

# 1 SymPy: Symbolic Computing in Python

## 2 Supplementary material

3 The supplementary material take a deeper look at certain topics in SymPy for which there  
4 was not enough room to discuss in the paper. Section 1 discusses the Gruntz algorithm, used to  
5 calculate limits in the SymPy. Sections 2–8 discuss in depth some selected submodules. Section 9  
6 discusses numerical simplification. Section 10 provides additional examples for topics, discussed  
7 in the main paper. In sections 11 and 12 the SymPy Gamma and the SymPy Live projects are  
8 introduced. Finally, section 14 has a brief comparison of SymPy with Wolfram Mathematica,  
9 and section 13 lists some projects that depend on SymPy.

10 As in the paper, all examples in the supplement assume that the following has been run:

```
11 >>> from sympy import *  
12 >>> x, y, z = symbols('x y z')
```

## 13 1 LIMITS: THE GRUNTZ ALGORITHM

14 SymPy calculates limits using the Gruntz algorithm, as described in [5]. The basic idea is as  
15 follows: any limit can be converted to a limit  $\lim_{x \rightarrow \infty} f(x)$  by substitutions like  $x \rightarrow \frac{1}{x}$ . Then  
16 the subexpression  $\omega$  (that converges to zero as  $x \rightarrow \infty$  faster than all other subexpressions) is  
17 identified in  $f(x)$ , and  $f(x)$  is expanded into a series with respect to  $\omega$ . Any positive powers  
18 of  $\omega$  converge to zero (while negative powers indicate an infinite limit) and any constant term  
19 independent of  $\omega$  determines the limit. When a constant term still depends on  $x$  the Gruntz  
20 algorithm is applied again until a final numerical value is obtained as the limit.

To determine the most rapidly varying subexpression, the comparability classes must first be defined, by calculating  $L$ :

$$L \equiv \lim_{x \rightarrow \infty} \frac{\log |f(x)|}{\log |g(x)|} \quad (1)$$

The relations  $<$ ,  $>$ , and  $\sim$  are defined as follows:  $f > g$  when  $L = \pm\infty$  (it is said that  $f$  is more rapidly varying than  $g$ , i.e.,  $f$  goes to  $\infty$  or 0 faster than  $g$ ),  $f < g$  when  $L = 0$  ( $f$  is less rapidly varying than  $g$ ) and  $f \sim g$  when  $L \neq 0, \pm\infty$  (both  $f$  and  $g$  are bounded from above and below by suitable integral powers of the other). Note that if  $f > g$ , then  $f > g^n$  for any  $n$ . Here are some examples of comparability classes:

$$\begin{aligned} 2 &< x < e^x < e^{x^2} < e^{e^x} \\ 2 &\sim 3 \sim -5 \\ x &\sim x^2 \sim x^3 \sim \frac{1}{x} \sim x^m \sim -x \\ e^x &\sim e^{-x} \sim e^{2x} \sim e^{x+e^{-x}} \\ f(x) &\sim \frac{1}{f(x)} \end{aligned}$$

The Gruntz algorithm is now illustrated with the following example:

$$f(x) = e^{x+2e^{-x}} - e^x + \frac{1}{x}. \quad (2)$$

21 First, the set of most rapidly varying subexpressions is determined—the so-called *mrv set*. For (2),  
22 the mrv set  $\{e^x, e^{-x}, e^{x+2e^{-x}}\}$  is obtained. These are all subexpressions of (2) and they all  
23 belong to the same comparability class. This calculation can be done using SymPy as follows:

```

24 >>> from sympy.series.gruntz import mrv
25 >>> mrv(exp(x+2*exp(-x))-exp(x) + 1/x, x)[0].keys()
26 dict_keys([exp(x + 2*exp(-x)), exp(x), exp(-x)])

```

Next, an arbitrary item  $\omega$  is taken from mrv set that converges to zero for  $x \rightarrow \infty$  and doesn't have subexpressions in the given mrv set. If such a term is not present in the mrv set (i.e., all terms converge to infinity instead of zero), the relation  $f(x) \sim \frac{1}{f(x)}$  can be used. In the considered case, only item  $\omega = e^{-x}$  can be accepted.

The next step is to rewrite the mrv set in terms of  $\omega = g(x)$ . Every element  $f(x)$  of the mrv set is rewritten as  $A\omega^c$ , where

$$c = \lim_{x \rightarrow \infty} \frac{\log f(x)}{\log g(x)}, \quad A = e^{\log f - c \log g} \quad (3)$$

Note that this step includes calculation of more simple limits, for instance

$$\lim_{x \rightarrow \infty} \frac{\log e^{x+2e^{-x}}}{\log e^{-x}} = \lim_{x \rightarrow \infty} \frac{x+2e^{-x}}{-x} = -1 \quad (4)$$

In this example we obtain the rewritten mrv set:  $\{\frac{1}{\omega}, \omega, \frac{1}{\omega}e^{2\omega}\}$ . This can be done in SymPy with

```

32 >>> from sympy.series.gruntz import mrv, rewrite
33 >>> m = mrv(exp(x+2*exp(-x))-exp(x) + 1/x, x)
34 >>> w = Symbol('w')
35 >>> rewrite(m[1], m[0], x, w)[0]
36 1/x + exp(2*w)/w - 1/w

```

Then the rewritten subexpressions are substituted back into  $f(x)$  in (2) and the result is expanded with respect to  $\omega$ :

$$f(x) = \frac{1}{x} - \frac{1}{\omega} + \frac{1}{\omega}e^{2\omega} = 2 + \frac{1}{x} + 2\omega + O(\omega^2) \quad (5)$$

Since  $\omega$  is from the mrv set, then in the limit as  $x \rightarrow \infty$ ,  $\omega \rightarrow 0$ , and so  $2\omega + O(\omega^2) \rightarrow 0$  in (5):

$$f(x) = \frac{1}{x} - \frac{1}{\omega} + \frac{1}{\omega}e^{2\omega} = 2 + \frac{1}{x} + 2\omega + O(\omega^2) \rightarrow 2 + \frac{1}{x} \quad (6)$$

In this example the result  $(2 + \frac{1}{x})$  still depends on  $x$ , so the above procedure is repeated until just a value independent of  $x$  is obtained. This is the final limit. In the above case the limit is 2, as can be verified by SymPy:<sup>1</sup>

```

40 >>> limit(exp(x+2*exp(-x))-exp(x) + 1/x, x, oo)
41 2

```

In general, when  $f(x)$  is expanded in terms of  $\omega$ , the following is obtained:

$$f(x) = \underbrace{O\left(\frac{1}{\omega^3}\right)}_{\infty} + \underbrace{\frac{C_{-2}(x)}{\omega^2}}_{\infty} + \underbrace{\frac{C_{-1}(x)}{\omega}}_{\infty} + C_0(x) + \underbrace{C_1(x)\omega}_0 + \underbrace{O(\omega^2)}_0 \quad (7)$$

The positive powers of  $\omega$  are zero. If there are any negative powers of  $\omega$ , then the result of the limit is infinity, otherwise the limit is equal to  $\lim_{x \rightarrow \infty} C_0(x)$ . The expression  $C_0(x)$  is always simpler than original  $f(x)$ , same is true for limits, arising in the rewrite stage (3), so the algorithm converges. A proof of this and further details on the algorithm are given in Gruntz's PhD thesis [5].

<sup>1</sup>To see intermediate steps, discussed above, interested readers can switch on debugging output by setting the environment variable `SYMPY_DEBUG=True`, before importing anything from the SymPy namespace.

## 47 2 SERIES

### 48 2.1 Series Expansion

49 SymPy is able to calculate the symbolic series expansion of an arbitrary series or expression  
50 involving elementary and special functions and multiple variables. For this it has two different  
51 implementations: the `series` method and Ring Series.

52 The first approach stores a series as an instance of the `Expr` class. Each function has its  
53 specific implementation of its expansion, which is able to evaluate the Puiseux series expansion  
54 about a specified point. For example, consider a Taylor expansion about 0:

```
55 >>> series(sin(x+y) + cos(x*y), x, 0, 2)
56 1 + sin(y) + x*cos(y) + O(x**2)
```

57 The newer and much faster approach called Ring Series makes use of the fact that a truncated  
58 Taylor series is simply a polynomial. Correspondingly, they may be represented by sparse  
59 polynomials which perform well in a under a wide range of cases. Ring Series also gives the user  
60 the freedom to choose the type of coefficients to use, resulting in faster operations on certain  
61 types.

62 For this, several low-level methods for expansion of trigonometric, hyperbolic and other  
63 elementary operations (like series inversion, calculating the  $n$ th root, etc.) are implemented  
64 using variants of the Newton Method [Brent and Zimmermann]. All these support Puiseux series  
65 expansion. The following example demonstrates the use of an elementary function that calculates  
66 the Taylor expansion of the sine of a series.

```
67 >>> from sympy.polys.ring_series import rs_sin
68 >>> R, t = ring('t', QQ)
69 >>> rs_sin(t**2 + t, t, 5)
70 -1/2*t**4 - 1/6*t**3 + t**2 + t
```

71 The function `sympy.polys.rs_series` makes use of these elementary functions to expand  
72 an arbitrary SymPy expression. It does so by following a recursive strategy of expanding the  
73 lowermost functions first and then composing them recursively to calculate the desired expansion.  
74 Currently, it only supports expansion about 0 and is under active development. Ring Series is  
75 several times faster than the default implementation with the speed difference increasing with  
76 the size of the series. The `sympy.polys.rs_series` takes as input any SymPy expression and  
77 hence there is no need to explicitly create a polynomial ring. An example demonstrating its use:

```
78 >>> from sympy.polys.ring_series import rs_series
79 >>> from sympy.abc import a, b
80 >>> rs_series(sin(a + b), a, 4)
81 -1/2*(sin(b))*a**2 + (sin(b)) - 1/6*a**3*(cos(b)) + a*(cos(b))
```

### 82 2.2 Formal Power Series

83 SymPy can be used for computing the formal power series of a function. The implementation  
84 is based on the algorithm described in the paper on formal power series [6]. The advantage of  
85 this approach is that an explicit formula for the coefficients of the series expansion is generated  
86 rather than just computing a few terms.

87 The following example shows how to use `fps`:

```
88 >>> f = fps(sin(x), x, x0=0)
89 >>> f.truncate(6)
90 x - x**3/6 + x**5/120 + O(x**6)
91 >>> f[15]
92 -x**15/1307674368000
```

### 2.3 Fourier Series

SymPy provides functionality to compute Fourier series of a function using the `fourier_series` function:

```
96 >>> L = symbols('L')
97 >>> expr = 2 * (Heaviside(x/L) - Heaviside(x/L - 1)) - 1
98 >>> f = fourier_series(expr, (x, 0, 2*L))
99 >>> f.truncate(3)
100 4*sin(pi*x/L)/pi + 4*sin(3*pi*x/L)/(3*pi) + 4*sin(5*pi*x/L)/(5*pi)
```

## 3 LOGIC

SymPy supports construction and manipulation of boolean expressions through the `sympy.logic` submodule. SymPy symbols can be used as propositional variables and subsequently be replaced with `True` or `False` values. Many functions for manipulating boolean expressions have been implemented in the `sympy.logic` submodule.

### 3.1 Constructing Boolean Expressions

A boolean variable can be declared as a SymPy `Symbol`. Python operators `&`, `|` and `~` are overridden when using SymPy objects to use the SymPy functionality for logical `And`, `Or`, and `Not`. Other logic functions are also integrated into SymPy, including `Xor` and `Implies`, which are constructed with `^` and `>>`, respectively. Expressions can therefore be constructed either by using the shortcut operator notation or by directly creating the relevant objects: `And()`, `Or()`, `Not()`, `Xor()`, `Implies()`, `Nand()`, `Nor()`, etc.:

```
113 >>> e = (x & y) | z
114 >>> e.subs({x: True, y: True, z: False})
115 True
```

### 3.2 CNF and DNF

Any boolean expression can be converted to conjunctive normal form, disjunctive normal form, or negation normal form. The API also exposes methods to check if a boolean expression is in any of the aforementioned forms.

```
120 >>> from sympy.logic.boolalg import is_dnf, is_cnf
121 >>> to_cnf((x & y) | z)
122 And(Or(x, z), Or(y, z))
123 >>> to_dnf(x & (y | z))
124 Or(And(x, y), And(x, z))
125 >>> is_cnf((x | y) & z)
126 True
127 >>> is_dnf((x & y) | z)
128 True
```

### 3.3 Simplification and Equivalence

The `sympy.logic` submodule supports simplification of given boolean expression by making deductions from the expression. Equivalence of two logical expressions can also be checked. In the case of equivalence, the function `bool_map` can be used to show which variables of the first expression correspond to which variables of the second one.

```
134 >>> a, b, c = symbols('a b c')
135 >>> e = a & (~a | ~b) & (a | c)
136 >>> simplify(e)
137 And(Not(b), a)
138 >>> e1 = a & (b | c)
139 >>> e2 = (x & y) | (x & z)
140 >>> bool_map(e1, e2)
141 (And(Or(b, c), a), {a: x, b: y, c: z})
```

### 142 3.4 SAT Solving

143 The submodule also supports satisfiability (SAT) checking of a given boolean expression. If an  
144 expression is satisfiable, it is possible to return a variable assignment which satisfies it. The  
145 API also supports listing all possible assignments. The SAT solver has a clause learning DPLL  
146 algorithm implemented with a watch literal scheme and VSIDS heuristic [9].

```
147 >>> satisfiable(a & (~a | b) & (~b | c) & ~c)
148 False
149 >>> satisfiable(a & (~a | b) & (~b | c) & c)
150 {a: True, b: True, c: True}
```

## 151 4 DIOPHANTINE EQUATIONS

152 Diophantine equations play a central role in number theory. A Diophantine equation has the  
153 form,  $f(x_1, x_2, \dots, x_n) = 0$  where  $n \geq 2$  and  $x_1, x_2, \dots, x_n$  are integer variables. If there are  $n$   
154 integers  $a_1, a_2, \dots, a_n$  such that  $x_1 = a_1, x_2 = a_2, \dots, x_n = a_n$  satisfies the above equation, the  
155 equation is said to be solvable.

156 Currently, the following five types of Diophantine equations can be solved using SymPy's  
157 Diophantine submodule ( $a_1, \dots, a_{n+1}$ ,  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ ,  $f$ , and  $k$  are explicitly given rational constants):

- 158 • Linear Diophantine equations:  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$
- 159 • General binary quadratic equation:  $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- 160 • Homogeneous ternary quadratic equation:  $ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0$
- 161 • Extended Pythagorean equation:  $a_1x_1^2 + a_2x_2^2 + \dots + a_nx_n^2 = a_{n+1}x_{n+1}^2$
- 162 • General sum of squares:  $x_1^2 + x_2^2 + \dots + x_n^2 = k$

163 The `diophantine` function factors the equation it is given (if possible), solves each factor  
164 separately, and combines the results to give a final solution set. The following examples illustrate  
165 some of the basic functionalities of the Diophantine submodule.

```
166 >>> from sympy.solvers.diophantine import *
167 >>> diophantine(2*x + 3*y - 5)
168 set([(3*t_0 - 5, -2*t_0 + 5)])
169
170 >>> diophantine(2*x + 4*y - 3)
171 set()
172
173 >>> diophantine(x**2 - 4*x*y + 8*y**2 - 3*x + 7*y - 5)
174 set([(2, 1), (5, 1)])
175
176 >>> diophantine(x**2 - 4*x*y + 4*y**2 - 3*x + 7*y - 5)
177 set([(-2*t**2 - 7*t + 10, -t**2 - 3*t + 5)])
178
179 >>> diophantine(3*x**2 + 4*y**2 - 5*z**2 + 4*x*y - 7*y*z + 7*z*x)
180 set([(-16*p**2 + 28*p*q + 20*q**2,
181 3*p**2 + 38*p*q - 25*q**2,
182 4*p**2 - 24*p*q + 68*q**2)])
183
184 >>> x1, x2, x3, x4, x5, x6 = symbols('x1, x2, x3, x4, x5, x6')
185 >>> diophantine(9*x1**2 + 16*x2**2 + x3**2 + 49*x4**2 + 4*x5**2 - 25*x6**2)
186 set([(70*t1**2 + 70*t2**2 + 70*t3**2 + 70*t4**2 - 70*t5**2, 105*t1*t5,
187 420*t2*t5, 60*t3*t5, 210*t4*t5,
188 42*t1**2 + 42*t2**2 + 42*t3**2 + 42*t4**2 + 42*t5**2)])
189
```

```

190 >>> a, b, c, d = symbols('a b c d')
191 >>> diophantine(a**2 + b**2 + c**2 + d**2 - 23)
192 set([(2, 3, 3, 1)])

```

## 193 5 SETS

194 SymPy supports representation of a wide variety of mathematical sets. This is achieved by first  
 195 defining abstract representations of atomic set classes and then combining and transforming  
 196 them using various set operations.

197 Each of the set classes inherits from the base class `Set` and defines methods to check  
 198 membership and calculate unions, intersections, and set differences. When these methods are  
 199 not able to evaluate to atomic set classes, they are represented as abstract unevaluated objects.

200 SymPy has the following atomic set classes:

- 201 • `EmptySet` represents the empty set  $\emptyset$ .
- 202 • `UniversalSet` is an abstract “universal set” of which everything is a member. The union of  
 203 the universal set with any set gives the universal set and the intersection gives the other  
 204 set itself.
- 205 • `FiniteSet` is functionally equivalent to Python’s built in `set` object. Its members can be  
 206 any SymPy object including other sets.
- 207 • `Integers` represents the set of integers  $\mathbb{Z}$ .
- 208 • `Naturals` represents the set of natural numbers  $\mathbb{N}$ , i.e., the set of positive integers.
- 209 • `Naturals0` represents the set of whole numbers  $\mathbb{N}_0$ , which are all the non-negative integers.
- 210 • `Range` represents a range of integers. A range is defined by specifying a start value, an end  
 211 value, and a step size. The enumeration of a `Range` object is functionally equivalent to  
 212 Python’s `range` except it supports infinite endpoints, allowing the representation of infinite  
 213 ranges.
- 214 • `Interval` represents an interval of real numbers. It is defined by giving the start and the  
 215 end points and by specifying if the interval is open or closed on the respective ends.

216 Other than unevaluated classes of `Union`, `Intersection`, and `Complement` operations, SymPy  
 217 has the following set classes.

- 218 • `ProductSet` defines the Cartesian product of two or more sets. The product set is useful  
 219 when representing higher dimensional spaces. For example, to represent a three-dimensional  
 220 space, SymPy uses the Cartesian product of three real sets.
- 221 • `ImageSet` represents the image of a function when applied to a particular set. The image  
 222 set of a function  $F$  with respect to a set  $S$  is  $\{F(x) \mid x \in S\}$ . SymPy uses image sets to  
 223 represent sets of infinite solutions of equations such as  $\sin(x) = 0$ .
- 224 • `ConditionSet` represents a subset of a set whose members satisfy a particular condition.  
 225 The subset of set  $S$  given by the condition  $H$  is  $\{x \mid H(x), x \in S\}$ . SymPy uses condition  
 226 sets to represent the set of solutions of equations and inequalities, where the equation or  
 227 the inequality is the condition and the set is the domain over which it is being solved.

228 A few other classes are implemented as special cases of the classes described above. The set of  
 229 real numbers, `Reals`, is implemented as a special case of `Interval`. `ComplexRegion` is implemented  
 230 as a special case of `ImageSet`. `ComplexRegion` supports both polar and rectangular representation  
 231 of regions on the complex plane.

## 6 STATISTICS

The `sympy.stats` submodule provides random variable types and methods for computing of statistical properties of expressions involving random variables, which can be either continuous or discrete, the latter ones being further divided into finite and infinite. The variables are associated with probability densities on corresponding domains and internally defined in terms of probability spaces. Apart from the possibility of defining the random variables from user supplied density distribution, SymPy provides definitions of most common distributions, including `Uniform`, `Poisson`, `Normal`, `Binomial`, `Bernoulli`, and many others.

Properties of random expressions can be calculated using, e.g., `expectation` (abbreviated `E`) and `variance` to calculate expectation and variance. Internally, these functions generate integrals and summations, which are automatically evaluated. The evaluation can be suppressed using `evaluate=False` keyword argument.

Conditions on random variables can be defined with inequalities, equalities, and logical operators and their overall probabilities are obtained using `P`. The features can be illustrated on a model of two dice throws:

```
>>> from sympy.stats import Die, P, E
>>> X, Y = Die("X"), Die("Y")
>>> P(Eq(X, 6) & Eq(Y, 6))
1/36
>>> P(X>Y)
5/12
```

The conditions can also be supplied as a second parameter to `E`, `P`, and other methods to calculate the property given the condition:

```
>>> E(X, X+Y<5)
5/3
```

Using the facilities of the `sympy.stats` submodule, one can, for example, calculate the well known properties of maxwellian velocity distribution

```
>>> from sympy.stats import Maxwell, density
>>> kT, m, x = symbols("kT m x", positive=True)
>>> v = Maxwell("v", sqrt(kT/m))
>>> E(v) # mean velocity
2*sqrt(2)*sqrt(kT)/(sqrt(pi)*sqrt(m))
>>> E(v, evaluate=False) # unevaluated mean velocity
Integral(sqrt(2)*m**(3/2)*v**3*exp(-m*v**2/(2*kT))/(sqrt(pi)*kT**(3/2)),
(v, 0, oo))
>>> E(m*v**2/2) # mean energy
3*kT/2
>>> solve(density(v)(x).diff(x), x)[0] # most probable velocity
sqrt(2)*sqrt(kT)/sqrt(m)
```

More information on the `sympy.stats` submodule can be found in [11].

## 7 CATEGORY THEORY

SymPy includes a submodule for dealing with categories—abstract mathematical objects representing classes of structures as classes of objects (points) and morphisms (arrows) between the objects. It was designed with the following two goals in mind:

1. automatic typesetting of diagrams given by a collection of objects and of morphisms between them, and
2. specification and semi-automatic derivation of properties using commutative diagrams.

As of version 1.0, SymPy only implements the first goal, while a partially working draft of implementation of the second goal is available at <https://github.com/scolobb/sympy/tree/ct4-commutativity>.

In order to achieve the two goals, the submodule `sympy.categories` defines several classes representing some of the essential concepts: objects, morphisms, categories, and diagrams. In category theory, the inner structure of objects is often discarded in the favor of studying the properties of morphisms, so the class `Object` is essentially a synonym of the class `Symbol`. There are several morphism classes which do not have a particular internal structure either, though an exception is `CompositeMorphism`, which essentially stores a list of morphisms.

The class `Diagram` captures the properties of morphisms. This class stores a family of morphisms, the corresponding source and target objects, and, possibly, some properties of the morphisms. Generally, no restrictions are imposed on what the properties may be—for example, one might use strings of the form “forall”, “exists”, “unique”, etc. Furthermore, the morphisms of a diagram are grouped into *premises* and *conclusions* in order to be able to represent logical implications of the form “for a collection of morphisms  $P$  with properties  $p : P \rightarrow \Omega$  (the premises), there exists a collection of morphisms  $C$  with properties  $c : C \rightarrow \Omega$  (the conclusions)”, where  $\Omega$  is the universal collection of properties. Finally, the class `Category` includes a collection of diagrams which are deemed commutative and which therefore define the properties of this category.

Automatic typesetting of diagrams takes a `Diagram` and produces L<sup>A</sup>T<sub>E</sub>X code using the `Xy-pic` package. Typesetting is done in two stages: layout and generation of `Xy-pic` code. The layout stage is taken care of by the class `DiagramGrid`, which takes a `Diagram` and lays out the objects in a grid, trying to reduce the average length of the arrows in the final picture. By default, `DiagramGrid` uses a series of triangle-based heuristics to produce a rectangular grid. A linear layout can also be imposed. Furthermore, groups of objects can be given; in this case, the groups will be treated as atomic cells, and the member objects will be typeset independently of the other objects.

The second phase of diagram typesetting consists in actually drawing the picture and is carried out by the class `XypicDiagramDrawer`. An example of a diagram automatically typeset by `DiagramGrid` and `XypicDiagramDrawer` is given in Figure 1.



**Figure 1.** An automatically typeset commutative diagram

As far as the second main goal of `sympy.categories` is concerned, the principal idea consists in automatically deciding whether a diagram is commutative or not, given a collection of “axioms”: diagrams known to be commutative. The implementation is based on graph embeddings (injective maps): whenever an embedding of a commutative diagram into a given diagram is found, one concludes that the subdiagram is commutative. Deciding commutativity of the whole diagram is therefore based (theoretically) on finding a “cover” of the target diagram by embeddings of the axioms. The naïve implementation proved to be prohibitively slow; a better optimized version is therefore in order, as well as application of heuristics.

## 8 TENSORS

Ongoing work to provide the capabilities of tensor computer algebra has so far produced the `sympy.tensor` submodule. It comprises three submodules whose purposes are quite different: `sympy.tensor.indexed` and `sympy.tensor.indexed_methods` support indexed symbols, `sympy.tensor.array` contains facilities to operate on symbolic  $N$ -dimensional arrays, and finally `sympy.`



321 `tensor.tensor` is used to define abstract tensors. The abstract tensors submodule is inspired  
 322 by `xAct` [8] and `Cadabra` [10]. Canonicalization based on the Butler-Portugal [7] algorithm  
 323 is supported in `SymPy`. Tensor support in `SymPy` is currently limited to polynomial tensor  
 324 expressions.

## 325 9 NUMERICAL SIMPLIFICATION

326 The `nsimplify` function in `SymPy` (a wrapper of `identify` in `mpmath`) attempts to find a simple  
 327 symbolic expression that evaluates to the same numerical value as the given input. It works  
 328 by applying a few simple transformations (including square roots, reciprocals, logarithms and  
 329 exponentials) to the input and, for each transformed value, using the PSLQ algorithm [4] to  
 330 search for a matching algebraic number or optionally a linear combination of user-provided base  
 331 constants (such as  $\pi$ ).

```
332 >>> t = 1 / (sin(pi/5)+sin(2*pi/5)+sin(3*pi/5)+sin(4*pi/5))*2
333 >>> nsimplify(t)
334 -2*sqrt(5)/5 + 1
335 >>> nsimplify(pi, tolerance=0.01)
336 22/7
337 >>> nsimplify(1.783919626661888, [pi], tolerance=1e-12)
338 pi/(-1/3 + 2*pi/3)
```

## 339 10 EXAMPLES

### 340 10.1 Simplification

- 341 • `expand`:

```
342 >>> expand((x + y)**3)
343 x**3 + 3*x**2*y + 3*x*y**2 + y**3
```

- 344 • `factor`:

```
345 >>> factor(x**3 + 3*x**2*y + 3*x*y**2 + y**3)
346 (x + y)**3
```

- 347 • `collect`:

```
348 >>> collect(y*x**2 + 3*x**2 - x*y + x - 1, x)
349 x**2*(y + 3) + x*(-y + 1) - 1
```

- 350 • `cancel`:

```
351 >>> cancel((x**2 + 2*x + 1)/(x**2 - 1))
352 (x + 1)/(x - 1)
```

- 353 • `apart`:

```
354 >>> apart((x**3 + 4*x - 1)/(x**2 - 1))
355 x + 3/(x + 1) + 2/(x - 1)
```

- 356 • `trigsimp`:

```
357 >>> trigsimp(cos(x)**2*tan(x) - sin(2*x))
358 -sin(2*x)/2
```

## 359 10.2 Polynomials

- 360 • Factorization:

```
361 >>> t = symbols('t')
362 >>> f = (2115*x**4*y + 45*x**3*z**3*t**2 - 45*x**3*t**2 -
363 ...      423*x*y**4 - 47*x*y**3 + 141*x*y*z**3 + 94*x*y*z*t -
364 ...      9*y**3*z**3*t**2 + 9*y**3*t**2 - y**2*z**3*t**2 +
365 ...      y**2*t**2 + 3*z**6*t**2 + 2*z**4*t**3 - 3*z**3*t**2 -
366 ...      2*z*t**3)
367 >>> factor(f)
368 (t**2*z**3 - t**2 + 47*x*y)*(2*t*z + 45*x**3 - 9*y**3 - y**2 +
369 3*z**3)
```

- 370 • Gröbner bases:

```
371 >>> x0, x1, x2 = symbols('x:3')
372 >>> I = [x0 + 2*x1 + 2*x2 - 1,
373 ...      x0**2 + 2*x1**2 + 2*x2**2 - x0,
374 ...      2*x0*x1 + 2*x1*x2 - x1]
375 >>> groebner(I, order='lex')
376 GroebnerBasis([7*x0 - 420*x2**3 + 158*x2**2 + 8*x2 - 7,
377 7*x1 + 210*x2**3 - 79*x2**2 + 3*x2,
378 84*x2**4 - 40*x2**3 + x2**2 + x2], x0, x1, x2, domain='ZZ',
379 order='lex')
```

- 380 • Root isolation:

```
381 >>> f = 7*z**4 - 19*z**3 + 20*z**2 + 17*z + 20
382 >>> intervals(f, all=True, eps=0.001)
383 ([],
384 [((-425/1024 - 625*I/1024, -1485/3584 - 2185*I/3584), 1),
385 [(-425/1024 + 2185*I/3584, -1485/3584 + 625*I/1024), 1),
386 [(3175/1792 - 2605*I/1792, 1815/1024 - 10415*I/7168), 1),
387 [(3175/1792 + 10415*I/7168, 1815/1024 + 2605*I/1792), 1]])
```

## 388 10.3 Solvers

- 389 • Single solution:

```
390 >>> solveset(x - 1, x)
391 {1}
```

- 392 • Finite solution set, quadratic equation:

```
393 >>> solveset(x**2 - pi**2, x)
394 {-pi, pi}
```

- 395 • No solution:

```
396 >>> solveset(1, x)
397 EmptySet()
```

- 398 • Interval solution:

```
399 >>> solveset(x**2 - 3 > 0, x, domain=S.Reals)
400 (-oo, -sqrt(3)) U (sqrt(3), oo)
```

- 401 • Infinitely many solutions:

```
402 >>> solveset(x - x, x, domain=S.Reals)
403 (-oo, oo)
404 >>> solveset(x - x, x, domain=S.Complexes)
405 S.Complexes
```

- 406 • Linear systems (linsolve)

```
407 >>> A = Matrix([[1, 2, 3], [4, 5, 6], [7, 8, 10]])
408 >>> b = Matrix([3, 6, 9])
409 >>> linsolve((A, b), x, y, z)
410 {(-1, 2, 0)}
411 >>> linsolve(Matrix((([1, 1, 1, 1], [1, 1, 2, 3])), (x, y, z))
412 {(-y - 1, y, 2)}
```

413 Below are examples of `solve` applied to problems not yet handled by `solveset`.

- 414 • Nonlinear (multivariate) system of equations (the intersection of a circle and a parabola):

```
415 >>> solve([x**2 + y**2 - 16, 4*x - y**2 + 6], x, y)
416 [(-2 + sqrt(14), -sqrt(-2 + 4*sqrt(14))),
417  (-2 + sqrt(14), sqrt(-2 + 4*sqrt(14))),
418  (-sqrt(14) - 2, -I*sqrt(2 + 4*sqrt(14))),
419  (-sqrt(14) - 2, I*sqrt(2 + 4*sqrt(14)))]
```

- 420 • Transcendental equations:

```
421 >>> solve((x + log(x))**2 - 5*(x + log(x)) + 6, x)
422 [LambertW(exp(2)), LambertW(exp(3))]
423 >>> solve(x**3 + exp(x))
424 [-3*LambertW((-1)**(2/3)/3)]
```

## 425 10.4 Matrices

- 426 • Matrix expressions

```
427 >>> m, n, p = symbols('m n p', integer=True)
428 >>> R = MatrixSymbol('R', m, n)
429 >>> S = MatrixSymbol('S', n, p)
430 >>> T = MatrixSymbol('T', m, p)
431 >>> U = R*S + 2*T
432 >>> U.shape
433 (m, p)
434 >>> U[0, 1]
435 2*T[0, 1] + Sum(R[0, _k]*S[_k, 1], (_k, 0, n - 1))
```

- 436 • Block Matrices

```
437 >>> n, m, l = symbols('n m l')
438 >>> X = MatrixSymbol('X', n, n)
439 >>> Y = MatrixSymbol('Y', m, m)
440 >>> Z = MatrixSymbol('Z', n, m)
441 >>> B = BlockMatrix([[X, Z], [ZeroMatrix(m, n), Y]])
442 >>> B
443 Matrix([
444 [X, Z],
```

```

445     [0, Y]])
446     >>> B[0, 0]
447     X[0, 0]
448     >>> B.shape
449     (m + n, m + n)

```

## 450 11 SYMPY GAMMA

451 SymPy Gamma is a simple web application that runs on Google App Engine. It executes and  
 452 displays the results of SymPy expressions as well as additional related computations, in a fashion  
 453 similar to that of Wolfram|Alpha. For instance, entering an integer will display its prime factors,  
 454 digits in the base-10 expansion, and a factorization diagram. Entering a function will display its  
 455 docstring; in general, entering an arbitrary expression will display its derivative, integral, series  
 456 expansion, plot, and roots.

457 SymPy Gamma also has several features beyond just computing the results using SymPy.

- 458 • SymPy Gamma displays integration and differentiation steps in detail, which can be viewed  
 459 in Figure 2:

Integral steps:

1. Rewrite the integrand:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

2. Let  $u = \cos(x)$ .

Then let  $du = -\sin(x)dx$  and substitute  $du$ :

$$\int -\frac{1}{u} du$$

- A. The integral of a constant times a function is the constant times the integral of the function:

$$\int -\frac{1}{u} du = - \int \frac{1}{u} du$$

- I. The integral of  $\frac{1}{u}$  is  $\log(u)$ .

So, the result is:  $-\log(u)$

Now substitute  $u$  back in:

$$-\log(\cos(x))$$

3. Add the constant of integration:

$$-\log(\cos(x)) + \text{constant}$$

The answer is:

$$-\log(\cos(x)) + \text{constant}$$

460

461 **Figure 2.** Integral steps of  $\tan(x)$

- 462 • SymPy Gamma displays the factor tree diagrams for different numbers.
- 463 • SymPy Gamma saves user search queries, and offers many such similar features for free,  
 464 which Wolfram|Alpha only offers to its paid users.

465 Every input query from the user on SymPy Gamma is first parsed by its own parser capable of  
 466 handling several different forms of function names which SymPy as a library does not support.

467 For instance, SymPy Gamma supports queries like `sin x`, whereas SymPy will only recognise  
468 `sin(x)`.

469 This parser converts the input query to the equivalent SymPy readable code, which is then  
470 processed by SymPy, and the result is finally printed with the built-in  $\text{\LaTeX}$  output and rendered  
471 by the SymPy Gamma web application.

## 472 12 SYMPY LIVE

473 SymPy Live is an online Python shell, which uses the Google App Engine to executes SymPy  
474 code. It is integrated in the SymPy documentation examples at <http://docs.sympy.org>.

475 This is accomplished by providing a HTML/JavaScript GUI for entering source code and  
476 visualization of output, and a server that evaluates the requested source code. It is an interactive  
477 AJAX shell that runs SymPy code using Python on the server.

## 478 13 OTHER PROJECTS THAT DEPEND ON SYMPY

479 There are several projects that depend on SymPy as a library for implementing a part of their  
480 functionality. Some of them are listed below:

- 481 • **Cadabra**: Cadabra is a CAS designed specifically for the resolution of problems encountered  
482 in field theory.
- 483 • **Octave Symbolic**: The Octave-Forge Symbolic package adds symbolic calculation features  
484 to GNU Octave. These include common CAS tools such as algebraic operations, calculus,  
485 equation solving, Fourier and Laplace transforms, variable precision arithmetic, and other  
486 features.
- 487 • **SymPy.jl**: Provides a Julia interface to SymPy using PyCall.
- 488 • **Mathics**: Mathics is a free, general-purpose online CAS featuring Mathematica compatible  
489 syntax and functions. It is backed by highly extensible Python code, relying on SymPy for  
490 most mathematical tasks.
- 491 • **Mathpix**: An iOS App, that detects handwritten math as input, and uses SymPy Gamma  
492 to evaluate the math input and generate the relevant steps to solve the problem.
- 493 • **IKFast**: IKFast is a robot kinematics compiler provided by OpenRAVE. It analytically  
494 solves robot inverse kinematics equations and generates optimized C++ files. It uses  
495 SymPy for its internal symbolic mathematics.
- 496 • **Sage**: A CAS, visioned to be a viable free open source alternative to Magma, Maple,  
497 Mathematica and MATLAB. Sage includes many open source mathematical libraries,  
498 including SymPy.
- 499 • **SageMathCloud**: SageMathCloud is a web-based cloud computing and course manage-  
500 ment platform for computational mathematics.
- 501 • **PyDy**: Multibody Dynamics with Python.
- 502 • **galgebra**: Geometric algebra (previously `sympy.galgebra`).
- 503 • **yt**: Python package for analyzing and visualizing volumetric data (`yt.units` uses SymPy).
- 504 • **SfePy** (Simple finite elements in Python), cf. [2], is a Python package for solving partial  
505 differential equations (PDEs) in 1D, 2D and 3D by the finite element (FE) method [14].  
506 SymPy is used within this package mostly for code generation and testing.
- 507 • **Quameon**: Quantum Monte Carlo in Python.
- 508 • **Lcapy**: Experimental Python package for teaching linear circuit analysis.

- 509 • **Quantum Programming in Python:** Quantum 1D Simple Harmonic Oscillator and  
510 Quantum Mapping Gate.
- 511 • **LaTeX Expression project:** Easy L<sup>A</sup>T<sub>E</sub>X typesetting of algebraic expressions in symbolic  
512 form with automatic substitution and result computation.
- 513 • **Symbolic statistical modeling:** Adding statistical operations to complex physical  
514 models.

## 515 14 COMPARISON WITH MATHEMATICA

516 Wolfram Mathematica is a popular proprietary CAS that features highly advanced algorithms,  
517 has a core written in C++ [13], and interprets its own programming language, Wolfram Language.

518 Analogous to Lisp S-expressions, Mathematica uses its own style of M-expressions, which  
519 are arrays of either atoms or other M-expressions. The first element of the expression identifies  
520 the type of the expression and is indexed by zero, and the first argument is indexed starting  
521 with one. In SymPy, expression arguments are stored in a Python tuple (that is, an immutable  
522 array), while the expression type is identified by the type of the object storing the expression.

523 Mathematica can associate attributes to its atoms. Attributes may define mathematical  
524 properties and behavior of the nodes associated to the atom. In SymPy, the usage of static class  
525 fields is roughly similar to Mathematica's attributes, though other programming patterns may  
526 also be used to achieve an equivalent behavior such as class inheritance.

527 Unlike SymPy, Mathematica's expressions are mutable: one can change parts of the expression  
528 tree without the need of creating a new object. The mutability of Mathematica expressions  
529 allows for a lazy updating of any references to a given data structure.

530 Products in Mathematica are determined by some built in node types, such as `Times`, `Dot`,  
531 and others. `Times` is a representation of the `*` operator, and is always meant to represent a  
532 commutative product operator. The other notable product is `Dot`, which represents the `.` operator.  
533 This product represents matrix multiplication. It is not commutative. Unlike Mathematica,  
534 SymPy determines commutativity with respect to multiplication from the expression type of the  
535 factors. Mathematica puts the `Orderless` attribute on the expression type.

536 Regarding associative expressions, SymPy handles associativity of sums and products by  
537 automatically flattening them, Mathematica specifies the `Flat` attribute on the expression type.

538 Mathematica relies heavily on pattern matching—even the so-called equivalent of function  
539 declaration is in reality the definition of a pattern generating an expression tree transformation  
540 on input expressions. Mathematica's pattern matching is sensitive to associative, commutative,  
541 and one-identity properties of its expression tree nodes. SymPy has various ways to perform  
542 pattern matching. All of them play a lesser role in the CAS than in Mathematica and are  
543 basically available as a tool to rewrite expressions. The differential equation solver in SymPy  
544 somewhat relies on pattern matching to identify differential equation types, but it is envisaged to  
545 replace that strategy with analysis of Lie symmetries in the future. Mathematica's real advantage  
546 is the ability to add (at runtime) new overloading to the expression builder or specific subnodes.  
547 Consider for example:

```
548 In[1]:= Unprotect[Plus]
549 Out[1]= {Plus}
550
551 In[2]:= Sin[x_]^2 + Cos[y_]^2 := 1
552
553 In[3]:= x + Sin[t]^2 + y + Cos[t]^2
554 Out[3]= 1 + x + y
```

555 This expression in Mathematica defines a substitution rule that overloads the functionality of  
556 the `Plus` node (the node for additions in Mathematica). A symbol with a trailing underscore is  
557 treated as a wildcard. Although one may wish to keep this identity unevaluated, this example  
558 clearly illustrates the potential to define one's own immediate transformation rules. In SymPy,  
559 the operations constructing the addition node in the expression tree are Python class constructors

and cannot be modified at runtime.<sup>2</sup> The way SymPy deals with extending the missing runtime overloadability functionality is by subclassing the node types: subclasses may redefine the class constructor to yield the proper extended functionality.

Unlike SymPy, Mathematica does not support type inheritance or polymorphism [3]. SymPy relies heavily on class inheritance, but for the most part, class inheritance is used to make sure that SymPy objects inherit the proper methods and implement the basic hashing system.

While Mathematica interprets nested lists as matrices whenever the sublists have the same length, matrices in SymPy are a type in their own right, allowing ordinary operators and functions (like multiplication and exponentiation) to be used as they traditionally are in mathematics.

```
>>> exp(Matrix([[1, 1],[0, 2]])) * Matrix([a, b])
Matrix([
[E*a + b*(-E + exp(2))],
[
b*exp(2)]])
```

Using the standard multiplication in Mathematica performs an element-wise product and calling the exponential function `Exp` on a matrix returns an element-wise exponentiation of its elements.

Unevaluated expressions in Mathematica can be achieved in various ways, most commonly with the `HoldForm` or `Hold` nodes, that block the evaluation of subnodes by the parser. Such a node cannot be expressed in Python because of greedy evaluation. Whenever needed in SymPy, it is necessary to add the parameter `evaluate=False` to all subnodes.

In Mathematica, the operator `==` returns a boolean whenever it is able to immediately evaluate the truth of the equality, otherwise it returns an `Equal` expression. In SymPy, `==` means structural equality and is always guaranteed to return a boolean expression. To express a mathematical equality in SymPy it is necessary to explicitly construct an instance of the `Equality` class.

SymPy, in accordance with Python (and unlike the usual programming convention), uses `**` to express the power operator, while Mathematica uses the more common `^`.

SymPy's use of floating-point numbers is similar to that of most other CAS's, including Maple and Maxima. By contrast, Mathematica uses a form of significance arithmetic [12] for approximate numbers. This offers further protection against numerical errors, although it comes with its own set of problems (for a critique of significance arithmetic, see Fateman [3]). Internally, SymPy's `evalf` method works similarly to Mathematica's significance arithmetic, but the semantics are isolated from the rest of the system.

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