Chapter 1

Introduction

1.1 Complex Numbers

x, yA complex number is an expression of the form $x + iy$ where $x, y \in \mathbb{R}$. (Here i denotes $\sqrt{-1}$ so that $i^2 = -1$. We denote the set of complex numbers by $\mathbb{C}$. We can represent $\mathbb{C}$ as the Argand diagram or complex plane by drawing the point $x + iy \in \mathbb{C}$ as the point with co-ordinates (x, y) in the plane $\mathbb{R}^2$ (see the Figure 1.2.1).

If $a + ib,\ c + id \in \mathbb{C}$ then we can add and multiply them as follows

$$\begin{aligned}(a + ib) + (c + id) &= (a + c) + i(b + d)\\(a + ib)(c + id) &= ac + iad + ibc + i^2bd\\&= (ac - bd) + i(ad + bc).\end{aligned}$$

Under these operation it is easily checked that $\mathbb{C}$ is a field.

Note that, $i^2 = -1$, so that the equation $z^2 + 1 = 0$ has roots in $\mathbb{C}$, where $\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$.

We shall often denote a complex number by z or w. suppose that $z = x + iy$ where $x, y \in \mathbb{R}$. we call x the real part of z and write $x = Re(z)$. We call y the imaginary part of z and write $y = Im(z)$.

Note: The imaginary part of $x + iy$ is y, and not iy.

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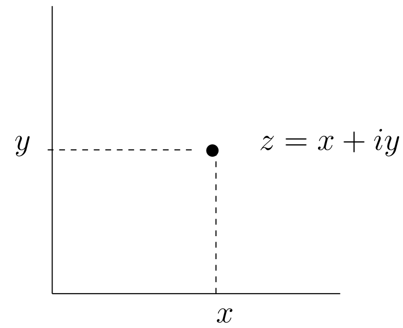

Figure 1.1: The Argand diagram or the complex plane.

We say that $z \in \mathbb{C}$ is real if $Im(z) = 0$ and we say that $z \in \mathbb{C}$ is imaginary if $Re(z) = 0$. In the complex plane, the set of real numbers corresponds to the x-axis (which we will often call the real axis) and the set of imaginary numbers corresponds to the y-axis (which we will often call the imaginary axis).

Theorem 1.1.1. *Complex numbers form a field.*

Proof. We have to show the following axioms.

1. Commutative law for addition: $z + w = w + z$.

2. Associative law for addition: $v + (w + z) = (v + w) + z$.

3. Additive identity: $z + (0, 0) = z$.

4. Additive inverse: $z + (-x, -y) = (0, 0)$.

5. Commutative law for multiplication: $z * w = w * z$.

6. Associative law for multiplication: $v * (w * z) = (v * w) * z$.

7. Multiplicative identity: $z * (1, 0) = z$.

8. Multiplicative inverse: $z * z^{-1} = (1, 0), \quad z \neq 0$.

9. The distributive law: $v * (w + z) = v * w + v * z$.

each of these is proved by simply writing it out and using the definition of the $*$ multiplication. We prove (8.) and (9.), Some of the others are in your homework.

Begin with (9.). Let $v = (a, b)$, $z = (x, y)$ and $w = (r, t)$. Observe by definition of $*$ and $+$ on $\mathbb{R}^2$,

$$\begin{aligned} v * (z + w) &= (a, b) * [(x, y) + (r, t)] \\ &= (a, b) * (x + r, y + t) \\ &= (a(x + r) - b(y + t), a(y + t) + b(x + r)) \\ &= (ax + ar - by - bt, ay + at + bx + br) \\ &= (ax - by, ay + bx) + (ar - bt, at + br) \\ &= (a, b) * (x, y) + (a, b) * (r, t) \\ &= v * z + v * w. \end{aligned}$$

Therefore (9.) is true for all $v, w, z \in \mathbb{R}^2$.

To prove (8.) we first must search out the formula for z^{-1}. Set it up as an algebra problem. We're given that $z = (x; y) \neq 0$ hence either $x \neq 0$ or $y \neq 0$. We would like to find $z^{-1} = (a, b)$ such that

$$(x, y) * (a, b) = (1, 0) \Rightarrow (ax - by, xb + ya) = (1, 0)$$

Thus by definition of vector equality,

$$ax - by = 1 \text{ and } xb + ya = 0$$

We'll need to consider several cases.

Case 1: $x \neq 0$ but $y = 0$ then $ax = 1$ hence $a = \dfrac{1}{x}$ and so $ya = 0$ and it follows $xb = 0$ hence $b = 0$ and we deduce $z^{-1} = \left(\dfrac{1}{x}, 0\right)$.

Case 2: $x = 0$ but $y \neq 0$ then $-by = 1$ hence $b = \dfrac{-1}{y}$ and so $xb = 0$ and it follows $ya = 0$ hence $a = 0$ and we deduce $z^{-1}1 = \left(0, \dfrac{-1}{y}\right)$.

Case 3: $x \neq 0$ and $y \neq 0$ so we can divide by both x and y without fear,

$$xb + ya = 0 \Rightarrow b = \frac{-ya}{x}$$

$$ax - by = 1 \Rightarrow ax + y^2a/x = 1 \Rightarrow a(x^2 + y^2) \Rightarrow a = \frac{x}{x^2 + y^2}$$

Substitute a into $b = \dfrac{-ya}{x}$,

$$b = \frac{-y}{x}\frac{x}{x^2 + y^2} = \frac{-y}{x^2 + y^2}$$

To summarize:

$$z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right).$$

The formula above solves $z^{-1} * z = (1,0)$ for all $z \in \mathbb{R}^2$ such that $x^2+y^2 \neq 0$. The proof of (8.) follows.

$\square$

Definition 1.1.1. (Conjugate)
If $z = x+iy,\ x, y \in \mathbb{R}$ then we define $\overline{z} = x-iy$ to be the complex conjugate of z.

Definition 1.1.2. (Modulus or absolute value)
Let $z = x+iy,\ x, y \in \mathbb{R}$. Then modulus or absolute value of z is

$$|z| = \sqrt{x^2+y^2}, \quad \geqslant 0.$$

The number $|z|$ is the distance between the origin and the point $z = (x, y)$. It is straight forward to check that $|\overline{z}| = |z|$ and that

$$z\overline{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2.$$

1.2 Polar Representation

Let r be the modulus of z (i.e., $r = |z|$), and let θ be the angle that the line from the origin to the complex number z makes with the positive x-axis. Then as Figure 1.2.(a) shows,

$$z = (r\cos\theta, r\sin\theta) = r(\cos\theta + i\sin\theta) \tag{1.2.1}$$

Definition 1.2.1. Identity (1.2.1) is known as a polar representation of z, and the values r and θ are called polar co-ordinates of z.

As Figure 1.2.(b) shows, θ can be any value for which the identities $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$ hold. for $z \neq 0$, the collection of all values of θ for which $z = r(\cos\theta + i\sin\theta)$ is denoted $arg\ \ z$.
i.e., $\theta = arg\ \ z, \quad \theta \in [0, 2\pi)$.

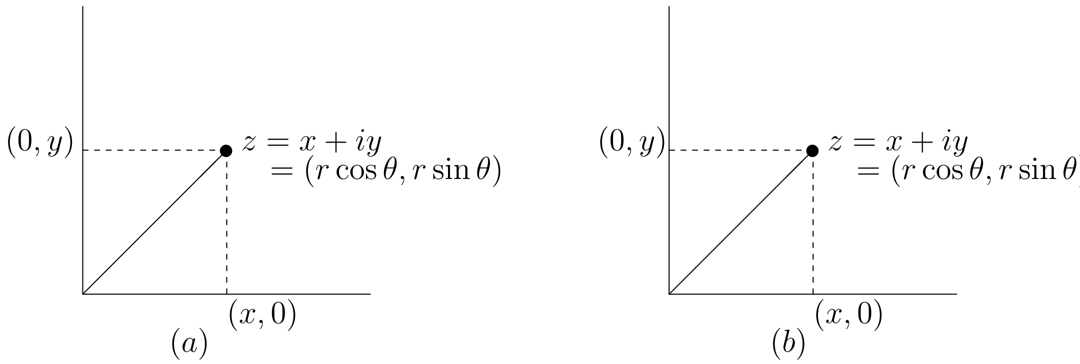

Figure 1.2: Polar representation of complex numbers

Note as $re^{i\theta} = re^{\theta+2n\pi}, n \in \mathbb{Z}$, for a given $z \in \mathbb{C}$, $arg\ z$ is not unique. Hence it has any one of an infinite number of real values differing by integral multiplies of 2π.

Thus for any $n \in \mathbb{Z}$,

$$\theta = \begin{cases} \tan^{-1}[\frac{y}{x}] + 2n\pi, & x \neq 0; \\ \frac{\pi}{2} + 2n\pi, & x = 0, \quad y > 0; \\ \frac{-\pi}{2} + 2n\pi, & x = 0, \quad y < 0; \\ \text{arbitrary}, & x = y = 0. \end{cases}$$

Definition 1.2.2. (Principal argument)
The principal argument of z is the single value of θ defined below:

$$\text{Arg}(z) = \theta \in \arg(z) \ \text{ such that } \ -\pi < \theta < \pi.$$

We may also use the notation $\text{Arg}(z) = \Theta$.

Note:

1. $\arg z = \text{Arg } z + 2n\pi, \quad n \in \mathbb{Z}$.
2. When z is a negative real number, $\text{Arg } z = \pi$.
3. The equation $e^{i\theta} := \cos\theta + i\sin\theta, \ \theta \in \mathbb{R}$ is known as Euler's formula.
4. Two non-zero complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ are equal if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2n\pi, \ n \in \mathbb{Z}$.

1.3 DeMoivre's Powers/Roots of complex numbers

De Moivre's Theorem may use to find the n^{th} root of a complex number $z = r\sin\theta + i\sin\theta$.

1.3.1 Power of complex number

De Moivre's Theorem states that if $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is any number then

$$z^n = (\cos\theta + i\sin\theta)^n = (\cos(n\theta) + i\sin(n\theta)). \tag{1.3.1}$$

1.3.2 Roots of complex number

Theorem 1.3.1. *If $z = r\cos\theta + i\sin\theta$ then it will have n distinct n^{th} roots, expressed as*

$$C_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \tag{1.3.2}$$

for $k = 0, 1, 2, ..., n-1$.

Then n distinct roots will have the same modulus, $r^{\frac{1}{n}}$, but will have different arguments determined by

$$\alpha = \frac{\theta + 2k\pi}{n}$$

for radians and

$$\alpha = \frac{\theta + k360^\circ}{n}$$

for degrees, where $k = 0, 1, 2, ..., n-1$.

Example 1.3.1.

Find the fourth roots of $-8 + i8\sqrt{3}$.

Solution: The modulus is
$$\sqrt{(-8)^2 + (8\sqrt{3})^2} = 16$$
and since the terminal side of θ contains the point $(-8, 8\sqrt{3})$, it must be in the second quadrant where
$$\cos\theta = \frac{-8}{16} = \frac{-1}{2} \text{ and } \theta = \cos^{-1}\left(\tfrac{-1}{2}\right) = 120°.$$
The four roots will then be evaluated by

$$16^{\frac{1}{4}}\left[\cos\left(\frac{120° + k\ 360°}{4}\right) + i\sin\left(\frac{120° + k\ 360°}{4}\right)\right]$$
where $k = 0, 1, 2, ..., n-1$ and since $n - 1 = 3$, then $k = 0, 1, 2$ and 3. The 4 roots are

$2[\cos 30° + i\sin 30°] = 2\left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = \sqrt{3} + i.$

$2[\cos 120° + i\sin 120°] = 2\left[\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right] = -1 + i\sqrt{3}.$

$2[\cos 210° + i\sin 210°] = 2\left[\frac{-\sqrt{3}}{2} - i\frac{1}{2}\right] = -\sqrt{3} - i.$

$2[\cos 300° + i\sin 300°] = 2\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = 1 - i\sqrt{3}.$

Note:

1. If c is any particular n^{th} root of z_0, the set of all n^{th} roots can be written $c,\ cw_n, cw_n^2, ..., cw_n^{n-1}$, where $\omega_n = e^{i\left(\frac{2\pi}{n}\right)}$.

2. When the value of α that is used in (1.3.2) is the principal value of $\arg z_0$ $(\pi < \alpha < \pi)$, the number c_0 is referred to as the principal n^{th} root of z_0.

3. When z_0 is a positive real number $\mathbb{R}$, its principal roots is $\sqrt[n]{R}$.

Proposition 1.3.2. *Let* $z, \in \mathbb{C}$. *Then*

1. $|z| = 0$ *iff* $z = 0$;
2. $|zw| = |z||w|$;
3. $\overline{z \pm w} = \overline{z} \pm \overline{w}$;
4. $\overline{zw} = \overline{zw}$;
5. $\overline{\left(\frac{1}{z}\right)} = \left(\frac{1}{\overline{z}}\right)$ *if* $z \neq 0$;
6. $z + \overline{z} = 2Re(z)$;
7. $z - \overline{z} = 2iIm(z)$;
8. $|\frac{1}{z}| = \frac{1}{|z|}$ *if* $z \neq 0$;
9. $|z + w| \leqslant |z| + |w|$;
10. $||z| - |w|| \leqslant |z - w|$.

Remark 1.3.1. The inequality $|z + w| \leqslant |z| + |w|$ is often called the triangular inequality. The inequality $||z| - |w|| \leqslant |z - w|$ is often called reverse triangular inequality.

Proof. Part *(a)*, *(b)* and *(c)* follows easily from the definition of $|z|$. We have *(e)* as an exercise. To see *(d)*, first note that if $z = x + iy$ then $\text{Re}(z) = x \leqslant \sqrt{x^2 + y^2} = |z|$. Then

$$\begin{aligned}
|z + w|^2 &= (z + w)\overline{(z + w)} \\
&= (z + w)(\overline{z} + \overline{w}) \\
&= z\overline{z} + w\overline{w} + z\overline{w} + \overline{z}w \\
&= |z|^2 + |w|^2 + z\overline{w} + \overline{z}w \\
&= |z|^2 + |w|^2 + z\overline{w} + \overline{\overline{z}w} \\
&\leqslant |z|^2 + |w|^2 + 2|z\overline{w}| \\
&= |z|^2 + |w|^2 + 2|z||\overline{w}| \\
&= |z|^2 + |w|^2 + 2|z||w| \\
&= (|z| + |w|)^2.
\end{aligned}$$

□

Chapter 2

Foundation of Complex analysis

2.1 Point set in the complex plane

We shall study functions of the form $f : S \to \mathbb{C}$, where S is a set in $\mathbb{C}$. In most situations, various properties of the point sets S play a crucial role in our study. We therefore begin by discussing various type of point set in the complex plane.

Let us consider a few examples of sets which frequently occur in our subsequent discussion.

Example 2.1.1.

Suppose that $z_0 \in \mathbb{C},\ r, R \in \mathbb{R}$ and $0 < r < R$. The set $\{z \in \mathbb{C} : |z - z_0| < R\}$ represent a disc, with center z_0 and radius R, and the set $\{z \in \mathbb{C} : r < |z - z_0| < R\}$ represents an annulus, with center z_0, inner radius r and outer radius R.

Example 2.1.2.

Suppose that $A < \; b \in \mathbb{R}$ and $A < B$. The set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \;\text{ and }\; x > A\}$ represents a half-plane, and the set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \;\text{ and }\; A < x < B\}$ represents a strip.

Example 2.1.3.

Suppose that $\alpha, \beta \in \mathbb{R}$ and $0 \leqslant< \alpha < \beta < 2\pi$. The set $\{z = r(\cos\theta + i\sin\theta) \in \mathbb{C} : r, \theta \in \mathbb{R} \;\text{ and }\; r > 0 \;\text{ and }\; \alpha < \theta < \beta\}$ represents a sector.

Definition 2.1.1. (Neighborhood)

Suppose that $z_0 \in \mathbb{C}$ and $x \in \mathbb{R}$, with $\epsilon > 0$. By an ϵ-neighborhood of z_0, we mean a disc of the form $\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$, with center z_0 and radius $\epsilon > 0$. We denote it by $N(z_0, \epsilon)$ or $N_\epsilon(z_0)$. That is

$$N(z_0, \epsilon) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}.$$

The deleted ϵ neighborhood of z_0 is defined to be the set of all points z in the plane such that $0 < |z - z_0| < \epsilon$. We shall denote it by $N^*(z_0, \epsilon) = \{z \in \mathbb{C} : 0 <$

$|z - z_0| < \epsilon\}$.

Here we denote that z_0 is removed from $N(z_0, \epsilon)$.

Example 2.1.4.

(i) $N(i, 1)$, the "1-neighborhood of i", is the interior of the circle $|z - i| = 1$.

i.e., $N(i, 1) = \{z \in \mathbb{Z} : |z - i| < 1\}$.

(ii) $N^*(0, \epsilon) = \{z \in \mathbb{Z} : 0 < |z| < 1\}$. i.e., it is the interior of the circle $|z| = \epsilon$ from which the center $z = 0$ has been removed.

Definition 2.1.2. (Interior Point and Exterior Point)

Suppose that S is a point set in $\mathbb{C}$. A point $z_0 \in S$ is said to be an interior point of S if there exist an ϵ-neighborhood of z_0 which is contained in S.

The point z_0 is called an exterior point of S when there exist a neighborhood of z_0 containing no points of S.

Definition 2.1.3. (Boundary Points)

A point $z_0 \in \mathbb{C}$ is said to be a boundary point of a set S if every ϵ-neighborhood of z_0 contains a point in S as well as a point not in S.

That is , z_0 is a boundary point of S if and only if $\forall \epsilon > 0,\ \exists\ N(z_0, \epsilon) \cap S \neq \Phi$ and $N(z_0, \epsilon) \cap S^c \neq \Phi$.

The set of all boundary points of s set S is called the boundary of S and is denoted by ∂S.

Example 2.1.5.

The annulus $\{z \in \mathbb{C} : r < |z - z_0| < R\}$, where $0 < r < R$, has boundary $C_1 \cup C_2$, where

$$C_1 = \{z \in \mathbb{C} : |z - z_0| = r\} \text{ and}$$
$$C_2 = \{z \in \mathbb{C} : |z - z_0| = R\}$$

are circles, with center z_0 and radius r and R respectively. Note that the annulus is connected and hence a domain. However, note that its boundary is made up of two separate pieces.

Definition 2.1.4. (Limits Points)

A point $z_0 \in \mathbb{C}$ is called a limit point or point of accumulation of a set $S \subseteq \mathbb{C}$ iff every deleted ϵ-neighborhood of z_0 contains points of S.

$$\text{i.e., } \forall \epsilon > 0,\ N^*(z_0, \epsilon) \cap S \neq \Phi.$$

Definition 2.1.5. (Open and Closed set)

The set $S \subseteq \mathbb{C}$ is said to be open if, every point of S is an interior point of S.

A set $S \subseteq \mathbb{C}$ which contains all its boundary points is said to be closed.

Example 2.1.6.

1. The set in Example 1.4.1 - 1.4.3 are open.
2. The punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ is open.
3. The disc $\{z \in \mathbb{C} : |z - z_0| \leqslant R\}$ is not open.

Definition 2.1.6. (Connected set)

An open set S is said to be connected if every two points $z_1,\ z_2 \in S$ can be joined by the union of a finite number of line segment lying in S.

Definition 2.1.7. (Bounded and Compact)

A set $S \subseteq \mathbb{C}$ is said to be bounded set or finite if there exists a real number M such that $|z| \leqslant M$ for every $z \in S$. A set $S \subseteq \mathbb{C}$ that is closed and bounded is said to be compact set if no such M exists, the set will be called unbounded.

Example 2.1.7.

The set $\{z \in \mathbb{C} : |z - z_0| \leqslant R\}$ is closed and bounded, hence compact. It is called the closed disc with center z_0 and radius R.

Example 2.1.8.

The set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \leqslant x \leqslant 1\}$ is closed but not bounded.

Example 2.1.9.

The square $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \leqslant x \leqslant 1 \text{ and } 0 < y < 1\}$ is bounded but not closed.

Definition 2.1.8. (Closure)
For any set S, we denote by $\overline{S}$ the closed region containing S and all its boundary points, and called $\overline{S}$ the closure of S.

i.e.,$\overline{S} = S \cup \partial S$.

Definition 2.1.9. (Domain)
Let $D \in \mathbb{C}$ be a non-empty set. Then we say that D is a domain if,

1. D is open;
2. every pair of point in D can be connected by a polynomial arc (i.e, one built up of line segments) lying entirely in D.

Definition 2.1.10. (Region)
A region is a domain together with all, some or none of its boundary points. A region which contains all its boundary points is said to be closed; if it contains none of its boundary points, the region is open.

Definition 2.1.11. (Open Region or domain)
An open connected set is called a domain.

Definition 2.1.12. The closure of an open region or domain is called a closed region.

2.2 Convergence of Sequence and Series

Theorem 2.2.1. *Let $w, z \in \mathbb{C}$ and $d(z, w) = |z - w|$ then $(\mathbb{C}, d)$ is a metric space.*

Proof. Do as exercise. □

Definition 2.2.1. A sequence z_n of complex numbers converges to z iff for each $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that $|z_n - z| < \epsilon, \quad \forall n > N$.

Theorem 2.2.2. *Let $z_n \in \mathbb{C}$ and write $z_n = x_n + iy_n,\ x_n,\ y_n \in \mathbb{R}$. Then z_n converges iff x_n and y_n converges.*

Proof. ($\Rightarrow$)

Suppose that $z_n \to z$ as $n \to \infty$.

Then $\exists N_0 \in \mathbb{N}$ such that $|z_n - z| < \epsilon,\ \forall n > N_0$.

i.e., $\exists N_0 \in \mathbb{N}$ such that $|(x_n + iy_n) - (x + iy)| < \epsilon,\ \forall n > N_0$,

i.e., $\exists N_0 \in \mathbb{N}$ such that $|(x_n - x) + i(y_n - y)| < \epsilon,\ \forall n > N_0$,

Since $|x_n - x| \leqslant |(x_n - x) + i(y_n - y)|$ and

$|y_n - y| \leqslant |(x_n - x) + i(y_n - y)|$

it follows that $|x_n - x| < \epsilon$ and $|y_n - y| < \epsilon,\ \forall n > N_0$

Since $\epsilon > 0$ is arbitrary it follows that

$\forall \epsilon > 0,\ \exists N_0 \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ and $|y_n - y| < \epsilon,\ \forall n > N_0$

Hence $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$.

Conversely assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Let $\epsilon > 0$ be given. Then $\exists N_1 \in \mathbb{N}$ such that $|x_n - x| < \frac{\epsilon}{2},\ \forall n > N_1$ and $\exists N_2 \in \mathbb{N}$ such that $|y_n - y| < \frac{\epsilon}{2},\ \forall n > N_2$.

Take $N_0 = \max\{N_1, N_2\}$.

Then $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2},\ \forall n > N_0$

$$\begin{aligned}
\text{Now } |(x_n - x) + i(y_n - y)| &\leqslant |(x_n - x)| + |i(y_n - y)| \\
&= |x_n - x| + |y_n - y| \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon,\ \forall n > N_0.
\end{aligned}$$

i.e., $|(x_n - x) + i(y_n - y)| < \epsilon,\ \forall n > N_0$.

Since $\epsilon > 0$ be arbitrary, it follows that

$\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ such that $|z_n - z| < \epsilon,\ \forall n > N_0$.

Hence $z_n \to z$ as $n \to \infty$. $\square$

Definition 2.2.2. (Cauchy sequence)

A sequence (z_n) is called Cauchy sequence if for every $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that $|z_n - z_m| < \epsilon, \forall n > m > N$

Theorem 2.2.3. $(\mathbb{C}, d)$ *is a complete metric space.*

Definition 2.2.3. An infinite series of complex numbers

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + ... + z_n + ...,$$

converges to a number S, called the sum of the series, if the sequence

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + ... + z_N, N \in \mathbb{N}$$

of partial sums converges to S, we then write

$$\sum_{n=1}^{\infty} z_n = S$$

. When a series does not converge, we say that it diverges.

Theorem 2.2.4. *Suppose that* $z_n = x_n + iy_n,\ n \in \mathbb{N}$ *and* $S = X + iY$. *Then* $\displaystyle\sum_{n=1}^{\infty} z_n = S$ *iff* $\displaystyle\sum_{n=1}^{\infty} x_n = X$ *and* $\displaystyle\sum_{n=1}^{\infty} y_n = Y$.

Proof. $(\Rightarrow)$

Suppose that $z_n = x_n + iy_n,\ n \in \mathbb{N}$ and $S = X + iY$.

Then by Definition 2.2.3, we have $\displaystyle\lim_{N\to\infty} S_N = S$,

i.e., $\lim_{N\to\infty}(z_1 + z_2 + ... + z_N) = X + iY$,

i.e., $\lim_{N\to\infty}\left[\displaystyle\sum_{n=1}^{N} x_n + i\sum_{n=1}^{N} y_n\right] = X + iY$,

i.e., $\displaystyle\lim_{N\to\infty}[X_N + iY_N] = X + iY$, $\left[\text{where } \ X_N = \displaystyle\sum_{n=1}^{N} x_n \quad \text{and} \quad \sum_{n=1}^{N} Y_n = y_n\right]$

i.e., $\displaystyle\lim_{N\to\infty} X_N = X$ and $\displaystyle\lim_{N\to\infty} Y_N = Y$

where $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$

Hence $X = \sum_{n=1}^{\infty} x_n$ and $Y = \sum_{n=1}^{\infty} y_n$.

($\Leftarrow$) Conversely suppose that $X = \sum_{n=1}^{\infty} x_n$ and $Y = \sum_{n=1}^{\infty} y_n$

Then $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$ such that $\lim_{N\to\infty} X_N = X$ and $\lim_{N\to\infty} Y_N = Y$

i.e., $\lim_{N\to\infty} [X_N + iY_N] = X + iY$,

i.e., $\lim_{N\to\infty} S_N = S$,

Hence $\sum_{n=1}^{\infty} z_n = S$. $\square$

2.3 Functions of a Complex Variable

Definition 2.3.1. (Single valued function)

A function f defined on $S \subseteq \mathbb{C}$, that is $f : S \to \mathbb{C}$, is a rule that assigns to each $z \in S$ a complex variable w. The number w is called the value of f at z and is denoted by $f(z)$, that is, $w = f(z)$.

The set is called the domain of definition of f. Here z is the independent variable and w the dependent variable. $f(z)$ is also known as the image of z under f, and z is called a pre image of w under f.

Definition 2.3.2. (Multiple valued function)

If corresponding to each value of z there is more than one value of w, then the complex function $w = f(z)$ is called a multiple-valued function.

For example, the square root function $w = f(z) = \sqrt{z}$ is a multiple valued function and two values of w exist for each value of z. These two values are $w_1 = +\sqrt{z}$ and $w_2 = -\sqrt{z}$.

2.4 Mapping(Transformation)

The geometric interpretation of a complex function can be through of as a process by which entire subsets of the z-plane are "mapped" or "transformed" onto corresponding subsets of the w- plane. This aspect of a complex function has generated the terms mapping and transformation as alternate names for "function".

Just as z can be expressed by its real and imaginary parts, $z = x + iy$, we write $f(z) = w = u+iv$, where u and v are real and imaginary parts of w, respectively. Doing so gives us the representation $w = f(z) = f(x, y) = u + iv$.

Because u and v depend on x and y, they can be considered to be real-valued functions of the real variable x and y; that is;

$$u = u(x, y) \text{ and } v = v(x, y).$$

Combining these ideas, we often write a complex function f in the form

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) \tag{2.4.1}$$

Figure 2.4.1 illustrates the notion of a function (mapping) using these symbols.

Note: Using $z = re^{i\theta}$ in the expression of a complex function f may be convenient. It gives us the polar representation,

$$f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \tag{2.4.2}$$

where u and v are real functions of the real variables r and θ.

Remark 2.4.1. For a given function f, the functions u and $v$defined here are different form those defined by equation 2.4.1 because the equation 2.4.1 involves Cartesian co-ordinates and equation 2.4.2 involves polar co-ordinates.

Example 2.4.1.

Consider the function $f : S \to \mathbb{C}$, given by $f(z) = z^2$, where $S = \{z \in \mathbb{C}/|z| < 2\}$. The the range of f is the open disc $f(s) = \{w \in \mathbb{C}/|z| < 4\}$.

Let $f(z) = w = z^2$. Take $z = re^{i\theta}$, where $0 \leqslant r = |z| < 2$ and $0 \leqslant \theta \leqslant 2\pi,\ w = r^2 e^{i2\theta} = r^2 \cos(2\theta) + i\sin(2\theta)$.
Hence each argument is doubled, indicating that the disc $|z| < 2$ is mapped onto $|w| < 4$ with each point of $0 < w < 4$ the image of two points of $0 < |z| < 2$.

2.5 Limits and Continuity

Definition 2.5.1. (Limits)
Let $f : S \subseteq \mathbb{C} \to \mathbb{C}$ and let z_0 be a fixed point in S or on the boundary of S. We say that the number l is the limit of $f(z)$ as z approaches z_0 and write $\lim\limits_{z \to z_0} = l$ iff in given any $\epsilon > 0$, one can find a, $\delta > 0$ so that whenever a point $z \in N^*(z_0, \delta) \subseteq S$, then $f(z) \in N(l, \epsilon)$. That is $\forall \epsilon > 0,\ \exists \delta > 0$ such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Notes:

1. δ usually depends on ϵ.
2. f need not even be defined at z_0.
3. The limit is independent of the manner in which $z \longrightarrow z_0$.

Example 2.5.1.

Show that if $f(z) = \dfrac{iz}{2}$ in the open disc $|z| < 1$, then $\lim\limits_{z \to 1} f(z) = \dfrac{i}{2}$.

Solution:

We note that the point $z = 1$ being on the boundary of the domain of definition.

Let $\epsilon > 0$ be given.

We have to show that $\exists\delta > 0$ such that $\left|f(z) - \dfrac{i}{2}\right| < \epsilon$ whenever $0 < |z-1| < \delta$

choose $\delta = 2\epsilon > 0$. So when $0 < |z-1| < \delta$

$$\left|f(z) - \frac{i}{2}\right| = \left|\frac{iz}{2} - \frac{i}{2}\right| = \frac{|z-1|}{2} < \frac{\delta}{2} = \epsilon$$

Hence as $\epsilon > 0$ is arbitrary it follows that $\forall\epsilon > 0$, $\exists\delta(= 2\epsilon) > 0$ such that $\left|f(z) - \dfrac{iz}{2}\right| < \epsilon$ whenever $0 < |z-1| < \delta$.

Example 2.5.2.

Show that if $f(z) = z^2$ then $\lim\limits_{z \longrightarrow z_0} = z_0^2$.

Solution:

Let $\epsilon > 0$ be given.

We have to show that $\exists\delta > 0$ such that $|z^2 - z_0^2| < \epsilon$ whenever $0 < |z - z_0| < \delta$

If $\delta \leqslant 1$, then $0 < |z - z_0| < \delta \Rightarrow |f(z) - z_0^2| = |z^2 - z_0^2| = |z - z_0||z + z_0|$

$$< \delta|z - z_0 + 2z_0| \leqslant \delta(|z - z_0 + 2|z_0|)$$
$$< \delta(1 + 2|z_0|).$$

Take $\delta = \min\{1, \dfrac{\epsilon}{1 + 2|z_0|}\}$. Then we have

$|f(z) - z_0^2| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Thus as $\epsilon > 0$ is arbitrary it follows that $\forall\epsilon > 0$, $\exists\delta > 0$ such that $|f(z) - z_0^2| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Example 2.5.3.

Show that $\lim\limits_{z\to 2i}(2x+iy^2)=4i, \quad z=x+iy.$

Solution:

Let $\epsilon>0$ be given.

We have to show that $\exists\delta>0$ such that $|2x+iy^2-4i|<\epsilon$ whenever $0<|z-2i|<\delta$.

Consider $|2x+iy^2-4i|\leqslant 2|x|+|y-2||y+2|$.
If $2|x|<\dfrac{\epsilon}{2}$ and $|y-2||y+2|<\dfrac{\epsilon}{2}$, then $|2x+iy^2-4i|<\epsilon$
first if these inequalities are satisfied if $|x|<\dfrac{\epsilon}{4}$
Now if $|y-2|<1$, then $|y+2|=|y-2+4|<1+4=5$
Hence if $|y-2|<\min\{1,\dfrac{\epsilon}{10}\}$, then $|y-2||y+2|<\dfrac{\epsilon}{10}\times 5=\dfrac{\epsilon}{2}$
Thus $|2x+iy^2-4i|\leqslant 2|x|+|y-2||y+2|<\dfrac{\epsilon}{2}+\dfrac{\epsilon}{2}=\epsilon$
An appropriate value of δ is now easily seen from the conditions $|x|<\dfrac{\epsilon}{4}$ and

$|y-2|<\min\{1,\dfrac{\epsilon}{10}\}$
i.e., we choose $\delta=\min\{1,\dfrac{\epsilon}{10}\}$
So whenever $0<|z-2i|<\delta$, we have $|2x+iy^2-4i|<\epsilon$

Since $\epsilon>0$ is arbitrary it follows that

$\forall\epsilon>0,\ \exists\delta>0$ such that $|f(z)-4i|<\epsilon$ whenever $0<|z-2i|<\delta$.

Theorem 2.5.1. *If a function has a limit at a point z_0, then its limit is unique.*

Proof. suppose that $\lim\limits_{z\to z_0}f(z)=l_1$ and $\lim\limits_{z\to z_0}f(z)=l_2$ where $l_1\neq l_2$
Take $\epsilon=\dfrac{1}{2}|l_1-l_2|>0$
Now by the definition of limit, for this ϵ, $\exists\delta>0$ such that $|f(z)-l_1<\epsilon|$ and $|f(z)-l_2<\epsilon|$ whenever $0<|z-z_0|<\delta$.

Now $|l_1-l_2| = |l_1-f(z)+f(z)-l_2| \leqslant |l_1-f(z)|+|f(z)-l_2| < \epsilon+\epsilon = 2\epsilon = |l_1-l_2|$, which is contradiction. Thus the limit of f is unique. $\square$

Example 2.5.4.

Show that $\lim\limits_{z\to z_0} \frac{\overline{z}}{z}$ does not exist.

Solution: Suppose the limit exists. Thus it is independent of the manner in which $z \to 0$

Let $z \to 0$ along the x axis. Then $y = 0$ and $z = x+iy = iy$ and $\overrightarrow{z} = x-iy = x$.

Thus $\lim\limits_{z\to z_0} \frac{\overline{z}}{z} = \lim\limits_{z\to z_0} \frac{x}{x} = 1$

Let $z \to 0$ along the y axis. Then $x = 0$ and $z = x+iy = iy$ and $\overrightarrow{z} = x-iy = -iy$.

Thus $\lim\limits_{z\to z_0} \frac{\overline{z}}{z} = \lim\limits_{z\to z_0} \frac{-iy}{iy} = -1$

Since the two approaches don't give the same answer, hence the theorem 2.5.1. $\lim\limits_{z\to z_0} \frac{\overline{z}}{z}$ doesn't exist.

Theorem 2.5.2. *Let $f(z) = u(x,y) + iv(x,y)$ be a complex function that is defined in some neighborhood of z_0, except perhaps at $z_0 = x_0 + iy_0$. Then $\lim\limits_{z\to z_0} f(z) = w_0 = u_0 + iv_0$ iff $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.*

Proof. Assume that $\lim\limits_{z\to z_0} f(z) = w_0 = u_0 + iv_0$.

Then $\forall\epsilon > 0,\ \exists\delta > 0$ s.t. $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$,

i.e., $\forall\epsilon > 0,\ \exists\delta > 0$ s.t. $|(u(x,y) - u_0) + i(v(x,y) - v_{0)}| < \epsilon$ whenever $0 < |(x - x_0) + i(y - y_0)| < \delta$

Since $|u(x,y) - u_0| \leqslant |(u(x,y) - u_0) + i(v(x,y) - v_{0)}|$ and $|v(x,y) - v_0| \leqslant |(u(x,y) - u_0) + i(v(x,y) - v_{0)}|$

it follows that $|u(x,y) - u_0| < \epsilon$ and $|v(x,y) - v_0| < \epsilon$

whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

That is, $\forall\epsilon > 0,\ \exists\delta > 0$ s.t. $|u(x,y) - u_0| < \epsilon$ and $|v(x,y) - v_0| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Hence $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

Conversely assume that $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

Then $\exists\delta_1 > 0$ s.t. $|u(x,y) - u_0| < \dfrac{\epsilon}{2}$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1$
and $\exists\delta_2 > 0$ s.t. $|v(x,y) - v_0| < \dfrac{\epsilon}{2}$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2$
Take $\delta = \min \delta_1, \delta_2$. Then

$\exists\delta > 0$ s.t.$|(u(x,y) - u_0) + v(u(x,y) - v_0)| \leqslant |(u(x,y) - u_0)| + |v(u(x,y) - v_0)| \leqslant \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$ whenever$0 < |(x+iy) - (x_0 + iy_0)| < \delta$
$\forall\epsilon > 0,\ \exists\delta > 0$ s.t. $|(u(x,y) - u_0) + i(v(x,y) - v_{0)}| < \epsilon$

whenever $0 < |(x - x_0) + i(y - y_0)| < \delta$

Hence $\lim\limits_{z\to z_0} f(z) = w_0 = u_0 + iv_0$

$\square$

Theorem 2.5.3. *Suppose that* $\lim\limits_{z\to z_0} f(z) = l$ *and* $\lim\limits_{z\to z_0} g(z) = m$. *Then*

1. $\lim\limits_{z\to z_0} (f(z) + g(z)) = l + m$;
2. $\lim\limits_{z\to z_0} f(z)g(z) = lm$;
3. $\lim\limits_{z\to z_0} \dfrac{1}{g(z)} = \dfrac{1}{m}$ *if* $m \neq 0$;
4. $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{1}{m}$ *if* $m \neq 0$;
5. $\lim\limits_{z\to z_0} f(z) = l$, *then* $\lim\limits_{z\to z_0} |f(z)| = |l|$.

Example 2.5.5.

Show that $\lim\limits_{z\to 1+i} f(z) = (z^2 - 2z + 1) = -1$.
Solution: We let $f(z) = z^2 - 2z + 1 = (x^2 - y^2 - 2x + 1) + i(2xy - 2y)$

Computing the limits for u and v, we obtain $\lim\limits_{(x,y)\to(1,1)} u(x,y) = 1-1-2+1 = -1$

and

$\lim\limits_{(x,y)\to(1,1)} v(x,y) = 2 - 2 = 0$
so our theorem 2.5.2 implies that

$$\lim_{z\to 1+i} f(z) = (z^2 - 2z + 1) = -1.$$

Definition 2.5.2. (Continuity)

Let $f(z)$ be a complex function of the complex variable z that is defined for all values of z in some neighborhood of z_0. We say that f is continuous at z_0 if the following three conditions are satisfied:

1. $\lim_{z \to z_0} f(z)$ exist
2. $f(z_0)$ exist
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

If a function is continuous at every point of a region S, it is said to be continuous on S.

Notes:

1. From (3.) we have $\forall \epsilon > 0, \exists \delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever$|z - z_0| < \delta$.
2. Equivalently, if $f(z)$ is continuous at z_0 we can write $\lim_{z \to z_0} f(z) = f(\lim_{z \to z_0} z)$.
3. Point in the z-plane where $f(z)$ fails to be continuous are discontinuities of $f(z)$, and $f(z)$ is said to be discontinuous at these point.
4. If $\lim_{z \to z_0} f(z)$ exists but is not equal to $f(z_0)$, then we call z_0 a removable discontinuity since by redefining $f(z_0)$ to be the same as $\lim_{z \to z_0} f(z)$ the function becomes continuous.

Example 2.5.6.

Let $f(z) = z^2$ for all $z \in \mathbb{C}$ then f is continuous at $z = i$.

For:

1. $f(i) = -1$, f is defined at $z = i$.
2. $\lim_{z \to i} f(z) = \lim_{z \to i} z^2 = -1$.
3. $\lim_{z \to i} f(z) = f(i)$

Hence f is continuous at $z = i$.

Example 2.5.7.

$$f(z) = \begin{cases} z^{2,} & z \neq i; \\ 0, & z = i. \end{cases}$$

For: We have $f(i) = 0$, so f is defined at $z = i$, $\lim_{z \to z_0} f(z) = -1$.
But $\lim_{z \to z_0} f(z) = -1 \neq 0 = f(i)$
So f is not continuous at $z = i$.

Theorem 2.5.4. *Let $f(z) = u(x, y) + iv(x, y)$ be defined in some neighborhood of z_0. Then f is continuous at $z_0 = x_0 + iy_0$ iff u and v are continuous at (x_0, y_0).*

Theorem 2.5.5. *suppose that f and g are continuous at the point z_0. Then The following functions are continuous at z_0:*

1. $f(z) \pm g(z)$;
2. $f(z)g(z)$;
3. $\dfrac{f(z)}{g(z)}$, *provided* $g(z) \neq 0$; *and*
4. *Composition* $f(g(z))$, *provided that* f *is continuous in a neighborhood of* $g(z_0)$.

Similar result hold for continuity in a region.

Example 2.5.8.

Show that the polynomial function given by $w = P(z) = a_0 + a_1 z + a_2 z^2 + ... a_n z^n$ is continuous at each point z_0 in the complex plane.

Theorem 2.5.6. *Suppose that*

1. $f(z) = u(x, y) + iv(x, y)$;

2. $f(z)$ *is defined at every point of a region* S;

3. $z_0 = x_0 + iy_0$ *is a point in* S.

Then $f(z)$ *is continuous at* z_0 *iff* $u(x, y)$ *and* $v(x, y)$ *are continuous at* (x_0, y_0). *That is* $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u(x_0, y_0)$ *and* $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v(x_0, y_0)$.

Let $f(z) = u(x, y) + iv(x, y)$ be continuous in a region S which is that closed and bounded. The function $|f(z)| = \sqrt{[u(x, y)]^2 + [v(x, y)]^2}$ is then continuous in S and reaches a maximum value some where inth that region. That is,f is bounded in S and $|f(z)|$ reaches a maximum value same where in S, that is, there exists a nonnegative real number M such that $|f(z)| \leqslant M \quad \forall z \in S$, where equality holds for at least one such z.

Theorem 2.5.7. *If* $f(z)$ *is continuous in a region* S *which is both closed and bounded then it is bounded in the region. i.e., there exists a non negative real number* M *such that* $|f(z)| \leqslant M$ *for all* z *in* S.

Definition 2.5.3. (Uniformly Continuous)
Let $f : S \subseteq \mathbb{C} \to \mathbb{C}$. Then $f(z)$ is said to be uniformly continuous in a region S if given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z_1) - f(z_2)| < \epsilon$ whenever $|z_1 - z_2| < \delta$ for every $z_1, z_2 \in S$.

Note:

1. Continuity of a function is defined at at point, whereas uniform continuity is defined on a set of points.

2. In defining continuity, we begin with a point z_0. Then for a given $\epsilon > 0$, we find $\delta > 0$ that depends on both z_0 and ϵ. In the case of uniform continuity, we begin with a set of points in S. Then for a given $\epsilon > 0$ we find a $\delta > 0$ that depends only on ϵ.

3. Uniform continuity implies continuity; that is, if a function is uniformly continuous on a set S, it is continuous at every points of S.

4. The converse of the proceeding property is not true; that is continuity of a function at every points of a set S does not imply f uniformly continuous on S.

Example 2.5.9.

Prove that $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$.

Solution:

Example 2.5.10.

Prove that $f(z) = \dfrac{1}{z}$ is not a uniformly continuous in the region $|z| < 1$.

Solution:

Chapter 3

Complex Differentiation

3.1 Introduction

Definition 3.1.1. Suppose that $S \subseteq \mathbb{C}$ is a domain. A function $f : S \to \mathbb{C}$ is said to be differentiable at $z_0 \in S$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In the case, we write

$$f^{'}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and call $f^{'}(z_0)$ the derivative of f at z_0.

If we let $w = f(z)$ and $\triangle w = f(z) - f(z_0)$, then we can use the Leibnitz notation $\frac{dw}{dz}$ for the derivative:

$$f^{'}(z_0) = \frac{dw}{dz} = \lim_{\triangle z \to 0} \frac{\triangle w}{\triangle z}$$

Note:

1. Let $h = \triangle z = z - z_0$. Then we can write the definition 3.1.1. as

$$f^{'}(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{z - z_0}.$$

2. If $z \neq z_0$, then

$$f(z) = \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) + f(z_0)$$

It follows from (3.1.1) and the arithmetic of limits that if $f^{'}(z_0)$ exists, then $f(z) \to f(z_0)$ as $z \to z_0$, so that f is continuous at z_0. In other words, differentiability at z_0 implies continuity at z_0.

Example 3.1.1.

If $f(z) = z^3$, show that $f^{'}(z) = 3z^2$.

Solution: By the definition,
$$\begin{aligned} f^{'}(z_0) &= \lim_{z \to z_0} \frac{z^3 - z_0^3}{z - z_0} \\ &= \lim_{z \to z_0} \frac{(z - z_0)(z_0^2 + zz_0 + z^2)}{z - z_0} \\ &= 3z_0^2. \end{aligned}$$

Example 3.1.2.

prove that if $f(z)$ is differentiable at z_0 then $f(z)$ is continuous at z_0.

Solution: Assume that $f^{'}(z_0)$ exists.

Now,
$$\begin{aligned} \lim_{z - z_0}[f(z) - f(z_0)] &= \lim_{z - z_0} \left[\frac{f(z) - f(z_0)}{(z - z_0)}(z - z_0)\right] \\ &= \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{(z - z_0)}\right] \lim_{z \to z_0}(z - z_0) \\ &= f^{'}(z_0).0 = 0 \end{aligned}$$

So, $\lim_{z \to z_0} f(z) = f(z_0)$

Hence $f(z)$ is continuous at z_0.

Remark 3.1.1. But continuity of $f(z)$ at a point implies not the differentiability of $f(z)$ at that point.

For:

Let $f(z) - |z^2|$. Then

$$\frac{dw}{dz} = \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z}}{\Delta z}$$

$$\Rightarrow \frac{dw}{dz} = \lim_{\Delta z \to 0} \left(\overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}\right) \tag{3.1.1}$$

when $z = 0$, then $\dfrac{dw}{dz} = \lim\limits_{\Delta z \to 0} \overline{\Delta z}$
Hence $\dfrac{dw}{dz} = 0$ at the origin $z_0 = 0$.

We now show that if $z \neq 0$ then limit does not exist in (3.1.1)

Suppose that limit in (3.1.1) exists. When $z \neq 0$. Then the limit may be found by letting the variable $\Delta z = \Delta x + i\Delta y$ approach 0 in any manner.

If $\Delta z \to 0$ through the real values

$\Delta z = \Delta x + i.0$. Then $\overline{\Delta z} = \Delta x$. So,
$\dfrac{dw}{dz} = \lim\limits_{\Delta z \to 0} \left(\overline{z} + \overline{\Delta z} + z\dfrac{\overline{\Delta z}}{\Delta z} \right) \dfrac{dw}{dz} = \lim\limits_{\Delta z \to 0} \left(\overline{z} + \Delta x + z\dfrac{\Delta x}{\Delta x} \right) = \overline{z} + z.$
When Δz approaches 0 through the pure imaginary values $\Delta z = 0 + i\Delta y$. Then

$\overline{\Delta z} = -i\Delta y = -\Delta z$ so,
$\dfrac{dw}{dz} = \lim\limits_{\Delta z \to 0} \left(\overline{z} + \Delta x + z\dfrac{\overline{\Delta z}}{\Delta z} \right) \dfrac{dw}{dz} = \lim\limits_{\Delta z \to 0} \left(\overline{z} - i\Delta y - z\dfrac{i\Delta y}{\Delta y} \right) = \overline{z} - z.$

Since a limit is unique by Theorem (2.5.1), it follows that $\overline{z} + z = \overline{z} - z$.
Thus $z = 0$, if $\dfrac{dw}{dz}$ exists, a contradiction to our assumption.
Hence $f(z)$ is differentiable only at the origin.

Now $f(z) = |z|^2 = x^2 + y^2 + i.0$

So, $u(x, y) = x^2 + y^2, v(x, y) = 0$

Since $u(x, y)$ and $v(x, y)$ are continuous at each point (x_0, y_0) in the plane, it follows that by Theorem (2.5.1), the function $f(z) = |z|^2$ is continuous at each point in the plane.

So, the continuous of a function at a point does not imply the existence of a derivative there.

3.2 Differentiation Formulas

Since the definition of derivative has the same form as in the real case, the elementary results concerning the derivatives of sums and products of functions can be used. In the following formulas the derivative of a function f at a point z is denoted by either $f^{'}(z)$ or $\dfrac{df}{dz}$.

Eg: Let c the complex constant, and let f be differentiable at a point z. Then

$\dfrac{dc}{dz} = 0, \quad \dfrac{d\ c\ f(z)}{dz} = c\ f^{'}(z)$, etc.

If $n \in \mathbb{Z}^+$ then $\dfrac{dz^n}{dz} = nz^{n-1}$. If n is a negative integer then the formula remains valid, provided $z \neq 0$.

Theorem 3.2.1. *If the function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and $g : D \subseteq \mathbb{C} \to \mathbb{C}$ are both differentiable at every point z of a set D. Then their sum, product and quotient are differentiable at every point of D at which they are defined and their derivatives are given by the following formulas:*

(a) $\dfrac{d}{dz}[f(z) \pm g(z)] = f^{'}(z) \pm g^{'}(z)$;

(b) $\dfrac{d}{dz}[f(z).g(z)] = f(z)g^{'}(z) + g(z)f^{'}(z)$;

(c) $\dfrac{d}{dz}\dfrac{f(z)}{g(z)} = \dfrac{f(z)g^{'}(z) - g(z)f^{'}(z)}{[g(z)]^2}$ *if* $g(z) \neq 0$;

(d) If f is differentiable at z_0 and that g is differentiable at the point $f(z_0)$. Then the function $F(z) = g(f(z))$ is differentiable at z_0, and $F^{'}(z_0) = g^{'}[f(z_0)]f^{'}(z_0)$. (Chain rule)

If we write $W = f(w)$ and $w = g(z)$, so that $W = F(z)$.

So, $\dfrac{dF}{dz} = \dfrac{dW}{dz} = \dfrac{dW}{dw} \times \dfrac{dw}{dz}$.

Eg: Find the derivative of $(2z^2 + i)^5$.

Let $w = 2z^2 + i$ and $W = w^5$

$\dfrac{d}{dz}(2z^2 + i)^5 = 5w^4.4z = 20z(2z^2 + i)^4$

3.2.1 Cauchy-Riemann Equations

Theorem 3.2.2. *Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f^{'}(z_0)$ exists at a point $z_0 = x_0 + iy_0$. Then the first order partial derivatives of $u(x, y)$ and $v(x, y)$ with respect to x and y must at (x_0, y_0), and satisfy the Cauchy-Riemann equations* $\dfrac{\partial u}{\partial x}(x_0, y_0) = \dfrac{\partial v}{\partial y}(x_0, y_0)$ *and* $\dfrac{\partial u}{\partial y}(x_0, y_0) = -\dfrac{\partial v}{\partial x}(x_0, y_0)$. *Also,* $f^{'}(z_0)$

is given in terms of these partial derivatives by either $f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$ *or* $f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) = -i[f'(z_0)] = \frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0)$.

Proof. Suppose that the derivatives

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{3.2.1}$$

exists. Write $z_0 = x_0 + iy_0$ and $\delta z = \Delta x + i\Delta y$.
Now $\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

$$= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + \Delta y} \tag{3.2.2}$$

By (3.2.1) and theorem (2.5.2), we obtain

$$Re[f'(z_0)] = \lim_{(\Delta x, \Delta y) \to (0,0)} Re\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \tag{3.2.3}$$

$$Im[f'(z_0)] = \lim_{(\Delta x, \Delta y) \to (0,0)} Im\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \tag{3.2.4}$$

let $(\Delta x, \Delta y) \to (0, 0)$ horizontally through the points $(\Delta x, 0)$

i.e., $\Delta z = (\Delta x, 0) = \Delta x$. Then from (3.2.2), (3.2.3) and (3.2.4), we get
$Re[f'(z_0)] = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$
$Im[f'(z_0)] = \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$
That is

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \tag{3.2.5}$$

where $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ denote the first order partial derivatives with respect to x of the forms u and v at (x_0, y_0).

Now let $(\Delta x, \Delta y) \to (0, 0)$ vertically through the points $(0, \Delta y)$, i.e.),$\Delta z = (0, \Delta y) = i\Delta y$. Then again from (3.2.2), (3.2.3) and (3.2.4) we get,
$Re[f'(z_0)] = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y}$
$Im[f'(z_0)] = -\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}$

That is

$$f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0) \qquad (3.2.6)$$

we have (3.2.5)≡ (3.2.6), so

$\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0)$

Hence

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0) \qquad (3.2.7)$$

□

Note:

The above theorem 3.2.2 provides the necessary conditions in the existence of $f'(z_0)$.

Example 3.2.1.

Let $f(z) = z^2 = x^2 - y^2 + i2xy = u(x, y) + iv(x, y)$.

we can show that derivative of $f(z)$ exists everywhere and that $f'(z) = 2z$. Now we verify the Cauchy-Riemann equation, that is

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Now $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$

Thus the Cauchy-Riemann equations are varified.

Further, $f'(z) = \frac{\partial u}{\partial x} = 2x + i2y = 2(x + iy) = 2z$.

Note:

Since the Cauchy-Riemann equations are necessary conditions in the existence of the derivative of a function f at a point z_0, they can often used to locate at which f does not have a derivative.

⇐(summary of theorem 3.2.2) $f'(z_0)$ exists, then u_x, u_y, v_x, and v_y exist at $z_0 = (x_0, y_0)$ and satisfy the Cauchy-Riemann equations

$u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$

Also, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - u_y(x_0, y_0)$.

Example 3.2.2.

Let $f(z) = |z|^2$

Then $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$.

So, $u_x(x, y) = 2x$ and $v_y(x, y) = 0$

and $u_y(x, y) = 2y$ and $v_x(x, y) = 0$.

The Cauchy-Riemann equations are not satisfied unless $x = y = 0$,

hence $f^{'}(z)$ does not exist if $z \neq 0$.

Note:

Satisfaction of the Cauchy-Riemann equations (3.2.7) at a point $z_0 = x_0 + iy_0$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point, i.e.), $f^{'}(z_0)$. We shall show next that we require also the continuity of the partial derivatives in (3.2.7), i.e.), $u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$ and $v_x(x, y)$ must also be continuous at z_0.

Theorem 3.2.3. *Let the function $f(x) = u(x, y)+iv(x, y)$ be defined through out some ϵ neighborhood of a point $z_0 = x_0+iy_0$. Suppose that the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood and that they are continuous at (x_0, y_0). Then if those partial derivatives satisfy the Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$ at (x_0, y_0), then the derivative $f^{'}(z_0)$ exists.*

i.e.), $f(z) = u + iv$ is defined in $N(z_0, \epsilon)$, $u_x\, u_y$, v_x and v_y exist everywhere in $N(z_0, \epsilon)$ and continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $\dfrac{\partial u}{\partial x}(x_0, y_0) = \dfrac{\partial v}{\partial y}(x_0, y_0)$ and $\dfrac{\partial u}{\partial y}(x_0, y_0) = -\dfrac{\partial v}{\partial x}(x_0, y_0)$, then $f^{'}(z_0)$ exists.

Proof. Let $z_0 = x_0 + iy_0 \in N(z_0, \epsilon)$ for Δx, Δy (real) so small that $z = (x_0 + \Delta x) + i(y_0 + \Delta y) \in N(z_0, \epsilon)$.

Consider $u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$

$$\begin{aligned} &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0 + \Delta x, y_0) + u(x_0 + \Delta x, y_0) - u(x_0, y_0) \\ &= \frac{\partial u}{\partial y}(x_0 + \Delta x, \eta)\Delta y + \frac{\partial u}{\partial x}(\xi, y_0)\Delta x, \end{aligned}$$

where $y_0 < \eta < y_0 + \Delta y$ and $x_0 < \xi < x_0 + \Delta x$ and we have used the Mean Value Theorem. Then as $\Delta z = (\Delta x, \Delta y) \to 0$

i.e.), $\Delta x, \Delta y \to 0,\ \eta \to y_0$ and $\xi \to x_0$, so by continuity of $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$
$\dfrac{\partial u}{\partial y}(x_0+\Delta x, \eta) \to \dfrac{\partial u}{\partial y}(x_0, y_0)$; $\dfrac{\partial u}{\partial x}(\xi, y_0) \to \dfrac{\partial u}{\partial x}(x_0, y_0)$.
Hence we can write

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) = \left[\frac{\partial u}{\partial y}(x_0, y_0) + \epsilon_1\right]\Delta y + \left[\frac{\partial u}{\partial x}(x_0, y_0) + \epsilon_2\right]\Delta x, \tag{3.2.8}$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

Similarly,

$$v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0) = \left[\frac{\partial v}{\partial y}(x_0, y_0) + \delta_1\right]\Delta y + \left[\frac{\partial v}{\partial x}(x_0, y_0) + \delta_2\right]\Delta x, \tag{3.2.9}$$

where $\delta_1, \delta_2 \to 0$ as $\Delta x, \Delta y \to 0$

$$\begin{aligned}
&\frac{f(x_0+\Delta z) - f(z_0)}{\Delta z} \\
&= \frac{[u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)] + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y} \\
&= \frac{\left[\frac{\partial u}{\partial x}(x_0, y_0)\Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\Delta y + \epsilon_1\Delta y + \epsilon_2\Delta x\right]}{\Delta x + i\Delta y} \\
&\quad + i\frac{\left[\frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + \delta_1\Delta y + \delta_2\Delta x\right]}{\Delta x + i\Delta y} \qquad (\because (3.2.8), (3.2.9)) \\
&= \frac{\left[\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)\right]\Delta x + \left[\frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0)\right]\Delta y + w_1\Delta y + w_2\Delta x}{\Delta x + i\Delta y}
\end{aligned}$$

where $w_i = \epsilon_i + i\delta i, \quad i = 1, 2.$

$$\begin{aligned}
&= \frac{\left[\frac{\partial u}{\partial x}(x_0, y_0) + i\dfrac{\partial v}{\partial x}(x_0, y_0)\right]\Delta x + \left[-\frac{\partial v}{\partial x}(x_0, y_0) + i\frac{\partial u}{\partial x}(x_0, y_0)\right]\Delta y + w_1\Delta y + w_2\Delta x}{\Delta x + i\Delta y} \\
&\qquad \text{(by the Cauchy-Riemann equations)} \\
&= \frac{\left[\frac{\partial u}{\partial x}(x_0, y_0) + i\dfrac{\partial v}{\partial x}(x_0, y_0)\right](\Delta x + i\Delta y) + w_1\Delta y + w_2\Delta x}{\Delta x + i\Delta y} \\
&= \left[\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)\right] + \frac{w_1\Delta y + w_2\Delta x}{\Delta x + i\Delta y}
\end{aligned}$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) + \gamma$$

where $|\gamma| = \left|\dfrac{w_1\Delta y + w_2\Delta x}{\Delta x + i\Delta y}\right| \leqslant \dfrac{|w_1||\Delta y| + |w_2||\Delta x|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$
$\leqslant |w_1| + |w_2| \to 0$

as $\Delta x, \Delta y \to 0$, since $\epsilon_i, \delta_i \to 0, \quad i = 1, 2$.
So, $f^{'}(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \dfrac{\partial u}{\partial x}(x_0, y_0) + i\dfrac{\partial v}{\partial x}(x_0, y_0)$ exists,
i.e., f is differentiable at z_0.

□

Example 3.2.3.

Let $f(z) = e^x(\cos y + i \sin y)$, where y is to be taken radius when $\cos y$ and $\sin y$ are evaluated.

Then $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$.
Since $\dfrac{\partial u}{\partial x} = e^x \cos y = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -e^x \sin y = -\dfrac{\partial v}{\partial x}$
every where and since $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ are every where continuous.
Thus by Theorem (3.2.3), $f^{'}(z)$ exists every where and
$f^{'}(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = e^x(\cos y + i \sin y)$.
Hence $f^{'}(z) = f(z)$.

Example 3.2.4.

Let $f(z) = |z|^2 = (x^2 + y^2) + i \times 0$ so,

$u(x, y) = x^2 + y^2$ and $v(x, y) = 0$.
$\dfrac{\partial u}{\partial x} = 2x, \quad \dfrac{\partial u}{\partial y} = 2y, \quad \dfrac{\partial v}{\partial x} = 0$ and $\dfrac{\partial v}{\partial y} = 0$
$\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ are every where continuous. But Cauchy-Riemann equations are not satisfied when $z \neq 0$.

So, $f^{'}(z)$ does not exist in $z \neq 0$

At $z = 0, \quad f^{'}(z) = 0 + i \ . \ 0 = 0$.

3.2.2 Analytic Function

A function f of the complex variable z is said to be analytic at a point $z_0 \in \mathbb{C}$ if it is differentiable not only at z_0 but also at each point z in some ϵ neighborhood of the point z_0.

Definition 3.2.1. A function f is said to be analytic in a region S if it is analytic at each point in S.

Note:

The term "homomorphic" is also used in the literature to denote analyticity.

Example 3.2.5.

Let $f(z) = z^2$.

Now by Example 3.1.2 in section 3.1,f is differentiable everywhere and hence f is analytic everywhere.

Example 3.2.6.

Let $f(z) = |z^2|$.

In section 3.1, we have shown f is differentiable only at $z = 0$, hence $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$ and not through out any neighborhood. So, $f(z) = |z^2|$ no where analytic.

Remark:

Analytic functions are usually defined on domains. If, however, we speak of an analytic function f on the closed disc $|z| \leqslant 1$, it is to be understood that f is analytic through out some domain containing that disc.

Definition 3.2.2. (Entire Function) An entire function is a function that is analytic at each point in the entire finite plane.

Example 3.2.7.

Let $P(z) = a_0 + a_1 z + ... + a_n z^n$, a polynomial.
Since the derivative of the polynomial $P(z)$ exists everywhere, it follows that every polynomial is an entire function.

Definition 3.2.3. (Singular Point or Singularity of f) If a function f fails to be analytic at a point z_0 but is analytic at some point in every ϵ neighborhood of z_0 is called a singular point or singularity of f.

Example 3.2.8.

Let $f(z) = \dfrac{1}{z}$, $z \neq 0$. Then $f'(z) = \dfrac{-1}{z^2}$, f is analytic at every point expect for $z = 0$, where it's not even defined. So, $z = 0$, a singular point.

Example 3.2.9.

Let $f(z) = |z^2|$
Since f has no singular points it is no where analytic.

Remark:
A necessary condition for a function f to be analytic in a region S is clearly the continuity of f through out S satisfying the Cauchy-Riemann equations is also necessary, but not sufficient.

Theorem 3.2.4. *(Necessary)*
Suppose that the function $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z_0 = x_0 = +iy_0$. Then the Cauchy-Riemann equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ hold at every point of some neighborhood of z_0.
Proof: The proof follows from theorem 3.2.2 and the definition of analyticity.

Theorem 3.2.5. *(Sufficient)*
Given $f(z) = u(x, y) + iv(x, y)$*, suppose that*

1. *The function* u*,* v *and their first partial derivatives* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ *and* $\frac{\partial v}{\partial y}$ *are all continuous throughout som neighborhood* $N(z_0, \epsilon)$ *of* $z_0 = x_0 + iy_0$*.*

2. *The Cauchy-Riemann equations* $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ *holds at every point of* $N(z_0, \epsilon)$*.*

Then $f(z)$ *is analytic at* z_0*.*

Proof: The proof follows from the Theorem 3.2.3 and the definition of analyticity.

Theorem 3.2.6. *Suppose that*

1. $f(z)$ *and* $g(z)$ *are analytic in a domain* S*.*

2. f *is analytic at every* $g(z)$ *for all* z *in* S

Then $(f \pm g)(z)$*,* $(fg)(z)$*,* $\left(\frac{f}{g}\right)(z)$*,* $g(z) \neq 0$*,* $(f \circ g)(z)$ *are also analytic functions at every point of* S *at which they are defined.*

proof:By Theorem 3.2.1 and the definition of analyticity.

Theorem 3.2.7. *Suppose* $u(x, y)$ *has partial derivatives* u_x *and* u_y *that vanish at every point of a region* S*. Then* u *is contain in* S*.*

Proof. Let (x_1, y_1), $(x_2, y_2) \in S$.

Then as S is a path connected set, so points P and Q can be connected by a path is contained in S.

Consider the line segment $P \to P'$. Let $(x+\Delta x, y+\Delta y)$ and (x,y) be points on this segment.

Then

$$\begin{aligned}\Delta u &= u(x+\Delta x, y+\Delta y) - (x,y)\\ &= u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y) + -u(x+\Delta x, y) - u(x,y)\\ &= \frac{\partial u}{\partial y}(x+\Delta x, \eta)\Delta y + \frac{\partial u}{\partial x}(\xi, y)\Delta x,\end{aligned}$$

where $y < \eta < y+\Delta y$ and $x < \xi < x+\Delta x$ and we have used the Mean Value Theorem. Since, however, $u_x$and u_y vanish identically in S, where $\Delta u = 0$ between each pair of successive vertices. Hence $u(x_1, y_1) = u(x_2, y_2)$. So, u is constant in S.

□

Example 3.2.10.

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a region S and $f'(z) = 0$ everywhere in S. Then f is constant throughout S.

Solution: Assume that $f'(z) = 0$everywhere in S.

Then $f'(z) = u_x + iv_x = v_y - iu_y = 0$ (f is analytic $\Rightarrow$ Cauchy-Riemann equations are satisfied by Theorem 3.2.4)

Thus $u_x = v_x = v_y = u_y = 0$

Hence by Theorem 3.2.7, u is constant in S.

So, $\exists$ real constant c_1 such that $u(x,y) = c_1$, throughout S

Similarly, $v(x,y) = c_2$. Hence $f(z) = c_1 + ic_2$ at each point in S.

So, f is constant everywhere is S.

3.2.3 Harmonic Function

Definition 3.2.4. A real valued function h of two real variables x and y, that is $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x,y)$, is said to be harmonic in a given region of the xy-plane if throughout that region it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{3.2.10}$$

Example 3.2.11.

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a region S. Show that the component functions u and v are harmonic in S.

Solution: We shall use the result that if a function of a complex variable is analytic at a point, then its real and imaginary components have continuous partial derivatives of all order at that point.

Since f is analytic in S, by Theorem 3.2.4 the first order partial derivatives of its components functions satisfy the Cauchy-Riemann equations throughout S. i.e.,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \tag{3.2.11}$$

Differentiate (3.2.11) w. r. t. x and y,

$$u_{xx} = v_{yx} \quad \text{and} \quad u_{yx} = -v_{xx} \tag{3.2.12}$$

$$u_{xy} = v_{yy} \quad \text{and} \quad u_{yy} = -v_{xy} \tag{3.2.13}$$

But the continuity of the partial derivatives ensures that $u_{xy} = u_{yx}$ and $v_{yx} = v_{xy}$. So, from (3.2.120 and (3.2.13) we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Hence, if a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a region S, its component functions u and v are harmonic in S.

Definition 3.2.5. If two functions u and v are harmonic in a region S and their first order partial derivatives satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ throughout S, then v is said to be a harmonic conjugate of v.

Note:

1. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a region S, then v is a harmonic conjugate of u. Conversely, if v is a harmonic conjugate of u in a region S, the function $f(z) = u(x, y) + iv(x, y)$is analytic in S.

2. A necessary and sufficient conditions for a function $f(z) = u(x, y)+iv(x, y)$ to be analytic in aregion S is that v be a harmonic conjugate of u in S.

3. If v is a harmonic conjugate of u in same region, it is not, in general, true that u is a harmonic conjugate of v there. Here follows an example for it.

Example 3.2.12.

Let $f(z) = z^2$. Then $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$
By Example 3.2.5 $f(z) = z^2$ is analytic everywhere. So, f is entire function. Then by Note 1. v is a harmonic conjugate of u throughout the plane.
Now let $g(z) = 2xy + i(x^2 - y^2 = v(x, y) + iu(x, y)$
Suppose that g is analytic at $z_0x_0 + iy_0$. Then by Theorem 3.2.4, the Cauchy-Riemann equations are satisfied,
i.e., $u_x = v_y,\ u_y = -v_x$
i.e., $2y = -2y$ and $2x = -2x$, which is not possible
So, $g(z) = 2xy + i(x^2 - y^2)$ is not analytic anywhere.
Thus u cannot be a harmonic conjugate of v by Note 1.

Example 3.2.13.

If two functions u and v to be harmonic conjugates of each other, then both u and v must be constant functions
Solution: Let v be a harmonic conjugate of u. Then $f(z) = u(x, y) + iv(x, y)$ is analytic hence u and v satisfy the Cauchy-Riemann equations, i.e.,

$$u_x = v_y, \quad \text{and} \quad u_y = -v_x \tag{3.2.14}$$

Now let u be a harmonic conjugate of v, then $g(z) = v(x, y)+iu(x, y)$ is analytic hence v and u satisfy the Cauchy-Riemann equations, i.e.,

$$v_x = u_y,\ v_y = -u_x \tag{3.2.15}$$

From (3.2.14) and (3.2.15) : $u_x = v_y = v_x = u_y = 0$

Hence $u(x, y) = c_1$ and $v(x, y) = c_2$ () by Theorem 3.2.7).

Remark:

If v is harmonic conjugate of u in a region S, then $-u$ is a harmonic conjugate of v in S, and conversely. That is, v is a harmonic conjugate of u in a region S iff $-u$ is a harmonic conjugate of v in S.

For: $f(z) = u(x, y) + iv(x, y)$ is analytic in S iff the function $-if(z) = v(x, y) - iu(x, y)$ is analytic there.

Note:

If a function u which is harmonic in a region of a certain type always has a harmonic conjugate. Thus, in such domains, every harmonic function is the real part of an analytic function. It is also that a harmonic conjugate, when it exists, is unique except an additive constant.

Example 3.2.14.

Find the harmonic conjugate of $\tan^{-1}\left(\frac{x}{y}\right)$ where $\pi < \tan^{-1}\left(\frac{x}{y}\right) < \pi$.

Solution: Write $u(x, y) = \tan^{-1}\left(\frac{x}{y}\right)$. Then by the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{y^2}{x^2 + y^2}\frac{1}{y} = \frac{y}{x^2 + y^2} = \frac{\partial v}{\partial y} \tag{3.2.16}$$

$$-\frac{\partial u}{\partial y} = -\frac{y^2}{x^2 + y^2}\frac{-x}{y^2} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial x} \tag{3.2.17}$$

Integrating (3.2.16) w.r.t. y,

$$v = \frac{1}{2}\log(x^2 + y^2) + \phi(x), \tag{3.2.18}$$

Differentiating (3.2.18) w.r..t. x,

$$\frac{\partial v}{\partial x} = \frac{x}{(x^2 + y^2)} + \phi^{'}(x) \tag{3.2.19}$$

From (3.2.17) and (3.2.19),

$\phi^{'}(x) = 0 \Rightarrow \quad \phi(x)$ is a constant, say C. Therefore,

$$v(x, y) = \frac{1}{2}\log(x^2 + y^2 + C$$

So, $v(x, y) = \frac{1}{2}\log(x^2 + y^2 + C$ is a harmonic conjugate of $u(x, y)$

The corresponding analytic function is $f(z) = \tan^{-1}\left(\frac{x}{y}\right) + i(\frac{1}{2}\log(x^2 + y^2 + C)$.

Chapter 4

Introduction to Special Functions

In this section, we shall generalize various functions that we have studied in real analysis to the complex domain.

4.1 The Exponential Function

Definition 4.1.1. Suppose that $z = x + iy$, where $x, y \in \mathbb{R}$. Then the exponential function e^z is defined for every $z \in \mathbb{C}$ by

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y) \tag{4.1.1}$$

If we write $e^z = u(x, y) + iv(x, y)$, then $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. It is easy to check that the Cauchy-Riemann equations are satisfied for every $z \in \mathbb{C}$, so that e^z is an **entire** function.

Furthermore,
$\dfrac{d}{dz}\, e^z = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = e^x(\cos y + i \sin y) = e^z$, everywhere in the z-plane.

Note:

Sometimes, for convenience, we use the notation $\exp(z)$ instead of e^z.

4.1.1 Properties of e^z

Example 4.1.1.

Prove that

1. $e^{(z_1+z_2)} = e^{z_1} \,.\, e^{z_2}$;

2. $e^{(z+2\pi i)} = e^z, \quad \forall z \in \mathbb{C}$; (the exponential function is periodic with a pure imaginary period of $2\pi i$)

3. $e^{(z_1-z_2)} = \dfrac{e^{z_1}}{e^{z_2}}$;

4. $\dfrac{1}{e^z} = e^{-z}$;

5. $(e^z)^n = e^{nz}, \quad n = 1, 2, ...$;

6. $|e^z| = e^x$ and $arg(e^z) = y + 2n\pi, \quad n = 0, \pm 1, \pm 2, ...$;

7. $e^z \neq 0$, for any $z \in \mathbb{C}$ (since $|e^z| > 0$ always.)

Proof. 1. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} e^{z_1}.e^{z_2} &= (e^{x_1}.e^{iy_1})(e^{x_2}.e^{iy_2}) \\ &= (e^{x_1}.e^{x_2})(e^{iy_1}.e^{iy_2}) \\ &= e^{(x_1+x_2)}\ e^{i(y_1+y_2)} \\ &= e^{[(x_1+x_2)+i(y_1+y_2)]} \\ &= e^{z_1+z_2}. \end{aligned}$$

2. $e^{z+2\pi i} = e^z.e^{2\pi i} = e^z \quad (\because e^{2\pi i} = 1)$

Thus $e^{z+2\pi i} = e^z$, $\forall z \in \mathbb{C}$, the exponential function is periodic with a pure imaginary period of $2\pi i$.

3. $$\begin{aligned} \frac{e^{z_1}}{e^{z_2}} = \frac{e^{x_1}.e^{iy_1}}{e^{x_2}.e^{iy_2}} &= e^{(x_1-x_2)}e^{i(y_1-y_2)} \\ &= e^{[(x_1-x_2)+i(y_1-y_2)]} \\ &= e^{z_1-z_2}. \end{aligned}$$

4. In 3., let $z_1 = 0$ and $z_2 = z$, then $\dfrac{e^0}{e^z} = e^{0-z}$,

i.e.), $\dfrac{1}{e^z} = e^{-z}, \quad \forall z \in \mathbb{C}$.

5. $e^{nz} = e^z.e^z....e^z, \quad \forall n \in \mathbb{N}.$

6. $|e^z| = |e^x.\ e^{iy}| = |e^x||e^{iy}| = e^x$ and $arg(e^z) = y + 2n\pi,\ n = 0, \pm1, \pm2, ...$

7. Since $|e^z| = e^x > 0$, so,

   $e^z \neq 0,\ \forall z \in \mathbb{C}$

$\square$

Remark:

When $w = e^z$, w can be written $w = \rho\ e^{i\phi}$, so

$$\rho\ e^{i\phi} = e^x\ e^{iy}, \quad \rho = e^x \text{ and } \phi = y$$

$$\therefore x = \log\rho, \quad y = \rho \tag{4.1.2}$$

Thinking of the equation $w = e^z$ as a transformation from z to the w-plane. We find that any given non-zero point $w = \rho\ e^{i\phi}$ is the image of the point

$$z = \log\rho + i\phi \tag{4.1.3}$$

By Ex 4.1.1 (7.) $e^z \neq 0$, so the point $w = 0$ cannot be the image of any point z under the transformation $w = e^z$. So, the range of the exponential function $w = e^z$ is the entire w-plane except for the origin $w = 0$.

Any point $w = \rho\ e^{i\phi}(\neq 0)$ is actually the image of an infinite number of points in the z-plan under the transformation $w = e^z$, in general, ϕ may have any one of the values $\phi = \Phi + 2n\pi,\ n = 0, \pm1, \pm2, ...$, where Φ denotes the principal values $(-\pi < \Phi \leqslant \pi)$ of $arg(w)$. So from equation (4.1.3) it follows that w is the image of all the points

$$z = \log\rho + i(\Phi + 2n\pi),\ n = 0, \pm1, \pm2, ... \tag{4.1.4}$$

Example 4.1.2.

Find all z such that $e^z = -1$.

Solution: Let $z = x + iy$. Then

$$e^x.e^{iy} = 1.e^{i\pi}.$$

$$\therefore e^x = e^0 \text{ and } y = \pi + 2n\pi,\ n \in \mathbb{Z}$$

$$x = 0 \text{ and } y = \pi + 2n\pi,\ n \in \mathbb{Z}.$$

$$\therefore z = (2n+1)\pi i,\ n \in \mathbb{Z}.$$

Alternatively, $\rho = e^x = 1$

$$\therefore \log \rho = 0 \text{ and } \Phi = \pi$$

So, by equation (4.1.4), $z = i(2n+1)\pi$.

4.2 Trigonometric Functions

Definition 4.2.1. Suppose that $z \in \mathbb{C}$. Then the trigonometric functions $\cos z$ and $\sin z$ are defined in terms of the exponential function by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{4.2.1}$$

Since the exponential function is an entire function, it follows easily from (4.3.1) that both $\cos z$ and $\sin z$ are entire functions

Furthermore, $\dfrac{d}{dz}\ \cos z = -\sin z$ and $\dfrac{d}{dz}\ \sin z = \cos z$

We can define the function $\tan z$, $\cot z$, $\sec z$ and $\operatorname{cosec} z$ in terms of the functions $\cos z$ and $\sin z$ as in real variables. However, note that these four functions are **not entire**.

We can deduce,

$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$

4.3 Hyperbolic Function

Definition 4.3.1. Suppose that $z \in \mathbb{C}$. Then the hyperbolic functions $\cosh z$ and $\sinh z$ are defined in terms of the exponential function by

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}. \tag{4.3.1}$$

Since the exponential function is an entire function, it follows easily from (4.3.1) that both $\cosh z$ and $\sinh z$ are entire functions.

Furthermore,

$$\frac{d}{dz}\cosh z = \sinh z$$

and

$$\frac{d}{dz}\sinh z = \cosh z$$

We can defined the function $\tanh z$, $\coth z$, $\operatorname{sech} z$ and $\operatorname{cosech} z$ in terms of the functions $\cosh z$ and $\sinh z$ in real variables.
However, note that these four functions are **not entire**.

Note also that compare (4.3.1) and (4.2.1), we obtain

$$\cosh z = \cos iz \text{ and } \sinh z = -i \sin iz$$

Example 4.3.1.

Prove that

1. $\sinh z = -i\sin(iz), \quad \sin z = i\sinh(iz)$

2. $\cosh z = \cos iz, \quad \cos z = \cosh(iz)$

3. $\cosh^2 z - \sinh^2 z = 1$

4. $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$
   $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

5. $\sinh z = \sinh x \cosh y + i \cosh x \sinh y$
   $\cosh z = \cosh x \cosh y + i \sinh x \sinh y$

6. $|\sinh z|^2 = \sinh^2 x + \sinh^2 y$

   $|\cosh z|^2 = \sinh^2 x + \cosh^2 y, \quad z = x + iy.$

4.4 Complex Logarithm

Definition 4.4.1. We define the logarithm function of a non-zero complex number $z = re^{i\theta}$ by

$$\log z = \log r + i\theta. \tag{4.4.1}$$

$$\log z = \log |z| + i\, arg z. \tag{4.4.2}$$

Note:

1. In the above definition for a specific $z \neq 0, \quad z = re^{i\theta}$, the value of θ is not unique.

2. If Θ denotes the principal value $(-\pi < \Theta \leq \pi)$ of θ, we can write $\theta = \Theta + 2n\pi,\ n \in \mathbb{Z}$, so (4.4.1) takes the form

$$\log z = \text{Log } |z| + i(\Theta + 2n\pi), \quad in\mathbb{Z} \tag{4.4.3}$$

Definition 4.4.2 (Principal value of the logarithm). The principal value of $\log z$ is the value obtain from (4.4.3) when $n = 0$ and is denoted by $\text{Log}\, z$ and defined by

$$\text{i.e., } \text{Log}\, z = \text{Log}\, r + i\Theta$$

$$\text{Log}\, z = \text{Log}\, |z| + i\text{Arg } \ z, \quad z \neq 0. \tag{4.4.4}$$

Note:

1. The logarithm function, $\text{Log}\, z$, is a single-valued function whose component functions are

$$u(r,\theta) = \text{Log}\, r \quad \text{and} \quad v(r,\theta) = \theta \tag{4.4.5}$$

is not continuous, and so not analytic, through out its domain of definition. $r > 0,\ -\pi < \theta \leq \pi$. The component functions u and v of $\operatorname{Log} z$ are, however, each continuous through out the domain $r > 0,\ -\pi < \theta < \pi$ consisting of all points in the plane expect $z = 0$ and points on the negative real axis.

The partial derivatives $u_r = \frac{1}{r},\ u_\theta = 0,\ v_r = 0$ and $v_\theta = 1$ are continuous every where in that domain, and they satisfy the polar form $u_r = \frac{1}{r} v_\theta$ and $\frac{1}{r} u_\theta = -v_r$ of the C-R equations there. So by theorem (3.2.3), the function $\operatorname{Log} z$ is analytic in the domain $r > 0,\ -\pi < \theta < \pi$. Also if

$$
\begin{aligned}
z = e^{i\theta},\quad \frac{d}{dz}\operatorname{Log} z &= e^{-i\theta}(u_r + i\, v_r) \\
&= e^{-i\theta}(\frac{1}{r} + i.0) \\
&= e^{-i\theta} r \\
&= \frac{1}{e^{i\theta}} \\
&= \frac{1}{z};
\end{aligned}
$$

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z},\quad |z| > 0, \quad -\pi < \operatorname{Arg} z < \pi.$$

2. The function $\operatorname{Log} z$, when restricted to the domain $r > 0,\ -\pi < \theta < \pi$, is a branch of the logarithm function (4.4.1). This branch is called the principal branch.

4.5 Properties of Logarithms

Example 4.5.1.

Show that $e^{\log z} = z,\ z \neq 0$.

Solution: Let $z = re^{i\theta}$. Then $\log z = \operatorname{Log} r + i\theta$, θ is any particular values of $\arg z$. Then

$$e^{\log z} = e^{\operatorname{Log} r + i\theta} = e^{\operatorname{Log} r} e^{i\theta} = r\, e^{i\theta} = z. \tag{4.5.1}$$

Example 4.5.2.

Shoe that $\log(e^z)$ is not always equal to z. But $\mathrm{Log}\,(e^z) = z, \quad -\pi < \mathrm{Im}\, z < \pi$.

Solution: We know that $|e^z| = e^x$ and $\arg(e^z)y + 2n\pi,\ n \in \mathbb{Z},\ z = x + iy$

So, $\log(e^z) = \mathrm{Log}|e^z| + i\arg(e^z) = x + i(y + 2n)\pi = z + i2n\pi,\ n \in \mathbb{Z}$.

Thus $\log(e^z)$ has an infinite number of values for any given z.

Now suppose that $z = x + iy$ such that $-\pi < y \leqslant \pi$ and that principal values of the logarithmic functions are taken. As $\arg(e^z) = y$, hence

$$\mathrm{Log}(e^z) = z, \quad -\pi < \mathrm{Im}\, z \leqslant \pi \quad (n = 0). \tag{4.5.2}$$

Example 4.5.3.

Show that $\log(z_1 z_2) = \log z_1 + \log z_2$.

Solution:
$$\begin{aligned} \log(z_1 z_2) &= \mathrm{Log}|z_1 z_2| + i\ \arg(z_1 z2) \\ &= \mathrm{Log}|z_1||z_2| + i\ \arg z_1 + i\ \arg z_2 \\ &= \mathrm{Log}|z_1| + i\ \arg z_1 + \mathrm{Log}|z_2| + i\ \arg z_2 \\ &= \log|z_1| + \log|z_2| \end{aligned}$$

$$\text{i.e.,} \qquad \log(z_1 z_2) = \log|z_1| + \log|z_2|. \tag{4.5.3}$$

Example 4.5.4.

Show that $\log\left(\dfrac{z_1}{z_2}\right) = \log z_1 - \log z_2$.

Solution:
$$\begin{aligned} \log\left(\frac{z_1}{z_2}\right) &= \mathrm{Log}\left|\frac{z_2}{z_2}\right| + i\arg\left(\frac{z_1}{z_2}\right) \\ &= [\mathrm{Log}|z_1| + i\ \arg z_1] - [\mathrm{Log}|z_2| + i\ \arg z_2] \\ &= \log z_1 - \log z_2. \end{aligned}$$

$$\text{i.e.,} \quad \log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 \tag{4.5.4}$$

Example 4.5.5.

Let $z_1 = z_2 = -1$. So, $z_1 z_2 = 1$. Now

$$\begin{aligned} \log(z_1 z_2) &= \mathrm{Log}|z_1 z_2| + i\arg(z_1 z_2) \\ &= 0 + i.0 = 0 \end{aligned}$$

$\log z_1 = \pi i$, and $\log z_2 = -\pi i$

$\log(-1) = (2n+1)\pi\ i,\ n \in \mathbb{Z}$

So,(4.5.3) is valid for $\log(z_1 z_2) = 0$. $\log z_1 = \pi\, i$, and $\log z_2 = -\pi\, i$.

Now, $\text{Log}(z_1 z_2) = \text{Log}|z_1 z_2| + i\text{Arg}(z_1 z_2)$

$= 0 + i.0 = 0$

$\text{Log}\, z_1 = \text{Log}\,|z_1| + i\ \text{Arg}(z_1)$

$= 0 + i\pi = \pi\, i \quad (\because \theta = \Theta + 2n\pi,\ n \in \mathbb{Z}, \quad -\pi < \Theta \leqslant \pi)$

Similarly, $\text{Log}\, z_2 = \text{Log}\,|z_2| + i\ \text{Arg}(z_2)$

$= 0 + i\pi = \pi\, i)$

So, $\text{Log}\, z_1 + \text{Log}\, z_2 = 2\pi\, i$.

Thus, (4.5.3) is not in general, valid when log is replaced everywhere by Log. A similar remark applied to (4.5.4).

Remark:

If $z \neq 0$, then

$$z^n = e^{n \log z}, \quad n \in \mathbb{Z} \tag{4.5.5}$$

For any value of $\log z$ that's taken. Similarly,

$$z^{\left[\frac{1}{n}\right]} = e^{\left[\frac{1}{n}\right] \log z}, \quad n \in \mathbb{N} \tag{4.5.6}$$

The terms of the RHS here has n distinct values and those values are the n^{th} roots of z.

Proof of (4.5.6):

For, write $z = r\, e^{i\Theta}$, where Θ is the principal value of $\text{agr}\, z$. So, by equation (4.4.3),

$\dfrac{1}{n} \log z = \dfrac{1}{n}\text{Log}\ \ r + i\dfrac{\Theta + 2k\pi}{n},\ k \in \mathbb{Z}$

$\therefore \exp\left(\dfrac{1}{n} \log z\right) = \exp\left[\dfrac{1}{n} \log r + i\dfrac{\Theta + 2k\pi}{n}\right], \quad k \in \mathbb{Z}$

$$i.e., \quad \exp\left(\frac{1}{n} \log z\right) = \sqrt[n]{r} \exp\left[i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)\right] \tag{4.5.7}$$

As $e^{\left[\frac{i2\pi k}{n}\right]}$ has values only when $k = 0, 1, ..., n-1$, RHS of (4.5.7) has only n values. RHS is an expression for the n^{th} roots of z, and so it can be written $z^{\frac{1}{n}}$.

4.6 Complex Exponents

Definition 4.6.1. Let $z \neq 0$ and let the exponent c is any complex number, z^c is defined by the equation

$$z^c = e^{c \log z} \tag{4.6.1}$$

where $\log z$ denotes the multiple-valued logarithmic function.

Example 4.6.1.

Calculate i^{-2i}

Solution: $i^{-2i} = e^{-2i\log(i)} = e^{-2i\left[\text{Log}|i| + i\left(\frac{\pi}{2} + 2n\pi\right)\right]}, \quad n \in \mathbb{Z}$

$$= e^{[(4n+1)\pi]}, \quad n \in \mathbb{Z}$$

Note:

If $z = r\,e^{i\theta}$ and $\alpha \in \mathbb{R}$, the function $\log z = \text{Log} r + i\theta, \quad r > 0,\ \alpha < \theta < \alpha + 2\pi$, is a single-valued and analytic in the indicated domain. When that branch of $\log z$ is used, it follows that the function $z^c = e^{c\log z}$is single-valued and analytic in the same domain. The derivative of such a branch of z^c is found by writing

$\dfrac{d}{dz}\, z^c = \dfrac{d}{dz} = e^{c\log z} e^{c\log z}\, \dfrac{c}{z} = c\dfrac{e^{c\log z}}{e^{\log z}} = ce^{(c-1)\log z} = c\, z^{c-1}$

Hence $c\, z^{c-1}$, $(|z| > 0,\ 0 < \text{arg} z < \alpha + 2\pi)$ is single-valued function. That is

$$\frac{d}{dz}\, z^c = c\, z^{c-1}, \quad (|z| > 0,\ 0 < \text{arg} z < \alpha + 2\pi) \tag{4.6.2}$$

The principal value of z^c occurs when $\log z$ is replaced by $\text{Log}\, z$ in (4.6.1), so

$$z^c = e^{c\,\text{Log}\, z} \tag{4.6.3}$$

Equation (4.6.3) also serves to defined the principal branch of the function z^c on the domain $(|z| > 0,\ 0 < \text{arg} z < \alpha + 2\pi)$.

Example 4.6.2.

Find the principal value of $(-i)^i$.

Solution: $(-i)^i = e^{i\,\text{Log}\,(-i)} = e^{i\left(\frac{-\pi}{2}\, i\right)} = e^{\frac{\pi}{2}}$.

Example 4.6.3.

Find the principal branch of $z^{\left[\frac{2}{3}\right]}$.

Solution: $z^{\left[\frac{2}{3}\right]} = e^{\left[\frac{2}{3}\operatorname{Log} z\right]} = \exp\left[\frac{2}{3}(\operatorname{Log} + i\theta)\right] = \sqrt[3]{r^2} \quad e^{\left(\frac{i2\theta}{3}\right)}$

Chapter 5

Complex Integration

Integration is an important and useful concept in elementary calculus. Integrals are extremely important in the study of function a variable too. The two dimensional nature of the complex plane suggests the consideration of integrals along arbitrary curves in $\mathbb{C}$ instead of only on segments of the real axis.

5.1 Integration of Functions defined on $[\alpha, \beta] \subseteq \mathbb{R}$ to $\mathbb{C}$

Definition 5.1.1. In the complex plane we describe the curve(arc) γ by the continuous complex valued function of a real variable $\gamma : w = w(t) = u(t)+iv(t),\ t \in [\alpha, \beta]$, that is, $\gamma : [\alpha, \beta] \to \mathbb{C}$, and $u(t)$ and $v(t)$ are real valued functions.

Note:

If $w(t) = u(t) + iv(t)$, then $w^{'}(t) = u^{'}(t) + iv^{'}(t)$.

Theorem 5.1.1. *(Mean Value Theorem) If $w(t)$ is differentiable on $[\alpha, \beta]$, then $w(\beta) - w(\alpha) = (\beta - \alpha)[u^{'}(\theta) + iv^{'}\phi(t)]$ where $\theta, \phi \in [\alpha, \beta]$.*

Proof. $w(\beta) - w(\alpha) = [u(\beta) - u(\alpha)] + i[v(\beta) - v(\alpha)]$

$$= (\beta - \alpha)u^{'}(\theta) + i(\beta - \alpha)v^{'}(\mu) \qquad \text{(By MVT for real}$$

valued function u and v)

$= (\beta - \alpha)[u'(\theta) - iv'(\theta)]$

By the usual Mean Value Theorem $\square$