A4.1 gives

$$H_2(X, \partial X; G) \cong H^1(X; G) \cong [X, Y].$$

In particular, suppose $p \in Y$ is a regular value of the maps $f, g\colon M \to Y$. If the surfaces $f^{-1}(p)$ and $g^{-1}(p)$ determine the same homology class in $H_2(M, \partial M; G)$, then f and g are homotopic by the above theorem. This fact is used in the Fiber Lemma.

A5 The Loop Theorem and Dehn's Lemma

5.1 History

One of the important results which appears in Stallings' Fibering Theorem is the famous Loop Theorem. A restriction of this result, known as Dehn's Lemma, was originally published in 1910 under M. Dehn (1878-1952), although a mistake was later found in his proof. The result stood unproven for nearly half a century before C. D. Papakyriakopoulos (1914-1976) published a proof in 1952. Papakyriakopoulos proved a stronger result, called the Loop Theorem. This result states that when an essential loop in the boundary of a manifold bounds a proper disk, we can find an essential loop in the boundary that bounds a properly embedded disk. In this section, we prove the Loop Theorem and conclude Dehn's Lemma as a corollary.

The key to Papakyriakopoulos' proof is the use of a *tower construction*, which reduces the singularities of the given disk by creating a tower of nontrivial double covers. At the top of this tower (i.e. after removing sufficiently many singularities), it is much easier to produce an embedding. This embedding percolates down the tower by surgering away any introduced singularities.

5.2 Loop Theorem

Proofs of the Loop Theorem can be found it many modern texts (see e.g. [4], [5], [10]). The following proof is based on Stallings' paper on the Loop Theorem [10], in which we utilize the tower construction of Papakyriakopoulos [9].

Theorem 5.1. (Loop Theorem) Let M be a 3-manifold such that $\pi_1(\partial M) \to \pi_1(M)$ is not injective. Then there is an embedding $f\colon (D^2, S^1) \to (M, \partial M)$ with $f(S^1)$ representing an essential loop in ∂M.

Proof. The first stage of the proof is to construct the tower mentioned above. Triangulate D and M, and take $f_0 \simeq f$ to be simplicial with respect to these triangulations. Let M_0 be a regular neighborhood of $f_0(D^2)$ whose boundary contains the surface $T_0 = \partial M \cap M_0$.

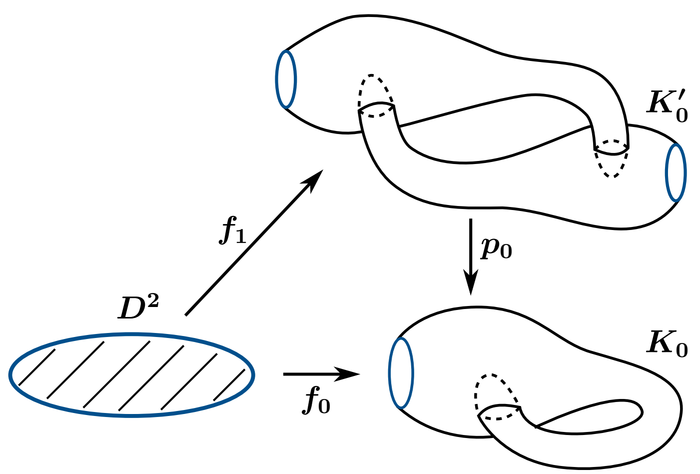


Figure A5.2: A potential nontrivial double cover of a proper disk with singularities.

Suppose that M_0 has a nontrivial double cover $p_0\colon M_0' \to M_0$. In such a case, there is an induced triangulation of M_0', where each simplex in M_0 lifts to a pair of simplices in the cover. Since D^2 is simply connected, we can also lift the map f_0 to a map f_1 which is simplicial with respect to the triangulation. Similar to the first layer, let $M_1 \subset M_0'$ be a regular neighborhood of $D_1 = f_1(D^2)$ whose boundary contains the surface $T_1 = \partial M \cap M_1$. Each subsequent layer is produced whenever the regular neighborhood M_i has a nontrivial double cover $p_i\colon M_i' \to M_i$. We should note that at each stage, the conjugacy class of $f_i|_{S^1}$ is nontrivial in T_i, lest its projection to $f_0|_{S^1}$ be trivial ∂M. We claim that the resulting tower is finite, namely, there is a regular neighborhood M_i of $D_i = f_i(D^2)$ which has no nontrivial double cover. We show this by examining the integer n_i of simplices in D_i, showing it is bounded and strictly increasing. Because each f_i is simplicial, it is clear that each n_i does not exceed the number of simplices in the triangulation of D^2. Consider a cover $p_i\colon M_i' \to M_i$ from the construction. The deformation retract from M_i onto D_i lifts to one from M_i' onto $D_i' = p_i^{-1}(D_i)$. By restricting the projection, there is an induced, nontrivial double cover $p_i'\colon D_i' \to D_i$. Since the fundamental group of a double cover is normal, there is a nontrivial simplicial automorphism $\tau_i\colon D_i' \to D_i'$ with no fixed points. Since $D_i' = D_{i+1} \cup \tau_i(D_{i+'})$ is connected, then $D_{i+1} \cap \tau_i(D_{i+1})$ is nonempty. In particular, there is a simplex σ in this intersection (and in particular in D_{i+1}) distinct from $\tau_i(\sigma)$. Since $\tau_i(\sigma)$ must be in D_{i+1}, this subcomplex contains two simplices which are identified in D_i. We conclude that $n_i < n_{i+1}$, and thus the tower must be finite.

$$
\begin{tikzcd}
 & D_{i+1} \arrow[r, hook] & M_{i+1} \arrow[r, hook] & M'_{i+1} \arrow[dl, "p_i"] \\
 & \vdots & \vdots & \\
 & D_1 \arrow[r, hook] & M_1 \arrow[r, hook] & M'_0 \arrow[dl, "p_0"] \\
D^2 \arrow[r, "f_0"] \arrow[ur, "f_1"] \arrow[uuur, "f_i"] & D_0 \arrow[r, hook] & M_0 \arrow[r, hook] & M
\end{tikzcd}
$$

We are now guaranteed a regular neighborhood M_i that has no nontrivial double cover. We will rescue an embedding of the disk into M_i. The first step is to show that the components of ∂M_i are all spheres. Since the only surface with trivial first homology group is the 2-sphere, this is accomplished by showing $H_1(\partial M_i; \mathbb{Z}_2) = 0$. Every double cover of M_i corresponds uniquely to an index two subgroup of $\pi_1(M_i)$, which when realized as a homomorphism $\pi_1(M_i) \to \mathbb{Z}_2$, factors uniquely through $H_1(M_i)$ because $\mathbb{Z}_2$ is abelian. Thus there is a bijection between double covers and $\operatorname{Hom}(H_1(M_i), \mathbb{Z}_2)$. Since M_i has no nontrivial double covers, this group is trivial. By Universal Coefficient Theorem

$$H^1(M_i; \mathbb{Z}_2) \cong \operatorname{Hom}(H_1(M_i; \mathbb{Z}), \mathbb{Z}_2) \oplus \operatorname{Ext}(H_0(M_i; \mathbb{Z}), \mathbb{Z}_2),$$
$$H_1(M_i; \mathbb{Z}_2) \cong (H_1(M_i; \mathbb{Z}) \otimes \mathbb{Z}_2) \oplus \operatorname{Tor}(H_0(M_i; \mathbb{Z}), \mathbb{Z}_2).$$

Since M_i is path-connected the latter term in both isomorphisms is trivial. The first line, which is trivial, then implies that $H_1(M_i; \mathbb{Z}) \cong \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_n}$ where all $p_1, \ldots, p_n$ are odd. Consequently, the first term in the second line vanishes, meaning $H_1(M_i; \mathbb{Z}_2)$ is trivial. Now, by Poincaré Duality $H_2(M_i, \partial M_i; \mathbb{Z}_2)$ is trivial, so the long exact sequence

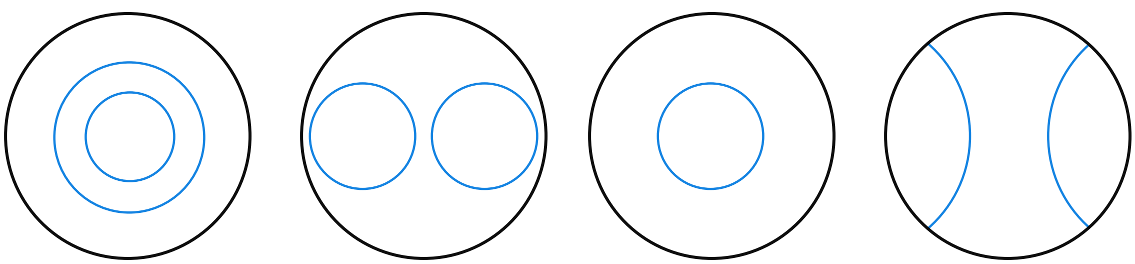
$$\cdots \to H_2(M_i, \partial M_i; \mathbb{Z}_2) \to H_1(\partial M_i; \mathbb{Z}_2) \to H_1(M_i; \mathbb{Z}_2) \to \cdots$$

yields that $H_1(\partial M_i; \mathbb{Z}_2) = 0$, as desired.

Consider the regular neighborhood T_i of $f_i(S^1)$, which by connectedness, is a subset of some sphere comprising ∂M_i. Note that $\pi_1(T_i, t_i)$ is normally generated by its boundary curves. So the conjugacy class of some boundary component of T_i is nontrivial, lest the conjugacy class of its core, $f_i(S^1)$, is trivial. This boundary component is an embedded curve in some component of ∂M_i, which we have discovered is a sphere. By the Jordan Curve Theorem, such an embedded curve separates this sphere into two components, and consequently bounds a disk in ∂M_i. Therefore we have rescued an embedding $g_i\colon (D, S^1) \to (M_i, T_i)$ where the conjugacy class of $g_i(S^1)$ is nontrivial in T_i.

Now that we have found an embedding g_i, we move down the tower by projecting downward with the covering map. This may introduce singularities, which in general position, take the form of finitely many transverse double curves meeting at finitely many triple points. Fortunately, the map $p_i g_i$ is a two-to-one map, so no triple points will occur. We claim that this map can be surgered to an embedding.

The introduced singularities can be better understood by looking at their preimages in the disk. They take the form double curves (Figure A5.3a. - A5.3c.) and double arcs (Figure A5LT.3d.). The three double curves can be further divided by the degree of the map (Figure A5.3c. has degree two, the others degree one).



A5.3a. *A5.3b.* *A5.3c.* *A5.3d.*

Figure A5.3: The four possible types of singularities introduced by projecting the embedding down the tower.

Since there are finitely many singularities, we can find an innermost curve α in D^2 (i.e. the disk bound by α contains no singularities of the map). We proceed as follows:

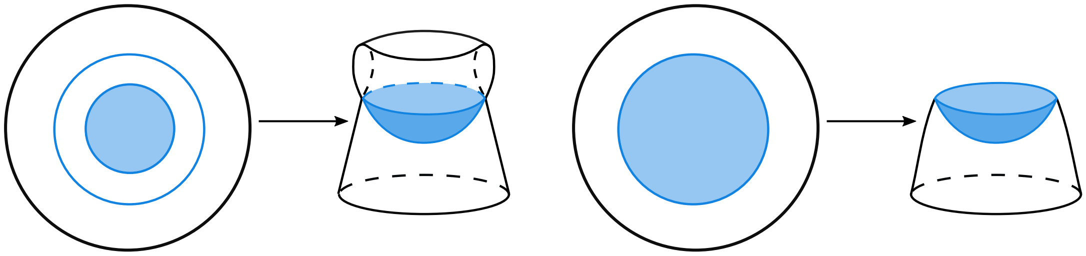


Figure A5.4a: If α is a singularity of the form (A5.3a.), then cut out the disk bound by the corresponding double curve of α, and replace it by the disk bound by α. The resulting map completely removes the singularity.

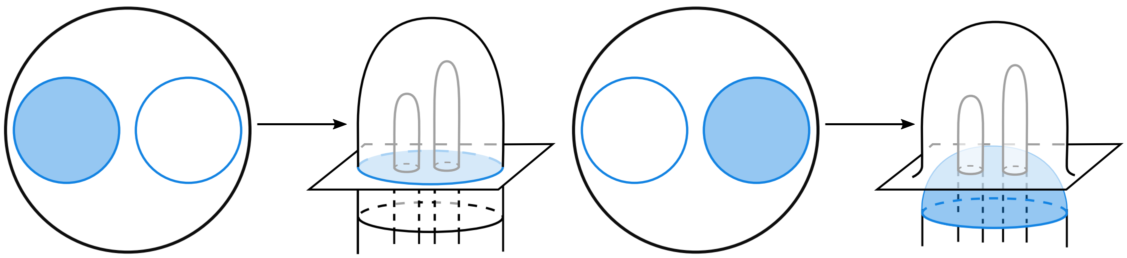


Figure A5.4b: Similarly for the singularity in (A5.3b.), we cut out and swap the disks bound by the double curves. The resulting map can be perturbed into general position, removing the singularity.

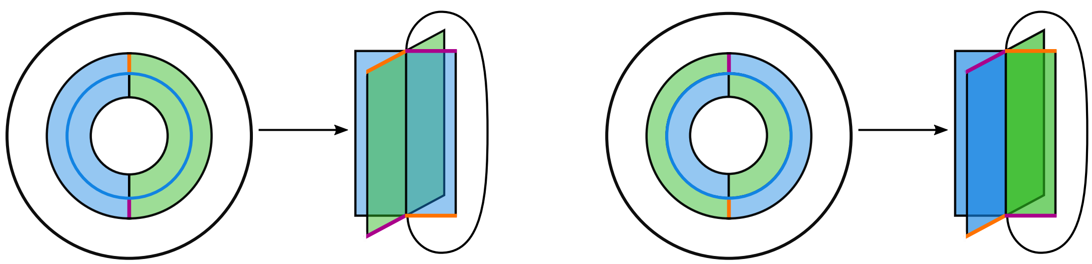


Figure A5.4c: If α is of the form (A5.3c.), we rotate the inside of an annular neighborhood of α by π radians, and jiggle the singularities away.

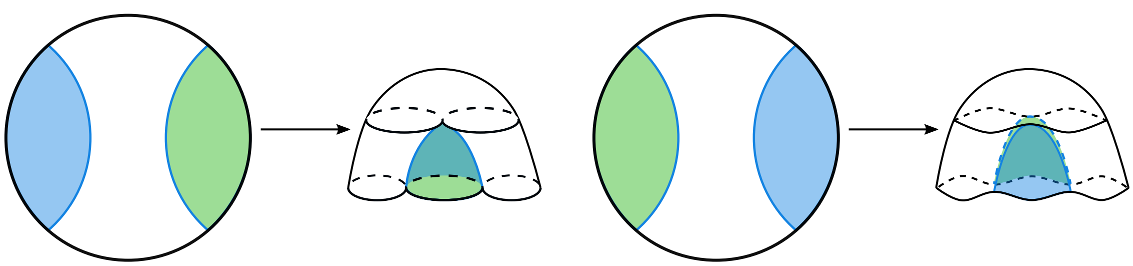


Figure A5.4d: Since the only remaining singularities are arcs, we can choose an innermost pair of arcs. The domain bound by these arcs can be cut out and switched, whereby we can perturb the map away from these singularities.

After finitely many surgeries on each of the finitely many towers, we produce the desired embedding. This concludes the proof of the Loop Theorem. □

A more general statement of the Loop Theorem holds, and the proof follows mutatis mutandis to the argument above. Indeed, Theorem A5.1 is obtained from the following theorem by setting $B = \partial M$ and N trivial.

THEOREM 5.2. (Loop Theorem) Let M be a 3-manifold, $B \subseteq \partial M$ a compact surface, and $N \lhd \pi_1(B)$ not contained in $\ker(\pi_1(B) \to \pi_1(M))$. Then there exists an embedding $f\colon (D^2, S^1) \to (M, B)$ such that $f|_{S^1}$ represents a conjugacy class in $\pi_1(B) \setminus N$.

5.3 Dehn's Lemma

THEOREM 5.3. (Dehn's Lemma) Let M be a 3-manifold and $f\colon (D^2, S^1) \to (M, \partial M)$ an embedding on a collar neighborhood A of S^1. Then $f|_A$ extends to an embedding on D^2.

Proof. We can take f to be smooth by the Whitney Approximation Theorem. The embedding on A produces an isotopy between $f|_{S^1}$ and some map $g_0\colon S^1 \to M$ (orange in Figure A5.5). Since $f(S^1)$ is an embedded submanifold of ∂M, it has a collar neighborhood $B \subset \partial M$ (gray). Then for N the trivial subgroup of $\pi_1(B)$, the Loop Theorem produces an embedding $g_1\colon (D^2, S^1) \to (M, B)$ that takes S^1 to an essential loop in B. The only essential loop in an annulus is the core circle, so there is an isotopy between $g_1|_{S^1}$ and g_0 (green). Since $g_1(D^2)$ is an embedded submanifold of M, it has a collar neighborhood in M. This neighborhood provides sufficient room to stack the isotopies $f \simeq g_0 \simeq g_1$ on g_1 (Figure), producing an embedding $g\colon (D^2, S^1) \to (M, \partial M)$ that agrees on A with f. □

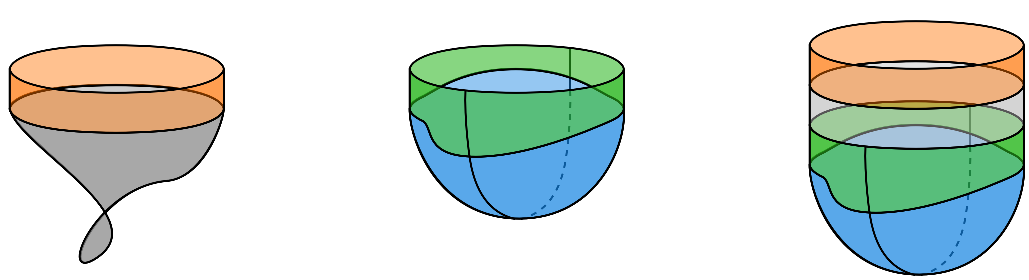


Figure A5.5: Stacking the isotopies on $f(A)$, we produce a proper embedding $D^2 \to M$.

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