

SYMPY: SYMBOLIC COMPUTING IN PYTHON

AARON MEURER*, MATEUSZ PAPROCKI†, ONDŘEJ ČERTÍK‡, MATTHEW ROCKLIN§,
 AMIT KUMAR¶, SERGIU IVANOV||, JASON K. MOORE#, SARTAJ SINGH††, THILINA
 RATHNAYAKE‡‡ SEAN VIG§§ BRIAN E. GRANGER¶¶ RICHARD P. MULLER|||
 FRANCESCO BONAZZI## HARSH GUPTA¹, SHIVAM VATS², FREDRIK JOHANSSON³,
 FABIAN PEDREGOSA⁴, ASHUTOSH SABOO⁵, ISURU FERNANDO⁶, SUMITH⁷, ROBERT
 CIMRMAN⁸, AND ANTHONY SCOPATZ⁹

1. Introduction. SymPy is a full featured computer algebra system (CAS) written in the Python programming language [33]. It is free and open source software, being licensed under the 3-clause BSD license [46]. The SymPy project was started by Ondřej Čertík in 2005, and it has since grown to over 500 contributors. Currently, SymPy is developed on GitHub using a bazaar community model [44]. The accessibility of the codebase and the open community model allow SymPy to rapidly respond to the needs of the community of users and developers.

Python is a dynamically typed programming language that has a focus on ease of use and readability. Due in part to this focus, it has become a popular language for scientific computing and data science, with a broad ecosystem of libraries [39]. SymPy is itself used by many libraries and tools to support research within a variety

*University of South Carolina, Columbia, SC 29201 (asmeurer@gmail.com).

†Continuum Analytics, Inc., Austin, TX 78701 (mattpap@gmail.com).

‡Los Alamos National Laboratory, Los Alamos, NM 87545 (certik@lanl.gov).

§Continuum Analytics, Inc., Austin, TX 78701 (mrocklin@gmail.com).

¶Delhi Technological University, Shahbad Daultpur, Bawana Road, New Delhi 110042, India (dtu.amit@gmail.com).

||Université Paris Est Créteil, 61 av. Général de Gaulle, 94010 Créteil, France (sergiu.ivanov@u-pec.fr).

#University of California, Davis, Davis, CA 95616 (jkm@ucdavis.edu).

††Indian Institute of Technology (BHU), Varanasi, Uttar Pradesh 221005, India (singhsartaj94@gmail.com).

‡‡University of Moratuwa, Bandaranayake Mawatha, Katubedda, Moratuwa 10400, Sri Lanka (thilinarmtb.10@cse.mrt.ac.lk).

§§University of Illinois at Urbana-Champaign, Urbana, IL 61801 (sean.v.775@gmail.com).

¶¶California Polytechnic State University, San Luis Obispo, CA 93407 (ellisonbg@gmail.com).

|||Center for Computing Research, Sandia National Laboratories, Albuquerque, NM 87185 (rmuller@sandia.gov).

##Max Planck Institute of Colloids and Interfaces, Am Mühlenberg 1, 14476 Potsdam, Germany (francesco.bonazzi@mpikg.mpg.de).

¹Indian Institute of Technology Kharagpur, Kharagpur, West Bengal 721302, India (hargup@protonmail.com).

²Indian Institute of Technology Kharagpur, Kharagpur, West Bengal 721302, India (shivamvats.iitkgp@gmail.com).

³INRIA Bordeaux-Sud-Ouest - LFANT project-team, 200 Avenue de la Vieille Tour, 33405 Talence, France (fredrik.johansson@gmail.com).

⁴INRIA - SIERRA project-team, 2 Rue Simone IFF, 75012 Paris, France (f@bianp.net).

⁵Birla Institute of Technology and Science, Pilani, K.K. Birla Goa Campus, NH 17B Bypass Road, Zuarinagar, Sancoale, Goa 403726, India (ashutosh.saboo@gmail.com).

⁶University of Moratuwa, Bandaranayake Mawatha, Katubedda, Moratuwa 10400, Sri Lanka (isuru.11@cse.mrt.ac.lk).

⁷Indian Institute of Technology Bombay, Powai, Mumbai, Maharashtra 400076, India (sumith@cse.iitb.ac.in).

⁸New Technologies - Research Centre, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic (cimrman3@ntc.zcu.cz).

⁹University of South Carolina, Columbia, SC 29201 (scopatz@cec.sc.edu).

domains, such as Sage [51] (pure mathematics), yt [55] (astronomy and astrophysics), PyDy [25] (multibody dynamics), and SfePy [19] (finite elements).

Unlike many CASs, SymPy does not invent its own programming language. Python itself is used both for the internal implementation and the end user interaction. The exclusive usage of a single programming language makes it easier for people already familiar with that language to use or develop SymPy. Simultaneously, it enables developers to focus on mathematics, rather than language design.

SymPy is designed with a strong focus on usability as a library. Extensibility is important in its application program interface (API) design. Thus, SymPy makes no attempt to extend the Python language itself. The goal is for users of SymPy to be able to include SymPy alongside other Python libraries in their workflow, whether that is in an interactive environment or programmatic use as part of a larger system.

As a library, SymPy does not have a built-in graphical user interface (GUI). However, SymPy exposes a rich interactive display system, including registering printers with Jupyter [41] frontends, including the Notebook and Qt Console, which will render SymPy expressions using MathJax [18] or L^AT_EX.

The remainder of this paper discusses key components of the SymPy software. Section 2 discusses the architecture of SymPy. Section 3 enumerates the features of SymPy and takes a closer look at some of the important ones. Following that, section 4 looks at the numerical features of SymPy and its dependency library, mpmath. Section 5 looks at the domain specific physics submodules for performing symbolic and numerical calculations in classical mechanics and quantum mechanics. Finally, section 6 concludes the paper and discusses future work.

2. Architecture. Software architecture is of central importance in any large software project because it establishes predictable patterns of usage and development [48]. This section describes the essential structural components of SymPy, provides justifications for the design decisions that have heretofore been made, and the provides example user-facing code as appropriate.

2.1. Basic Usage. SymPy requires that all variable names be defined prior to use because it is built on Python. The following statement imports all SymPy functions into the global Python namespace. From here on, all examples in this paper assume that this statement has been executed.

```
>>> from sympy import *
```

Symbolic variables, called symbols, must be defined and assigned to Python variables before they can be used. This is typically done through the `symbols` function, which may create multiple symbols in a single function call. For instance,

```
>>> x, y, z = symbols('x y z')
```

creates three symbols representing variables named x , y , and z . In this particular instance, these symbols are all assigned to Python variables of the same name. However, the user is free to assign them to different Python variables, while representing the same symbol, such as `a, b, c = symbols('x y z')`. In order to minimize potential confusion, though, all examples in this paper will assume that the symbols x , y , and z have been assigned to Python variables identical to their symbolic names.

Expressions are created from symbols using Python mathematical syntax. Note that in Python, exponentiation is represented by the `**` binary infix operator. For instance, the following Python code creates the expression $(x^2 - 2x + 3)/y$.

```
>>> (x**2 - 2*x + 3)/y
```

```
(x**2 - 2*x + 3)/y
```

Importantly, SymPy expressions are immutable. This simplifies the design of SymPy by allowing expression interning. It also enables expressions to be hashed and stored in Python dictionaries, thereby permitting features such as caching.

2.2. The Core. A computer algebra system (CAS) represents mathematical expressions as data structures. For example, the mathematical expression $x + y$ is represented as a tree with three nodes, $+$, x , and y , where x and y are ordered children of $+$. As users manipulate mathematical expressions with traditional mathematical syntax, the CAS manipulates the underlying data structures. Automated optimizations and computations such as integration, simplification, etc. are all functions that consume and produce expression trees.

In SymPy every symbolic expression is an instance of a Python `Basic` class, a superclass of all SymPy types providing common methods to all SymPy tree-elements, such as traversals. The children of a node in the tree are held in the `args` attribute. A terminal or leaf node in the expression tree has empty `args`.

For example, consider the expression $xy + 2$:

```
>>> expr = x*y + 2
By order of operations, the parent of the expression tree for expr is an addition, so it
is of type Add. The child nodes of expr are 2 and x*y.
>>> type(expr)
<class 'sympy.core.add.Add'>
>>> expr.args
(2, x*y)
Traversing further into the expression tree grants the full expression. For example,
the first child node, given by expr.args[0], is 2. Its class is Integer, and it has empty
an args tuple, indicating that it is a leaf node.
>>> expr.args[0]
2
>>> type(expr.args[0])
<class 'sympy.core.numbers.Integer'>
>>> expr.args[0].args
()
```

A useful way to view an expression tree is with the `srepr` function. This returns a string representation of an expression as valid Python code with all the nested class constructor calls to create the given expression.

```
>>> srepr(expr)
"Add(Mul(Symbol('x'), Symbol('y')), Integer(2))"
```

Every SymPy expression satisfies a key identity invariant:

```
expr.func(*expr.args) == expr
```

This means that expressions are rebuildable from their `args`.¹ Note that in SymPy the `==` operator represents exact structural equality, not mathematical equality. This allows any two expressions to be tested if they are equal to one another as expression trees, and not only they are equivalent.

Python allows classes to override mathematical operators. The Python interpreter translates the above $x*y + 2$ to, roughly, `(x.__mul__(y)).__add__(2)`. Both `x` and `y`, returned from the `symbols` function, are `Symbol` instances. The `2` in the expression is processed by Python as a literal, and is stored as Python's built in `int` type. When `2` is passed to the `__add__` method of `Symbol`, it is converted to the SymPy type `Integer(2)`

¹`expr.func` is used instead of `type(expr)` to allow the function of an expression to be distinct from its actual Python class. In most cases the two are the same.

before being stored in the resulting expression tree. In this way, SymPy expressions can be built in the natural way using Python operators and numeric literals.

2.3. Assumptions. SymPy performs logical inference through its assumptions system. The assumptions system allows users to specify that symbols have certain common mathematical properties, such as being positive, imaginary, or integral. SymPy is careful to never perform simplifications on an expression unless the assumptions allow them. For instance, the identity $\sqrt{t^2} = t$ holds if t is nonnegative ($t \geq 0$). If t is real, the identity $\sqrt{t^2} = |t|$ holds. However, for general complex t , no such identity holds.

By default, SymPy performs all calculations assuming that symbols are complex valued. This assumption makes it easier to treat mathematical problems in full generality.

```
>>> t = Symbol('t')
>>> sqrt(t**2)
sqrt(t**2)
```

By assuming the most general case, that symbols are complex by default, SymPy avoids performing mathematically invalid operations. However, in many cases users will wish to simplify expressions containing terms like $\sqrt{t^2}$.

Assumptions are set on `Symbol` objects when they are created. For instance `Symbol('t', positive=True)` will create a symbol named `t` that is assumed to be positive.

```
>>> t = Symbol('t', positive=True)
>>> sqrt(t**2)
t
```

Some of the common assumptions that SymPy allows are `positive`, `negative`, `real`, `nonpositive`, `nonnegative`, `real`, `integer`, and `commutative`.² Assumptions on any object can be checked with the `is_assumption` attributes, like `t.is_positive`.

Assumptions are only needed to restrict a domain so that certain simplifications can be performed. They are not required to make the domain match the input of a function. For instance, one can create the object $\sum_{n=0}^m f(n)$ as `Sum(f(n), (n, 0, m))` without setting `integer=True` when creating the `Symbol` object `n`.

The assumptions system additionally has deductive capabilities. The assumptions use a three-valued logic using the Python built in objects `True`, `False`, and `None`. `None` represents the “unknown” case. This could mean that the given assumption could be either true or false under the given information, for instance, `Symbol('x', real=True).is_positive` will give `None` because a real symbol might be positive or it might not. It could also mean not enough is implemented to compute the given fact. For instance, `(pi + E).is_irrational` gives `None`, because SymPy does not know how to determine if $\pi + e$ is rational or irrational. Indeed, this case is an open problem in mathematics.

Basic implications between the facts are used to deduce assumptions. For instance, the assumptions system knows that being an integer implies being rational, so `Symbol('x', integer=True).is_rational` returns `True`. Furthermore, expressions compute the assumptions on themselves based on the assumptions of their arguments. For instance, if `x` and `y` are both created with `positive=True`, then `(x + y).is_positive` will be `True`.

²If A and B are Symbols created with `commutative=False` then SymPy will keep $A \cdot B$ and $B \cdot A$ distinct.

2.4. Extensibility. While the core of SymPy is relatively small, it has been extended to a wide variety of domains by a broad range of contributors. This is due in part because the same language, Python, is used both for the internal implementation and the external usage by users. All of the extensibility capabilities available to users are also utilized by SymPy itself. This eases the transition pathway from SymPy user to SymPy developer.

The typical way to create a custom SymPy object is to subclass an existing SymPy class, usually `Basic`, `Expr`, or `Function`. All SymPy classes used for expression trees³ should be subclasses of the base class `Basic`, which defines some basic methods for symbolic expression trees. `Expr` is the subclass for mathematical expressions that can be added and multiplied together. Instances of `Expr` typically represent complex numbers, but may also include other “rings” like matrix expressions. Not all SymPy classes are subclasses of `Expr`. For instance, logic expressions such as `And(x, y)` are subclasses of `Basic` but not of `Expr`.

The `Function` class is a subclass of `Expr` which makes it easier to define mathematical functions called with arguments. This includes named functions like `sin(x)` and `log(x)` as well as undefined functions like `f(x)`. Subclasses of `Function` should define a class method `eval`, which returns values for which the function should be automatically evaluated, and `None` for arguments that should not be automatically evaluated.

Many SymPy functions perform various evaluations down the expression tree. Classes define their behavior in such functions by defining a relevant `_eval_*` method. For instance, an object can indicate to the `diff` function how to take the derivative of itself by defining the `_eval_derivative(self, x)` method, which may in turn call `diff` on its args. The most common `_eval_*` methods relate to the assumptions. `_eval_is_assumption` defines the assumptions for *assumption*.

As an example of the notions presented in this section, Listing 1 presents a minimal version of the gamma function $\Gamma(x)$ from SymPy, which evaluates itself on positive integer arguments, has the positive and real assumptions defined, can be rewritten in terms of factorial with `gamma(x).rewrite(factorial)`, and can be differentiated. `fdiff` is a convenience method for subclasses of `Function`. `fdiff` returns the derivative of the function without considering the chain rule. `self.func` is used throughout instead of referencing `gamma` explicitly so that potential subclasses of `gamma` can reuse the methods.

Listing 1: A stripped down version of `sympy.gamma`.

```

194 from sympy import Integer, Function, floor, factorial, polygamma
195
196 class gamma(Function)
197     @classmethod
198     def eval(cls, arg):
199         if isinstance(arg, Integer) and arg.is_positive:
200             return factorial(arg - 1)
201
202     def _eval_is_positive(self):
203         x = self.args[0]
204         if x.is_positive:

```

³Some internal classes, such as those used in the polynomial module, do not follow this rule for efficiency reasons.

```

205         return True
206     elif x.is_noninteger:
207         return floor(x).is_even
208
209     def _eval_is_real(self):
210         x = self.args[0]
211         # noninteger means real and not integer
212         if x.is_positive or x.is_noninteger:
213             return True
214
215     def _eval_rewrite_as_factorial(self, z):
216         return factorial(z - 1)
217
218     def fdiff(self, argindex=1):
219         from sympy.core.function import ArgumentIndexError
220         if argindex == 1:
221             return self.func(self.args[0])*polygamma(0, self.args[0])
222         else:
223             raise ArgumentIndexError(self, argindex)

```

224 The gamma function implemented in SymPy has many more capabilities than the
225 above listing, such as evaluation at rational points and series expansion.

226 **3. Features.** SymPy has an extensive feature set that is too encompassing for
227 complete, in-depth coverage in this paper. However, calculus and other bedrock areas
228 receive their own subsections here. Additionally, Table 1 gives a compact listing of
229 all major capabilities present in the SymPy codebase. This grants a sampling from
230 the breadth of topics and application domains that SymPy services. Unless stated
231 otherwise, all features noted in Table 1 are symbolic in nature. Numeric features are
232 discussed in Section 4.

Table 1: SymPy Features and Descriptions

Feature	Description
Calculus	Algorithms for computing derivatives, integrals, and limits.
Category Theory	Representation of objects, morphisms, and diagrams. Tools for drawing diagrams with Xy-pic.
Code Generation	Enables generation of compilable and executable code in a variety of different programming languages directly from expressions. Target languages include C, Fortran, Julia, JavaScript, Mathematica, MATLAB and Octave, Python, and Theano.
Combinatorics & Group Theory	Implements permutations, combinations, partitions, subsets, various permutation groups (such as polyhedral, Rubik, symmetric, and others), Gray codes [38], and Prufer sequences [13].
Concrete Math	Summation, products, tools for determining whether summation and product expressions are convergent, absolutely convergent, hypergeometric, and other properties. May also compute Gosper’s normal form [43] for two univariate polynomials.

Cryptography		Represents block and stream ciphers, including shift, Affine, substitution, Vigenere's, Hill's, bifid, RSA, Kid RSA, linear-feedback shift registers, and Elgamal encryption
Differential Geometry	Ge-	Classes to represent manifolds, metrics, tensor products, and coordinate systems in Riemannian and pseudo-Riemannian geometries [52].
Geometry		Allows the creation of 2D geometrical entities, such as lines and circles. Enables queries on these entities, such as asking the area of an ellipse, checking for collinearity of a set of points, or finding the intersection between two lines.
Lie Algebras		Represents Lie algebras and root systems.
Logic		Boolean expression, equivalence testing, satisfiability, and normal forms.
Matrices		Tools for creating matrices of symbols and expressions. This is capable of both sparse and dense representations and performing symbolic linear algebraic operations (e.g., inversion and factorization).
Matrix Expressions	Expres-	Matrices with symbolic dimensions (unspecified entries). Block matrices.
Number Theory		Prime number generation, primality testing, integer factorization, continued fractions, Egyptian fractions, modular arithmetic, quadratic residues, partitions, binomial and multinomial coefficients, prime number tools, and integer factorization.
Plotting		Hooks for visualizing expressions via matplotlib [30] or as text drawings when lacking a graphical back-end. 2D function plotting, 3D function plotting, and 2D implicit function plotting are supported.
Polynomials		Computes polynomial algebras over various coefficient domains. Functionality ranges from the simple (e.g., polynomial division) to the advanced (e.g., Gröbner bases [10] and multivariate factorization over algebraic number domains).
Printing		Functions for printing SymPy expressions in the terminal with ASCII or Unicode characters and converting SymPy expressions to L ^A T _E X and MathML.
Quantum Mechanics	Me-	Quantum states, bra-ket notation, operators, basis sets, representations, tensor products, inner products, outer products, commutators, anticommutators, and specific quantum system implementations.
Series		Implements series expansion, sequences, and limit of sequences. This includes Taylor, Laurent, and Puiseux series as well as special series, such as Fourier and formal power series.
Sets		Representations of empty, finite, and infinite sets. This includes special sets such as for all natural, integer, and complex numbers. Operations on sets such as union, intersection, Cartesian product, and building sets from other sets.

Simplification	Functions for manipulating and simplifying expressions. Includes algorithms for simplifying hypergeometric functions, trigonometric expressions, rational functions, combinatorial functions, square root denesting, and common subexpression elimination.
Solvers	Functions for symbolically solving equations algebraically, systems of equations, both linear and non-linear, inequalities, ordinary differential equations, partial differential equations, Diophantine equations, and recurrence relations.
Special Functions	Implements a number of well known special functions, including Dirac delta, Gamma, Beta, Gauss error functions, Fresnel integrals, Exponential integrals, Logarithmic integrals, Trigonometric integrals, Bessel, Hankel, Airy, B-spline, Riemann Zeta, Dirichlet eta, polylogarithm, Lerch transcendent, hypergeometric, elliptic integrals, Mathieu, Jacobi polynomials, Gegenbauer polynomial, Chebyshev polynomial, Legendre polynomial, Hermite polynomial, Laguerre polynomial, and spherical harmonic functions.
Statistics	Support for a random variable type as well as the ability to declare this variable from prebuilt distribution functions such as Normal, Exponential, Coin, Die, and other custom distributions [45].
Tensors	Symbolic manipulation of indexed objects.
Vectors	Provides basic vector math and differential calculus with respect to 3D Cartesian coordinate systems.

3.1. Simplification. The generic way to simplify an expression is by calling the `simplify` function. It must be emphasized that simplification is not an unambiguously defined mathematical operation [17]. The `simplify` function applies several simplification routines along with heuristics to make the output expression as “simple” as possible.

It is often preferable to apply more directed simplification functions. These apply very specific rules to the input expression and are typically able to make guarantees about the output. Take for instance the `factor` function, which given a polynomial with rational coefficients in several variables, is guaranteed to produce a factorization into irreducible factors. Table 2 lists common simplification functions.

Table 2: Some SymPy Simplification Functions

<code>expand</code>	expand the expression
<code>factor</code>	factor a polynomial into irreducibles
<code>collect</code>	collect polynomial coefficients
<code>cancel</code>	rewrite a rational function as p/q with common factors canceled
<code>apart</code>	compute the partial fraction decomposition of a rational function
<code>trigsimp</code>	simplify trigonometric expressions [23]

Substitutions are performed through the `.subs` method.

```
>>> (sin(x) + x**2 + 1).subs(x, y + 1)
```


245 `(y + 1)**2 + sin(y + 1) + 1`

246 **3.2. Calculus.** Integrals are calculated with the `integrate` function. SymPy im-
 247 plements a combination of the Risch algorithm [16], table lookups, a reimplementa-
 248 tion of Manuel Bronstein’s “Poor Man’s Integrator” [15], and an algorithm for computing
 249 integrals based on Meijer G-functions. These allow SymPy to compute a wide variety
 250 of indefinite and definite integrals.

```
251 >>> integrate(sin(x), x)
252 -cos(x)
253 >>> integrate(sin(x), (x, 0, 1))
254 -cos(1) + 1
```

255 Derivatives are computed with the `diff` function. Derivatives are computed re-
 256 cursively using the various differentiation rules.

```
257 >>> diff(sin(x)*exp(x), x)
258 exp(x)*sin(x) + exp(x)*cos(x)
```

259 Summations and products are computed with `summation` and `product`, respec-
 260 tively. Summations are computed using a combination of Gosper’s algorithm, an
 261 algorithm that uses Meijer G-functions, and heuristics. Products are computed via a
 262 suite of heuristics.

263 Limits are computed with the `limit` function. The limit module implements the
 264 Gruntz algorithm [27] for computing symbolic limits (a description of the Gruntz
 265 algorithm can be found in the supplement). For example, the following computes
 266 $\lim_{x \rightarrow \infty} x \sin(\frac{1}{x}) = 1$ (note that ∞ is `oo` in SymPy).

```
267 >>> limit(x*sin(1/x), x, oo)
268 1
```

269 As a more complex example, SymPy computes $\lim_{x \rightarrow 0} \left(2e^{\frac{1-\cos(x)}{\sin(x)}} - 1 \right)^{\frac{\sinh(x)}{\operatorname{atan}^2(x)}} = e$.

```
270 >>> limit((2*E**((1-cos(x))/sin(x))-1)**(sinh(x)/atan(x)**2), x, 0)
271 E
```

272 Integrals, derivatives, summations, products, and limits that cannot be computed
 273 return unevaluated objects. These can also be created directly if the user chooses.

```
274 >>> integrate(x**x, x)
275 Integral(x**x, x)
```

276 **3.3. Polynomials.** SymPy implements a suite of algorithms for polynomial ma-
 277 nipulation, which ranges from relatively simple algorithms for doing arithmetic of
 278 polynomials, to advanced methods for factoring multivariate polynomials into irre-
 279 ducibles, symbolically determining real and complex root isolation intervals, or com-
 280 puting Gröbner bases.

281 Polynomial manipulation is useful on its own. Within SymPy, though, it is mostly
 282 used indirectly as a tool in other areas of the library. In fact, many mathematical
 283 problems in symbolic computing are first expressed using entities from the symbolic
 284 core, preprocessed, and then transformed into a problem in the polynomial algebra,
 285 where generic and efficient algorithms are used to solve the problem and solutions to
 286 the original problem are recovered. This is a common scheme in symbolic integration
 287 or summation algorithms.

288 SymPy implements dense and sparse polynomial representations.⁴ Both are used
 289 in the univariate and multivariate cases. The dense representation is the default for

⁴In a dense representation, the coefficients for all terms up to the degree of each variable are stored in memory. In a sparse representation, only the nonzero coefficients are stored.

univariate polynomials. For multivariate polynomials, the choice of representation is based on the application. The most common case for the sparse representation is algorithms for computing Gröbner bases (Buchberger, F4, and F5). This is because different monomial orderings can be expressed easily in this representation. However, algorithms for computing multivariate GCDs or factorizations, at least those currently implemented in SymPy, are better expressed when the representation is dense. The dense multivariate representation is specifically a recursively dense representation, where polynomials in $K[x_0, x_1, \dots, x_n]$ are viewed as a polynomials in $K[x_0][x_1] \dots [x_n]$. Note that despite this, the coefficient domain K , can be a multivariate polynomial domain as well. The dense recursive representation in Python becomes inefficient when the number of variables gets high.

Some examples for the `sympy.polys` module can be found in the supplement.

3.4. Printers. SymPy has a rich collection of expression printers for displaying expressions to the user. By default, an interactive Python session will render the `str` form of an expression, which has been used in all the examples in this paper so far. The `str` form of an expression is valid Python and roughly matches what a user would type to enter the expression.

```
>>> phi0 = Symbol('phi0')
>>> str(Integral(sqrt(phi0), phi0))
'Integral(sqrt(phi0), phi0)'
```

Expressions can be printed with 2D, monospace fonts via `pprint`. This uses Unicode characters to render mathematical symbols such as integral signs, square roots, and parentheses. Greek letters and subscripts in symbol names that have Unicode code points associated with them are also rendered automatically.

```
>>> pprint(Integral(sqrt(phi0 + 1), phi0))
```

$$\int \sqrt{\phi_0 + 1} \, d(\phi_0)$$

Alternately, the `use_unicode=False` flag can be set, which causes the expression to be printed using only ASCII characters.

```
>>> pprint(Integral(sqrt(phi0 + 1), phi0), use_unicode=False)
```

```
/
|
| _____
| \ /  phi0 + 1  d(phi0)
|
/
```

The function `latex` returns a \LaTeX representation of an expression.

```
>>> print(latex(Integral(sqrt(phi0 + 1), phi0)))
\int \sqrt{\phi_0 + 1}\, d\phi_0
```

Users are encouraged to run the `init_printing` function at the beginning of interactive sessions, which automatically enables the best pretty printing supported by their environment. In the Jupyter Notebook or Qt Console [41], the \LaTeX printer is used to render expressions using MathJax or \LaTeX , if it is installed on the system. The 2D text representation is used otherwise.

Other printers such as MathML are also available. SymPy uses an extensible printer subsystem for customizing the printing for any given printer, and for custom objects to define their printing behavior for any printer. The code generation capabilities of SymPy use this subsystem to convert expressions into code in various target

336 programming languages.

337 **3.5. Solvers.** SymPy has a module of equation solvers for symbolic equations.
338 There are two functions for solving algebraic equations in SymPy: `solve` and `solveset`.
339 `solveset` has several design changes with respect to the older `solve` function. This dis-
340 tinction is present in order to resolve the usability issues with the previous `solve` func-
341 tion API while maintaining backward compatibility with earlier versions of SymPy.
342 `solveset` only requires the necessary input information from the user. The function
343 signatures of `solve` and `solveset` are

344 `solve(f, *symbols, **flags)`

345 `solveset(f, symbol, domain=S.Complexes)`

346 The `domain` parameter is typically either `S.Complexes` (the default) or `S.Reals`, which
347 causes it to only return real solutions.

348 Additionally, `solve` has an inconsistent output API for various types of inputs. For
349 instance, depending on the input, sometimes it returns a Python list and sometimes it
350 returns a Python dictionary. On the other hand, the `solveset` has a canonical output
351 API. `solveset` always returns a SymPy set object.

352 Both functions implicitly assume that expressions are equal to 0. For instance,
353 `solveset(x - 1, x)` solves $x - 1 = 0$ for x .

354 `solveset` is under active development as a planned replacement for `solve`. There
355 are certain features which are implemented in `solve` that are not yet implemented in
356 `solveset`. Notably, these include nonlinear multivariate system and transcendental
357 equations.

358 More examples of `solveset` and `solve` can be found in the supplement.

359 **3.6. Matrices.** Computations on matrices with symbolic entries are important
360 for many algorithms within SymPy, as well as being an important feature in its own
361 right.

362 `>>> A = Matrix(2, 2, [x, x + y, y, x])`

363 `>>> A`

364 `Matrix([`

365 `[x, x + y],`

366 `[y, x])`

367 SymPy matrices support common symbolic linear algebra manipulations, includ-
368 ing matrix addition, multiplication, exponentiation, computing determinants, solving
369 linear systems, and computing inverses using LU decomposition, LDL decomposi-
370 tion, Gauss-Jordan elimination, Cholesky decomposition, Moore-Penrose pseudoin-
371 verse, and adjugate matrix.

372 All operations are computed symbolically. For instance, eigenvalues are computed
373 by generating the characteristic polynomial using the Berkowitz algorithm and then
374 solving it using polynomial routines.

375 `>>> A.eigenvals()`

376 `{x - sqrt(y*(x + y)): 1, x + sqrt(y*(x + y)): 1}`

377 Internally these matrices store the elements as a list of lists (LIL), making it a
378 dense representation.⁵ For storing sparse matrices, the `SparseMatrix` class can be
379 used. Sparse matrices store the elements in a dictionary of keys (DOK) format.

380 SymPy also supports matrices with symbolic dimension values. `MatrixSymbol`
381 represents a matrix with dimensions $m \times n$, where m and n can be symbolic. Ma-

⁵Similar to the polynomials module, dense here means that all entries are stored in memory, contrasted with a sparse representation where only nonzero entries are stored.

trix addition and multiplication, scalar operations, matrix inverse, and transpose are stored symbolically as matrix expressions.

Block matrices are also implemented in SymPy. `BlockMatrix` elements can be any matrix expression, including explicit matrices, matrix symbols, and other block matrices. All functionalities of matrix expressions are also present in `BlockMatrix`.

When symbolic matrices are combined with the assumptions module for logical inference, they provide powerful reasoning over invertibility, semi-definiteness, orthogonality, etc., which are valuable in the construction of numerical linear algebra systems.

More examples for `Matrix` and `BlockMatrix` can be found in the supplement.

4. Numerics. Floating point numbers in SymPy are represented by the `Float` class, which represents an arbitrary-precision binary floating-point number by storing its value and precision (in bits). This representation is distinct from the Python built in `float` type, which is a wrapper around machine `double` types and uses a fixed precision (53-bit).

Because Python `float` literals are limited in precision, strings should be used to input precise decimal values:

```
>>> Float(1.1)
1.100000000000000
>>> Float(1.1, 30) # precision equivalent to 30 digits
1.10000000000000008881784197001
>>> Float("1.1", 30)
1.100000000000000000000000000000000
```

The `evalf` method converts a constant symbolic expression to a `Float` with the specified precision, here 25 digits:

```
>>> (pi + 1).evalf(25)
4.141592653589793238462643
```

`Float` numbers do not track their *accuracy*, and should be used with caution within symbolic expressions since familiar dangers of floating-point arithmetic apply [26]. A notorious case is that of catastrophic cancellation:

```
>>> cos(exp(-100)).evalf(25) - 1
0
```

Applying the `evalf` method to the whole expression solves this problem. Internally, `evalf` estimates the number of accurate bits of the floating-point approximation for each sub-expression, and adaptively increases the working precision until the estimated accuracy of the final result matches the sought number of decimal digits:

```
>>> (cos(exp(-100)) - 1).evalf(25)
-6.919482633683687653243407e-88
```

The `evalf` method works with complex numbers and supports more complicated expressions, such as special functions, infinite series, and integrals. The internal error tracking does not provide rigorous error bounds (in the sense of interval arithmetic) and cannot be used to accurately track uncertainty in measurement data; the sole purpose is to mitigate loss of accuracy that typically occurs when converting symbolic expressions to numerical values.

4.1. The mpmath library. The implementation of arbitrary-precision floating-point arithmetic is supplied by the `mpmath` library, which originally was developed as a SymPy module but subsequently has been moved to a standalone pure Python package. The basic datatypes in `mpmath` are `mpf` and `mpc`, which respectively act as multiprecision substitutes for Python's `float` and `complex`. The floating-point

tum mechanics along with support for manipulating physical quantities with units.

5.1. Classical Mechanics.

5.1.1. Vector Algebra. The `sympy.physics.vector` package provides reference frame, time, and space aware vector and dyadic objects that allow for three-dimensional operations such as addition, subtraction, scalar multiplication, inner and outer products, and cross products. Both of these objects can be written in very compact notation that make it easy to express the vectors and dyadics in terms of multiple reference frames with arbitrarily defined relative orientations. The vectors are used to specify the positions, velocities, and accelerations of points, orientations, angular velocities, and angular accelerations of reference frames, and force and torques. The dyadics are essentially reference frame aware 3×3 tensors. The vector and dyadic objects can be used for any one-, two-, or three-dimensional vector algebra and they provide a strong framework for building physics and engineering tools.

The following Python code demonstrates how a vector is created using the orthogonal unit vectors of three reference frames that are oriented with respect to each other and the result of expressing the vector in the A frame. The B frame is oriented with respect to the A frame using Z-X-Z Euler Angles of magnitude π , $\frac{\pi}{2}$, and $\frac{\pi}{3}$ rad, respectively, whereas the C frame is oriented with respect to the B frame through a simple rotation about the B frame's X unit vector through $\frac{\pi}{2}$ rad.

```
>>> from sympy.physics.vector import ReferenceFrame
>>> A = ReferenceFrame('A')
>>> B = ReferenceFrame('B')
>>> C = ReferenceFrame('C')
>>> B.orient(A, 'body', (pi, pi / 3, pi / 4), 'zxz')
>>> C.orient(B, 'axis', (pi / 2, B.x))
>>> v = 1 * A.x + 2 * B.z + 3 * C.y
>>> v
A.x + 2*B.z + 3*C.y
>>> v.express(A)
A.x + 5*sqrt(3)/2*A.y + 5/2*A.z
```

5.1.2. Mechanics. The `sympy.physics.mechanics` package utilizes the `sympy.physics.vector` package to populate time aware particle and rigid body objects to fully describe the kinematics and kinetics of a rigid multi-body system. These objects store all of the information needed to derive the ordinary differential or differential algebraic equations that govern the motion of the system, i.e., the equations of motion. These equations of motion abide by Newton's laws of motion and can handle arbitrary kinematic constraints or complex loads. The package offers two automated methods for formulating the equations of motion based on Lagrangian Dynamics [32] and Kane's Method [31]. Lastly, there are automated linearization routines for constrained dynamical systems [42].

5.2. Quantum Mechanics. The `sympy.physics.quantum` package has extensive capabilities for performing symbolic quantum mechanics, using Python objects to represent the different mathematical objects relevant in quantum theory [47]: states (bras and kets), operators (unitary, Hermitian, etc.), and basis sets, as well as operations on these objects such as representations, tensor products, inner products, outer products, commutators, and anticommutators. The base objects are designed in the most general way possible to enable any particular quantum system to be implemented by subclassing the base operators and defining the relevant class methods to provide

```

528 system specific logic.
529     Symbolic quantum operators and states may be defined, and one can perform a
530 full range of operations with them:
531 >>> from sympy.physics.quantum import Commutator, Dagger, Operator
532 >>> from sympy.physics.quantum import Ket, qapply
533 >>> A = Operator('A')
534 >>> B = Operator('B')
535 >>> C = Operator('C')
536 >>> D = Operator('D')
537 >>> a = Ket('a')
538 >>> comm = Commutator(A, B)
539 >>> comm
540 [A,B]
541 >>> qapply(Dagger(comm*a)).doit()
542 -<a|*(Dagger(A)*Dagger(B) - Dagger(B)*Dagger(A))
543 Commutators can be expanded using common commutator identities:
544 >>> Commutator(C+B, A*D).expand(commutator=True)
545 -[A,B]*D - [A,C]*D + A*[B,D] + A*[C,D]
546     On top of this set of base objects, a number of specific quantum systems have
547 been implemented in a fully symbolic framework. These include:
548     • Many of the exactly solvable quantum systems, including simple harmonic
549 oscillator states and raising/lowering operators, infinite square well states,
550 and 3D position and momentum operators and states.
551     • Second quantized formalism of non-relativistic many-body quantum mechan-
552 ics [22].
553     • Quantum angular momentum [56]. Spin operators and their eigenstates can
554 be represented in any basis and for any quantum numbers. A rotation opera-
555 tor representing the Wigner-D matrix, which may be defined symbolically or
556 numerically, is also implemented to rotate spin eigenstates. Functionality for
557 coupling and uncoupling of arbitrary spin eigenstates is provided, including
558 symbolic representations of Clebsch-Gordon coefficients and Wigner symbols.
559     • Quantum information and computing [37]. Multidimensional qubit states,
560 and a full set of one- and two-qubit gates are provided and can be represented
561 symbolically or as matrices/vectors. With these building blocks, it is possible
562 to implement a number of basic quantum algorithms including the quantum
563 Fourier transform, quantum error correction, quantum teleportation, Grover's
564 algorithm, dense coding, etc. In addition, any quantum circuit may be plotted
565 using the circuit_plot function (Figure 1).
566     Here are a few short examples of the quantum information and computing capa-
567 bilities in sympy.physics.quantum. Start with a simple four-qubit state and flip the
568 second qubit from the right using a Pauli-X gate:
569 >>> from sympy.physics.quantum.qubit import Qubit
570 >>> from sympy.physics.quantum.gate import XGate
571 >>> q = Qubit('0101')
572 >>> q
573 |0101>
574 >>> X = XGate(1)
575 >>> qapply(X*q)
576 |0111>
577 Qubit states can also be used in adjoint operations, tensor products, inner/outer

```



```

578 products:
579 >>> Dagger(q)
580 <0101|
581 >>> ip = Dagger(q)*q
582 >>> ip
583 <0101|0101>
584 >>> ip.doit()
585 1
586 Quantum gates (unitary operators) can be applied to transform these states and then
587 classical measurements can be performed on the results:
588 >>> from sympy.physics.quantum.qubit import measure_all
589 >>> from sympy.physics.quantum.gate import H, X, Y, Z
590 >>> c = H(0)*H(1)*Qubit('00')
591 >>> c
592 H(0)*H(1)*|00>
593 >>> q = qapply(c)
594 >>> measure_all(q)
595 [(|00>, 1/4), (|01>, 1/4), (|10>, 1/4), (|11>, 1/4)]

```

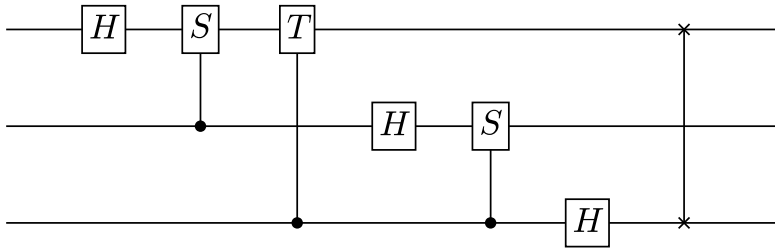


Fig. 1: The circuit diagram for a three-qubit quantum Fourier transform generated by SymPy.

```

596 Lastly, the following example demonstrates creating a three-qubit quantum Fourier
597 transform, decomposing it into one- and two-qubit gates, and then generating a circuit
598 plot for the sequence of gates (see Figure 1).
599 >>> from sympy.physics.quantum.qft import QFT
600 >>> from sympy.physics.quantum.circuitplot import circuit_plot
601 >>> fourier = QFT(0,3).decompose()
602 >>> fourier
603 SWAP(0,2)*H(0)*C((0),S(1))*H(1)*C((0),T(2))*C((1),S(2))*H(2)
604 >>> c = circuit_plot(fourier, nqubits=3)

```

605 **6. Conclusion and future work.** SymPy is a robust computer algebra system
606 that provides a wide spectrum of features both in traditional computer algebra and
607 in a plethora of scientific disciplines. This allows SymPy to be used in a first-class
608 way with other Python projects, including the scientific Python stack. Unlike many
609 other CASs, SymPy is designed to be used in an extensible way: both as an end-user

application and as a library.

SymPy expressions are immutable trees of Python objects. SymPy uses Python both as the internal language and the user language. This permits users to access to the same methods that the library implements in order to extend it for their needs. Additionally, SymPy has a powerful assumptions system for declaring and deducing mathematical properties on expressions.

SymPy has submodules for many areas of mathematics. This includes functions for simplifying expressions, performing common calculus operations, pretty printing expressions, solving equations, and representing symbolic matrices. Other included areas are discrete math, concrete math, plotting, geometry, statistics, polynomials, sets, series, vectors, combinatorics, group theory, code generation, tensors, Lie algebras, cryptography, and special functions. Additionally, SymPy contains submodules targeting certain specific domains, such as classical mechanics and quantum mechanics. This breadth of domains has been engendered by a strong and vibrant user community. Anecdotally, these users likely chose SymPy because of its ease of access.

Some of the planned future work for SymPy includes work on improving code generation, improvements to the speed of SymPy, improving the assumptions system, and improving the solvers module.

Work is being done on an assumptions subsystem, distinct from the one discussed in section 2.3. The new system stores assumption predicates separate from objects, and uses a SAT solver to do inference.

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9. Supplement.

9.1. Limits: The Gruntz Algorithm. SymPy calculates limits using the Gruntz algorithm, as described in [27]. The basic idea is as follows: any limit can be converted to a limit $\lim_{x \rightarrow \infty} f(x)$ by substitutions like $x \rightarrow \frac{1}{x}$. Then the most varying subexpression ω (that converges to zero as $x \rightarrow \infty$ the fastest from all subexpressions) is identified in $f(x)$, and $f(x)$ is expanded into a series with respect to ω . Any positive powers of ω converge to zero. If there are negative powers of ω , then the limit is infinite. The constant term (independent of ω , but could depend on x) then determines the limit (one might need to recursively apply the Gruntz algorithm on this term to determine the limit).

To determine the most varying subexpression, the comparability classes must first be defined, by calculating L :

$$(1) \quad L \equiv \lim_{x \rightarrow \infty} \frac{\log |f(x)|}{\log |g(x)|}$$

And then operations $<$, $>$ and \sim are defined as follows: $f > g$ when $L = \pm\infty$ (it is said that f is more rapidly varying than g , i.e., f goes to ∞ or 0 faster than g , f is greater than any power of g), $f < g$ when $L = 0$ (f is less rapidly varying than g) and $f \sim g$ when $L \neq 0, \pm\infty$ (both f and g are bounded from above and below by suitable integral powers of the other). Here are some examples of comparability classes:

$$\begin{aligned} 2 &< x < e^x < e^{x^2} < e^{e^x} \\ 2 &\sim 3 \sim -5 \\ x &\sim x^2 \sim x^3 \sim \frac{1}{x} \sim x^m \sim -x \\ e^x &\sim e^{-x} \sim e^{2x} \sim e^{x+e^{-x}} \end{aligned}$$

$$f(x) \sim \frac{1}{f(x)}$$

The Gruntz algorithm is now illustrated on the following example:

$$(2) \quad f(x) = e^{x+2e^{-x}} - e^x + \frac{1}{x}.$$

The goal is to calculate $\lim_{x \rightarrow \infty} f(x)$. First the set of most rapidly varying subexpressions is determined, the so called *mrsv set*. For (2), the following mrsv set $\{e^x, e^{-x}, e^{x+2e^{-x}}\}$ is obtained. These are all subexpressions of (2) and they all belong to the same comparability class. This calculation can be done using SymPy as follows:

```
>>> from sympy.series.gruntz import mrsv
>>> mrsv(exp(x+2*exp(-x))-exp(x) + 1/x, x)[0].keys()
dict_keys([exp(x + 2*exp(-x)), exp(x), exp(-x)])
```

Next any item ω is taken from mrsv that converges to zero for $x \rightarrow \infty$. The item $\omega = e^{-x}$ is obtained. If such a term is not present in the mrsv set (i.e., all terms converge to infinity instead of zero), the relation $f(x) \sim \frac{1}{f(x)}$ can be used.

Next step is to rewrite the mrsv in terms of ω : $\{\frac{1}{\omega}, \omega, \frac{1}{\omega}e^{2\omega}\}$. Then the original subexpressions are substituted back into $f(x)$ and expanded with respect to ω :

$$(3) \quad f(x) = \frac{1}{x} - \frac{1}{\omega} + \frac{1}{\omega}e^{2\omega} = 2 + \frac{1}{x} + 2\omega + O(\omega^2)$$

Since ω is from the mrsv set, then in the limit $x \rightarrow \infty$ it is $\omega \rightarrow 0$ and so $2\omega + O(\omega^2) \rightarrow 0$ in (3):

$$(4) \quad f(x) = \frac{1}{x} - \frac{1}{\omega} + \frac{1}{\omega}e^{2\omega} = 2 + \frac{1}{x} + 2\omega + O(\omega^2) \rightarrow 2 + \frac{1}{x}$$

Since the result $(2 + \frac{1}{x})$ still depends on x , the above procedure is iterated on the result until just a number (independent of x) is obtained, which is the final limit. In the above case the limit is 2, as can be verified by SymPy:

```
>>> limit(exp(x+2*exp(-x))-exp(x) + 1/x, x, oo)
2
```

In general, when $f(x)$ is expanded in terms of ω , it is obtained:

$$(5) \quad f(x) = \underbrace{O\left(\frac{1}{\omega^3}\right)}_{\infty} + \underbrace{\frac{C_{-2}(x)}{\omega^2}}_{\infty} + \underbrace{\frac{C_{-1}(x)}{\omega}}_{\infty} + \underbrace{C_0(x)}_0 + \underbrace{C_1(x)\omega}_{0} + \underbrace{O(\omega^2)}_0$$

The positive powers of ω are zero. If there are any negative powers of ω , then the result of the limit is infinity, otherwise the limit is equal to $\lim_{x \rightarrow \infty} C_0(x)$. The expression $C_0(x)$ is simpler than $f(x)$ and so the algorithm always converges. A proof of this, as well as further details are given in Gruntz's PhD thesis [27].

9.2. Series.

9.2.1. Series Expansion. SymPy is able to calculate the symbolic series expansion of an arbitrary series or expression involving elementary and special functions and multiple variables. For this it has two different implementations: the `series` method and `Ring Series`.

The first approach stores a series as an object of the `Basic` class. Each function has its specific implementation of its expansion which is able to evaluate the Puiseux series expansion about a specified point. For example, consider a Taylor expansion about 0:

```

801 >>> from sympy import symbols, series
802 >>> x, y = symbols('x, y')
803 >>> series(sin(x+y) + cos(x*y), x, 0, 2)
804 1 + sin(y) + x*cos(y) + O(x**2)

```

805 The newer and much faster^[1] approach called Ring Series makes use of the ob-
806 servation that a truncated Taylor series, is in fact a polynomial. Ring Series uses the
807 efficient representation and operations of sparse polynomials. The choice of sparse
808 polynomials is deliberate as it performs well in a wider range of cases than a dense
809 representation. Ring Series gives the user the freedom to choose the type of coeffi-
810 cients he wants to have in his series, allowing the use of faster operations on certain
811 types.

812 For this, several low level methods for expansion of trigonometric, hyperbolic and
813 other elementary functions like inverse of a series, calculating n th root, etc, are imple-
814 mented using variants of the Newton Method [14]. All these support Puiseux series
815 expansion. The following example demonstrates the use of an elementary function
816 that calculates the Taylor expansion of the sine of a series.

```

817 >>> from sympy import ring
818 >>> from sympy.polys.ring_series import rs_sin
819 >>> R, t = ring('t', QQ)
820 >>> rs_sin(t**2 + t, t, 5)
821 -1/2*t**4 - 1/6*t**3 + t**2 + t

```

822 The function `sympy.polys.rs_series` makes use of these elementary functions to
823 expand an arbitrary SymPy expression. It does so by following a recursive strategy
824 of expanding the lower most functions first and then composing them recursively to
825 calculate the desired expansion. Currently, it only supports expansion about 0 and
826 is under active development. Ring Series is several times faster than the default
827 implementation with the speed difference increasing with the size of the series. The
828 `sympy.polys.rs_series` takes as input any SymPy expression and hence there is no
829 need to explicitly create a polynomial ring. An example demonstrating its use:

```

830 >>> from sympy.polys.ring_series import rs_series
831 >>> from sympy.abc import a, b
832 >>> from sympy import sin, cos
833 >>> rs_series(sin(a + b), a, 4)
834 -1/2*(sin(b))*a**2 + (sin(b)) - 1/6*a**3*(cos(b)) + a*(cos(b))

```

835 **9.2.2. Formal Power Series.** SymPy can be used for computing the Formal
836 Power Series of a function. The implementation is based on the algorithm described
837 in the paper on Formal Power Series [28]. The advantage of this approach is that an
838 explicit formula for the coefficients of the series expansion is generated rather than
839 just computing a few terms.

840 The following example shows how to use `fps`:

```

841 >>> f = fps(sin(x), x, x0=0)
842 >>> f.truncate(6)
843 x - x**3/6 + x**5/120 + O(x**6)
844 >>> f[15]
845 -x**15/1307674368000

```

846 **9.2.3. Fourier Series.** SymPy provides functionality to compute Fourier se-
847 ries of a function using the `fourier_series` function. Under the hood, this function
848 computes a_0 , a_n , b_n coefficients using standard integration formulas.

849 Here's an example on how to compute Fourier series in SymPy:

```

850 >>> L = symbols('L')
851 >>> expr = 2 * (Heaviside(x/L) - Heaviside(x/L - 1)) - 1
852 >>> f = fourier_series(expr, (x, 0, 2*L))
853 >>> f.truncate(3)
854 4*sin(pi*x/L)/pi + 4*sin(3*pi*x/L)/(3*pi) + 4*sin(5*pi*x/L)/(5*pi)

```

855 **9.3. Logic.** SymPy supports construction and manipulation of boolean expressions through the `logic` module. SymPy symbols can be used as propositional variables and also be substituted as `True` or `False`. A good number of manipulation features for boolean expressions have been implemented in the `logic` module.

859 **9.3.1. Constructing boolean expressions.** A boolean variable can be declared as a SymPy symbol. Python operators `&`, `|` and `~` are overridden when using SymPy objects to use the SymPy functionality for logical `And`, `Or`, and `negate`. Other logic functions are also integrated into SymPy, including `Xor` and `Implies`, which are constructed with `^` and `>>`, respectively. The above are just a shorthand, expressions can also be constructed by directly creating the relevant objects: `And()`, `Or()`, `Not()`, `Xor()`, `Nand()`, `Nor()`, etc.

```

866 >>> from sympy import *
867 >>> x, y, z = symbols('x y z')
868 >>> e = (x & y) | z
869 >>> e.subs({x: True, y: True, z: False})
870 True

```

871 **9.3.2. CNF and DNF.** Any boolean expression can be converted to conjunctive normal form, disjunctive normal form, and negation normal form. The API also exposes methods to check if a boolean expression is in any of the above mentioned forms.

```

875 >>> from sympy.logic.boolalg import is_dnf, is_cnf
876 >>> x, y, z = symbols('x y z')
877 >>> to_cnf((x & y) | z)
878 And(Or(x, z), Or(y, z))
879 >>> to_dnf(x & (y | z))
880 Or(And(x, y), And(x, z))
881 >>> is_cnf((x | y) & z)
882 True
883 >>> is_dnf((x & y) | z)
884 True

```

885 **9.3.3. Simplification and Equivalence.** The module supports simplification of given boolean expression by making deductions from the expression. Equivalence of two logical expressions can also be checked. In the case of equivalence, it is possible to return the mapping of variables in two expressions so as to represent the same logical behavior.

```

890 >>> from sympy import *
891 >>> a, b, c, x, y, z = symbols('a b c x y z')
892 >>> e = a & (~a | ~b) & (a | c)
893 >>> simplify(e)
894 And(Not(b), a)
895 >>> e1 = a & (b | c)
896 >>> e2 = (x & y) | (x & z)
897 >>> bool_map(e1, e2)

```



```
898 (And(Or(b, c), a), {a: x, b: y, c: z})
```

899 **9.3.4. SAT solving.** The module also supports satisfiability (SAT) checking of
900 a given boolean expression. If satisfiable, it is possible to return a model for which the
901 expression is satisfiable. The API also supports returning all possible models. The
902 SAT solver has a clause learning DPLL algorithm implemented with a watch literal
903 scheme and VSIDS heuristic[36].

```
904 >>> from sympy import *
905 >>> a, b, c = symbols('a b c')
906 >>> satisfiable(a & (~a | b) & (~b | c) & ~c)
907 False
908 >>> satisfiable(a & (~a | b) & (~b | c) & c)
909 {a: True, b: True, c: True}
```

910 **9.4. Diophantine Equations.** Diophantine equations play a central and an im-
911 portant role in number theory. A Diophantine equation has the form, $f(x_1, x_2, \dots, x_n) =$
912 0 where $n \geq 2$ and x_1, x_2, \dots, x_n are integer variables. If we can find n integers
913 a_1, a_2, \dots, a_n such that $x_1 = a_1, x_2 = a_2, \dots, x_n = a_n$ satisfies the above equation, we
914 say that the equation is solvable.

915 Currently, the following five types of Diophantine equations can be solved using
916 SymPy's Diophantine module.

- 917 • Linear Diophantine equations: $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$
- 918 • General binary quadratic equation: $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- 919 • Homogeneous ternary quadratic equation: $ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0$
- 920 • Extended Pythagorean equation: $a_1x_1^2 + a_2x_2^2 + \dots + a_nx_n^2 = a_{n+1}x_{n+1}^2$
- 921 • General sum of squares: $x_1^2 + x_2^2 + \dots + x_n^2 = k$

922 When an equation is fed into Diophantine module, it factors the equation (if
923 possible) and solves each factor separately. Then, all the results are combined to
924 create the final solution set. The following examples illustrate some of the basic
925 functionalities of the Diophantine module.

```
926 >>> from sympy.solvers.diophantine import *
927 >>> diophantine(2*x + 3*y - 5)
928 set([(3*t_0 - 5, -2*t_0 + 5)])
929
930 >>> diophantine(2*x + 4*y - 3)
931 set()
932
933 >>> diophantine(x**2 - 4*x*y + 8*y**2 - 3*x + 7*y - 5)
934 set([(2, 1), (5, 1)])
935
936 >>> diophantine(x**2 - 4*x*y + 4*y**2 - 3*x + 7*y - 5)
937 set([(-2*t**2 - 7*t + 10, -t**2 - 3*t + 5)])
938
939 >>> diophantine(3*x**2 + 4*y**2 - 5*z**2 + 4*x*y - 7*y*z + 7*z*x)
940 set([(-16*p**2 + 28*p*q + 20*q**2,
941 3*p**2 + 38*p*q - 25*q**2,
942 4*p**2 - 24*p*q + 68*q**2)])
943
944 >>> from sympy.abc import a, b, c, d, e, f
945 >>> diophantine(9*a**2 + 16*b**2 + c**2 + 49*d**2 + 4*e**2 - 25*f**2)
946 set([(70*t1**2 + 70*t2**2 + 70*t3**2 + 70*t4**2 - 70*t5**2, 105*t1*t5,
```

```

947 420*t2*t5, 60*t3*t5, 210*t4*t5,
948 42*t1**2 + 42*t2**2 + 42*t3**2 + 42*t4**2 + 42*t5**2))
949
950 >>> diophantine(a**2 + b**2 + c**2 + d**2 + e**2 + f**2 - 112)
951 set([(8, 4, 4, 4, 0, 0)])

```

952 **9.5. Sets.** SymPy supports representation of a wide variety of mathematical
953 sets. This is achieved by first defining abstract representations of atomic set classes
954 and then combining and transforming them using various set operations.

955 Each of the set classes inherits from the base class `Set` and defines methods to
956 check membership and calculate unions, intersections, and set differences. When these
957 methods are not able to evaluate to atomic set classes, they are represented as abstract
958 unevaluated objects.

959 SymPy has the following atomic set classes:

- 960 • `EmptySet` represents the empty set \emptyset .
- 961 • `UniversalSet` is an abstract “universal set” for which everything is a member.
962 The union of the universal set with any set gives the universal set and the
963 intersection gives the other set itself.
- 964 • `FiniteSet` is functionally equivalent to Python’s built in `set` object. Its mem-
965 bers can be any SymPy object including other sets.
- 966 • `Integers` represents the set of integers \mathbb{Z} .
- 967 • `Naturals` represents the set of natural numbers \mathbb{N} , i.e., the set of positive
968 integers.
- 969 • `Naturals0` represents the set of whole numbers \mathbb{N}^0 , which are all the non-
970 negative integers.
- 971 • `Range` represents a range of integers. A range is defined by specifying a start
972 value, an end value, and a step size. The enumeration of a `Range` object
973 is functionally equivalent to Python’s `range` except it supports infinite end-
974 points, allowing the representation of infinite ranges.
- 975 • `Interval` represents an interval of real numbers. It is specified by giving the
976 start and end point and specifying if it is open or closed in the respective
977 ends.

978 Other than unevaluated classes of `Union`, `Intersection`, and `Complement` opera-
979 tions, we have following set classes.

- 980 • `ProductSet` defines the Cartesian product of two or more sets. The product
981 set is useful when representing higher dimensional spaces. For example, to
982 represent a three-dimensional space, we simply take the Cartesian product of
983 three real sets.
- 984 • `ImageSet` represents the image of a function when applied to a particular set.
985 The image set of a function F with respect to a set S is $\{F(x)|x \in S\}$.
986 SymPy uses image sets to represent sets of infinite solutions equations such
987 as $\sin(x) = 0$.
- 988 • `ConditionSet` represents a subset of a set whose members satisfies a particular
989 condition. The condition set of the set S with respect to the condition H is
990 $\{x|H(x), x \in S\}$. SymPy uses condition sets to represent the set of solutions
991 of equations and inequalities, where the equation or the inequality is the
992 condition and the set is the domain being solved over.

993 A few other classes are implemented as special cases of the classes described above.
994 The set of real numbers, `Reals`, is implemented as a special case of `Interval` over
995 the interval $(-\infty, \infty)$. `ComplexRegion` is implemented as a special case of `ImageSet`.

`ComplexRegion` supports both polar and rectangular representation of regions on the complex plane.

9.6. Category Theory. SymPy includes a basic version of the module for dealing with categories — abstract mathematical objects representing classes of structures as classes of objects (points) and morphisms (arrows) between the objects. This version of the module was designed with the following two goals in mind:

1. automatic typesetting of diagrams given by a collection of objects and of morphisms between them, and
2. specification and (semi-)automatic derivation of properties using commutative diagrams.

At of version 1.0, SymPy only implements the first goal, while a (very partially working) draft of implementation of the second goal is available at [2].

In order to achieve the two goals, the module `categories` defines several classes representing some of the essential concepts: objects, morphisms, categories, and diagrams. In category theory, the inner structure of objects is often discarded in the favor of studying the properties of morphisms, so the class `Object` is essentially a synonym of the class `Symbol`. There are several morphism classes which do not have a particular internal structure either, though an exception is `CompositeMorphism`, which essentially stores a list of morphisms.

To capture the properties of morphisms, the class `Diagram` is expected to be used. This class stores a family of morphisms, the corresponding source and target objects, and, possibly, some properties of the morphisms. Generally, no restrictions are imposed on what the properties may be — for example, one might use strings of the form “forall”, “exists”, “unique”, etc. Furthermore, the morphisms of a diagram are grouped into *premises* and *conclusions*, in order to be able to represent logical implications of the form “for a collection of morphisms P with properties $p : P \rightarrow \Omega$ (the premises), there exists a collection of morphisms C with properties $c : C \rightarrow \Omega$ (the conclusions),” where Ω is the universal collection of properties. Finally, the class `Category` includes a collection of diagrams which are deemed commutative and which therefore define the properties of this category.

Automatic typesetting of diagrams takes a `Diagram` and produces L^AT_EX code using the Xy-pic package. Typesetting is done in two stages: layout and generation of Xy-pic code. The layout stage is taken care of by the class `DiagramGrid`, which takes a `Diagram` and lays out the objects in a grid, trying to reduce the average length of the arrows in the final picture. By default, `DiagramGrid` uses a series of triangle-based heuristics to produce a rectangular grid. A linear layout can also be imposed. Furthermore, groups of objects can be given; in this case, the groups will be treated as atomic cells, and the member objects will be typeset independently of the other objects.

The second phase of diagram typesetting consists of actually drawing the picture and is carried out by the class `XypicDiagramDrawer`. An example of a diagram automatically typeset by `DiagramGrid` and `XypicDiagramDrawer` is given in Figure 2.

As far as the second main goal of the module is concerned, a (non-working) draft of an implementation is at [2]. The principal idea consists of automatically deciding whether a diagram is commutative or not, given a collection of “axioms” — diagrams *known* to be commutative. The implementation is based on graph embeddings (injective maps): whenever an embedding of a commutative diagram into a given diagram is found, one concludes that the subdiagram is commutative. Deciding commuta-

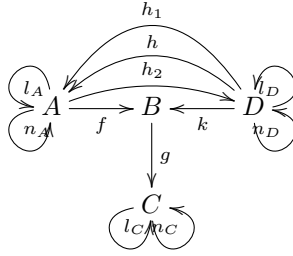


Fig. 2: An automatically typeset commutative diagram

tivity of the whole diagram is therefore based (theoretically) on finding a “cover” of the target diagram by embeddings of the axioms. The naïve implementation proved to be prohibitively slow; a better optimized version is therefore in order, as well as application of heuristics.

Contributions to automatic inference of commutativity of diagrams are welcome. The source code (both the one in master and in `ct4-commutativity`) is extensively documented. Even more extensive explanations (including some literary chatter) are given at [3].

9.7. SymPy Gamma. SymPy Gamma is a simple web application that runs on Google App Engine. It executes and displays the results of SymPy expressions as well as additional related computations, in a fashion similar to that of Wolfram|Alpha. For instance, entering an integer will display its prime factors, digits in the base-10 expansion, and a factorization diagram. Entering a function will display its docstring; in general, entering an arbitrary expression will display its derivative, integral, series expansion, plot, and roots.

SymPy Gamma also has several additional features than just computing the results using SymPy.

- SymPy Gamma displays integration and differentiation steps in detail, which can be viewed in Figure 3:

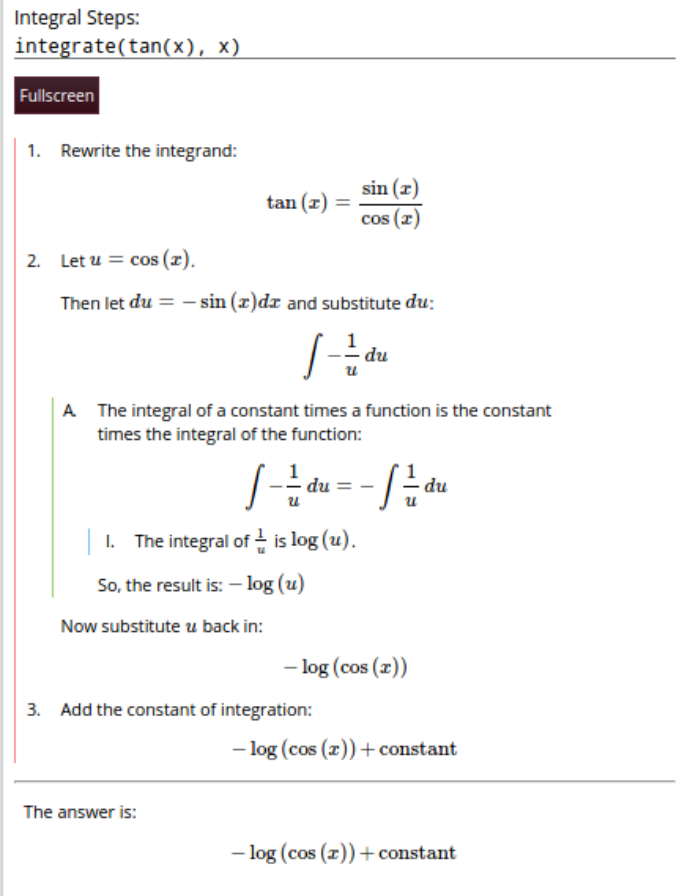


Fig. 3: Integral steps of $\tan(x)$

- SymPy Gamma displays the factor tree diagrams for different numbers.
- SymPy Gamma saves user search queries, and offers many such similar features for free, which Wolfram|Alpha only offers to its paid users.

Every input query from the user on SymPy Gamma is first parsed by its own parser, which handles several different forms of function names, which SymPy as a library does not support. For instance, SymPy Gamma supports queries like `sin x`, whereas SymPy does not support this, and supports only `sin(x)`.

This parser converts the input query to the equivalent SymPy readable code, which is then eventually processed by SymPy, and the result is finally printed with the built-in LaTeX output and rendered on the SymPy Gamma web-application.

9.8. SymPy Live. SymPy Live is an online Python shell, which runs on Google App Engine, that executes SymPy code. It is integrated in the SymPy documentation examples, located at this [link](#).

This is accomplished by providing a HTML/JavaScript GUI for entering source code and visualization of output, and a server that evaluates the requested source code. It is an interactive AJAX shell that runs SymPy code using Python on the server.

Certain Features of SymPy Live:

- It supports the exact same syntax as SymPy, hence it can be used easily to test for outputs from various SymPy expressions.
- It can be run as a standalone app or in an existing app as an admin-only handler, and can also be used for system administration tasks, as an interactive way to try out APIs, or as a debugging aid during development.
- It can also be used to plot figures ([link](#)), and execute all kinds of expressions that SymPy can evaluate.
- SymPy Live also renders the output in LaTeX for pretty-printing the output.

9.9. Comparison with Mathematica. Wolfram Mathematica is a popular proprietary CAS. It features highly advanced algorithms. Mathematica has a core implemented in C++ [8] which interprets its own programming language (known as Wolfram language).

Analogous to Lisp’s S-expressions, Mathematica uses its own style of M-expressions, which are arrays of either atoms or other M-expression. The first element of the expression identifies the type of the expression and is indexed by zero, whereas the first argument is indexed by one. Notice that SymPy expression arguments are stored in a Python tuple (that is, an immutable array), while the expression type is identified by the type of the object storing the expression.

Mathematica can associate attributes to its atoms. Attributes may define mathematical properties and behavior of the nodes associated to the atom. In SymPy, the usage of static class fields is roughly similar to Mathematica’s attributes, though other programming patterns may also be used to achieve an equivalent behavior, such as class inheritance.

Unlike SymPy, Mathematica’s expressions are mutable, that is one can change parts of the expression tree without the need of creating a new object. The mutability of Mathematica allows for a lazy updating of any references to that data structure.

Products in Mathematica are determined by some built-in node types, such as `Times`, `Dot`, and others. `Times` is a representation of the `*` operator, and is always meant to represent a commutative product operator. The other notable product is `Dot`, which represents the `.` operator. This product represents matrix multiplication, it is not commutative. In general, SymPy uses the same node for both scalar and matrix multiplication, the only exception being with abstract matrix symbols. Unlike Mathematica, SymPy determines commutativity with respect to multiplication from the factor’s expression type. Mathematica puts the `Orderless` attribute on the expression type.

Regarding associative expressions, SymPy handles associativity by making associative expressions inherit the class `AssocOp`, while Mathematica specifies the `Flat` [4] attribute on the expression type.

Mathematica relies heavily on pattern matching — even the so-called equivalent of function declaration is in reality the definition of a pattern matching generating an expression tree transformation on input expressions. Mathematica’s pattern matching is sensitive to associative [4], commutative [5], and one-identity [6] properties of its expression tree nodes [7]. SymPy has various ways to perform pattern matching. All of them play a lesser role in the CAS than in Mathematica and are basically available as a tool to rewrite expressions. The differential equation solver in SymPy somewhat relies on pattern matching to identify the kind of differential equation, but it is envisaged to replace that strategy with analysis of Lie symmetries in the future. Mathematica’s real advantage is the ability to add new overloading to the expression builder at runtime, or for specific subnodes. Consider for example:

```

1133 In[1]:= Unprotect[Plus]
1134
1135 Out[1]= {Plus}
1136
1137 In[2]:= Sin[x_]^2 + Cos[y_]^2 := 1
1138
1139 In[3]:= x + Sin[t]^2 + y + Cos[t]^2
1140
1141 Out[3]= 1 + x + y

```

This expression in Mathematica defines a substitution rule that overloads the functionality of the `Plus` node (the node for additions in Mathematica). The trailing underscore after a symbol means that it is to be considered a wildcard. This example may not be practical, one may wish to keep this identity unevaluated. Nevertheless, it clearly illustrates the potential to define one's own immediate transformation rules. In SymPy, the operations constructing the addition node in the expression tree are Python class constructors and cannot be modified at runtime.⁶ The way SymPy deals with extending the missing runtime overloadability functionality is by subclassing the node types. Subclasses may redefine the class constructor to yield the proper extended functionality.

Unlike SymPy, Mathematica does not support type inheritance or polymorphism [20]. SymPy relies heavily on class inheritance, but for the most part, class inheritance is used to make sure that SymPy objects inherit the proper methods and implement the basic hashing system. Associativity of expressions can be achieved by inheriting the class `AssocOp`, which may appear a more cumbersome operation than Mathematica's attribute setting.

Matrices in SymPy are types on their own. In Mathematica, nested lists are interpreted as matrices whenever the sublists have the same length. The main difference to SymPy is that ordinary operators and functions do not get generalized the same way as used in traditional mathematics. Using the standard multiplication in Mathematica performs an element-wise product, this is compatible with Mathematica's convention of commutativity of `Times` nodes. Matrix product is expressed by the `dot` operator, or the `Dot` node. The same is true for the other operators, and even functions, most notably calling the exponential function `Exp` on a matrix returns an element-wise exponentiation of its elements. The real matrix exponential is available through the `MatrixExp` function.

Unevaluated expressions in Mathematica can be achieved in various ways, most commonly with the `HoldForm` or `Hold` nodes, that block the evaluation of subnodes by the parser. Note that such a node cannot be expressed in Python, because of greedy evaluation. Whenever needed in SymPy, it is necessary to add the parameter `evaluate=False` to all subnodes, or put the input expression in a string.

In Mathematica, the operator `==` returns a boolean whenever it is able to immediately evaluate the truth of the equality, otherwise it returns an `Equal` expression. In SymPy, `==` means structural equality and is always guaranteed to return a boolean expression. To express an equality in SymPy it is necessary to explicitly construct an object of the `Equality` class.

SymPy, in accordance with Python and unlike the usual programming convention, uses `**` to express the power operator, while Mathematica uses the more common `^`.

⁶In reality, Python supports monkey patching, nonetheless, it is a discouraged programming pattern.

SymPy’s use of floating-point numbers is similar to that of most other CASs, including Maple and Maxima. By contrast, Mathematica uses a form of significance arithmetic [49] for approximate numbers. This offers further protection against numerical errors, although it comes with its own set of problems (for a critique of significance arithmetic, see Fateman [20]). Internally, SymPy’s `evalf` method works similarly to Mathematica’s significance arithmetic, but the semantics are isolated from the rest of the system.

9.10. Other Projects that use SymPy. There are several projects that use SymPy as a library for implementing a part of their project, or even as a part of back-end for their application as well.

Some of them are listed below:

- **Cadabra**: Cadabra is a symbolic computer algebra system (CAS) designed specifically for the solution of problems encountered in field theory.
- **Octave Symbolic**: The Octave-Forge Symbolic package adds symbolic calculation features to GNU Octave. These include common Computer Algebra System tools such as algebraic operations, calculus, equation solving, Fourier and Laplace transforms, variable precision arithmetic and other features.
- **SymPy.jl**: Provides a Julia interface to SymPy using PyCall.
- **Mathics**: Mathics is a free, general-purpose online CAS featuring Mathematica compatible syntax and functions. It is backed by highly extensible Python code, relying on SymPy for most mathematical tasks.
- **Mathpix**: An iOS App, that uses Artificial Intelligence to detect handwritten math as input, and uses SymPy Gamma, to evaluate the math input and generate the relevant steps to solve the problem.
- **IKFast**: IKFast is a robot kinematics compiler provided by OpenRAVE. It analytically solves robot inverse kinematics equations and generates optimized C++ files. It uses SymPy for its internal symbolic mathematics.
- **Sage**: A CAS, visioned to be a viable free open source alternative to Magma, Maple, Mathematica and MATLAB.
- **SageMathCloud**: SageMathCloud is a web-based cloud computing and course management platform for computational mathematics.
- **PyDy**: Multibody Dynamics with Python.
- **galgebra**: Geometric algebra (previously `sympy.galgebra`).
- **yt**: Python package for analyzing and visualizing volumetric data (`yt.units` uses SymPy).
- **SfePy**: Simple finite elements in Python, see section 9.11.1.
- **Quameon**: Quantum Monte Carlo in Python.
- **Lcapy**: Experimental Python package for teaching linear circuit analysis.
- **Quantum Programming in Python**: Quantum 1D Simple Harmonic Oscillator and Quantum Mapping Gate.
- **LaTeX Expression project**: Easy LaTeX typesetting of algebraic expressions in symbolic form with automatic substitution and result computation.
- **Symbolic statistical modeling**: Adding statistical operations to complex physical models.

9.11. Project Details. Below we provide particular examples of SymPy use in some of the projects listed above.

9.11.1. SfePy. **SfePy** (Simple finite elements in Python), cf. [19] is a Python package for solving partial differential equations (PDEs) in 1D, 2D and 3D by the

finite element (FE) method [57]. SymPy is used within this package mostly for code generation and testing, namely:

- generation of the hierarchical FE basis module, involving generation and symbolic differentiation of 1D Legendre and Lobatto polynomials, constructing the FE basis polynomials [50] and generating the C code;
- generation of symbolic conversion formulas for various groups of elastic constants [24]: provide any two of the Young's modulus, Poisson's ratio, bulk modulus, Lamé's first parameter, shear modulus (Lamé's second parameter) or longitudinal wave modulus and get the other ones;
- simple physical unit conversions, generation of consistent unit sets;
- testing FE solutions using method of manufactured (analytical) solutions: the differential operator of a PDE is symbolically applied and a symbolic right-hand side is created, evaluated in quadrature points, and subsequently used to obtain a numerical solution that is then compared to the analytical one;
- testing accuracy of 1D, 2D and 3D numerical quadrature formulas (cf. [9]) by generating polynomials of suitable orders, integrating them, and comparing the results with those obtained by the numerical quadrature.

9.12. Tensors. Ongoing work to provide the capabilities of tensor computer algebra has so far produced the `tensor` module. It is composed of three separated sub-modules, whose purposes are quite different: `tensor.indexed` and `tensor.indexed_methods` support indexed symbols, `tensor.array` contains facilities to operator on symbolic N -dimensional arrays, and finally `tensor.tensor` is used to define abstract tensors. The abstract tensors subsection is inspired by xAct [35] and Cadabra [40]. Canonicalization based on the Butler-Portugal [34] algorithm is supported in SymPy. It is currently limited to polynomial tensor expressions.

9.13. Numerical simplification. The `nsimplify` function in SymPy (a wrapper of `identify` in `mpmath`) attempts to find a simple symbolic expression that evaluates to the same numerical value as the given input. It works by applying a few simple transformations (including square roots, reciprocals, logarithms and exponentials) to the input and, for each transformed value, using the PSLQ algorithm [21] to search for a matching algebraic number or optionally a linear combination of user-provided base constants (such as π).

```
>>> t = 1 / (sin(pi/5)+sin(2*pi/5)+sin(3*pi/5)+sin(4*pi/5))**2
>>> nsimplify(t)
-2*sqrt(5)/5 + 1
>>> nsimplify(pi, tolerance=0.01)
22/7
>>> nsimplify(1.783919626661888, [pi], tolerance=1e-12)
pi/(-1/3 + 2*pi/3)
```

9.14. Examples.

9.14.1. Simplification.

```
expand
>>> expand((x + y)**3)
x**3 + 3*x**2*y + 3*x*y**2 + y**3
factor
>>> factor(x**3 + 3*x**2*y + 3*x*y**2 + y**3)
(x + y)**3
collect
>>> collect(y*x**2 + 3*x**2 - x*y + x - 1, x)
```

```

1276 x**2*(y + 3) + x*(-y + 1) - 1
1277 cancel
1278 >>> cancel((x**2 + 2*x + 1)/(x**2 - 1))
1279 (x + 1)/(x - 1)
1280 apart
1281 >>> apart((x**3 + 4*x - 1)/(x**2 - 1))
1282 x + 3/(x + 1) + 2/(x - 1)
1283 trigsimp
1284 >>> trigsimp(cos(x)**2*tan(x) - sin(2*x))
1285 -sin(2*x)/2

1286 9.14.2. Polynomials. Factorization:
1287 >>> t = symbols("t")
1288 >>> f = (2115*x**4*y + 45*x**3*z**3*t**2 - 45*x**3*t**2 - 423*x*y**4 -
1289 ...      47*x*y**3 + 141*x*y*z**3 + 94*x*y*z*t - 9*y**3*z**3*t**2 +
1290 ...      9*y**3*t**2 - y**2*z**3*t**2 + y**2*t**2 + 3*z**6*t**2 +
1291 ...      2*z**4*t**3 - 3*z**3*t**2 - 2*z*t**3)
1292 >>> factor(f)
1293 (t**2*z**3 - t**2 + 47*x*y)*(2*t*z + 45*x**3 - 9*y**3 - y**2 + 3*z**3)
1294 Gröbner bases:
1295 >>> x0, x1, x2 = symbols('x:3')
1296 >>> I = [x0 + 2*x1 + 2*x2 - 1,
1297 ...      x0**2 + 2*x1**2 + 2*x2**2 - x0,
1298 ...      2*x0*x1 + 2*x1*x2 - x1]
1299 >>> groebner(I, order='lex')
1300 GroebnerBasis([7*x0 - 420*x2**3 + 158*x2**2 + 8*x2 - 7,
1301 7*x1 + 210*x2**3 - 79*x2**2 + 3*x2,
1302 84*x2**4 - 40*x2**3 + x2**2 + x2], x0, x1, x2, domain='ZZ', order='lex')
1303 Root isolation:
1304 >>> f = 7*z**4 - 19*z**3 + 20*z**2 + 17*z + 20
1305 >>> intervals(f, all=True, eps=0.001)
1306 ([],
1307 [((-425/1024 - 625*I/1024, -1485/3584 - 2185*I/3584), 1),
1308 ((-425/1024 + 2185*I/3584, -1485/3584 + 625*I/1024), 1),
1309 ((3175/1792 - 2605*I/1792, 1815/1024 - 10415*I/7168), 1),
1310 ((3175/1792 + 10415*I/7168, 1815/1024 + 2605*I/1792), 1)])

1311 9.14.3. Solvers. Single solution:
1312 >>> solveset(x - 1, x)
1313 {1}
1314 Finite solution set, quadratic equation:
1315 >>> solveset(x**2 - pi**2, x)
1316 {-pi, pi}
1317 No solution:
1318 >>> solveset(1, x)
1319 EmptySet()
1320 Interval solution:
1321 >>> solveset(x**2 - 3 > 0, x, domain=S.Reals)
1322 (-oo, -sqrt(3)) U (sqrt(3), oo)
1323 Infinitely many solutions:
1324 >>> solveset(x - x, x, domain=S.Reals)

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1325 (-oo, oo)
1326 >>> solveset(x - x, x, domain=S.Complexes)
1327 S.Complexes
1328 Linear systems (linsolve)
1329 >>> A = Matrix([[1, 2, 3], [4, 5, 6], [7, 8, 10]])
1330 >>> b = Matrix([3, 6, 9])
1331 >>> linsolve((A, b), x, y, z)
1332 {(-1, 2, 0)}
1333 >>> linsolve(Matrix(([1, 1, 1, 1], [1, 1, 2, 3])), (x, y, z))
1334 {(-y - 1, y, 2)}
1335     Below are examples of solve applied to problems not yet handled by solveset.
1336 Nonlinear (multivariate) system of equations (the intersection of a circle and a parabola):
1337 >>> solve([x**2 + y**2 - 16, 4*x - y**2 + 6], x, y)
1338 [(-2 + sqrt(14), -sqrt(-2 + 4*sqrt(14))),
1339  (-2 + sqrt(14), sqrt(-2 + 4*sqrt(14))),
1340  (-sqrt(14) - 2, -I*sqrt(2 + 4*sqrt(14))),
1341  (-sqrt(14) - 2, I*sqrt(2 + 4*sqrt(14)))]
1342 Transcendental equations:
1343 >>> solve((x + log(x))**2 - 5*(x + log(x)) + 6, x)
1344 [LambertW(exp(2)), LambertW(exp(3))]
1345 >>> solve(x**3 + exp(x))
1346 [-3*LambertW((-1)**(2/3)/3)]

1347 9.14.4. Matrices. Matrix expressions
1348 >>> m, n, p = symbols("m, n, p", integer=True)
1349 >>> R = MatrixSymbol("R", m, n)
1350 >>> S = MatrixSymbol("S", n, p)
1351 >>> T = MatrixSymbol("T", m, p)
1352 >>> U = R*S + 2*T
1353 >>> U.shape
1354 (m, p)
1355 >>> U[0, 1]
1356 2*T[0, 1] + Sum(R[0, _k]*S[_k, 1], (_k, 0, n - 1))
1357 Block Matrices
1358 >>> n, m, l = symbols('n m l')
1359 >>> X = MatrixSymbol('X', n, n)
1360 >>> Y = MatrixSymbol('Y', m, m)
1361 >>> Z = MatrixSymbol('Z', n, m)
1362 >>> B = BlockMatrix([[X, Z], [ZeroMatrix(m, n), Y]])
1363 >>> B
1364 Matrix([
1365 [X, Z],
1366 [0, Y]])
1367 >>> B[0, 0]
1368 X[0, 0]
1369 >>> B.shape
1370 (m + n, m + n)

```