

SymPy: Symbolic Computing in Python

Aaron Meurer¹, Christopher P. Smith², Mateusz Paprocki³, Ondřej Čertík⁴, Sergey B. Kirpichev⁵, Matthew Rocklin⁶, AMiT Kumar⁷, Sergiu Ivanov⁸, Jason K. Moore⁹, Sartaj Singh¹⁰, Thilina Rathnayake¹¹, Sean Vig¹², Brian E. Granger¹³, Richard P. Muller¹⁴, Francesco Bonazzi¹⁵, Harsh Gupta¹⁶, Shivam Vats¹⁷, Fredrik Johansson¹⁸, Fabian Pedregosa¹⁹, Matthew J. Curry²⁰, Andy R. Terrel²¹, Štěpán Roučka²², Ashutosh Saboo²³, Isuru Fernando²⁴, Sumith Kulal²⁵, Robert Cimrman²⁶, and Anthony Scopatz²⁷

¹University of South Carolina, Columbia, SC 29201 (asmeurer@gmail.com).

²Polar Semiconductor, Inc., Bloomington, MN 55425 (smichr@gmail.com).

³Continuum Analytics, Inc., Austin, TX 78701 (mattpap@gmail.com).

⁴Los Alamos National Laboratory, Los Alamos, NM 87545 (certik@lanl.gov).

⁵Moscow State University, Faculty of Physics, Leninskie Gory, Moscow, 119991, Russia (skirpichev@gmail.com).

⁶Continuum Analytics, Inc., Austin, TX 78701 (mrocklin@gmail.com).

⁷Delhi Technological University, Shahbad Daultpur, Bawana Road, New Delhi 110042, India (dtu.amit@gmail.com).

⁸Université Paris Est Créteil, 61 av. Général de Gaulle, 94010 Créteil, France (sergiu.ivanov@u-pec.fr).

⁹University of California, Davis, Davis, CA 95616 (jkm@ucdavis.edu).

¹⁰Indian Institute of Technology (BHU), Varanasi, Uttar Pradesh 221005, India (singhsartaj94@gmail.com).

¹¹University of Moratuwa, Bandaranayake Mawatha, Katubedda, Moratuwa 10400, Sri Lanka (thilinarmtb.10@cse.mrt.ac.lk).

¹²University of Illinois at Urbana-Champaign, Urbana, IL 61801 (sean.v.775@gmail.com).

¹³California Polytechnic State University, San Luis Obispo, CA 93407 (ellisonbg@gmail.com).

¹⁴Center for Computing Research, Sandia National Laboratories, Albuquerque, NM 87185 (rmuller@sandia.gov).

¹⁵Max Planck Institute of Colloids and Interfaces, Department of Theory and Bio-Systems, Science Park Golm, 14424 Potsdam, Germany (francesco.bonazzi@mpikg.mpg.de).

¹⁶Indian Institute of Technology Kharagpur, Kharagpur, West Bengal 721302, India (hargup@protonmail.com).

¹⁷Indian Institute of Technology Kharagpur, Kharagpur, West Bengal 721302, India (shivamvats.iitkgp@gmail.com).

¹⁸INRIA Bordeaux-Sud-Ouest – LFANT project-team, 200 Avenue de la Vieille Tour, 33405 Talence, France (fredrik.johansson@gmail.com).

¹⁹INRIA – SIERRA project-team, 2 Rue Simone IFF, 75012 Paris, France (f@bianp.net).

²⁰Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131 (mattjcurry@gmail.com).

²¹Fashion Metric, Inc, Austin, TX 78681 (andy.terrel@gmail.com).

²²Faculty of Mathematics and Physics, Charles University in Prague, V Holešovičkách 2, 180 00 Praha, Czech Republic (stepan.roucka@mff.cuni.cz).

²³Birla Institute of Technology and Science, Pilani, K.K. Birla Goa Campus, NH 17B Bypass Road, Zuarinagar, Sancoale, Goa 403726, India (ashutosh.saboo96@gmail.com).

²⁴University of Moratuwa, Bandaranayake Mawatha, Katubedda, Moratuwa 10400, Sri Lanka (isuru.11@cse.mrt.ac.lk).

²⁵Indian Institute of Technology Bombay, Powai, Mumbai, Maharashtra 400076, India (sumith@cse.iitb.ac.in).

²⁶New Technologies – Research Centre, University of West Bohemia, Univerzitní 8, 306

55 **ABSTRACT**

56 SymPy is an open source computer algebra system written in pure Python. It is built with a focus on
57 extensibility and ease of use, through both interactive and programmatic applications. These characteristics
58 have led SymPy to become a popular symbolic library for the scientific Python ecosystem. This paper
59 presents the architecture of SymPy, a description of its features, and a discussion of select domain specific
60 modules. The supplementary materials provide additional examples and further outline details of the
61 architecture and features of SymPy.

62 **Keywords:** symbolic, Python, computer algebra system

63 **1 INTRODUCTION**

64 SymPy is a full featured computer algebra system (CAS) written in the Python [27] programming
65 language. It is free and open source software, licensed under the 3-clause BSD license [39].
66 The SymPy project was started by Ondřej Čertík in 2005, and it has since grown to over 500
67 contributors. Currently, SymPy is developed on GitHub using a bazaar community model [35].
68 The accessibility of the codebase and the open community model allow SymPy to rapidly respond
69 to the needs of users and developers.

70 Python is a dynamically typed programming language that has a focus on ease of use and
71 readability. Due in part to this focus, it has become a popular language for scientific computing
72 and data science, with a broad ecosystem of libraries [30]. SymPy is itself used by many libraries
73 and tools to support research within a variety of domains, such as Sage [42] (pure mathematics),
74 yt [47] (astronomy and astrophysics), PyDy [15] (multibody dynamics), and SfePy [10] (finite
75 elements).

76 Unlike many CASs, SymPy does not invent its own programming language. Python itself
77 is used both for the internal implementation and end user interaction. By using the operator
78 overloading functionality of Python, SymPy follows the embedded domain specific language
79 paradigm proposed by Hudak [20]. The exclusive usage of a single programming language makes
80 it easier for people already familiar with that language to use or develop SymPy. Simultaneously,
81 it enables developers to focus on mathematics, rather than language design. SymPy officially
82 supports Python 2.6, 2.7 and 3.2–3.5.

83 SymPy is designed with a strong focus on usability as a library. Extensibility is important in
84 its application program interface (API) design. Thus, SymPy makes no attempt to extend the
85 Python language itself. The goal is for users of SymPy to be able to include SymPy alongside
86 other Python libraries in their workflow, whether that be in an interactive environment or as a
87 programmatic part in a larger system.

88 As a library, SymPy does not have a built-in graphical user interface (GUI). However,
89 SymPy exposes a rich interactive display system, and supports registering display formatters
90 with Jupyter [24] frontends, including the Notebook and Qt Console, which will render SymPy
91 expressions using MathJax [9] or L^AT_EX.

92 The remainder of this paper discusses key components of the SymPy library. Section 2
93 discusses the architecture of SymPy. Section 3 enumerates the features of SymPy and takes
94 a closer look at some of the important ones. The section 4 looks at the numerical features of
95 SymPy and its dependency library, mpmath. Section 5 looks at the domain specific physics
96 submodules for performing symbolic and numerical calculations in classical mechanics and
97 quantum mechanics. Conclusions and future directions for SymPy are given in section 6. All
98 examples in this paper use SymPy version 1.0, examples in section 4.1 use mpmath version 0.19.

99 The following statement imports all SymPy functions into the global Python namespace.¹

¹`import *` has been used here to aid the readability of the paper, but is best to avoid such wildcard import statements in production code, as they make it unclear which names are present in the namespace. Furthermore,

100 From here on, all examples in this paper assume that this statement has been executed²:

```
101 >>> from sympy import *
```

102 All examples could be tested on the SymPy Live instance, that is an online Python shell,
103 which uses the Google App Engine to execute SymPy code.

104 2 ARCHITECTURE

105 Software architecture is of central importance in any large software project because it establishes
106 predictable patterns of usage and development [41]. This section describes the essential structural
107 components of SymPy, provides justifications for the design decisions that have been made, and
108 gives example user-facing code as appropriate.

109 2.1 Basic Usage

110 Symbolic variables, called symbols, must be defined and assigned to Python variables before they
111 can be used. This is typically done through the `symbols` function, which may create multiple
112 symbols in a single function call. For instance,

```
113 >>> x, y, z = symbols('x y z')
```

114 creates three symbols representing variables named x , y , and z . In this particular instance, these
115 symbols are all assigned to Python variables of the same name. However, the user is free to
116 assign them to different Python variables, while representing the same symbol, such as `a`, `b`,
117 `c = symbols('x y z')`. In order to minimize potential confusion, though, all examples in this
118 paper will assume that the symbols x , y , and z have been assigned to Python variables identical
119 to their symbolic names.

120 Expressions are created from symbols using Python's mathematical syntax. For instance, the
121 following Python code creates the expression $(x^2 - 2x + 3)/y$. Note that the expression remains
122 unevaluated: it is represented symbolically.

```
123 >>> (x**2 - 2*x + 3)/y  
124 (x**2 - 2*x + 3)/y
```

125 Importantly, SymPy expressions are immutable. This simplifies the design of SymPy by
126 allowing expression interning. It also enables expressions to be hashed, that is used to implement
127 caching in SymPy.

128 2.2 The Core

129 A computer algebra system stores mathematical expressions as data structures. For example,
130 the mathematical expression $x + y$ is represented as a tree with three nodes, $+$, x , and y ,
131 where x and y are ordered children of $+$. As users manipulate mathematical expressions
132 with traditional mathematical syntax, the CAS manipulates the underlying data structures.
133 Automated optimizations and computations such as integration, simplification, etc. are all
134 functions that consume and produce expression trees.

135 In SymPy every symbolic expression is an instance of a Python `Basic` class³, a superclass
136 of all SymPy types providing common methods to all SymPy tree-elements, such as traversals.
137 The children of a node in the tree are held in the `args` attribute. A terminal or leaf node in the
138 expression tree has empty `args`.

139 For example, consider the expression $xy + 2$:

```
140 >>> expr = x*y + 2
```

imported names could clash with already existing imports from another package. For example, SymPy, the standard Python `math` library, and NumPy all define the `exp` function, but only the SymPy one will work with SymPy symbolic expressions.

²Three greater-than signs denote standard Python prompt for interactive session.

³Some internal classes, such as those used in the polynomial module, do not follow this rule for efficiency reasons.

141 By order of operations, the parent of the expression tree for `expr` is an addition, so it is of type
142 `Add`. The child nodes of `expr` are 2 and `x*y`.

```
143 >>> type(expr)
144 <class 'sympy.core.add.Add'>
145 >>> expr.args
146 (2, x*y)
```

147 Descending further down into the expression tree yields the full expression. For example,
148 the next child node (given by `expr.args[0]`) is 2. Its class is `Integer`, and it has an empty `args`
149 tuple, indicating that it is a leaf node.

```
150 >>> expr.args[0]
151 2
152 >>> type(expr.args[0])
153 <class 'sympy.core.numbers.Integer'>
154 >>> expr.args[0].args
155 ()
```

156 Symbols or symbolic constants, like e or π are another examples of leaf nodes.

```
157 >>> exp(1)
158 E
159 >>> exp(1).args
160 ()
161 >>> x.args
162 ()
```

163 A useful way to view an expression tree is using the `srepr` function, which returns a string
164 representation of an expression as valid Python code⁴ with all the nested class constructor calls
165 to create the given expression.

```
166 >>> srepr(expr)
167 "Add(Mul(Symbol('x'), Symbol('y')), Integer(2))"
```

168 Every SymPy expression satisfies a key identity invariant:

```
169 expr.func(*expr.args) == expr
```

170 This means that expressions are rebuildable from their `args`.⁵ Note that in SymPy the `==`
171 operator represents exact structural equality, not mathematical equality. This allows testing if
172 any two expressions are equal to one another as expression trees. For example, even though
173 $(x+1)^2$ and x^2+2x+1 are equal mathematically, SymPy gives

```
174 >>> (x + 1)**2 == x**2 + 2*x + 1
175 False
```

176 because they are different as expression trees (the former is a `Pow` object and the latter is an `Add`
177 object).

178 Python allows classes to override mathematical operators. The Python interpreter translates
179 the above `x*y + 2` to, roughly, `(x.__mul__(y)).__add__(2)`. Both `x` and `y`, returned from the
180 `symbols` function, are `Symbol` instances. The 2 in the expression is processed by Python as a
181 literal, and is stored as Python's built in `int` type. When 2 is passed to the `__add__` method
182 of `Symbol`, it is converted to the SymPy type `Integer(2)` before being stored in the resulting
183 expression tree. In this way, SymPy expressions can be built in the natural way using Python
184 operators and numeric literals.

⁴ The `dotprint` function from the `sympy.printing.dot` package prints output to dot format, which can be rendered with Graphviz to visualize expression trees graphically.

⁵`expr.func` is used instead of `type(expr)` to allow the function of an expression to be distinct from its actual Python class. In most cases the two are the same.

2.3 Assumptions

SymPy performs logical inference through its assumptions system. The assumptions system allows users to specify that symbols have certain common mathematical properties, such as being positive, imaginary, or integral. SymPy is careful to never perform simplifications on an expression unless the assumptions allow them. For instance, the identity $\sqrt{t^2} = t$ holds if t is nonnegative ($t \geq 0$). However, for general complex t , no such identity holds.

By default, SymPy performs all calculations assuming that symbols are complex valued. This assumption makes it easier to treat mathematical problems in full generality.

```
>>> t = Symbol('t')
>>> sqrt(t**2)
sqrt(t**2)
```

By assuming the most general case, that t is complex by default, SymPy avoids performing mathematically invalid operations. However, in many cases users will wish to simplify expressions containing terms like $\sqrt{t^2}$.

Assumptions are set on `Symbol` objects when they are created. For instance `Symbol('t', positive=True)` will create a symbol named t that is assumed to be positive.

```
>>> t = Symbol('t', positive=True)
>>> sqrt(t**2)
t
```

Some of the common assumptions that SymPy allows are `positive`, `negative`, `real`, `nonpositive`, `integer`, `prime` and `commutative`⁶. Assumptions on any object can be checked with the `is_assumption` attributes, like `t.is_positive`.

Assumptions are only needed to restrict a domain so that certain simplifications can be performed. They are not required to make the domain match the input of a function. For instance, one can create the object $\sum_{n=0}^m f(n)$ as `Sum(f(n), (n, 0, m))` without setting `integer=True` when creating the `Symbol` object n .

The assumptions system additionally has deductive capabilities. The assumptions use a three-valued logic using the Python built in objects `True`, `False`, and `None`. Note that `False` is returned if the SymPy object doesn't or can't have the assumption. For example, both `I.is_real` and `I.is_prime` return `False` for the imaginary unit I .

`None` represents the “unknown” case. This could mean that given assumptions do not unambiguously specify the truth of an attribute. For instance, `Symbol('x', real=True).is_positive` will give `None` because a real symbol might be positive or negative. The `None` could also mean that not enough is known or implemented to compute the given fact. For instance, `(pi + E).is_irrational` gives `None`, because determining whether $\pi + e$ is rational or irrational is an open problem in mathematics [26].

Basic implications between the facts are used to deduce assumptions. For instance, the assumptions system knows that being an integer implies being rational.

```
>>> i = Symbol('i', integer=True)
>>> i.is_rational
True
```

Furthermore, expressions compute the assumptions on themselves based on the assumptions of their arguments. For instance, if x and y are both created with `positive=True`, then `(x + y).is_positive` will be `True` whereas `(x - y).is_positive` will be `None`.

2.4 Extensibility

While the core of SymPy is relatively small, it has been extended to a wide variety of domains by a broad range of contributors. This is due, in part, to the fact that the same language, Python, is used both for the internal implementation and the external usage by users. All of

⁶SymPy assumes that two expressions A and B commute with each other multiplicatively, that is, $A \cdot B = B \cdot A$, unless they both have `commutative=False`. Many algorithms in SymPy require special consideration to work correctly with noncommutative products.

the extensibility capabilities available to users are also utilized by SymPy itself. This eases the transition pathway from SymPy user to SymPy developer.

The typical way to create a custom SymPy object is to subclass an existing SymPy class, usually `Basic`, `Expr`, or `Function`. As it was stated before, almost all SymPy classes used for expression trees should be subclasses of the base class `Basic`. `Expr` is the subclass for mathematical expressions (including `Symbol`) that can be added and multiplied together. Instances of `Expr` typically represent complex numbers, but may also include other “rings” like matrix expressions. Not all SymPy classes are subclasses of `Expr`. For instance, logic expressions such as `And(x, y)` are subclasses of `Basic` but not of `Expr`.

The `Function` class is a subclass of `Expr` which makes it easier to define mathematical functions called with arguments. This includes named functions like $\sin(x)$ and $\log(x)$ as well as undefined functions like $f(x)$. Subclasses of `Function` should define a class method `eval`, which returns a canonical form of the function application (usually an instance of some other class, i.e. a `Number`) or `None`, if for given arguments that function should not be automatically evaluated.

Many SymPy functions perform various evaluations down the expression tree. Classes define their behavior in such functions by defining a relevant `_eval_*` method. For instance, an object can indicate to the `diff` function how to take the derivative of itself by defining the `_eval_derivative(self, x)` method, which may in turn call `diff` on its `args`. (Subclasses of `Function` should implement `fdiff` method instead, it returns the derivative of the function without considering the chain rule.) The most common `_eval_*` methods relate to the assumptions: `_eval_is_assumption` is used to deduce *assumption* on the object.

As an example of the notions presented in this section, Listing 1 presents a minimal version of the gamma function $\Gamma(x)$ from SymPy, which evaluates itself on positive integer arguments, has the positive and real assumptions defined, can be rewritten in terms of factorial with `gamma(x).rewrite(factorial)`, and can be differentiated. `self.func` is used throughout instead of referencing `gamma` explicitly so that potential subclasses of `gamma` can reuse the methods.

Listing 1. A minimal implementation of `sympy.gamma`.

```
from sympy import Integer, Function, floor, factorial, polygamma

class gamma(Function)
    @classmethod
    def eval(cls, arg):
        if isinstance(arg, Integer) and arg.is_positive:
            return factorial(arg - 1)

    def _eval_is_positive(self):
        x = self.args[0]
        if x.is_positive:
            return True
        elif x.is_noninteger:
            return floor(x).is_even

    def _eval_is_real(self):
        x = self.args[0]
        # noninteger means real and not integer
        if x.is_positive or x.is_noninteger:
            return True

    def _eval_rewrite_as_factorial(self, z):
        return factorial(z - 1)

    def fdiff(self, argindex=1):
        from sympy.core.function import ArgumentIndexError
        if argindex == 1:
```

```

286         return self.func(self.args[0])*polygamma(0, self.args[0])
287     else:
288         raise ArgumentIndexError(self, argindex)

```

289 The gamma function implemented in SymPy has many more capabilities than the above listing,
290 such as evaluation at rational points and series expansion.

291 3 FEATURES

292 Although SymPy's extensive feature set cannot be covered in-depth in this paper, calculus and
293 other bedrock areas are discussed in their own subsections. Additionally, Table 1 gives a compact
294 listing of all major capabilities present in the SymPy codebase. This grants a sampling from the
295 breadth of topics and application domains that SymPy services. Unless stated otherwise, all
296 features noted in Table 1 are symbolic in nature. Numeric features are discussed in Section 4.

Table 1. SymPy Features and Descriptions

Feature	Description
Calculus	Algorithms for computing derivatives, integrals, and limits.
Category Theory	Representation of objects, morphisms, and diagrams. Tools for drawing diagrams with Xy-pic.
Code Generation	Generation of compilable and executable code in a variety of different programming languages from expressions directly. Target languages include C, Fortran, Julia, JavaScript, Mathematica, MATLAB and Octave, Python, and Theano.
Combinatorics & Group Theory	Permutations, combinations, partitions, subsets, various permutation groups (such as polyhedral, Rubik, symmetric, and others), Gray codes [29], and Prufer sequences [4].
Concrete Math	Summation, products, tools for determining whether summation and product expressions are convergent, absolutely convergent, hypergeometric, and for determining other properties; computation of Gosper's normal form [34] for two univariate polynomials.
Cryptography	Block and stream ciphers, including shift, Affine, substitution, Vigenère's, Hill's, bifid, RSA, Kid RSA, linear-feedback shift registers, and Elgamal encryption.
Differential Geometry	Representations of manifolds, metrics, tensor products, and coordinate systems in Riemannian and pseudo-Riemannian geometries [43].
Geometry	Representations of 2D geometrical entities, such as lines and circles. Enables queries on these entities, such as asking the area of an ellipse, checking for collinearity of a set of points, or finding the intersection between objects.
Lie Algebras	Representations of Lie algebras and root systems.
Logic	Boolean expressions, equivalence testing, satisfiability, and normal forms.
Matrices	Tools for creating matrices of symbols and expressions. Both sparse and dense representations, as well as symbolic linear algebraic operations (e.g., inversion and factorization), are supported.
Matrix Expressions	Matrices with symbolic dimensions (unspecified entries). Block matrices.
Number Theory	Prime number generation, primality testing, integer factorization, continued fractions, Egyptian fractions, modular arithmetic, quadratic residues, partitions, binomial and multinomial coefficients, prime number tools, hexadecimal digits of π , and integer factorization.
Plotting	Hooks for visualizing expressions via matplotlib [21] or as text drawings when lacking a graphical back-end. 2D function plotting, 3D function plotting, and 2D implicit function plotting are supported.

Polynomials	Polynomial algebras over various coefficient domains. Functionality ranges from simple operations (e.g., polynomial division) to advanced computations (e.g., Gröbner bases [1] and multivariate factorization over algebraic number domains).
Printing	Functions for printing SymPy expressions in the terminal with ASCII or Unicode characters and converting SymPy expressions to L ^A T _E X and MathML.
Quantum Mechanics	Quantum states, bra-ket notation, operators, basis sets, representations, tensor products, inner products, outer products, commutators, anticommutators, and specific quantum system implementations.
Series	Series expansion, sequences, and limits of sequences. This includes Taylor, Laurent, and Puiseux series as well as special series, such as Fourier and formal power series.
Sets	Representations of empty, finite, and infinite sets (including special sets such as for all natural, integer, and complex numbers). Operations on sets such as union, intersection, Cartesian product, and building sets from other sets are supported.
Simplification	Functions for manipulating and simplifying expressions. Includes algorithms for simplifying hypergeometric functions, trigonometric expressions, rational functions, combinatorial functions, square root denesting, and common subexpression elimination.
Solvers	Functions for symbolically solving equations, systems of equations, both linear and non-linear, inequalities, ordinary differential equations, partial differential equations, Diophantine equations, and recurrence relations.
Special Functions	Implementations of a number of well known special functions, including Dirac delta, Gamma, Beta, Gauss error functions, Fresnel integrals, Exponential integrals, Logarithmic integrals, Trigonometric integrals, Bessel, Hankel, Airy, B-spline, Riemann Zeta, Dirichlet eta, polylogarithm, Lerch transcendent, hypergeometric, elliptic integrals, Mathieu, Jacobi polynomials, Gegenbauer polynomial, Chebyshev polynomial, Legendre polynomial, Hermite polynomial, Laguerre polynomial, and spherical harmonic functions.
Statistics	Support for a random variable type as well as the ability to declare this variable from prebuilt distribution functions such as Normal, Exponential, Coin, Die, and other custom distributions [38].
Tensors	Symbolic manipulation of indexed objects.
Vectors	Basic operations on vectors and differential calculus with respect to 3D Cartesian coordinate systems.

3.1 Simplification

The generic way to simplify an expression is by calling the `simplify` function. It must be emphasized that simplification is not a rigorously defined mathematical operation [8]. The `simplify` function applies several simplification routines along with heuristics to make the output expression “simple”.⁷

It is often preferable to apply more directed simplification functions. These apply very specific rules to the input expression and are typically able to make guarantees about the output. For instance, the `factor` function, given a polynomial with rational coefficients in several variables, is guaranteed to produce a factorization into irreducible factors. Table 2 lists common simplification functions.

Table 2. Some SymPy Simplification Functions

⁷The `measure` parameter of the `simplify` function lets the user specify the Python function used to determine how complex an expression is. The default `measure` function returns the total number of operations in the expression.

<code>expand</code>	expand the expression
<code>factor</code>	factor a polynomial into irreducibles
<code>collect</code>	collect polynomial coefficients
<code>cancel</code>	rewrite a rational function as p/q with common factors canceled
<code>apart</code>	compute the partial fraction decomposition of a rational function
<code>trigsimp</code>	simplify trigonometric expressions [14]
<code>hyperexpand</code>	expand hypergeometric functions [36, 37]

3.2 Calculus

SymPy provides all the basic operations of calculus, such as calculating limits, derivatives, integrals, or summations.

Limits are computed with the `limit` function, using the Gruntz algorithm [18] for computing symbolic limits and heuristics (a description of the Gruntz algorithm may be found in the supplement). For example, the following computes $\lim_{x \rightarrow \infty} x \sin(\frac{1}{x}) = 1$. Note that SymPy denotes ∞ as `oo`.

```
>>> limit(x*sin(1/x), x, oo)
1
```

As a more complex example, SymPy computes

$$\lim_{x \rightarrow 0} \left(2e^{\frac{1 - \cos(x)}{\sin(x)}} - 1 \right)^{\frac{\sinh(x)}{\operatorname{atan}^2(x)}} = e.$$

```
>>> limit((2*E**((1-cos(x))/sin(x))-1)**(sinh(x)/atan(x)**2), x, 0)
E
```

Derivatives are computed with the `diff` function, which recursively uses the various differentiation rules.

```
>>> diff(sin(x)*exp(x), x)
exp(x)*sin(x) + exp(x)*cos(x)
```

Integrals are calculated with the `integrate` function. SymPy implements a combination of the Risch algorithm [6], table lookups, a reimplement of Manuel Bronstein’s “Poor Man’s Integrator” [5], and an algorithm for computing integrals based on Meijer G-functions [36, 37]. These allow SymPy to compute a wide variety of indefinite and definite integrals. The Meijer G-function algorithm and the Risch algorithm are respectively demonstrated below by the computation of

$$\int_0^\infty e^{-st} \log(t) dt = -\frac{\log(s) + \gamma}{s}$$

and

$$\int \frac{-2x^2(\log(x) + 1)e^{x^2} + (e^{x^2} + 1)^2}{x(e^{x^2} + 1)^2(\log(x) + 1)} dx = \log(\log(x) + 1) + \frac{1}{e^{x^2} + 1}.$$

```
>>> s, t = symbols('s t', positive=True)
>>> integrate(exp(-s*t)*log(t), (t, 0, oo)).simplify()
-(log(s) + EulerGamma)/s
>>> integrate((-2*x**2*(log(x) + 1)*exp(x**2) +
... (exp(x**2) + 1)**2)/(x*(exp(x**2) + 1)**2*(log(x) + 1)), x)
log(log(x) + 1) + 1/(exp(x**2) + 1)
```

Summations are computed with `summation` using a combination of Gosper’s algorithm [17], an algorithm that uses Meijer G-functions [36, 37], and heuristics. Products are computed with `product` function via a suite of heuristics.

```

331 >>> i, n = symbols('i n')
332 >>> summation(2**i, (i, 0, n - 1))
333 2**n - 1
334 >>> summation(i*factorial(i), (i, 1, n))
335 n*factorial(n) + factorial(n) - 1

```

Integrals, derivatives, summations, products, and limits that cannot be computed return unevaluated objects. These can also be created directly if the user chooses.

```

338 >>> integrate(x**x, x)
339 Integral(x**x, x)
340 >>> Sum(2**i, (i, 0, n - 1))
341 Sum(2**i, (i, 0, n - 1))

```

3.3 Polynomials

SymPy implements a suite of algorithms for polynomial manipulation, which ranges from relatively simple algorithms for doing arithmetic of polynomials, to advanced methods for factoring multivariate polynomials into irreducibles, symbolically determining real and complex root isolation intervals, or computing Gröbner bases.

Polynomial manipulation is useful in its own right. Within SymPy, though, it is mostly used indirectly as a tool in other areas of the library. In fact, many mathematical problems in symbolic computing are first expressed using entities from the symbolic core, preprocessed, and then transformed into a problem in the polynomial algebra, where generic and efficient algorithms are used to solve the problem. The solutions to the original problem are subsequently recovered from the results. This is a common scheme in symbolic integration or summation algorithms.

SymPy implements dense and sparse polynomial representations.⁸ Both are used in the univariate and multivariate cases. The dense representation is the default for univariate polynomials. For multivariate polynomials, the choice of representation is based on the application. The most common case for the sparse representation is algorithms for computing Gröbner bases (Buchberger, F4, and F5) [7, 11, 12]. This is because different monomial orderings can be expressed easily in this representation. However, algorithms for computing multivariate GCDs or factorizations, at least those currently implemented in SymPy [31], are better expressed when the representation is dense. The dense multivariate representation is specifically a recursively-dense representation, where polynomials in $K[x_0, x_1, \dots, x_n]$ are viewed as a polynomials in $K[x_0][x_1] \dots [x_n]$. Note that despite this, the coefficient domain K , can be a multivariate polynomial domain as well. The dense recursive representation in Python gets inefficient as the number of variables increases.

Some examples for the `sympy.polys` module can be found in the supplement.

3.4 Printers

SymPy has a rich collection of expression printers. By default, an interactive Python session will render the `str` form of an expression, which has been used in all the examples in this paper so far. The `str` form of an expression is valid Python and roughly matches what a user would type to enter the expression.

```

371 >>> phi0 = Symbol('phi0')
372 >>> str(Integral(sqrt(phi0), phi0))
373 'Integral(sqrt(phi0), phi0)'

```

Expressions can be printed in 2D with monospace fonts via `pprint`. Unicode characters are used for rendering mathematical symbols such as integral signs, square roots, and parentheses. Greek letters and subscripts in symbol names that have Unicode code points associated are also rendered automatically.

⁸In a dense representation, the coefficients for all terms up to the degree of each variable are stored in memory. In a sparse representation, only the nonzero coefficients are stored.

```

378 >>> pprint(Integral(sqrt(phi0 + 1), phi0))
379
380

```

$$\int \sqrt{\varphi_0 + 1} \, d(\varphi_0)$$

Alternately, the `use_unicode=False` flag can be set, which causes the expression to be printed using only ASCII characters.

```

381 >>> pprint(Integral(sqrt(phi0 + 1), phi0), use_unicode=False)
382 /
383 |
384 | _____
385 | \ / phi0 + 1 d(phi0)
386 |
387 /

```

The function `latex` returns a \LaTeX representation of an expression.

```

388 >>> print(latex(Integral(sqrt(phi0 + 1), phi0)))
389 \int \sqrt{\phi_{0} + 1}\, d\phi_{0}

```

Users are encouraged to run the `init_printing` function at the beginning of interactive sessions, which automatically enables the best pretty printing supported by their environment. In the Jupyter Notebook or Qt Console [32], the \LaTeX printer is used to render expressions using MathJax or \LaTeX , if it is installed on the system. The 2D text representation is used otherwise.

Other printers such as MathML are also available. SymPy uses an extensible printer subsystem for customizing any given printer, and allows custom objects to define their printing behavior for any printer. The code generation functionality of SymPy relies on this subsystem to convert expressions into code in various target programming languages.

3.5 Solvers

SymPy has a module of equation solvers that can handle ordinary differential equations, recurrence relationships, Diophantine equations, and algebraic equations. There is also rudimentary support for simple partial differential equations.

There are two functions for solving algebraic equations in SymPy: `solve` and `solveset`. `solveset` has several design changes with respect to the older `solve` function. This distinction is present in order to resolve the usability issues with the previous `solve` function API while maintaining backward compatibility with earlier versions of SymPy. `solveset` only requires essential input information from the user. The function signatures of `solve` and `solveset` are

```

409 solve(f, *symbols, **flags)
410 solveset(f, symbol, domain=S.Complexes)

```

The `domain` parameter is typically either `S.Complexes` (the default) or `S.Reals`; the latter causes `solveset` to only return real solutions.

An important difference between the two functions is that the output API of `solve` varies with input (sometimes returning a Python list and sometimes a Python dictionary) whereas `solveset` always returns a SymPy set object.

Both functions implicitly assume that expressions are equal to 0. For instance, `solveset(x - 1, x)` solves $x - 1 = 0$ for x .

`solveset` is under active development as a planned replacement for `solve`. There are certain features which are implemented in `solve` that are not yet implemented in `solveset`, including multivariate systems, and some transcendental equations.

More examples of `solveset` and `solve` can be found in the supplement.

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```

522 >>> mpmath.mp.dps = 15
523 >>> mpmath.meijerg([], [0], [[-0.5, -1, -1.5], []], 10000)
524 mpf('2.4392576907199564e-94')

```

525 Equivalently, with SymPy's interface this function can be evaluated as:

```

526 >>> meijerg([], [0], [[-S(1)/2, -1, -S(3)/2], []], 10000).evalf()
527 2.43925769071996e-94

```

528 Symbolic integration and summation often produce hypergeometric and Meijer G-function
529 closed forms (see Subsection 3.2); numerical evaluation of such special functions is a useful
530 complement to direct numerical integration and summation.

531 5 DOMAIN SPECIFIC SUBMODULES

532 SymPy includes several packages that allow users to solve domain specific problems. For example,
533 a comprehensive physics package is included that is useful for solving problems in mechanics,
534 optics, and quantum mechanics along with support for manipulating physical quantities with
535 units.

536 5.1 Classical Mechanics

537 One of the core domains that SymPy supports is the physics of classical mechanics. This is in
538 turn separated into two distinct components: vector algebra and mechanics.

539 5.1.1 Vector Algebra

540 The `sympy.physics.vector` submodule provides reference frame-, time-, and space-aware vector
541 and dyadic objects that allow for three-dimensional operations such as addition, subtraction,
542 scalar multiplication, inner and outer products, and cross products. Both of these objects can
543 be written in very compact notation that make it easy to express the vectors and dyadics in
544 terms of multiple reference frames with arbitrarily defined relative orientations. The vectors
545 are used to specify the positions, velocities, and accelerations of points; orientations, angular
546 velocities, and angular accelerations of reference frames; and forces and torques. The dyadics are
547 essentially reference frame-aware 3×3 tensors [44]. The vector and dyadic objects can be used
548 for any one-, two-, or three-dimensional vector algebra, and they provide a strong framework for
549 building physics and engineering tools.

550 The following Python code demonstrates how a vector is created using the orthogonal unit
551 vectors of three reference frames that are oriented with respect to each other, and the result of
552 expressing the vector in the A frame. The B frame is oriented with respect to the A frame using
553 Z-X-Z Euler Angles of magnitude π , $\frac{\pi}{2}$, and $\frac{\pi}{3}$, respectively, whereas the C frame is oriented with
554 respect to the B frame through a simple rotation about the B frame's X unit vector through $\frac{\pi}{2}$.

```

555 >>> from sympy.physics.vector import ReferenceFrame
556 >>> A = ReferenceFrame('A')
557 >>> B = ReferenceFrame('B')
558 >>> C = ReferenceFrame('C')
559 >>> B.orient(A, 'body', (pi, pi/3, pi/4), 'zxz')
560 >>> C.orient(B, 'axis', (pi/2, B.x))
561 >>> v = 1*A.x + 2*B.z + 3*C.y
562 >>> v
563 A.x + 2*B.z + 3*C.y
564 >>> v.express(A)
565 A.x + 5*sqrt(3)/2*A.y + 5/2*A.z

```

566 5.1.2 Mechanics

567 The `sympy.physics.mechanics` package utilizes the `sympy.physics.vector` package to populate
568 time-aware particle and rigid-body objects to fully describe the kinematics and kinetics of a
569 rigid multi-body system. These objects store all of the information needed to derive the ordinary
570 differential or differential algebraic equations that govern the motion of the system, i.e., the

equations of motion. These equations of motion abide by Newton's laws of motion and can handle arbitrary kinematic constraints or complex loads. The package offers two automated methods for formulating the equations of motion based on Lagrangian Dynamics [25] and Kane's Method [23]. Lastly, there are automated linearization routines for constrained dynamical systems [33].

5.2 Quantum Mechanics

The `sympy.physics.quantum` package has extensive capabilities for performing symbolic quantum mechanics, using Python objects to represent the different mathematical objects relevant in quantum theory [40]: states (bras and kets), operators (unitary, Hermitian, etc.), and basis sets, as well as operations on these objects such as representations, tensor products, inner products, outer products, commutators, and anticommutators. The base objects are designed in the most general way possible to enable any particular quantum system to be implemented by subclassing the base operators and defining the relevant class methods to provide system-specific logic.

Symbolic quantum operators and states may be defined, and one can perform a full range of operations with them.

```
>>> from sympy.physics.quantum import Commutator, Dagger, Operator
>>> from sympy.physics.quantum import Ket, qapply
>>> A = Operator('A')
>>> B = Operator('B')
>>> C = Operator('C')
>>> D = Operator('D')
>>> a = Ket('a')
>>> comm = Commutator(A, B)
>>> comm
[A,B]
>>> qapply(Dagger(comm*a)).doit()
-<a|*(Dagger(A)*Dagger(B) - Dagger(B)*Dagger(A))
```

Commutators can be expanded using common commutator identities:

```
>>> Commutator(C+B, A*D).expand(commutator=True)
-[A,B]*D - [A,C]*D + A*[B,D] + A*[C,D]
```

On top of this set of base objects, a number of specific quantum systems have been implemented in a fully symbolic framework. These include:

- Many of the exactly solvable quantum systems, including simple harmonic oscillator states and raising/lowering operators, infinite square well states, and 3D position and momentum operators and states.
- Second quantized formalism of non-relativistic many-body quantum mechanics [13].
- Quantum angular momentum [48]. Spin operators and their eigenstates can be represented in any basis and for any quantum numbers. A rotation operator representing the Wigner-D matrix, which may be defined symbolically or numerically, is also implemented to rotate spin eigenstates. Functionality for coupling and uncoupling of arbitrary spin eigenstates is provided, including symbolic representations of Clebsch-Gordon coefficients and Wigner symbols.
- Quantum information and computing [28]. Multidimensional qubit states, and a full set of one- and two-qubit gates are provided and can be represented symbolically or as matrices/vectors. With these building blocks, it is possible to implement a number of basic quantum algorithms including the quantum Fourier transform, quantum error correction, quantum teleportation, Grover's algorithm, dense coding, etc. In addition, any quantum circuit may be plotted using the `circuit_plot` function (Figure 1).

Here are a few short examples of the quantum information and computing capabilities in `sympy.physics.quantum`. Start with a simple four-qubit state and flip the second qubit from the right using a Pauli-X gate:

```

621 >>> from sympy.physics.quantum.qubit import Qubit
622 >>> from sympy.physics.quantum.gate import XGate
623 >>> q = Qubit('0101')
624 >>> q
625 |0101>
626 >>> X = XGate(1)
627 >>> qapply(X*q)
628 |0111>

629 Qubit states can also be used in adjoint operations, tensor products, inner/outer products:

630 >>> Dagger(q)
631 <0101|
632 >>> ip = Dagger(q)*q
633 >>> ip
634 <0101|0101>
635 >>> ip.doit()
636 1

637 Quantum gates (unitary operators) can be applied to transform these states and then classical
638 measurements can be performed on the results:

639 >>> from sympy.physics.quantum.qubit import measure_all
640 >>> from sympy.physics.quantum.gate import H, X, Y, Z
641 >>> c = H(0)*H(1)*Qubit('00')
642 >>> c
643 H(0)*H(1)*|00>
644 >>> q = qapply(c)
645 >>> measure_all(q)
646 [(|00>, 1/4), (|01>, 1/4), (|10>, 1/4), (|11>, 1/4)]

```

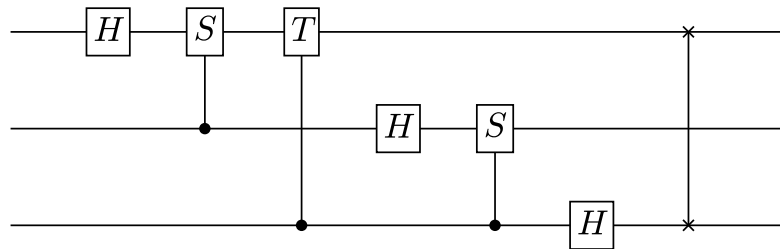


Figure 1. The circuit diagram for a three-qubit quantum Fourier transform generated by SymPy.

```

647 Lastly, the following example demonstrates creating a three-qubit quantum Fourier transform,
648 decomposing it into one- and two-qubit gates, and then generating a circuit plot for the sequence
649 of gates (see Figure 1).

650 >>> from sympy.physics.quantum.qft import QFT
651 >>> from sympy.physics.quantum.circuitplot import circuit_plot
652 >>> fourier = QFT(0,3).decompose()
653 >>> fourier
654 SWAP(0,2)*H(0)*C((0),S(1))*H(1)*C((0),T(2))*C((1),S(2))*H(2)
655 >>> c = circuit_plot(fourier, nqubits=3)

```


6 CONCLUSION AND FUTURE WORK

SymPy is a robust computer algebra system that provides a wide spectrum of features both in traditional computer algebra and in a plethora of scientific disciplines. This allows SymPy to be used in a first-class way with other Python projects, including the scientific Python stack. Unlike many other CASs, SymPy is designed to be used in an extensible way: both as an end-user application and as a library.

SymPy uses Python both as the internal language and the user language. This permits users to access to the same methods that the library implements in order to extend it for their needs. Additionally, SymPy has a powerful assumptions system for declaring and deducing mathematical properties of expressions.

SymPy has modules for many areas of mathematics. This includes functions for simplifying expressions, performing common calculus operations, pretty printing expressions, solving equations, and representing symbolic matrices. Other included areas are discrete math, concrete math, plotting, geometry, statistics, polynomials, sets, series, vectors, combinatorics, group theory, code generation, tensors, Lie algebras, cryptography, and special functions. Additionally, SymPy contains modules targeting certain specific domains, such as classical mechanics and quantum mechanics. This breadth of domains has been engendered by a strong and vibrant user community. Anecdotally, these users likely chose SymPy because of its ease of access.

Some of the planned future work for SymPy includes work on improving code generation, improvements to the speed of SymPy (one area of work in this direction is SymEngine, a C++ symbolic manipulation library that is planned to be usable as a alternative core for SymPy), improving the assumptions system, and improving the solvers module.

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