ONE HOUR THIRTY MINUTES

A list of constants is enclosed.

UNIVERSITY OF MANCHESTER

Statistical Mechanics

15 May 2019, 2:00 - 3:30

Answer **ALL** parts of question 1 and **TWO** other questions

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Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. a) State the form of the Boltzmann distribution for a system with discrete microstates i, associated with energies ε_i ($i = 1, 2, 3, \ldots$).

State the definition of Gibbs entropy, and show that $F = E - TS$, where E is the average energy of the system and F the Helmholtz free energy.

[5 marks]

b) Starting from the thermodynamic definition $F = E - TS$, derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V ,$$

for a system with a fixed number of particles. [You do not need to derive the fundamental relation of thermodynamics.]

[5 marks]

c) What are the SI units of the following quantities:

(i) entropy, S;
(ii) statistical weight, Ω;
(iii) grand partition function, $\mathcal{Z}$;
(iv) grand potential, Φ;
(v) spectral density, $u(\lambda)$?

[5 marks]

d) State the expression for the Bose-Einstein distribution function as a function of energy, and sketch its graph for a non-zero temperature.

Explain what it represents, and what values the chemical potential can take.

[5 marks]

e) You are given the density of states for a quantum particle in a volume V,

$$\frac{\mathrm{d}n}{\mathrm{d}k} = g\frac{Vk^2}{2\pi^2},$$

where g is the spin degeneracy.

Use this density of states to derive an expression for the Fermi temperature of a degenerate gas of electrons as a function of their number density.

[5 marks]

2. (a) Explain briefly what is meant by the ergodic hypothesis, and the principle of equal *a priori* probabilities.

[5 marks]

(b) A paramagnetic solid consists of N spin-$\frac{1}{2}$ magnetic dipoles, of magnetic moment μ. These can orient themselves in either of two directions, parallel or antiparallel to the external magnetic field B. What is the energy of the system if n dipoles are oriented parallel to the field? Give an expression for the statistical weight $\Omega(N, n)$ of this macrostate.

[5 marks]

(c) Two of the systems described in part (b), each with $N = 20$ dipoles, are initially isolated from each other and from the surroundings, and have internal energies $E = -16\mu B$ and zero respectively. How many of the spins are parallel and antiparallel to the magnetic field in each system? Show that the systems have entropies $5.25k_B$ and $12.13k_B$ respectively. (Do not use Stirling's approximation.)

[5 marks]

(d) The systems in part (c) are brought into thermal contact so that energy can flow freely between them. What is the total energy of the combined system? Calculate the statistical weight of the combined system, and hence find the increase in entropy as a result of bringing the two systems into contact.

[5 marks]

(e) State the definition of temperature in the microcanonical ensemble.

A solid with $N = 10^{23}$ dipoles is in a magnetic field of field strength $B = 1$ T. Each dipole has magnetic moment $\mu = 10^{-23}$ JT^{-1}.

It is found that when the internal energy of the solid is increased by 0.01 J, the entropy increases by 3.33×10^{-5} JK^{-1}. What is the temperature of the solid?

[5 marks]

3. (a) Briefly explain what types of physical systems are described by the canonical and grand-canonical ensembles respectively.

For each of the two ensembles state the associated thermodynamic potential and its natural variables.

[6 marks]

(b) Consider a system of N distinguishable particles, which can each be in states with energies $\varepsilon_n = n\varepsilon$, where $n = 0, 1, 2, \ldots$; the energy spacing $\varepsilon > 0$ is a fixed constant. The system is in contact with a heat bath.

(i) Specify the microstates and macrostates of this system.

[3 marks]

(ii) Calculate the canonical single-particle partition function, Z_1, and show that the mean energy of the N-particle system is

$$\langle E \rangle = \frac{N\varepsilon}{e^{\beta\varepsilon} - 1}.$$

[4 marks]

(iii) Find the low-temperature limit of this expression, and give a physical interpretation. Also obtain the behaviour of $\langle E \rangle$ for large temperatures.

[3 marks]

(iv) For $\varepsilon = 1$ eV, find the temperature for which the mean energy per particle is 2 eV.

[2 marks]

(c) Now consider a grand-canonical ensemble of distinguishable particles with single-particle energy states $\varepsilon_n = n\varepsilon$ $(n = 0, 1, 2, \ldots)$.

(i) Define the grand partition function, and show that it can be written in the form

$$\mathcal{Z} = \frac{1}{1 - e^{\beta\mu} Z_1}.$$

[3 marks]

(ii) Calculate the mean particle number in the system for the case in which energy spacing, temperature and chemical potential are chosen such that $\beta\varepsilon = \ln 2$ and $\beta\mu = -\ln 4$.

[4 marks]

4. In this question you may use without proof the integral

$$\int_0^\infty \frac{1}{ae^x - 1}\,\mathrm{d}x = \ln\frac{a}{a-1} \qquad (0 < a < 1).$$

(a) Describe the phenomenon of Bose-Einstein condensation, and give one example of a Bose-Einstein condensation seen in the laboratory.

[5 marks]

(b) The wave function for a non-relativistic spinless particle in *two* dimensions, confined to a square of area A, is given by

$$\psi(x, y) = C \sin(k_x x)\sin(k_y y),$$

where C is a constant.

(i) Show that the density of states in k-space is

$$\frac{\mathrm{d}n}{\mathrm{d}k} = \frac{Ak}{2\pi}.$$

[5 marks]

(ii) Derive the density of states $\dfrac{\mathrm{d}n}{\mathrm{d}\varepsilon}$ for such a particle.

[5 marks]

(c) A system contains N identical non-interacting spinless bosons, each of mass m moving non-relativistically in two dimensions, confined to a square of area A.

(i) Show that the number of particles in the ground state is given by

$$N_0 = N - \frac{Am}{2\pi\hbar^2}\int_0^\infty \frac{1}{e^{(\varepsilon-\mu)/(k_B T)} - 1}\mathrm{d}\varepsilon,$$

and explain clearly what the different terms in this expression represent.

[5 marks]

(ii) Use this result to determine whether or not Bose-Einstein condensation can occur in this system at a non-zero temperature.

[5 marks]

END OF EXAMINATION PAPER