

# 1 SymPy: symbolic computing in Python

## 2 Supplementary material

3 The supplementary material takes a deeper look at certain topics in SymPy ~~for~~ which there  
4 was not enough room to discuss in the paper. Section 1 discusses the Gruntz algorithm, used to  
5 calculate limits in ~~the~~ SymPy. Sections 2–8 discuss in depth some selected submodules. Section 9  
6 discusses numerical simplification. Section 10 provides additional examples for topics discussed  
7 in the main paper. In section 11 the SymPy Gamma project is discussed. Finally, section 12 has  
8 a brief comparison of SymPy with Wolfram Mathematica.

9 As in the paper, all examples in the supplement assume that the following has been run:

```
10 >>> from sympy import *  
11 >>> x, y, z = symbols('x y z')
```

## 12 1 LIMITS: THE GRUNTZ ALGORITHM

13 SymPy calculates limits using the Gruntz algorithm, as described in [6]. The basic idea is as  
14 follows: any limit can be converted to a limit  $\lim_{x \rightarrow \infty} f(x)$  by substitutions like  $x \rightarrow \frac{1}{x}$ . Then  
15 the subexpression  $\omega$  (that converges to zero as  $x \rightarrow \infty$  faster than all other subexpressions) is  
16 identified in  $f(x)$ , and  $f(x)$  is expanded into a series with respect to  $\omega$ . Any positive powers  
17 of  $\omega$  converge to zero (while negative powers indicate an infinite limit) and any constant term  
18 independent of  $\omega$  determines the limit. When a constant term still ~~dependends~~ depends on  $x$   
19 the Gruntz algorithm is applied again until a final numerical value is obtained as the limit.

To determine the most rapidly varying subexpression, the comparability classes must first be defined, by calculating  $L$ :

$$L \equiv \lim_{x \rightarrow \infty} \frac{\log|f(x)|}{\log|g(x)|} \quad (1)$$

The relations  $<$ ,  $>$ , and  $\sim$  are defined as follows:  $f > g$  when  $L = \pm\infty$  (it is said that  $f$  is more rapidly varying than  $g$ , i.e.,  $f$  goes to  $\infty$  or 0 faster than  $g$ ),  $f < g$  when  $L = 0$  ( $f$  is less rapidly varying than  $g$ ) and  $f \sim g$  when  $L \neq 0, \pm\infty$  (both  $f$  and  $g$  are bounded from above and below by suitable integral powers of the other). Note that if  $f > g$ , then  $f > g^n$  for any  $n$ . Here are some examples of comparability classes:

$$\begin{aligned} 2 &< x < e^x < e^{x^2} < e^{e^x} \\ 2 &\sim 3 \sim -5 \\ x &\sim x^2 \sim x^3 \sim \frac{1}{x} \sim x^m \sim -x \\ e^x &\sim e^{-x} \sim e^{2x} \sim e^{x+e^{-x}} \\ f(x) &\sim \frac{1}{f(x)} \end{aligned}$$

The Gruntz algorithm is now illustrated with the following example:

$$f(x) = e^{x+2e^{-x}} - e^x + \frac{1}{x}. \quad (2)$$

20 First, the set of most rapidly varying subexpressions is determined—the so-called *mrsv set*. For (2),  
21 the mrsv set  $\{e^x, e^{-x}, e^{x+2e^{-x}}\}$  is obtained. These are all subexpressions of (2) and they all  
22 belong to the same comparability class. This calculation can be done using SymPy as follows:

```
23 >>> from sympy.series.gruntz import mrsv  
24 >>> mrsv(exp(x+2*exp(-x))-exp(x) + 1/x, x)[0].keys()  
25 dict_keys([exp(x + 2*exp(-x)), exp(x), exp(-x)])
```

Next, an arbitrary item  $\omega$  is taken from mrv set that converges to zero for  $x \rightarrow \infty$  and doesn't have subexpressions in the given mrv set. If such a term is not present in the mrv set (i.e., all terms converge to infinity instead of zero), the relation  $f(x) \sim \frac{1}{f(x)}$  can be used. In the considered case, only the item  $\omega = e^{-x}$  can be accepted.

The next step is to rewrite the mrv set in terms of  $\omega = g(x)$ . Every element  $f(x)$  of the mrv set is rewritten as  $A\omega^c$ , where

$$c = \lim_{x \rightarrow \infty} \frac{\log f(x)}{\log g(x)}, \quad A = e^{\log f - c \log g} \quad (3)$$

Note that this step includes calculation of more simple limits, for instance

$$\lim_{x \rightarrow \infty} \frac{\log e^{x+2e^{-x}}}{\log e^{-x}} = \lim_{x \rightarrow \infty} \frac{x+2e^{-x}}{-x} = -1 \quad (4)$$

In this example we obtain the rewritten mrv set:  $\{\frac{1}{\omega}, \omega, \frac{1}{\omega}e^{2\omega}\}$ . This can be done in SymPy with

```
>>> from sympy.series.gruntz import mrv, rewrite
>>> m = mrv(exp(x+2*exp(-x))-exp(x) + 1/x, x)
>>> w = Symbol('w')
>>> rewrite(m[1], m[0], x, w)[0]
1/x + exp(2*w)/w - 1/w
```

Then the rewritten subexpressions are substituted back into  $f(x)$  in (2) and the result is expanded with respect to  $\omega$ :

$$f(x) = \frac{1}{x} - \frac{1}{\omega} + \frac{1}{\omega}e^{2\omega} = 2 + \frac{1}{x} + 2\omega + O(\omega^2) \quad (5)$$

Since  $\omega$  is from the mrv set, then in the limit as  $x \rightarrow \infty$ ,  $\omega \rightarrow 0$ , and so  $2\omega + O(\omega^2) \rightarrow 0$  in (5):

$$f(x) = \frac{1}{x} - \frac{1}{\omega} + \frac{1}{\omega}e^{2\omega} = 2 + \frac{1}{x} + 2\omega + O(\omega^2) \rightarrow 2 + \frac{1}{x} \quad (6)$$

In this example the result  $(2 + \frac{1}{x})$  still depends on  $x$ , so the above procedure is repeated until just a value independent of  $x$  is obtained. This is the final limit. In the above case the limit is 2, as can be verified by SymPy:<sup>1</sup>

```
>>> limit(exp(x+2*exp(-x))-exp(x) + 1/x, x, oo)
2
```

In general, when  $f(x)$  is expanded in terms of  $\omega$ , the following is obtained:

$$f(x) = \underbrace{O\left(\frac{1}{\omega^3}\right)}_{\infty} + \underbrace{\frac{C_{-2}(x)}{\omega^2}}_{\infty} + \underbrace{\frac{C_{-1}(x)}{\omega}}_{\infty} + C_0(x) + \underbrace{C_1(x)\omega}_0 + \underbrace{O(\omega^2)}_0 \quad (7)$$

The positive powers of  $\omega$  are zero. If there are any negative powers of  $\omega$ , then the result of the limit is infinity, otherwise the limit is equal to  $\lim_{x \rightarrow \infty} C_0(x)$ . The expression  $C_0(x)$  is always simpler than original  $f(x)$ , and the same is true for limits arising in the rewrite stage (3), so the algorithm converges. A proof of this and further details on the algorithm are given in Gruntz's PhD thesis [6].

<sup>1</sup>To see the intermediate steps discussed above, interested readers can switch on debugging output by setting the environment variable `SYMPY_DEBUG=True`, before importing anything from the SymPy namespace.

## 2 SERIES

### 2.1 Series Expansion

SymPy is able to calculate the symbolic series expansion of an arbitrary series or expression involving elementary and special functions and multiple variables. For this it has two different implementations: the `series` method and Ring Series.

The first approach stores a series as an instance of the `Expr` class. Each function has its specific implementation of its expansion, which is able to evaluate the Puiseux series expansion about a specified point. For example, consider a Taylor expansion about 0:

```
>>> series(sin(x+y) + cos(x*y), x, 0, 2)
1 + sin(y) + x*cos(y) + O(x**2)
```

The newer and much faster approach called Ring Series makes use of the fact that a truncated Taylor series is simply a polynomial. Correspondingly, it may be represented by a sparse polynomial, which performs well in a wide range of cases. Ring Series also gives the user the freedom to choose the type of coefficients to use, resulting in faster operations on certain types.

For this, several low-level methods for expansion of trigonometric, hyperbolic and other elementary operations (like series inversion, calculating the  $n$ th root, etc.) are implemented using variants of the Newton Method [1]. All these support Puiseux series expansion. The following example demonstrates the use of an elementary function that calculates the Taylor expansion of the sine of a series.

```
>>> from sympy.polys.ring_series import rs_sin
>>> R, t = ring('t', QQ)
>>> rs_sin(t**2 + t, t, 5)
-1/2*t**4 - 1/6*t**3 + t**2 + t
```

The function `sympy.polys.rs_series` makes use of these elementary functions to expand an arbitrary SymPy expression. It does so by following a recursive strategy of expanding the lowermost functions first and then composing them recursively to calculate the desired expansion. Currently, it only supports expansion about 0 and is under active development. Ring Series is several times faster than the default implementation with the speed difference increasing with the size of the series. The `sympy.polys.rs_series` takes as input any SymPy expression and hence there is no need to explicitly create a polynomial ring. An example demonstrating its use:

```
>>> from sympy.polys.ring_series import rs_series
>>> from sympy.abc import a, b
>>> rs_series(sin(a + b), a, 4)
-1/2*(sin(b))*a**2 + (sin(b)) - 1/6*a**3*(cos(b)) + a*(cos(b))
```

### 2.2 Formal Power Series

SymPy can be used for computing the formal power series of a function. The implementation is based on the algorithm described in the paper on formal power series [7]. The advantage of this approach is that an explicit formula for the coefficients of the series expansion is generated rather than just computing a few terms.

The following example shows how to use `fps`:

```
>>> f = fps(sin(x), x, x0=0)
>>> f.truncate(6)
x - x**3/6 + x**5/120 + O(x**6)
>>> f[15]
-x**15/1307674368000
```

## 2.3 Fourier Series

SymPy provides functionality to compute Fourier series of a function using the `fourier_series` function:

```
95 >>> L = symbols('L')
96 >>> expr = 2 * (Heaviside(x/L) - Heaviside(x/L - 1)) - 1
97 >>> f = fourier_series(expr, (x, 0, 2*L))
98 >>> f.truncate(3)
99 4*sin(pi*x/L)/pi + 4*sin(3*pi*x/L)/(3*pi) + 4*sin(5*pi*x/L)/(5*pi)
```

## 3 LOGIC

SymPy supports construction and manipulation of boolean expressions through the `sympy.logic` submodule. SymPy symbols can be used as propositional variables and subsequently be replaced with `True` or `False` values. Many functions for manipulating boolean expressions have been implemented in the `sympy.logic` submodule.

### 3.1 Constructing Boolean Expressions

A boolean variable can be declared as a SymPy `Symbol`. The Python operators `&`, `|` and `~` are overridden when using SymPy objects to use the SymPy functionality for logical `And`, `Or`, and `Not`. Other logic functions are also integrated into SymPy, including `Xor` and `Implies`, which are constructed with `^` and `>>`, respectively. Expressions can therefore be constructed either by using the shortcut operator notation or by directly creating the relevant objects: `And()`, `Or()`, `Not()`, `Xor()`, `Implies()`, `Nand()`, `Nor()`, etc.:

```
112 >>> e = (x & y) | z
113 >>> e.subs({x: True, y: True, z: False})
114 True
```

### 3.2 CNF and DNF

Any boolean expression can be converted to conjunctive normal form, disjunctive normal form, or negation normal form. The API also exposes methods to check if a boolean expression is in any of the aforementioned forms.

```
119 >>> from sympy.logic.boolalg import is_dnf, is_cnf
120 >>> to_cnf((x & y) | z)
121 And(Or(x, z), Or(y, z))
122 >>> to_dnf(x & (y | z))
123 Or(And(x, y), And(x, z))
124 >>> is_cnf((x | y) & z)
125 True
126 >>> is_dnf((x & y) | z)
127 True
```

### 3.3 Simplification and Equivalence

The `sympy.logic` submodule supports simplification of given boolean expression by making deductions from the expression. Equivalence of two logical expressions can also be checked. In the case of equivalence, the function `bool_map` can be used to show which variables of the first expression correspond to which variables of the second one.

```
133 >>> a, b, c = symbols('a b c')
134 >>> e = a & (~a | ~b) & (a | c)
135 >>> simplify(e)
136 And(Not(b), a)
137 >>> e1 = a & (b | c)
138 >>> e2 = (x & y) | (x & z)
139 >>> bool_map(e1, e2)
140 (And(Or(b, c), a), {a: x, b: y, c: z})
```

### 141 3.4 SAT Solving

142 The submodule also supports satisfiability (SAT) checking of a given boolean expression. If an  
143 expression is satisfiable, it is possible to return a variable assignment which satisfies it. The  
144 API also supports listing all possible assignments. The SAT solver has a clause learning DPLL  
145 algorithm implemented with a watch literal scheme and VSIDS heuristic [10].

```
146 >>> satisfiable(a & (~a | b) & (~b | c) & ~c)
147 False
148 >>> satisfiable(a & (~a | b) & (~b | c) & c)
149 {a: True, b: True, c: True}
```

## 150 4 DIOPHANTINE EQUATIONS

151 Diophantine equations play a central role in number theory. A Diophantine equation has the  
152 form,  $f(x_1, x_2, \dots, x_n) = 0$  where  $n \geq 2$  and  $x_1, x_2, \dots, x_n$  are integer variables. If there are  $n$   
153 integers  $a_1, a_2, \dots, a_n$  such that  $x_1 = a_1, x_2 = a_2, \dots, x_n = a_n$  satisfies the above equation, the  
154 equation is said to be solvable.

155 Currently, the following five types of Diophantine equations can be solved using SymPy's  
156 Diophantine submodule ( $a_1, \dots, a_{n+1}$ ,  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ ,  $f$ , and  $k$  are explicitly given rational constants,  
157  $x_1, \dots, x_{n+1}$ ,  $x$ ,  $y$ , and  $z$  are unknown variables):

- 158 • Linear Diophantine equations:  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$
- 159 • General binary quadratic equation:  $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- 160 • Homogeneous ternary quadratic equation:  $ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0$
- 161 • Extended Pythagorean equation:  $a_1x_1^2 + a_2x_2^2 + \dots + a_nx_n^2 = a_{n+1}x_{n+1}^2$
- 162 • General sum of squares:  $x_1^2 + x_2^2 + \dots + x_n^2 = k$

163 The `diophantine` function factors the equation it is given (if possible), solves each factor sep-  
164 arately, and combines the results to give a final solution set. Solutions may include parametrized  
165 variables (over the integers). The following examples illustrate some of the basic functionalities  
166 of the Diophantine submodule.

```
167 >>> from sympy.solvers.diophantine import *
168 >>> diophantine(2*x + 3*y - 5)
169 set([(3*t_0 - 5, -2*t_0 + 5)])
170
171 >>> diophantine(2*x + 4*y - 3)
172 set()
173
174 >>> diophantine(x**2 - 4*x*y + 8*y**2 - 3*x + 7*y - 5)
175 set([(2, 1), (5, 1)])
176
177 >>> diophantine(x**2 - 4*x*y + 4*y**2 - 3*x + 7*y - 5)
178 set([(-2*t**2 - 7*t + 10, -t**2 - 3*t + 5)])
179
180 >>> diophantine(3*x**2 + 4*y**2 - 5*z**2 + 4*x*y - 7*y*z + 7*z*x)
181 set([(-16*p**2 + 28*p*q + 20*q**2,
182 3*p**2 + 38*p*q - 25*q**2,
183 4*p**2 - 24*p*q + 68*q**2)])
184
185 >>> x1, x2, x3, x4, x5, x6 = symbols('x1 x2 x3 x4 x5 x6')
186 >>> diophantine(9*x1**2 + 16*x2**2 + x3**2 + 49*x4**2 + 4*x5**2 - 25*x6**2)
187 set([(70*t1**2 + 70*t2**2 + 70*t3**2 + 70*t4**2 - 70*t5**2, 105*t1*t5,
188 420*t2*t5, 60*t3*t5, 210*t4*t5, 42*t1**2 + 42*t2**2 + 42*t3**2 + 42*t4**2 +
```

```

189 42*t5**2)])
190
191 >>> a, b, c, d = symbols('a b c d')
192 >>> diophantine(a**2 + b**2 + c**2 + d**2 - 23)
193 set([(2, 3, 3, 1)])

```

## 194 5 SETS

195 SymPy supports representation of a wide variety of mathematical sets. This is achieved by first  
 196 defining abstract representations of atomic set classes and then combining and transforming  
 197 them using various set operations.

198 Each of the set classes inherits from the base class `Set` and defines methods to check  
 199 membership and calculate unions, intersections, and set differences. When these methods are  
 200 not able to evaluate to atomic set classes, they are represented as abstract unevaluated objects.

201 SymPy has the following atomic set classes:

- 202 • `EmptySet` represents the empty set  $\emptyset$ .
- 203 • `UniversalSet` is an abstract “universal set” of which everything is a member. The union of  
 204 the universal set with any set gives the universal set and the intersection gives the other  
 205 set itself.
- 206 • `FiniteSet` is functionally equivalent to Python’s built in `set` object. Its members can be  
 207 any SymPy object including other sets.
- 208 • `Integers` represents the set of integers  $\mathbb{Z}$ .
- 209 • `Naturals` represents the set of natural numbers  $\mathbb{N}$ , i.e., the set of positive integers.
- 210 • `Naturals0` represents the set of whole numbers  $\mathbb{N}_0$ , which are all the non-negative integers.
- 211 • `Range` represents a range of integers. A range is defined by specifying a start value, an end  
 212 value, and a step size. The enumeration of a `Range` object is functionally equivalent to  
 213 Python’s `range` except it supports infinite endpoints, allowing the representation of infinite  
 214 ranges.
- 215 • `Interval` represents an interval of real numbers. It is defined by giving the start and the  
 216 end points and by specifying if the interval is open or closed on the respective ends.

217 In addition to unevaluated classes for the basic `Union`, `Intersection`, and `Complement` set  
 218 operations, SymPy has the following set classes.

- 219 • `ProductSet` defines the Cartesian product of two or more sets. The product set is useful  
 220 when representing higher dimensional spaces. For example, to represent a three-dimensional  
 221 space, SymPy uses the Cartesian product of three real sets.
- 222 • `ImageSet` represents the image of a function when applied to a particular set. The image  
 223 set of a function  $F$  with respect to a set  $S$  is  $\{F(x) \mid x \in S\}$ . SymPy uses image sets to  
 224 represent sets of infinite solutions of equations such as  $\sin(x) = 0$ .
- 225 • `ConditionSet` represents a subset of a set whose members satisfy a particular condition.  
 226 The subset of set  $S$  given by the condition  $H$  is  $\{x \mid H(x), x \in S\}$ . SymPy uses condition  
 227 sets to represent the set of solutions of equations and inequalities, where the equation or  
 228 the inequality is the condition and the set is the domain over which it is being solved.

229 A few other classes are implemented as special cases of the classes described above. The  
 230 set of real numbers, `Reals`, is implemented as a special case of `Interval`. `ComplexRegion` is  
 231 implemented as a special case of `ImageSet`. `ComplexRegion` supports both the polar and rectangular  
 232 representation of regions in the complex plane.

## 6 STATISTICS

The `sympy.stats` submodule provides random variable types and methods for computing statistical properties of expressions involving random variables, which can be either continuous or discrete, the latter ones being further divided into finite and infinite. The variables are associated with probability densities on corresponding domains and internally defined in terms of probability spaces. Apart from the possibility of defining the random variables from a user supplied density distribution, SymPy provides definitions of most common distributions, including `Uniform`, `Poisson`, `Normal`, `Binomial`, `Bernoulli`, and many others.

Properties of random expressions can be calculated using, e.g., `expectation` (abbreviated `E`) and `variance` to calculate expectation and variance. Internally, these functions generate integrals and summations, which are automatically evaluated. The evaluation can be suppressed using `evaluate=False` keyword argument.

Conditions on random variables can be defined with inequalities, equalities, and logical operators and their overall probabilities are obtained using `P`. The features can be illustrated on a model of two dice throws:

```
>>> from sympy.stats import Die, P, E
>>> X, Y = Die("X"), Die("Y")
>>> P(Eq(X, 6) & Eq(Y, 6))
1/36
>>> P(X>Y)
5/12
```

The conditions can also be supplied as a second parameter to `E`, `P`, and other methods to calculate the property given the condition:

```
>>> E(X, X+Y<5)
5/3
```

Using the facilities of the `sympy.stats` submodule, one can, for example, calculate the well known properties of the Maxwellian velocity distribution.

```
>>> from sympy.stats import Maxwell, density
>>> kT, m, t = symbols("kT m t", positive=True)
>>> v = Maxwell("v", sqrt(kT/m))
>>> E(v) # mean velocity
2*sqrt(2)*sqrt(kT)/(sqrt(pi)*sqrt(m))
>>> E(v, evaluate=False) # unevaluated mean velocity
Integral(sqrt(2)*m**(3/2)*v**3*exp(-m*v**2/(2*kT))/(sqrt(pi)*kT**(3/2)),
(v, 0, oo))
>>> E(m*v**2/2) # mean energy
3*kT/2
>>> solve(density(v)(t).diff(t), t)[0] # most probable velocity
sqrt(2)*sqrt(kT)/sqrt(m)
```

More information on the `sympy.stats` submodule can be found in [12].

## 7 CATEGORY THEORY

SymPy includes a submodule for dealing with categories—abstract mathematical objects representing classes of structures as classes of objects (points) and morphisms (arrows) between the objects. It was designed with the following two goals in mind:

1. automatic typesetting of diagrams given by a collection of objects and of morphisms between them, and
2. specification and semi-automatic derivation of properties using commutative diagrams.

As of version 1.0, SymPy only implements the first goal, while a partially working draft of implementation of the second goal is available at <https://github.com/scolobb/sympy/tree/ct4-commutativity>.

In order to achieve the two goals, the submodule `sympy.categories` defines several classes representing some of the essential concepts: objects, morphisms, categories, and diagrams. In category theory, the inner structure of objects is often discarded in the favor of studying the properties of morphisms, so the class `Object` is essentially a synonym of the class `Symbol`. There are several morphism classes which do not have a particular internal structure either, though an exception is `CompositeMorphism`, which essentially stores a list of morphisms.

The class `Diagram` captures the properties of morphisms. This class stores a family of morphisms, the corresponding source and target objects, and, possibly, some properties of the morphisms. Generally, no restrictions are imposed on what the properties may be—for example, one might use strings of the form “forall”, “exists”, “unique”, etc. Furthermore, the morphisms of a diagram are grouped into *premises* and *conclusions* in order to be able to represent logical implications of the form “for a collection of morphisms  $P$  with properties  $p : P \rightarrow \Omega$  (the premises), there exists a collection of morphisms  $C$  with properties  $c : C \rightarrow \Omega$  (the conclusions)”, where  $\Omega$  is the universal collection of properties. Finally, the class `Category` includes a collection of diagrams which are deemed commutative and which therefore define the properties of this category.

Automatic typesetting of diagrams takes a `Diagram` and produces  $\text{\LaTeX}$  code using the `Xy-pic` package [13]. Typesetting is done in two stages: layout and generation of `Xy-pic` code. The layout stage is taken care of by the class `DiagramGrid`, which takes a `Diagram` and lays out the objects in a grid, trying to reduce the average length of the arrows in the final picture. By default, `DiagramGrid` uses a series of triangle-based heuristics to produce a rectangular grid. A linear layout can also be imposed. Furthermore, groups of objects can be given; in this case, the groups will be treated as atomic cells, and the member objects will be typeset independently of the other objects.

The second phase of diagram typesetting consists in actually drawing the picture and is carried out by the class `XypicDiagramDrawer`. An example of a diagram automatically typeset by `DiagramGrid` and `XypicDiagramDrawer` is given in Figure 1.



**Figure 1.** An automatically typeset commutative diagram typeset in `Xy-pic` automatically by `XypicDiagramDrawer`.

As far as the second main goal of `sympy.categories` is concerned, the principal idea consists in automatically deciding whether a diagram is commutative or not, given a collection of “axioms”: diagrams known to be commutative. The implementation is based on graph embeddings (injective maps): whenever an embedding of a commutative diagram into a given diagram is found, one concludes that the subdiagram is commutative. Deciding commutativity of the whole diagram is therefore based (theoretically) on finding a “cover” of the target diagram by embeddings of the axioms. The naïve implementation proved to be prohibitively slow; a better optimized version is therefore in order, as well as application of heuristics.

## 8 TENSORS

Ongoing work to provide the capabilities of tensor computer algebra has so far produced the `sympy.tensor` submodule. It comprises three submodules whose purposes are quite different: `sympy.tensor.indexed` and `sympy.tensor.indexed_methods` support indexed symbols, `sympy.`



321 `tensor.array` contains facilities to operate on symbolic  $N$ -dimensional arrays, and finally `sympy`.  
 322 `tensor.tensor` is used to define abstract tensors. The abstract tensors submodule is inspired  
 323 by xAct [9] and Cadabra [11]. Canonicalization based on the Butler-Portugal [8] algorithm  
 324 is supported in SymPy. Tensor support in SymPy is currently limited to polynomial tensor  
 325 expressions.

## 326 9 NUMERICAL SIMPLIFICATION

327 The `nsimplify` function in SymPy (a wrapper of `identify` in `mpmath`) attempts to find a simple  
 328 symbolic expression that evaluates to the same numerical value as the given input. It works  
 329 by applying a few simple transformations (including square roots, reciprocals, logarithms and  
 330 exponentials) to the input and, for each transformed value, using the PSLQ algorithm [5] to  
 331 search for a matching algebraic number or optionally a linear combination of user-provided base  
 332 constants (such as  $\pi$ ).

```
333 >>> t = 1 / (sin(pi/5)+sin(2*pi/5)+sin(3*pi/5)+sin(4*pi/5))*2
334 >>> nsimplify(t)
335 -2*sqrt(5)/5 + 1
336 >>> nsimplify(pi, tolerance=0.01)
337 22/7
338 >>> nsimplify(1.783919626661888, [pi], tolerance=1e-12)
339 pi/(-1/3 + 2*pi/3)
```

## 340 10 EXAMPLES

341 This section provides some additional examples for the features listed in the paper.

### 342 10.1 Simplification

- 343 • `expand`:

```
344 >>> expand((x + y)**3)
345 x**3 + 3*x**2*y + 3*x*y**2 + y**3
```

- 346 • `factor`:

```
347 >>> factor(x**3 + 3*x**2*y + 3*x*y**2 + y**3)
348 (x + y)**3
```

- 349 • `collect`:

```
350 >>> collect(y*x**2 + 3*x**2 - x*y + x - 1, x)
351 x**2*(y + 3) + x*(-y + 1) - 1
```

- 352 • `cancel`:

```
353 >>> cancel((x**2 + 2*x + 1)/(x**2 - 1))
354 (x + 1)/(x - 1)
```

- 355 • `apart`:

```
356 >>> apart((x**3 + 4*x - 1)/(x**2 - 1))
357 x + 3/(x + 1) + 2/(x - 1)
```

- 358 • `trigsimp`:

```
359 >>> trigsimp(cos(x)**2*tan(x) - sin(2*x))
360 -sin(2*x)/2
```

- 361 • `hyperexpand` (showing  ${}_2F_1\left(\begin{smallmatrix} 1, 1 \\ 2 \end{smallmatrix} \middle| -x \right) = \frac{\log(x+1)}{x}$ ):

```
362 >>> hyperexpand(hyper([1, 1], [2], -x))
363 log(x + 1)/x
```

## 364 10.2 Polynomials

- 365 • Factorization:

```
366 >>> t = symbols('t')
367 >>> f = (2115*x**4*y + 45*x**3*z**3*t**2 - 45*x**3*t**2 -
368 ...      423*x*y**4 - 47*x*y**3 + 141*x*y*z**3 + 94*x*y*z*t -
369 ...      9*y**3*z**3*t**2 + 9*y**3*t**2 - y**2*z**3*t**2 +
370 ...      y**2*t**2 + 3*z**6*t**2 + 2*z**4*t**3 - 3*z**3*t**2 -
371 ...      2*z*t**3)
372 >>> factor(f)
373 (t**2*z**3 - t**2 + 47*x*y)*(2*t*z + 45*x**3 - 9*y**3 - y**2 +
374 3*z**3)
```

- 375 • Gröbner bases:

```
376 >>> x0, x1, x2 = symbols('x0 x1 x2')
377 >>> I = [x0 + 2*x1 + 2*x2 - 1,
378 ...      x0**2 + 2*x1**2 + 2*x2**2 - x0,
379 ...      2*x0*x1 + 2*x1*x2 - x1]
380 >>> groebner(I, order='lex')
381 GroebnerBasis([7*x0 - 420*x2**3 + 158*x2**2 + 8*x2 - 7,
382 7*x1 + 210*x2**3 - 79*x2**2 + 3*x2,
383 84*x2**4 - 40*x2**3 + x2**2 + x2], x0, x1, x2, domain='ZZ',
384 order='lex')
```

- 385 • Root isolation:

```
386 >>> f = 7*z**4 - 19*z**3 + 20*z**2 + 17*z + 20
387 >>> intervals(f, all=True, eps=0.001)
388 ([],
389 [((-425/1024 - 625*I/1024, -1485/3584 - 2185*I/3584), 1),
390 [(-425/1024 + 2185*I/3584, -1485/3584 + 625*I/1024), 1),
391 [(3175/1792 - 2605*I/1792, 1815/1024 - 10415*I/7168), 1),
392 [(3175/1792 + 10415*I/7168, 1815/1024 + 2605*I/1792), 1]])
```

## 393 10.3 Solvers

- 394 • Single solution:

```
395 >>> solveset(x - 1, x)
396 {1}
```

- 397 • Finite solution set, quadratic equation:

```
398 >>> solveset(x**2 - pi**2, x)
399 {-pi, pi}
```

- 400 • No solution:

```
401 >>> solveset(1, x)
402 EmptySet()
```

- 403 • Interval solution:

```
404 >>> solveset(x**2 - 3 > 0, x, domain=S.Reals)
405 (-oo, -sqrt(3)) U (sqrt(3), oo)
```

- 406 • Infinitely many solutions:

```
407 >>> solveset(x - x, x, domain=S.Reals)
408 (-oo, oo)
409 >>> solveset(x - x, x, domain=S.Complexes)
410 S.Complexes
```

- 411 • Linear systems (linsolve)

```
412 >>> A = Matrix([[1, 2, 3], [4, 5, 6], [7, 8, 10]])
413 >>> b = Matrix([3, 6, 9])
414 >>> linsolve((A, b), x, y, z)
415 {(-1, 2, 0)}
416 >>> linsolve(Matrix((([1, 1, 1, 1], [1, 1, 2, 3])), (x, y, z))
417 {(-y - 1, y, 2)}
```

418 Below are examples of `solve` applied to problems not yet handled by `solveset`.

- 419 • Nonlinear (multivariate) system of equations (the intersection of a circle and a parabola):

```
420 >>> solve([x**2 + y**2 - 16, 4*x - y**2 + 6], x, y)
421 [(-2 + sqrt(14), -sqrt(-2 + 4*sqrt(14))),
422  (-2 + sqrt(14), sqrt(-2 + 4*sqrt(14))),
423  (-sqrt(14) - 2, -I*sqrt(2 + 4*sqrt(14))),
424  (-sqrt(14) - 2, I*sqrt(2 + 4*sqrt(14)))]
```

- 425 • Transcendental equations:

```
426 >>> solve((x + log(x))**2 - 5*(x + log(x)) + 6, x)
427 [LambertW(exp(2)), LambertW(exp(3))]
428 >>> solve(x**3 + exp(x))
429 [-3*LambertW((-1)**(2/3)/3)]
```

## 430 10.4 Matrices

- 431 • Matrix expressions

```
432 >>> m, n, p = symbols('m n p', integer=True)
433 >>> R = MatrixSymbol('R', m, n)
434 >>> S = MatrixSymbol('S', n, p)
435 >>> T = MatrixSymbol('T', m, p)
436 >>> U = R*S + 2*T
437 >>> U.shape
438 (m, p)
439 >>> U[0, 1]
440 2*T[0, 1] + Sum(R[0, _k]*S[_k, 1], (_k, 0, n - 1))
```

- 441 • Block Matrices

```
442 >>> n, m, l = symbols('n m l')
443 >>> X = MatrixSymbol('X', n, n)
444 >>> Y = MatrixSymbol('Y', m, m)
445 >>> Z = MatrixSymbol('Z', n, m)
446 >>> B = BlockMatrix([[X, Z], [ZeroMatrix(m, n), Y]])
447 >>> B
448 Matrix([
449 [X, Z],
```

```

450     [0, Y]])
451     >>> B[0, 0]
452     X[0, 0]
453     >>> B.shape
454     (m + n, m + n)

```

## 455 11 SYMPY GAMMA

456 **SymPy Gamma** is a simple web application that runs on the Google App Engine [3]. It executes  
 457 and displays the results of SymPy expressions as well as additional related computations, in  
 458 a fashion similar to that of Wolfram|Alpha. For instance, entering an integer will display its  
 459 prime factors, digits in the base-10 expansion, and a factorization diagram. Entering a function  
 460 will display its docstring; in general, entering an arbitrary expression will display its derivative,  
 461 integral, series expansion, plot, and roots.

462 SymPy Gamma also has several features beyond just computing the results using SymPy.

- 463 • SymPy Gamma displays integration and differentiation steps in detail, as demonstrated in  
 464 Figure 2.

Integral steps:

1. Rewrite the integrand:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

2. Let  $u = \cos(x)$ .

Then let  $du = -\sin(x)dx$  and substitute  $du$ :

$$\int -\frac{1}{u} du$$

- A. The integral of a constant times a function is the constant times the integral of the function:

$$\int -\frac{1}{u} du = - \int \frac{1}{u} du$$

- I. The integral of  $\frac{1}{u}$  is  $\log(u)$ .

So, the result is:  $-\log(u)$

Now substitute  $u$  back in:

$$-\log(\cos(x))$$

3. Add the constant of integration:

$$-\log(\cos(x)) + \text{constant}$$

The answer is:

$$-\log(\cos(x)) + \text{constant}$$

465

466

**Figure 2.** Integral steps of  $\tan(x)$ .

- 467 • SymPy Gamma displays the factor tree diagrams for different numbers.
- 468 • SymPy Gamma saves user search queries.

469 Every input query from the user on SymPy Gamma is first parsed by its own parser capable of  
 470 handling several different forms of function names which SymPy as a library does not support.  
 471 For instance, SymPy Gamma supports queries like `sin x`, whereas SymPy will only recognise  
 472 `sin(x)`.

473 This parser converts the input query to the equivalent SymPy readable code, which is then  
474 processed by SymPy, and the result is finally printed with the built-in MathJax [2] output and  
475 rendered by the SymPy Gamma web application.

## 476 12 COMPARISON WITH MATHEMATICA

477 Wolfram Mathematica is a popular proprietary CAS that features highly advanced algorithms,  
478 has a core written in C++ [15], and interprets its own programming language, Wolfram Language.

479 Analogous to Lisp S-expressions, Mathematica uses its own style of M-expressions, which  
480 are arrays of either atoms or other M-expressions. The first element of the expression identifies  
481 the type of the expression and is indexed by zero, and the first argument is indexed starting  
482 with one. In SymPy, expression arguments are stored in a Python tuple (that is, an immutable  
483 array), while the expression type is identified by the type of the object storing the expression.

484 Mathematica can associate attributes to its atoms. Attributes may define mathematical  
485 properties and behavior of the nodes associated to the atom. In SymPy, the usage of static class  
486 fields is roughly similar to Mathematica's attributes, though other programming patterns may  
487 also be used to achieve an equivalent behavior such, as class inheritance.

488 Unlike SymPy, Mathematica's expressions are mutable: one can change parts of the expression  
489 tree without the needing to create a new object. The mutability of Mathematica expressions  
490 allows for a lazy updating of any references to a given data structure.

491 Products in Mathematica are determined by some built in node types, such as `Times`, `Dot`,  
492 and others. `Times` is a representation of the `*` operator, and is always meant to represent a  
493 commutative product operator. The other notable product is `Dot`, which represents the `.` operator.  
494 This product represents matrix multiplication. It is not commutative. Unlike Mathematica,  
495 SymPy determines commutativity with respect to multiplication from the expression type of the  
496 factors. Mathematica puts the `Orderless` attribute on the expression type.

497 Regarding associative expressions, SymPy handles associativity of sums and products by  
498 automatically flattening them. Mathematica specifies the `Flat` attribute on the expression type.

499 Mathematica relies heavily on pattern matching—even the so-called equivalent of function  
500 declaration is in reality the definition of a pattern generating an expression tree transformation  
501 on input expressions. Mathematica's pattern matching is sensitive to associative, commutative,  
502 and one-identity properties of its expression tree nodes. SymPy has various ways to perform  
503 pattern matching. All of them play a lesser role in the CAS than in Mathematica and are  
504 basically available as a tool to rewrite expressions. The differential equation solver in SymPy  
505 somewhat relies on pattern matching to identify differential equation types, but it is envisaged to  
506 replace that strategy with analysis of Lie symmetries in the future. Mathematica's real advantage  
507 is the ability to add (at runtime) new overloading to the expression builder or specific subnodes.  
508 Consider for example:

```
509 In[1]:= Unprotect[Plus]
510 Out[1]= {Plus}
511
512 In[2]:= Sin[x_]^2 + Cos[y_]^2 := 1
513
514 In[3]:= x + Sin[t]^2 + y + Cos[t]^2
515 Out[3]= 1 + x + y
```

516 This expression in Mathematica defines a substitution rule that overloads the functionality of  
517 the `Plus` node (the node for additions in Mathematica). A symbol with a trailing underscore is  
518 treated as a wildcard. Although one may wish to keep this identity unevaluated, this example  
519 clearly illustrates the potential to define one's own immediate transformation rules. In SymPy,  
520 the operations constructing the addition node in the expression tree are Python class constructors  
521 and cannot be modified at runtime.<sup>2</sup> The way SymPy deals with extending the missing runtime  
522 overloadability is by subclassing the node types: subclasses may redefine the class constructor to  
523 yield the proper extended functionality.

---

<sup>2</sup>Python does support monkey patching, but it is a discouraged programming pattern.

524 Unlike SymPy, Mathematica does not support type inheritance or polymorphism [4]. SymPy  
 525 relies heavily on class inheritance, but for the most part, class inheritance is used to make sure  
 526 that SymPy objects inherit the proper methods and implement the basic hashing system.

527 While Mathematica interprets nested lists as matrices whenever the sublists have the same  
 528 length, matrices in SymPy are a type in their own right, allowing ordinary operators and functions  
 529 (like multiplication and exponentiation) to be used as they traditionally are in mathematics.

```
530 >>> a, b = symbols('a b')
531 >>> exp(Matrix([[1, 1], [0, 2]])) * Matrix([a, b])
532 Matrix([
533 [E*a + b*(-E + exp(2))],
534 [
           b*exp(2)]])
```

535 Using the standard multiplication in Mathematica performs an element-wise product and  
 536 calling the exponential function `Exp` on a matrix returns an element-wise exponentiation of its  
 537 elements.

538 Unevaluated expressions in Mathematica can be achieved in various ways, most commonly  
 539 with the `HoldForm` or `Hold` nodes, that block the evaluation of subnodes by the parser. Such a  
 540 node cannot be expressed in Python because of greedy evaluation. Whenever needed in SymPy,  
 541 it is necessary to add the parameter `evaluate=False` to all subnodes.

542 In Mathematica, the operator `==` returns a boolean whenever it is able to immediately evaluate  
 543 the truth of the equality, otherwise it returns an `Equal` expression. In SymPy, `==` means structural  
 544 equality and is always guaranteed to return a boolean expression. To express a mathematical  
 545 equality in SymPy it is necessary to explicitly construct an instance of the `Equality` class.

546 SymPy uses `**` to express the power operator, while Mathematica uses `^`.

547 SymPy's use of floating-point numbers is similar to that of most other CAS's, including  
 548 Maple and Maxima. By contrast, Mathematica uses a form of significance arithmetic [14]  
 549 for approximate numbers. This offers further protection against numerical errors, although it  
 550 comes with its own set of problems (for a critique of significance arithmetic, see Fateman [4]).  
 551 Internally, SymPy's `evalf` method works similarly to Mathematica's significance arithmetic, but  
 552 the semantics are isolated from the rest of the system.

## 553 REFERENCES

- 554 [1] Brent, R. P. and Zimmermann, P. (2010). *Modern Computer Arithmetic*. Cambridge  
 555 Monographs on Computational and Applied Mathematics. Cambridge University Press, version  
 556 0.5.1 edition.
- 557 [2] Cervone, D. (2012). Mathjax: a platform for mathematics on the web. *Notices of the AMS*,  
 558 59(2):312–316.
- 559 [3] Ciurana, E. (2009). Google app engine. *Developing with Google App Engine*, pages 1–10.
- 560 [4] Fateman, R. J. (1992). A review of Mathematica. *Journal of Symbolic Computation*,  
 561 13(5):545–579.
- 562 [5] Ferguson, H. R. P., Bailey, D. H., and Arno, S. (1999). Analysis of PSLQ, an integer relation  
 563 finding algorithm. *Mathematics of Computation*, 68(225):351–369.
- 564 [6] Gruntz, D. (1996). *On Computing Limits in a Symbolic Manipulation System*. PhD thesis,  
 565 Swiss Federal Institute of Technology, Zürich, Switzerland.
- 566 [7] Gruntz, D. and Koepf, W. (1993). Formal power series.
- 567 [8] Manssur, L. R. U., Portugal, R., and Svaite, B. F. (2002). Group-theoretic approach for  
 568 symbolic tensor manipulation. *International Journal of Modern Physics C*, 13.
- 569 [9] Martín-García, J. (2002-2016). xAct, efficient tensor computer algebra.
- 570 [10] Moskewicz, M., Madigan, C., and Malik, S. (2008). Method and system for efficient  
 571 implementation of boolean satisfiability. US Patent 7,418,369.
- 572 [11] Peeters, K. (2007). Cadabra: a field-theory motivated symbolic computer algebra system.  
 573 *Computer Physics Communications*.
- 574 [12] Rocklin, M. and Terrel, A. R. (2012). Symbolic statistics with SymPy. *Computing in Science  
 575 and Engineering*, 14.

- 576 [13] Rose, K. H. (1999). Xy-pic user's guide.
- 577 [14] Sofroniou, M. and Spaletta, G. (2005). Precise numerical computation. *Journal of Logic*  
578 *and Algebraic Programming*, 64(1):113–134.
- 579 [15] Wolfram, S. (2003). *The Mathematica Book*. Wolfram Media, Champaign, IL, USA, fifth  
580 edition.