Math Insight

Multiplying matrices and vectors

Matrix-vector product

To define multiplication between a matrix A and a vector $\mathbf{x}$ (i.e., the matrix-vector product), we need to view the vector as a column matrix. We define the matrix-vector product only for the case when the number of columns in A equals the number of rows in $\mathbf{x}$. So, if A is an $m \times n$ matrix (i.e., with n columns), then the product $A\mathbf{x}$ is defined for $n \times 1$ column vectors $\mathbf{x}$. If we let $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is an $m \times 1$ column vector. In other words, the number of rows in A (which can be anything) determines the number of rows in the product $\mathbf{b}$.

The general formula for a matrix-vector product is

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

Although it may look confusing at first, the process of matrix-vector multiplication is actually quite simple. One takes the dot product of $\mathbf{x}$ with each of the rows of A. (This is why the number of columns in A has to equal the number of components in $\mathbf{x}$.) The first component of the matrix-vector product is the dot product of $\mathbf{x}$ with the first row of A, etc. In fact, if A has only one row, the matrix-vector product is really a dot product in disguise.

For example, if

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & 1 \end{bmatrix}$$

and $\mathbf{x} = (2, 1, 0)$, then

$$
\begin{aligned}
A\mathbf{x} &= \begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot 1 - 1 \cdot 1 + 0 \cdot 2 \\ 2 \cdot 0 - 1 \cdot 3 + 0 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -3 \end{bmatrix}.
\end{aligned}
$$

Matrix-matrix product

Since we view vectors as column matrices, the matrix-vector product is simply a special case of the matrix-matrix product (i.e., a product between two matrices). Just like for the matrix-vector product, the product AB between matrices A and B is defined only if the number of *columns* in A equals the number of *rows* in B. In math terms, we say we can multiply an $m \times n$ matrix A by an $n \times p$ matrix B. (If p happened to be 1, then B would be an $n \times 1$ column vector and we'd be back to the matrix-vector product.)

The product AB is an $m \times p$ matrix which we'll call C, i.e., $AB = C$. To calculate the product B, we view B as a bunch of $n \times 1$ column vectors lined up next to each other:

$$
\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} \dots \begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{np} \end{bmatrix} \end{bmatrix}
$$

Then each column of C is the matrix-vector product of A with the respective column of B. In other words, the component in the ith row and jth column of C is the dot product between the ith row of A and the jth column of B. In math, we write this component of C as $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

An example help makes the process clear. Let A be the 2×3 matrix

$$
A = \begin{bmatrix} 0 & 4 & -2 \\ -4 & -3 & 0 \end{bmatrix}
$$

and B be the 3×2 matrix

$$B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

Then,

$$\begin{aligned}
AB &= \begin{bmatrix} 0 & 4 & -2 \\ -4 & -3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 \cdot 0 + 4 \cdot 1 - 2 \cdot 2 & 0 \cdot 1 + 4 \cdot (-1) - 2 \cdot 3 \\ -4 \cdot 0 - 3 \cdot 1 + 0 \cdot 2 & -4 \cdot 1 - 3 \cdot (-1) + 0 \cdot 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 + 4 - 4 & 0 - 4 - 6 \\ 0 - 3 + 0 & -4 + 3 + 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -10 \\ -3 & -1 \end{bmatrix}.
\end{aligned}$$

Want more examples?

See also

Vectors in arbitrary dimensions
Introduction to matrices
Matrix and vector multiplication examples
Matrices and determinants for multivariable calculus
Dot product in matrix notation

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