The Delay Lines Experiment

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This experiment was performed in collaboration with *Rashed Aljasmi*

Abstract

The signal through a delay line composed of inductors and capacitors transmits like waves and reflection occurs at the termination. The various parameters such as the voltage in the circuit and the frequency of the input signal at varying terminating impedances were measured to find the time delay α, attenuation τ, cut-off frequency ω_0, whose relations are closely connected to the inductor and capacitor in the delay line. Sine signal wave was injected at the input of the line. $\alpha = 0.98 \pm 0.01,\ \tau = 1.0 \pm 10\%\, \mu s,\ \omega_0 = 2.06 \times 10^6 \pm 3\%\, rad/s$, are the computed values of the line. Besides, this report investigated the rationale behind the sinusoidal look of the graph of voltage against changing frequency.

1 Introduction

Fundamental to different transmission lines, a lumped-element delay line, which is constituted by more fundamental elements: inductors and capacitors in the circuit, provides more insight into the properties of other similarly structured lines. This includes but not limited to twin-lead cable and coaxial cable, both of which are analogous to the delay line when time delay of the line can be ignored and when the line is infinitely long [1]. In this experiment, the delay line used consists of inductors $L = 1mH \pm 5\%$ and capacitors $C = 1nF \pm 1\%$, as shown in Figure 1. On the left side is the input wave of some frequency. On the right side, the line is terminated by some impedance, the wave is then reflected with an attenuated amplitude and rejoin with the input wave.

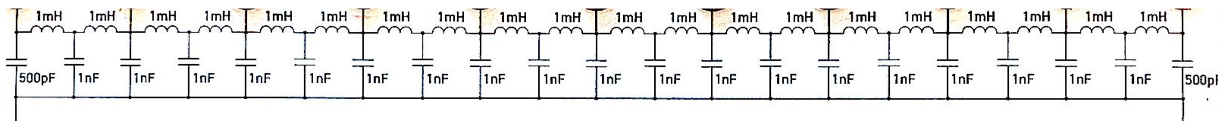


Figure 1: The delay line used in the experiment is composed of inductors and capacitors that are arranged as duplicates of one single unit section. There are 20 sections in this delay line. At two sides of the line, the capacitors are $L/2 = 500pF$.

2 Theory

The cut-off frequency of the line is defined as [2]

$$\omega_0 = 2/\sqrt{LC} \tag{1}$$

The wave can only transmit through the line when the frequency of the wave is below the cut-off frequency. The characteristic impedance of the line is [2]

$$Z_\pi = \frac{\sqrt{L/C}}{\sqrt{1 - \omega^2/{\omega_0}^2}} \simeq \sqrt{L/C}(\omega <<< \omega_0) \tag{2}$$

The time delay per section τ is [2]

$$\tau = \phi/\omega \approx 2/\omega_0 = \sqrt{LC}(\omega << \omega_0) \tag{3}$$

At section n, two waves are travelling in opposite direction, one is from left to right, with voltage and current defined as [3]

$$V_n = Ae^{i(\omega t - n\phi)}, I_n = V_n/Z_\pi \tag{4}$$

whereas the other wave is travelling from right to left with a reflection coefficient r,

$$V_n = r * Ae^{i(\omega t + n\phi)}, I_n = -r * V_n/Z_\pi \tag{5}$$

Add Eq. 4 and Eq. 5 together, and plug in $Z_L = V_n/I_n$, where Z_L is the terminating impedance(load) connected on the right end of the line,

$$r = \frac{Z_L - Z_\pi}{Z_L + Z_\pi} \tag{6}$$

The inductors and capacitors are not ideal electronics in practice, thus the waves transmitting along will be attenuated with attenuation per section α [2]

$$\alpha = 1 - \frac{R}{2Z_\pi} \tag{7}$$

where R is the resistance of the inductors. $R \simeq 20\Omega$.

3 Experimental approach & results

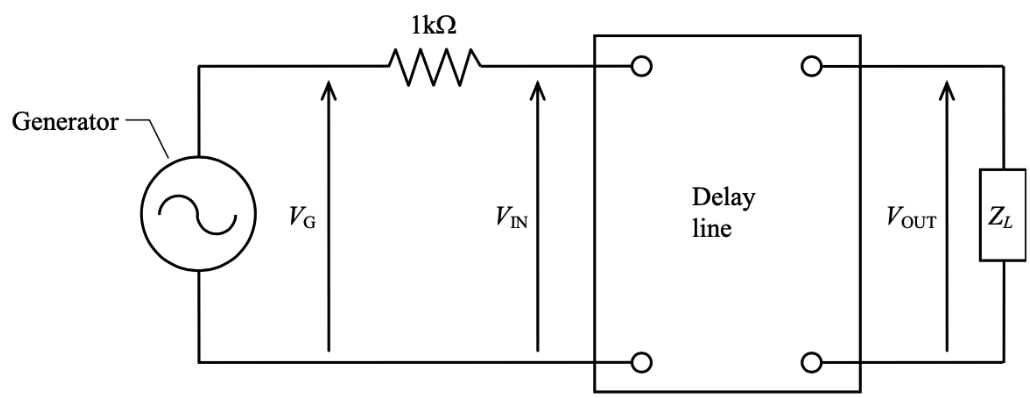


Figure 2: Schematic diagram showing the overall structure of the apparatus, including the function generator, $1k\Omega$ resistor at the input, a terminating impedance Z_L at the end, three voltage measuring positions and most importantly, the delay line of Figure 1 connected.

The setup is illustrated in Figure 2. Note that the voltages V_G, V_{IN} and V_{OUT} were measured with an oscilloscope showing the corresponding peak to peak value from its waveform displayed on the screen. The frequency of the wave from the generator was shown directly on the equipment. Sine wave was injected into the circuit.

3.1

V_{IN} was measured as a function of frequency f ranging from 1 to 100kHz when $Z_L = \infty$(open circuit at Z_L), $Z_L = 0$(short circuit), $Z_L = 0.03\mu F$, $Z_L = 0.5k\Omega$, $Z_L = 1k\Omega$ or $Z_L = 1.5k\Omega$ respectively.
Figures 3 and 4 show the two separate plots of the V_{IN} against f in different Z_L.

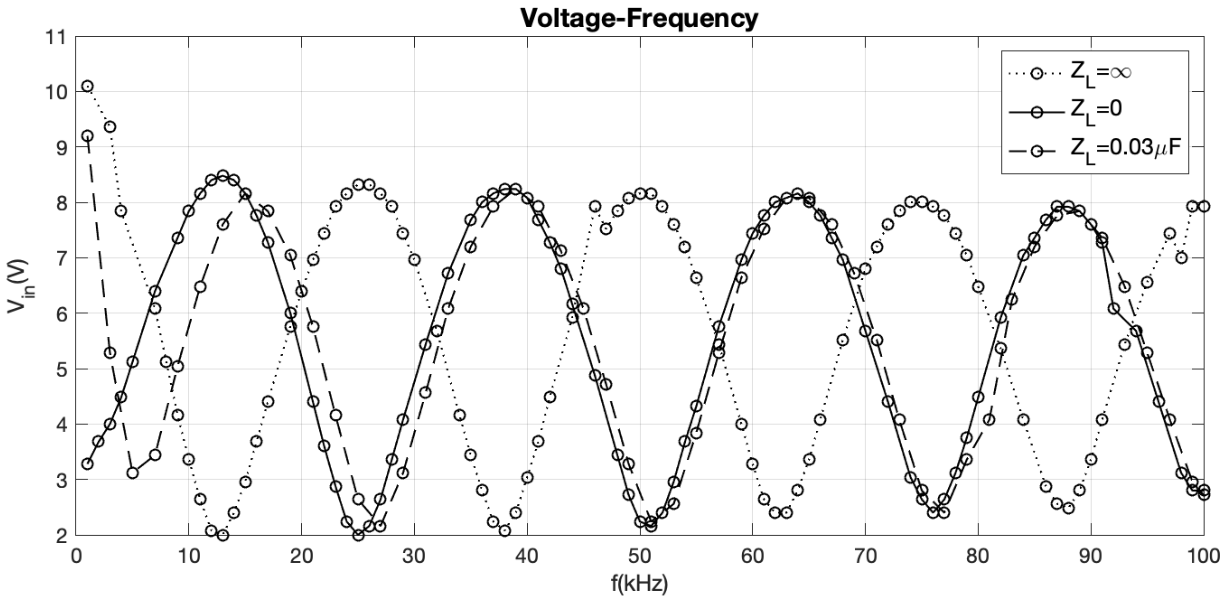


Figure 3: Combined input voltage response to frequency when $Z_L = \infty$, $Z_L = 0$, $Z_L = 0.03\mu F$. The graphs seem to be periodic and look sinusoidal. Further analysis regarding this is in Section 4.

3.2

V_{IN} and V_{OUT} were measured as a function of frequency f ranging from 1 to 1MHz when $Z_\pi = Z_L$.
The measurements were used to plot V_{OUT}/V_{IN} against f in logarithmic scale as shown in Figure 5.

4 Analysis

Periodicity of $V_{IN} - f$ curves As seen from Figures 3 and 4, the $V_{IN} - f$ curves are in periodic shape. The periodicity can be deduced by examining the property of the wave transmission. The propagation of waves in the delay lines has interference between the input wave(from the generator) and the reflected wave(reflected by the termination). At max(min) points, the waves are in(exactly out) phase, which constitutes constructive(destructive) interference that results in max(min) voltage at input [2]. As long as there exists the creation of the waves from the generator, one can observe such a periodic behaviour of V_{IN} over increasing frequency.

Time delay To find the time delay, one can take advantage of the extremum points from curves in Figure 3 in which the successive max or min points have a phase difference of the waves exactly at $\phi = 2\pi$. Hence, according to Eq. 3,

$$\tau = \Delta\phi/\Delta\omega = 2\pi/2\pi\Delta f = 1/\Delta f (\omega << \omega_0) \tag{8}$$

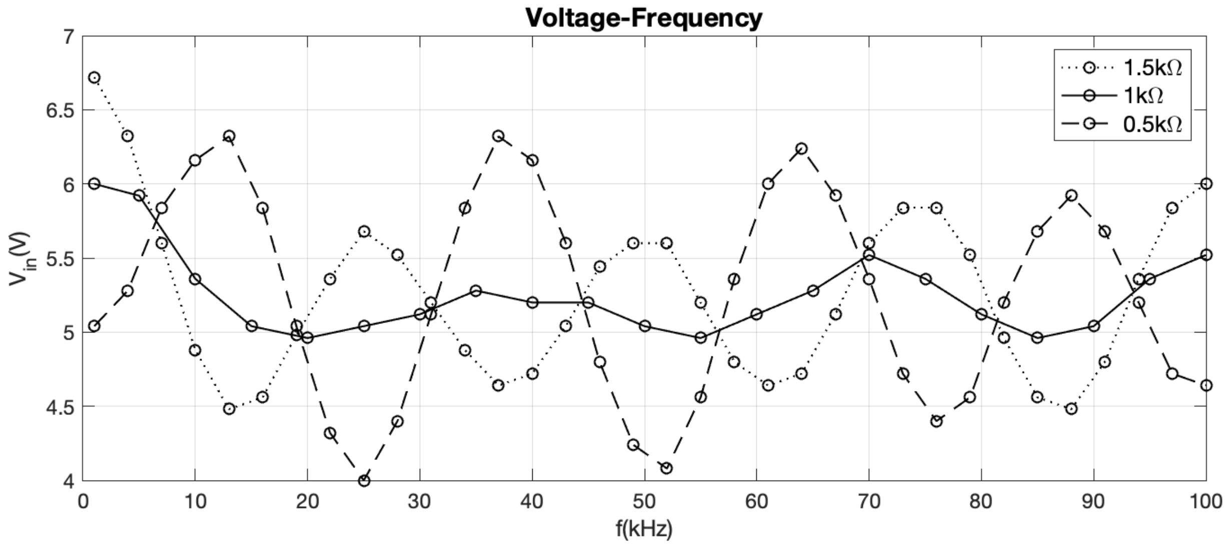


Figure 4: Combined input voltage response to frequency when $Z_L = 0.5k\Omega$, $Z_L = 1k\Omega$, $Z_L = 1.5k\Omega$. Same as in Figure 3, the curves look periodic and sinusoidal.

Apparently, Δf represents the frequency difference on consecutive max/min points.

Attenuation Finding attenuation factor employs the same logic as in finding τ. At peaks, the amplitude of V_{IN} should be a simple addition of the input and reflected wave. Twice of the amplitude of the input wave is expected. However, with the introduction of the attenuation, amplitudes at the peaks and dips A w.r.t. A_0 (input voltage) are,

$$A = A_0(1 + \alpha^{40}), A = A_0(1 - \alpha^{40}) \tag{9}$$

The appearance of "40" as exponents is due the $2 \times 20 = 40$ sections through which the reflected wave has transmitted.

Characteristic impedance Z_π The reflection coefficient $r = 0$ when $Z_\pi = Z_L$ according to Eq. 6, indicating that no reflection occurs. V_{IN} would therefore stay constant. From Figure 4, when $Z_L = 1k\Omega$, it tends to follow this requirement. Based on the other two curves, $Z_\pi = 1 \pm 0.5\ k\Omega$, agreeing with the theoretical ones that $Z_\pi = 1k\Omega \pm 25\Omega$. Nevertheless, the $1k\Omega$ curve still exists some fluctuation probably resulted from the periodic change in Z_π over f.

Cut-off frequency From Figure 5, a sharp drop in $log(V_{OUT}/V_{IN})$ indicates the location of cut-off frequency f_0, as explained in Section 2, beyond f_0, the wave can no longer propagate in the line.

Computed values of τ, α, f_0 By applying the methods and the discussions above, Eqs. 8 and 9 and Figures 3, 4, 5, time delay per section, attenuation per section, cut-off frequency, are found to be $\alpha = 0.98 \pm 0.01$, $\tau = 1.0 \pm 10\% \mu s$, $\omega_0 = 2.06 \times 10^6 \pm 3\% rad/s$. The theoretical values

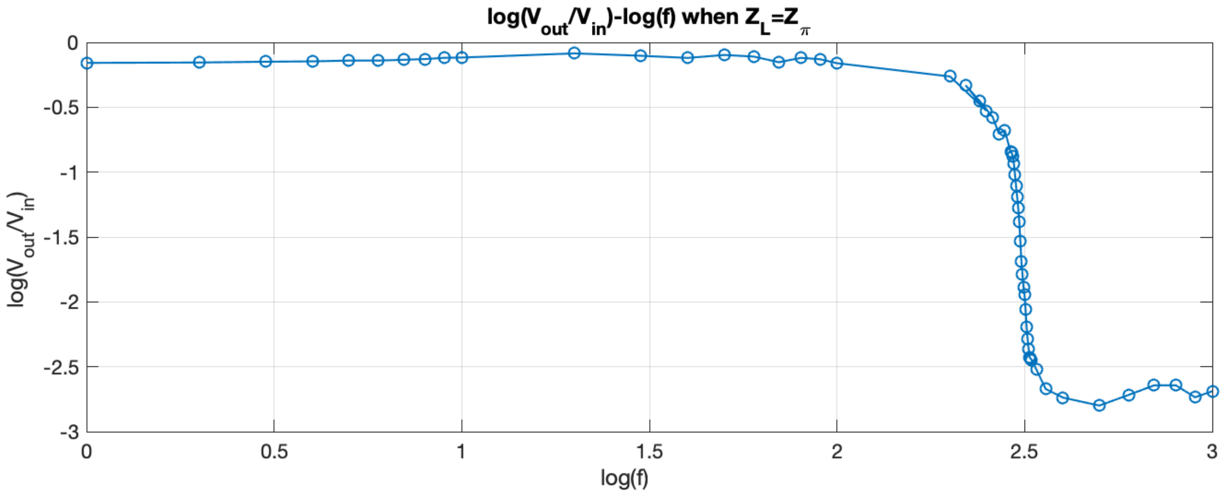


Figure 5: Plot of $log(V_{OUT}/V_{IN})$ over $log(f)$ The curve has a sharp decrease in the curve around $log(f) \simeq 2.5$ and is flat elsewhere. More points were recorded near the huge drop around $log(f) \simeq 2.5$ to make the graph less ambiguous and more practical to use.

of these are obtained from Eqs. 1, 3, 7 as $\alpha \approx 0.99$, $\tau = 1.0 \pm 2.5\% \mu s$, $\omega_0 = 2 \times 10^6 \pm 2.5\% rad/s$. The measured values do agree with the theoretical ones within the uncertainty.

Error Since the computation of these values largely relies on the figures plotted, which were made by repetitive measurements of similar variables, the random error by far is contributed from the locating process of either max/min points or the limiting points of the curves. $\pm 0.1kHz$ is a reasonable error to f when locating the extremum points. Repetitious measurements were made in successive points to avoid a systematic error.

Verify the hypothesis As mentioned before in Section 3, the $V - f$ curves look like sinusoidal curves, however, this is only a postulate which needs proof. The following illustrates a compact picture of this proof. Wave on delay lines can be expressed as [4]

$$a(z,t) = Re[A(0)e^{az}e^{j(\omega t - kz)}] = Re[A(z)e^{j\omega t}] \tag{10}$$

where ω, k, z, t, i are angular frequency, wave number, position and time, imaginary unit. $e^{-\alpha z}$ represents attenuation. $\delta = \alpha/k$.
Define operator $P(z)$ [4],

$$P(z) = e^{-(jk+\alpha)z} \tag{11}$$

This is useful because $A(x + d) = P(d)A(x)$. Using the following identities,

$$\begin{aligned} a(0) &= c + \Gamma_0 b(0),\ b(0) = P(1)b(1) \\ a(1) &= P(1)a(0),\ b(1) = \Gamma_1 a(1), \\ a(x) &= P(x)a(0),\ b(x) = P(1-x)b(1) = b(0)P(-x) \end{aligned} \tag{12}$$

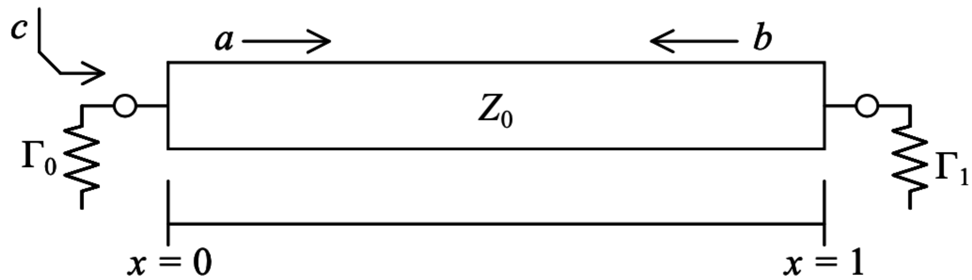


Figure 6: Unit length delay line Z_0. Γ_0, Γ_1 are reflection coefficients on the ends defined in Eq. 6, a, b, c are the input, output voltage and wave from the generator [4].

The above is obvious by the definition of the operator. The voltage along the line is then $V(x) = a(x) + b(x)$, combines with Eq. 12 [5],

$$V(x) = \frac{P(x-1) + \Gamma_1 P(1-x)}{P(-1) - \Gamma_0 \Gamma_1 P(1)} \tag{13}$$

To get V_{IN}, simply let $x = 0$ and $\Gamma_0 = 0$, the latter is because the $1k\Omega$ resistor placed at the input equal to the characteristic impedance Z_π. Hence,

$$V_{IN} = c\frac{P(-1) - P(1)}{P(-1)} = \frac{e^{jk(1-j\delta)} - e^{-jk(1-j\delta)}}{e^{jk(1-j\delta)}} \tag{14}$$

Let $\Gamma_1 = -1$ so it represents a short circuit $Z_L = 0$. Take the real part,

$$V_{IN} = c[1 - e^{-2\alpha} cos(2k)] \tag{15}$$

Since wave number k is a function of frequency f, we conclude that V_{IN} is a sinusoidal function of f. Further analysis shows a high relevance of the data V_{IN}, f and the fitting function of form $f(x) = A + Bcos(Kx)$ with $R^2 = 0.9593$.

5 Conclusions

The inspection through the wave transmission properties in the lumped delay lines gives an idea of how the frequency of the corresponding wave and the loaded impedance of the line influence the propagation of signals. It provides the implicit verification that the signals transmit through the line with wave-like properties. A deep dive into the principles behind the sinusoidal look of the $V_{IN} - f$ curves from postulate to actual proof gives a glance at the in-depth theoretical background of this experiment.

References

[1] B. Bleaney and B. Bleaney, *Electricity and Magnetism, Volume 1: Third Edition*. Electricity and Magnetism, Oxford University Press, 2013.

[2] F. R. Connor, *Wave transmission.* Introductory topics in electronics and telecommunication, London: Edward Arnold, 1972.

[3] F. T. Ulaby and U. Ravaioli, *Fundamentals of applied electromagnetics.* Boston: Pearson, 2015.

[4] F. R. Rice, “Transmission line resonance due to reflections (1-d cavity resonances).”

[5] F. R. Rice, “Lumped-parameter delay line.”

Reasoning behind the sinusoidal look of the $V_{in}-f$ curve in experiment with sine wave.

wave on delay lines can be expressed as

$$a(z,t) = Re[A(0)e^{-\alpha z}e^{j(\omega t - kz)}] = Re[A(z)e^{j\omega t}]$$

ω: angular frequency

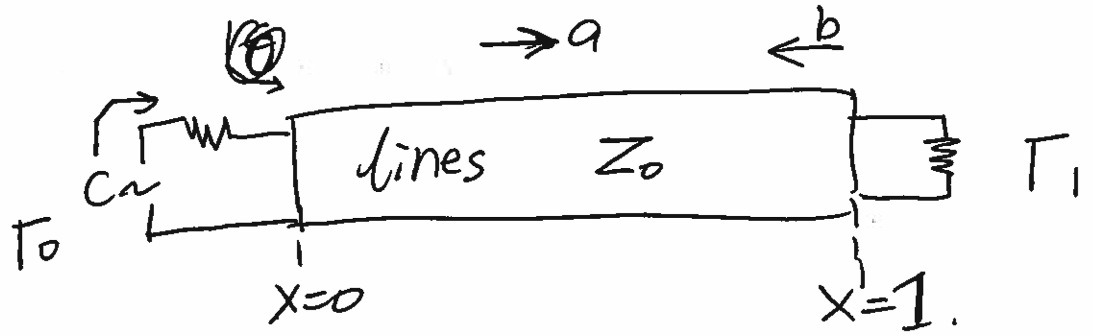
k: wave number.

z, t position and time.

$$A(z)=A(0)e^{-(jk+\alpha)z}. \qquad \xrightarrow{+z}$$

define $\boxed{p(z)=e^{-jk(1-j\delta)z}}$ $\quad \delta=\frac{\alpha}{k}$.

$e^{-\alpha z}$: represents attenuation.



Γ_0, Γ_1 are reflection coefficient on both ends.

c is the wave from generator.

a is input wave voltage $\quad$ ~~$a(0)=c+\Gamma_0 b_0$~~

b is output wave voltage

derivation from caltech website

http://www.sophphx.caltech.edu/Physics_6/Appendix_A_trans_line.pdf

Use. $a(0)=c+\Gamma_0 b(0)$; $a(1)=p(1)a(0)$

$b(0)=p(1)b(1)$

$b(1)=\Gamma_1 a(1)$

$a(x)=p(x)a(0)$ $\quad b(x)=p(1-x)b(1)=b(0)p(-x)$

Voltage along the line can be deduced as $V(x)=a(x)+b(x)$

$$\Rightarrow V(x)=c\frac{p(x-1)+\Gamma_1 p(1-x)}{p(-1)-\Gamma_0\Gamma_1 p(1)}$$, to get V_{in}, Let $x=0$.

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Since we have $1k\Omega$ resistor place at input and $Z_1 = Z_L$
$\Gamma_0 = 0$.

Hence $V_{in} = C\dfrac{P(-1)-P(1)}{P(-1)} = \dfrac{e^{jk(1-j\delta)} - e^{-jk(1-j\delta)}}{e^{jk(1-j\delta)}}$

Let $\Gamma_1 = -1$, represents short circuit, $Z_L = 0$.

Take the Real part using Mathematica,

we found $V_{in} = c[1 - e^{-2k\delta}\cos(2k)]$

As $k\delta = \alpha$. k is a function of frequency.

We conclude that $V_{in}-f$ is a sinusodial curve.

$V_{in} = c\cdot[1 + e^{-2k\delta}\cos(2k)]$ when $\Gamma_1 = 1$.

Fitting with our data, we found it had a high relation.

Results

General model:
f(x) = A+B*cos(K*x)
Coefficients (with 95% confidence bound
A = 5.59 (5.488, 5.692)
B = 2.904 (2.77, 3.038)
K = 0.251 (0.2501, 0.252)

Goodness of fit:
SSE: 16.83
R-square: 0.9593
Adjusted R-square: 0.9583
RMSE: 0.4616

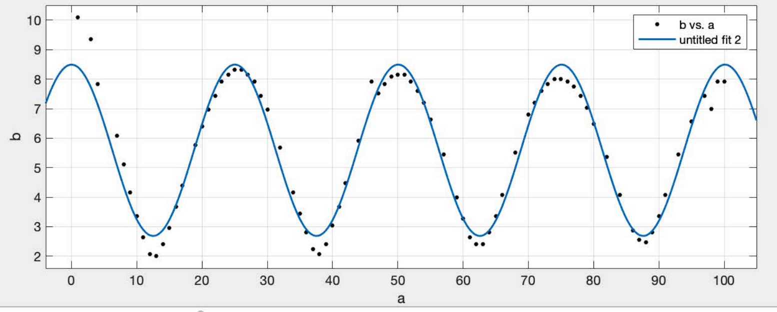

Table of Fits

Fit name ▲	Data	Fit type	SSE	R-square	DFE	Adj R-sq	RMSE	# Coeff	Validation Da...	Validation SSE	Validation R
untitled fi...	y vs. x	A-B*cos(K*x)	10.2361	0.9733	78	0.9726	0.3623	3			
untitled fi...	b vs. a	A+B*cos(K*x)	16.8317	0.9593	79	0.9583	0.4616	3			

computation ~~as~~ shown below.

```
In[•]:= ComplexExpand[(Exp[I*k*(1 - I*δ)] - Exp[-I*k*(1 - I*δ)]) / Exp[I*k*(1 - I*δ)]]
        复展开                虚数单位     虚数单位 指…  虚数单位      虚数单位      虚数单位     虚数单位
```

Out[•]= $-2\,\mathrm{Im}\left[e^{-2k\delta}\,\mathrm{Cos}[k]\,\mathrm{Sin}[k]\right] + \mathrm{Re}\left[\mathrm{Cos}[k]^2 - e^{-2k\delta}\,\mathrm{Cos}[k]^2 + \mathrm{Sin}[k]^2 + e^{-2k\delta}\,\mathrm{Sin}[k]^2\right]$

```
In[•]:= Simplify[Cos[k]^2 - e^(-2kδ) Cos[k]^2 + Sin[k]^2 + e^(-2kδ) Sin[k]^2]
        化简
```

Out[•]= $1 - e^{-2k\delta}\,\mathrm{Cos}[2k]$

```
In[•]:= ComplexExpand[(Exp[I*k*(1 - I*δ)] + Exp[-I*k*(1 - I*δ)]) / Exp[I*k*(1 - I*δ)]]
        复展开                虚数单位     虚数单位 指…  虚数单位      虚数单位      虚数单位     虚数单位
```

```
In[•]:= Re[Cos[k]^2 + e^(-2kδ) Cos[k]^2 - 2 i e^(-2kδ) Cos[k] Sin[k] + Sin[k]^2 - e^(-2kδ) Sin[k]^2]
        实部                                          余弦    正弦
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Out[•]= $2\,\mathrm{Im}\left[e^{-2k\delta}\,\mathrm{Cos}[k]\,\mathrm{Sin}[k]\right] + \mathrm{Re}\left[\mathrm{Cos}[k]^2 + e^{-2k\delta}\,\mathrm{Cos}[k]^2 + \mathrm{Sin}[k]^2 - e^{-2k\delta}\,\mathrm{Sin}[k]^2\right]$

```
In[•]:= Simplify[Cos[k]^2 + e^(-2kδ) Cos[k]^2 + Sin[k]^2 - e^(-2kδ) Sin[k]^2]
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Out[•]= $1 + e^{-2k\delta}\,\mathrm{Cos}[2k]$

Project (2021 summer)

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A. Angular Momentum Operator in Spherical Coordinates

The angular momentum of an object is,

$$\mathbf{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{1}$$

Whereas in terms of operators, $\boldsymbol{r}, \boldsymbol{p}$ are the position operator and momentum operator expressed in spherical coordinates as,

$$\boldsymbol{r} = r\hat{\boldsymbol{r}} \tag{2}$$

$$\boldsymbol{p} = -i\hbar\nabla = -i\hbar\left(\hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \tag{3}$$

Note: Eq.(3) is a result of $p_x = -i\hbar\frac{\partial}{\partial x}, \quad p_y = -i\hbar\frac{\partial}{\partial y}, \quad p_z = -i\hbar\frac{\partial}{\partial z}$. Hence,

$$\mathbf{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (-i\hbar\nabla) = -i\hbar(\boldsymbol{r} \times \nabla) \tag{4}$$

With Eqs.(2),(3) in hand, we now express $\mathbf{L}$ in spherical coordinates as follows,

$$\mathbf{L} = -i\hbar(\boldsymbol{r} \times \nabla) = -i\hbar\begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ r & 0 & 0 \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\end{vmatrix} = i\hbar\left(\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right) \tag{5}$$

Using the relations,

$$\hat{\boldsymbol{\theta}} = (\cos\theta\cos\phi)\hat{\boldsymbol{x}} + (\cos\theta\sin\phi)\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{z}} \tag{6}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}} \tag{7}$$

$\mathbf{L}$ can be expressed in Cartesian Coordinates as,

$$\mathbf{L} = i\hbar\left((\cos\phi\cot\theta\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta})\hat{\boldsymbol{x}} + (\sin\phi\cot\theta\frac{\partial}{\partial\phi} - \cos\phi\frac{\partial}{\partial\theta})\hat{\boldsymbol{y}} - (\frac{\partial}{\partial\phi})\hat{\boldsymbol{z}}\right) \tag{8}$$

Then the three components of $\mathbf{L}$ are,

$$L_x = i\hbar\left(\cos\phi\cot\theta\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta}\right) \tag{9}$$

$$L_y = i\hbar\left(\sin\phi\cot\theta\frac{\partial}{\partial\phi} - \cos\phi\frac{\partial}{\partial\theta}\right) \tag{10}$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \tag{11}$$

Since raising and lowering operators are $L_\pm = L_x \pm iL_y$, it follows that,

$$L_\pm = \hbar\left(\cot\theta(\mp\sin\phi + i\cos\phi)\frac{\partial}{\partial\phi} + (\pm\cos\phi + i\sin\phi)\frac{\partial}{\partial\theta}\right) \tag{12}$$

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*Electronic address:

with the Euler's Equation $e^{\pm i\phi} = \cos\phi \pm i\sin\phi$ and some manipulations on $\pm$ sign, the above is simplified as,

$$L_\pm = \hbar\left(i\cot\theta(\pm\sin\phi + \cos\phi)\frac{\partial}{\partial\phi} \pm (\cos\phi \pm i\sin\phi)\frac{\partial}{\partial\theta}\right) \tag{13}$$

$$L_\pm = \hbar e^{\pm i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right) \tag{14}$$

There are several ways of computing L^2 in spherical coordinates. The following briefly introduce one way, and elaborate another way in particular. The key here is to apply each operator twice.

Since $L^2 \equiv {L_x}^2 + {L_y}^2 + {L_z}^2$, With Eqs.(9),(10),(11), L^2 can be found. The calculation might be complex, but fortunately it can be simplified by previously derived $L_\pm$ as follows[1, Eq. 4.112],

$$\begin{aligned} L_+L_- &= (L_x + iL_y)(L_x - iL_y) = {L_x}^2 + {L_y}^2 + i(L_yL_x - L_xL_y) \\ &= L^2 - {L_z}^2 + \hbar L_z \end{aligned} \tag{15}$$

So,

$$L^2 = L_+L_- + {L_z}^2 - \hbar L_z \tag{16}$$

Thus the calculation of L^2 becomes more hassle-free with what has been done above.

Now we illustrate a straightforward calculation from the beginning (Eq.5). L^2 can be treated as applying the operator $\mathbf{L}$ on a function twice.

$$L^2 \equiv \mathbf{L}\cdot\mathbf{L} = -\hbar^2\left(\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right)\cdot\left(\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right) \tag{17}$$

$$= -\hbar^2\left[\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\cdot\frac{\partial}{\partial\phi}\left(\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\phi}\right) + \hat{\boldsymbol{\phi}}\cdot\frac{\partial}{\partial\theta}\left(\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right) - \hat{\boldsymbol{\phi}}\cdot\frac{\partial}{\partial\theta}\left(\frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\frac{\partial}{\partial\phi}\right) - \frac{\hat{\boldsymbol{\theta}}}{\sin\theta}\cdot\frac{\partial}{\partial\phi}\left(\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right)\right] \tag{18}$$

Be aware of the order in which the partial derivative is applied in expanding the equation. They are NOT commutative. Changing the order would ultimately change the expression.
Continue expanding the above with chain rule and simplify. Notice that we have

$$\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1 \tag{19}$$

$$\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = 0 \tag{20}$$

$$\hat{\boldsymbol{\theta}}\cdot\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} = \hat{\boldsymbol{\phi}}\cdot\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\theta} = \hat{\boldsymbol{\phi}}\cdot\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = 0 \tag{21}$$

$$\hat{\boldsymbol{\theta}}\cdot\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} = -\cos\theta \tag{22}$$

Eqs.(21),(22) are proved by transforming $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ into xyz-coordinates. So Eq.(18) becomes,

$$L^2 = -\hbar^2\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right) \tag{23}$$

This is more commonly written as

$$L^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{24}$$

Now,

$$\begin{aligned} [L^2, L_z] &= L^2L_z - L_zL^2 \\ &= i\hbar^3\left[\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi}\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right)\right] = 0 \end{aligned} \tag{25}$$

B. Spherical Harmonics

The spherical harmonics $Y_l^m(\theta,\phi)$ are defined as[1, p. 137],

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta) \tag{26}$$

where P_l^m is called the associated Legendre function($|m| < l$). Y_l^m are augular parts of solutions to the Time-Independent Schrodinger Equation of the form:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi \tag{27}$$

where $\psi = \psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$.

First few examples of spherical harmonics are illustrated here ($l = 0,1,2$).

$$\begin{array}{lll}
Y_0^0 = \frac{1}{2}\sqrt{\frac{1}{\pi}} & Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\phi}\sin\theta & Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta \\
Y_1^1 = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin\theta & Y_2^{-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{-2i\phi}\sin^2\theta & Y_2^{-1} = \frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{-i\phi}\sin\theta\cos\theta \\
Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1) & Y_2^1 = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{i\phi}\sin\theta\cos\theta & Y_2^2 = -\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\phi}\sin^2\theta
\end{array}$$

Apparently, spherical harmonics are complex. To visualize, we plot their real and imaginary parts separately on a sphere in Fig. 1.

1. *Vector spherical harmonics*

The three vectors spherical harmonics are,

$$\mathbf{U}_{lm}(\theta,\phi) = |\mathbf{r}|\nabla Y_{lm}(\theta,\phi), \tag{28}$$

$$\mathbf{V}_{lm}(\theta,\phi) = \mathbf{r}\times\nabla Y_{lm}(\theta,\phi), \tag{29}$$

$$\mathbf{Y}_{lm}(\theta,\phi) = \frac{\mathbf{r}}{|\mathbf{r}|}Y_{lm}(\theta,\phi). \tag{30}$$

We'll make a vector plot to visualize their real and imaginary parts separately on a sphere in the same manner as the density plot.

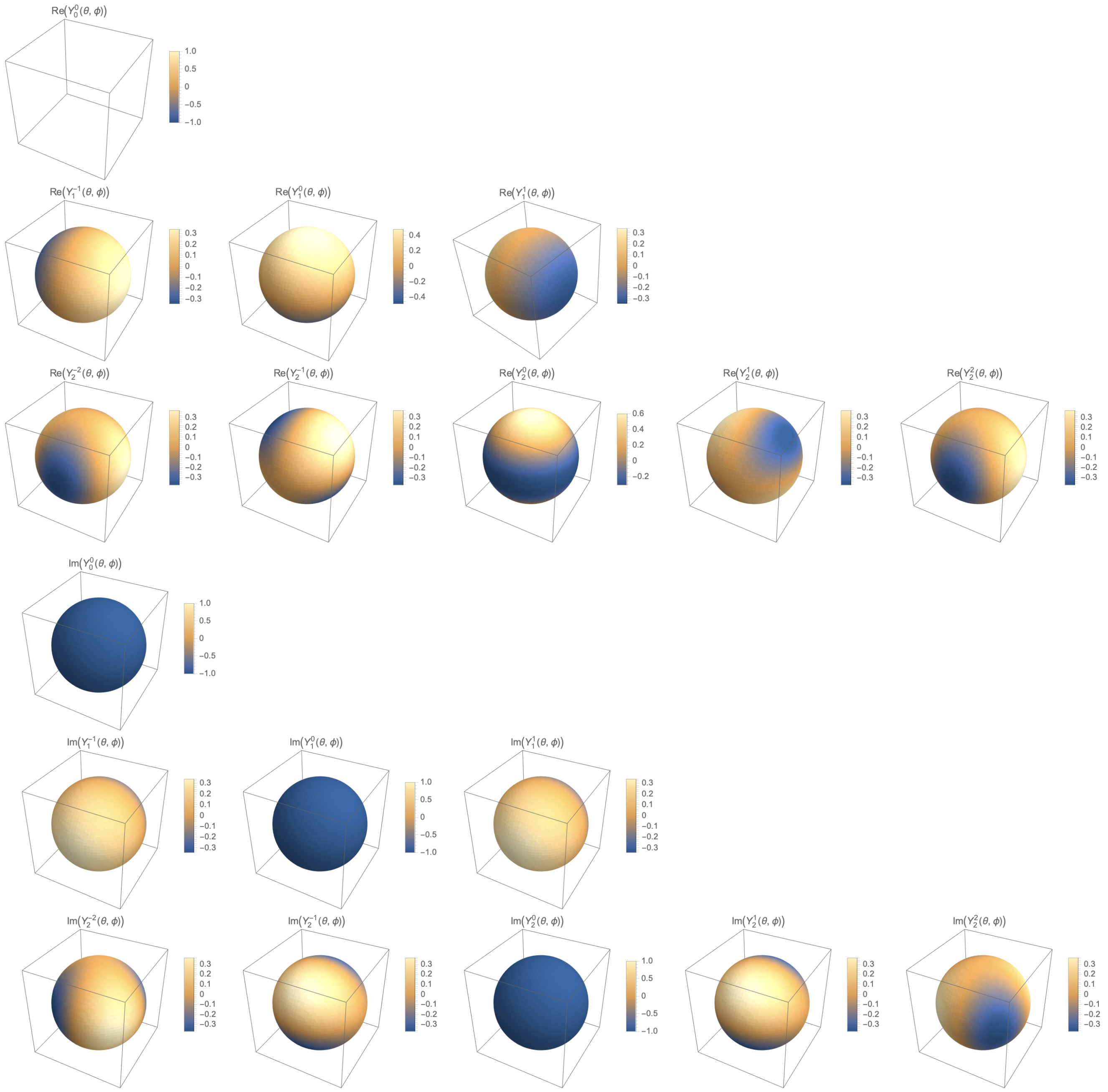
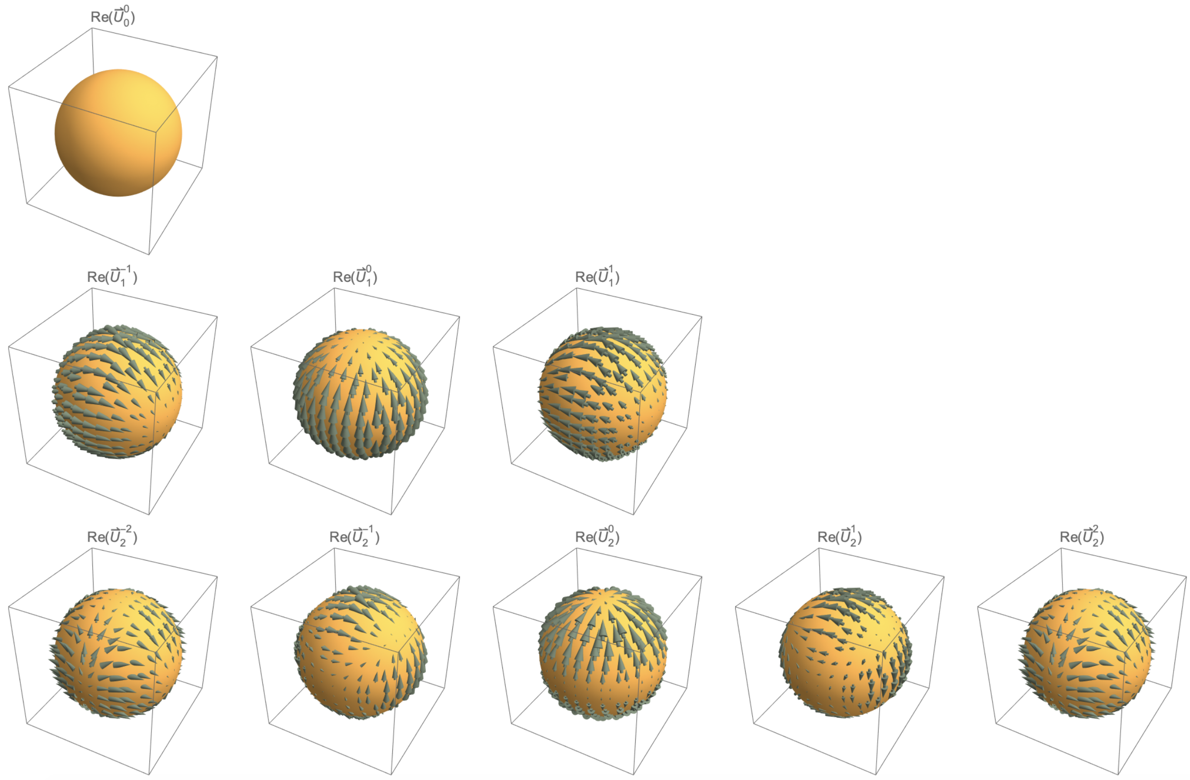
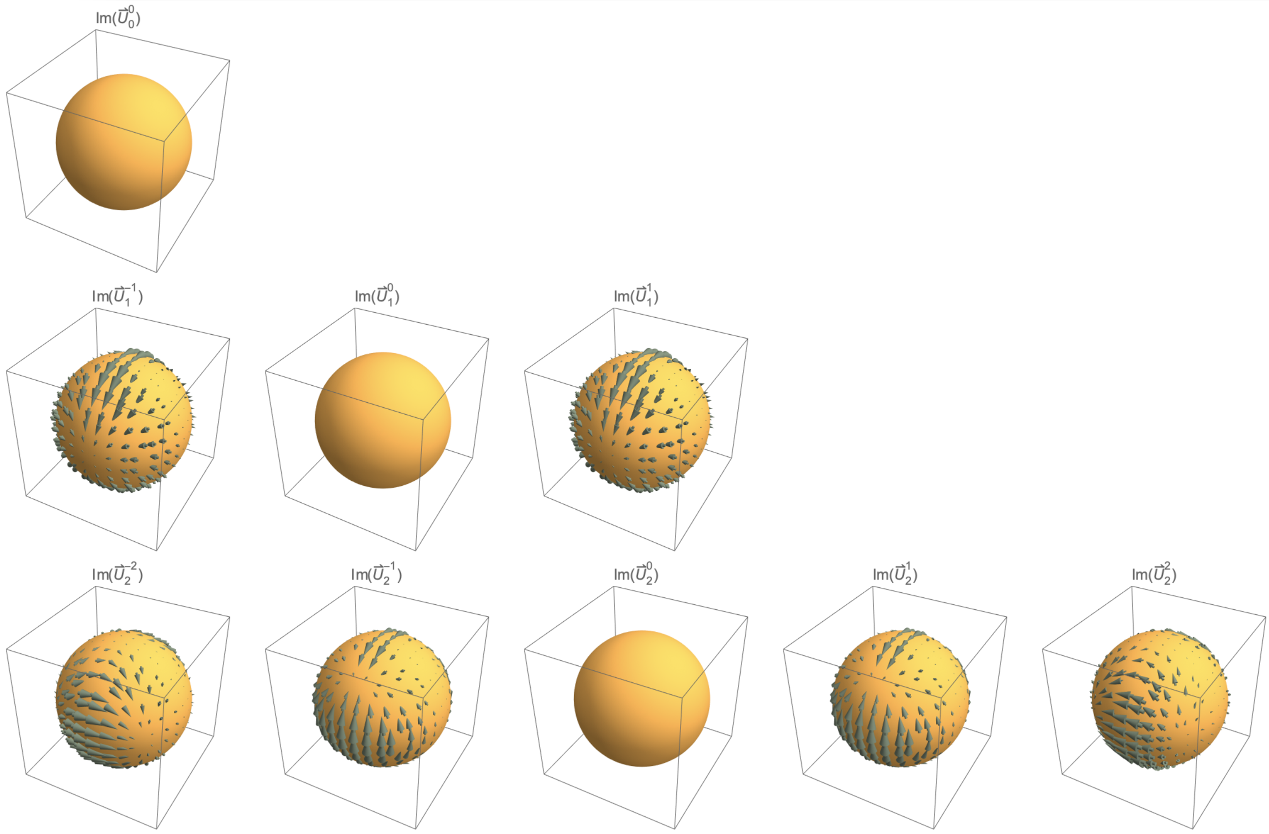
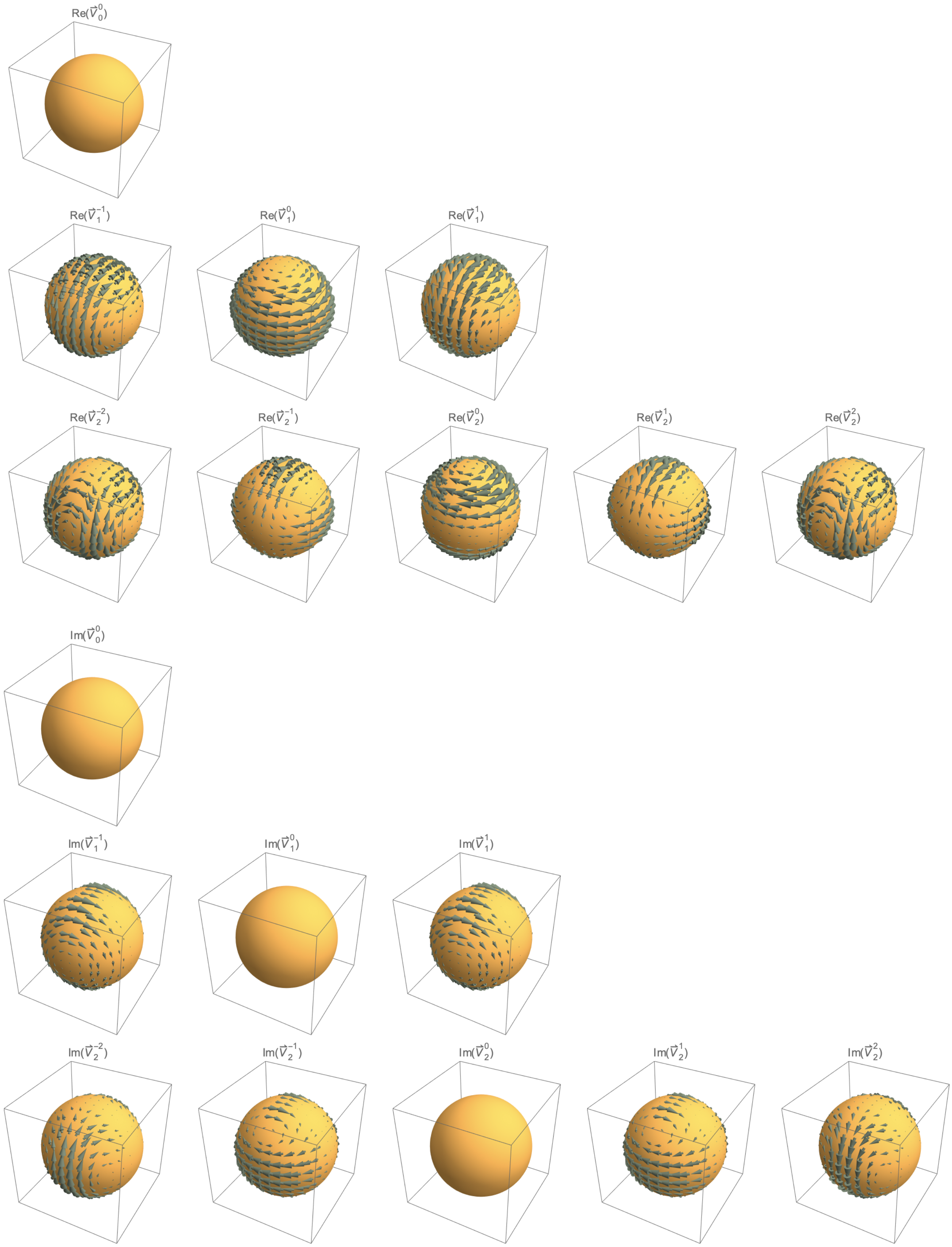
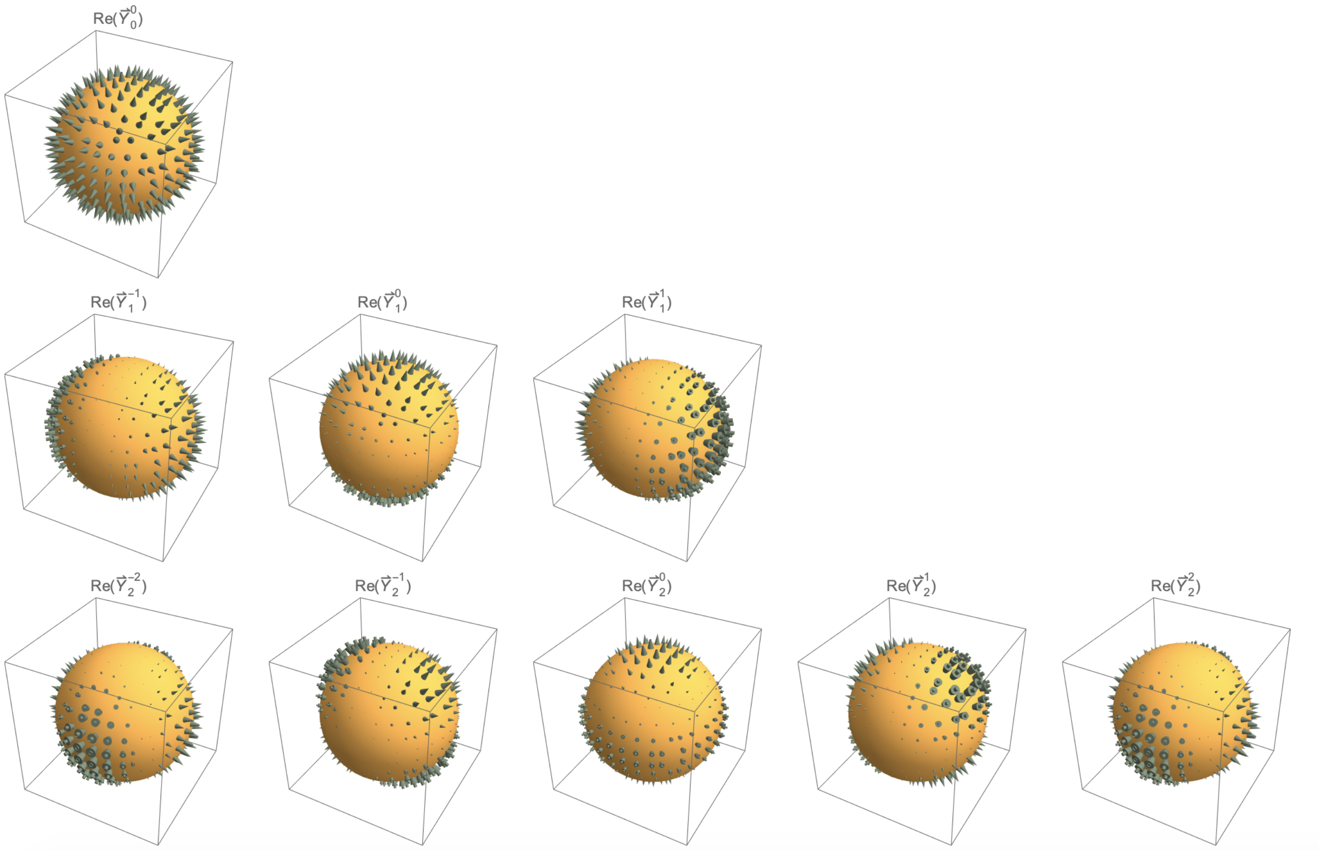


FIG. 1: Density 3D plots of the real and imaginary parts of spherical harmonics $Y_l^m(\theta,\phi)$ when $l = 0, 1, 2$ and $-l \leq m \leq l$. Lighter represents a higher value(density).


Re($\vec{U}_0^0$)
Re($\vec{U}_1^{-1}$)
Re($\vec{U}_1^0$)
Re($\vec{U}_1^1$)
Re($\vec{U}_2^{-2}$)
Re($\vec{U}_2^{-1}$)
Re($\vec{U}_2^0$)
Re($\vec{U}_2^1$)
Re($\vec{U}_2^2$)


Im($\vec{U}_0^0$)
Im($\vec{U}_1^{-1}$)
Im($\vec{U}_1^0$)
Im($\vec{U}_1^1$)
Im($\vec{U}_2^{-2}$)
Im($\vec{U}_2^{-1}$)
Im($\vec{U}_2^0$)
Im($\vec{U}_2^1$)
Im($\vec{U}_2^2$)


$\mathrm{Re}(\vec{V}_0^0)$
$\mathrm{Re}(\vec{V}_1^{-1})$
$\mathrm{Re}(\vec{V}_1^0)$
$\mathrm{Re}(\vec{V}_1^1)$
$\mathrm{Re}(\vec{V}_2^{-2})$
$\mathrm{Re}(\vec{V}_2^{-1})$
$\mathrm{Re}(\vec{V}_2^0)$
$\mathrm{Re}(\vec{V}_2^1)$
$\mathrm{Re}(\vec{V}_2^2)$
$\mathrm{Im}(\vec{V}_0^0)$
$\mathrm{Im}(\vec{V}_1^{-1})$
$\mathrm{Im}(\vec{V}_1^0)$
$\mathrm{Im}(\vec{V}_1^1)$
$\mathrm{Im}(\vec{V}_2^{-2})$
$\mathrm{Im}(\vec{V}_2^{-1})$
$\mathrm{Im}(\vec{V}_2^0)$
$\mathrm{Im}(\vec{V}_2^1)$
$\mathrm{Im}(\vec{V}_2^2)$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$\mathrm{Re}(\vec{Y}_1^{-1})$
$\mathrm{Re}(\vec{Y}_1^{0})$
$\mathrm{Re}(\vec{Y}_1^{1})$
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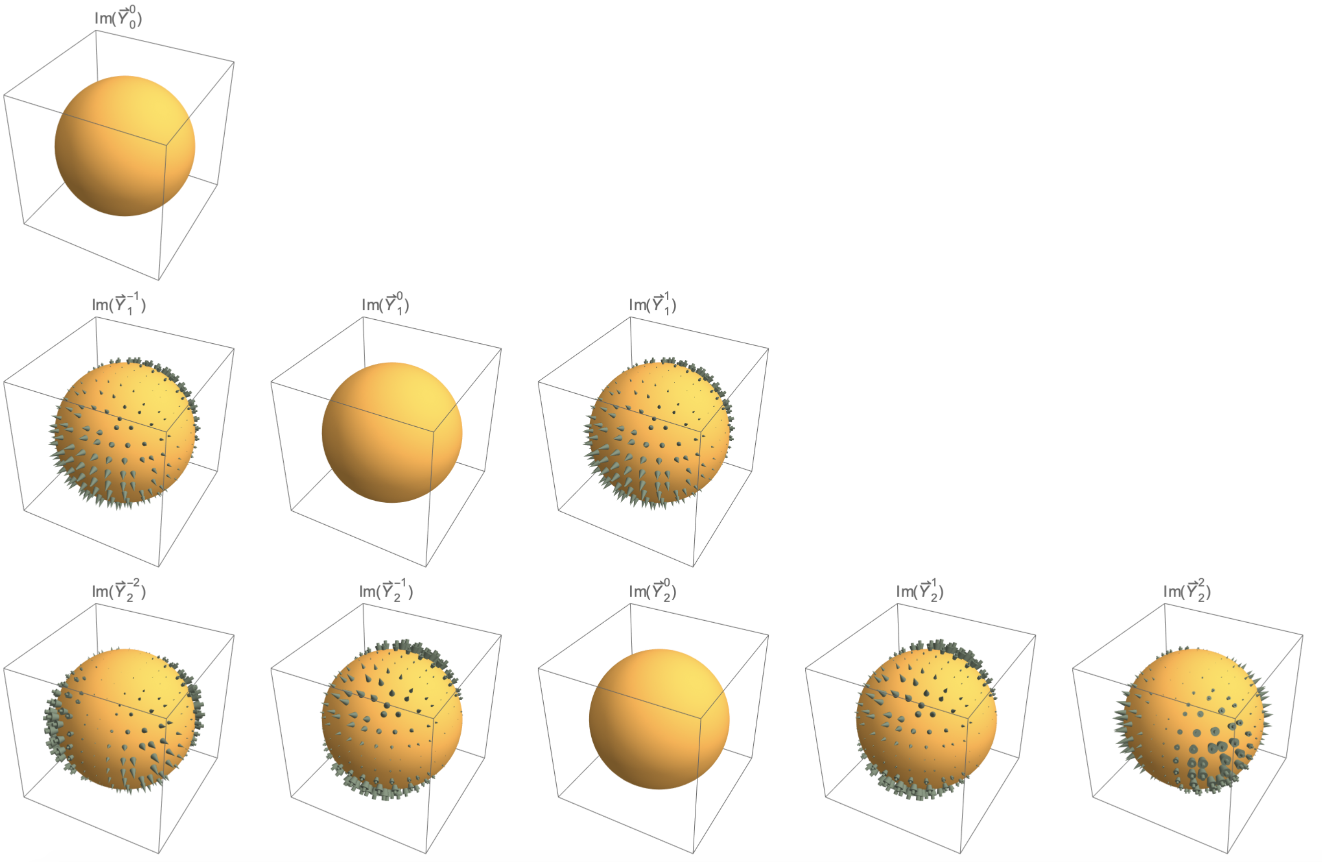
$\mathrm{Im}(\vec{Y}_0^0)$
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$\mathrm{Im}(\vec{Y}_2^{-1})$
$\mathrm{Im}(\vec{Y}_2^{0})$
$\mathrm{Im}(\vec{Y}_2^{1})$
$\mathrm{Im}(\vec{Y}_2^{2})$

C. Maxwell's Equations in Spherical Coordinates

1. Derivatives of unit bases

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial r}=0 \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial r}=0 \qquad \frac{\partial \hat{\boldsymbol{\phi}}}{\partial r}=0 \tag{31}$$

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta}=\hat{\boldsymbol{\theta}} \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta}=-\hat{\boldsymbol{r}} \qquad \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta}=0 \tag{32}$$

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial \phi}=\hat{\boldsymbol{\phi}}\sin\theta \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi}=\hat{\boldsymbol{\phi}}\cos\theta \qquad \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}=-(\hat{\boldsymbol{r}}\sin\theta+\hat{\boldsymbol{\theta}}\cos\theta) \tag{33}$$

Either by transforming unit bases into Cartesian coordinates plus some substitution or, by a geometric approach, we can get the above results.

2. Del, div, curl

First we define del operator ∇ with scalar field $u=u(r,\theta,\phi)$ as

$$du=\nabla u\cdot d\boldsymbol{r} \tag{34}$$

where

$$du=\frac{\partial u}{\partial r}dr+\frac{\partial u}{\partial \theta}d\theta+\frac{\partial u}{\partial \phi}d\phi \tag{35}$$

$$d\boldsymbol{r}=\hat{\boldsymbol{r}}dr+\hat{\boldsymbol{\theta}}rd\theta+\hat{\boldsymbol{\phi}}r\sin\theta d\phi \tag{36}$$

Plug the above two expressions into Eq.(34), then we get the vector ∇u expressed as

$$\nabla u=\hat{\boldsymbol{r}}\frac{\partial u}{\partial r}+\frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial u}{\partial \theta}+\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial u}{\partial \phi} \tag{37}$$

Hence,

$$\nabla=\hat{\boldsymbol{r}}\frac{\partial}{\partial r}+\frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}+\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{38}$$

If $\boldsymbol{V}$ is a vector field such that $\boldsymbol{V}=V_r\hat{\boldsymbol{r}}+V_\theta\hat{\boldsymbol{\theta}}+V_\phi\hat{\boldsymbol{\phi}}$,then

$$\nabla\cdot\boldsymbol{V}=\left(\hat{\boldsymbol{r}}\frac{\partial}{\partial r}+\frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}+\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot(V_r\hat{\boldsymbol{r}}+V_\theta\hat{\boldsymbol{\theta}}+V_\phi\hat{\boldsymbol{\phi}}) \tag{39}$$

$$\nabla\times\boldsymbol{V}=\left(\hat{\boldsymbol{r}}\frac{\partial}{\partial r}+\frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}+\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\times(V_r\hat{\boldsymbol{r}}+V_\theta\hat{\boldsymbol{\theta}}+V_\phi\hat{\boldsymbol{\phi}}) \tag{40}$$

expanding, rearranging, with the derivatives summarized in Section C 1 (similar to the process in Eq.(18)) we found,

$$\nabla\cdot\boldsymbol{V}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2V_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(V_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} \tag{41}$$

$$\nabla\times\boldsymbol{V}=\frac{\hat{\boldsymbol{r}}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(V_\phi\sin\theta\right)-\frac{\partial V_\theta}{\partial\phi}\right]+\frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\left[\frac{\partial V_r}{\partial\phi}-\sin\theta\frac{\partial}{\partial r}\left(rV_\phi\right)\right]+\frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial}{\partial r}\left(rV_\theta\right)-\frac{\partial V_r}{\partial\theta}\right] \tag{42}$$

Maxwell's equations in vacuum with Lorentz–Heaviside unit are:

$$\nabla \cdot \mathbf{E} = 0 \tag{43}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{44}$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{45}$$
$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{46}$$

If $\mathbf{E} = E_r \hat{\boldsymbol{r}} + E_\theta \hat{\boldsymbol{\theta}} + E_\phi \hat{\boldsymbol{\phi}}$ and $\mathbf{B} = B_r \hat{\boldsymbol{r}} + B_\theta \hat{\boldsymbol{\theta}} + B_\phi \hat{\boldsymbol{\phi}}$, with Eq.(41), we re-express Maxwell's equations in spherical coordinates. The first two Eqs. are saclar,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(E_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial \phi} = 0 \tag{47}$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(B_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi} = 0 \tag{48}$$

The latter two are vector Eqs. and thus having three scalar Eqs. to express each.
we have to find the time derivatives of $\mathbf{E}, \mathbf{B}$. For $\mathbf{E}$,

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\left(E_r \hat{\boldsymbol{r}} + E_\theta \hat{\boldsymbol{\theta}} + E_\phi \hat{\boldsymbol{\phi}}\right) \tag{49}$$

and

$$\dot{\hat{\boldsymbol{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}} + \sin\theta\dot{\phi}\hat{\boldsymbol{\phi}} \tag{50}$$
$$\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\boldsymbol{r}} + \cos\theta\dot{\phi}\hat{\boldsymbol{\phi}} \tag{51}$$
$$\dot{\hat{\boldsymbol{\phi}}} = -\sin\theta\dot{\phi}\hat{\boldsymbol{r}} - \cos\theta\dot{\phi}\hat{\boldsymbol{\theta}} \tag{52}$$

Using the above and combine Eqs.(42)(45)(46), we have the remaining two Maxwell's equations (two vector Eqs. = six scalar Eqs.),

$$\begin{aligned}
\frac{c}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(E_\phi\sin\theta\right) - \frac{\partial E_\theta}{\partial\phi}\right] &= -\dot{B}_r + \dot{\theta}B_\theta + \sin\theta\dot{\phi}B_\phi \\
\frac{c}{r\sin\theta}\left[\frac{\partial E_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}\left(rE_\phi\right)\right] &= -\dot{B}_\theta - \dot{\theta}B_r + \cos\theta\dot{\phi}B_\phi \\
\frac{c}{r}\left[\frac{\partial}{\partial r}\left(rE_\phi\right) - \frac{\partial E_r}{\partial\theta}\right] &= -\dot{B}_\phi - \sin\theta\dot{\phi}B_r - \cos\theta\dot{\phi}B_\theta
\end{aligned} \tag{53}$$

$$\begin{aligned}
\frac{c}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(B_\phi\sin\theta\right) - \frac{\partial B_\theta}{\partial\phi}\right] &= \dot{E}_r - \dot{\theta}E_\theta - \sin\theta\dot{\phi}E_\phi \\
\frac{c}{r\sin\theta}\left[\frac{\partial B_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}\left(rB_\phi\right)\right] &= \dot{E}_\theta + \dot{\theta}E_r - \cos\theta\dot{\phi}E_\phi \\
\frac{c}{r}\left[\frac{\partial}{\partial r}\left(rB_\phi\right) - \frac{\partial B_r}{\partial\theta}\right] &= \dot{E}_\phi + \sin\theta\dot{\phi}E_r + \cos\theta\dot{\phi}E_\theta
\end{aligned} \tag{54}$$

I. REAL REPRESENTATION AND ELECTROSTATIC FIELD

The complex spherical harmonics $Y_l^m(\theta,\phi)$ are defined as[1, p. 137],

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}\,P_l^m(\cos\theta) \tag{55}$$

where P_l^m is called the associated Legendre function($|m| < l$). Let us define the real spherical harmonics $Y_{lm}(\theta, \phi)$:

$$Y_{lm} = \begin{cases} \frac{(-1)^m}{\sqrt{2}\mathrm{i}}\left(Y_l^{|m|} - Y_l^{|m|*}\right) & \text{if } m < 0 \\ Y_l^0 & \text{if } m = 0 \\ \frac{(-1)^m}{\sqrt{2}}\left(Y_l^m + Y_l^{m*}\right) & \text{if } m > 0 \end{cases} \tag{56}$$

$$= \begin{cases} \sqrt{2}(-1)^m \Im\left[Y_l^{|m|}\right] & \text{if } m < 0 \\ Y_l^0 & \text{if } m = 0 \\ \sqrt{2}(-1)^m \Re\left[Y_l^m\right] & \text{if } m > 0 \end{cases} \tag{57}$$

$$= \begin{cases} (-1)^m\sqrt{2}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_\ell^{|m|}(\cos\theta)\sin(|m|\varphi) & \text{if } m < 0 \\ \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta) & \text{if } m = 0 \\ (-1)^m\sqrt{2}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)\cos(m\varphi) & \text{if } m > 0 \end{cases} . \tag{58}$$

The corresponding normalization relations for these real spherical harmonics are

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} Y_{lm}Y_{l'm'}\mathrm{d}\Omega = \delta_{ll'}\delta_{mm'} \tag{59}$$

The three vectors spherical harmonics are,

$$\mathbf{U}_{lm}(\theta, \phi) = r\nabla Y_{lm}(\theta, \phi) \tag{60}$$
$$\mathbf{V}_{lm}(\theta, \phi) = \mathbf{r} \times \nabla Y_{lm}(\theta, \phi) \tag{61}$$
$$\mathbf{Y}_{lm}(\theta, \phi) = Y_{lm}(\theta, \phi)\hat{\mathbf{r}} \tag{62}$$

The additional theorem for spherical harmonics does not change:

$$\sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_{lm}(\theta, \phi)Y_{lm}(\theta', \phi') = P_l(\cos\gamma). \tag{63}$$

Among the above equation, γ is the included angle between the two points (θ, ϕ) and (θ', ϕ') on a sphere. Thus we have the expression for the Coulomb potential as follow

$$\sum_{lm} \frac{4\pi}{2l+1}\frac{r'^l}{r^{l+1}} Y_{lm}(\theta, \phi)Y_{lm}(\theta', \phi') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \quad r' < r. \tag{64}$$

The potential for an electrostatic field in the spherical coordinate can be expressed by an integral

$$\Phi(\mathbf{r}) = \frac{1}{4\pi}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\mathrm{d}V' \tag{65}$$

Using the following formula

$$-\nabla^2 \frac{1}{|\mathbf{r}' - \mathbf{r}|} = 4\pi\delta^3(\mathbf{r}' - \mathbf{r}), \tag{66}$$

we can see Eq. (65) can give the Poisson equation

$$-\nabla^2\Phi(\mathbf{r}) = \rho(\mathbf{r}), \tag{67}$$

which is just the result of Gaussian theorem

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \rho(\mathbf{r}) \tag{68}$$

under the curl-free condition for the electrostatic field.

Now, if we apply the multi-pole expansion to the electric potential function, we can obtain

$$\Phi(\mathbf{r}) = \sum_{lm} \frac{Y_{lm}(\theta, \phi)}{(2l+1)r^{l+1}} q_{lm}, \tag{69}$$

$$q_{lm} = \int \rho(\mathbf{r}')r'^l Y_{lm}(\theta', \phi')dV'. \tag{70}$$

Note that we should restrict $\rho(\mathbf{r}')$ to distribute in the region $r' < r$. The electric field is the gradient of the potential function:

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}), \tag{71}$$

$$= \sum_{lm} \frac{q_{lm}}{(2l+1)r^{l+2}}[(l+1)\mathbf{Y}_{lm} - \mathbf{U}_{lm}], \tag{72}$$

$$q_{lm} = \int \rho(\mathbf{r}')r'^{l} Y_{lm}(\theta', \phi')dV'. \tag{73}$$

We define the spherical multi-pole field as

$$\mathbf{E}_{lm}(\mathbf{r}) = (l+1)\mathbf{Y}_{lm} - \mathbf{U}_{lm}. \tag{74}$$

Based on physics, we should know $\mathbf{E}(\mathbf{r})$ and $\mathbf{E}_{lm}(\mathbf{r})$ do not contain any zeros at far field places. Therefore, we are able to define a winding number of a multi-pole field over an origin-centered sphere.

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[1] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics* (Cambridge University Press, 2018).

Spherical electromagnetic field

I. MAXWELL EQUATION IN VACUUM

The aim of us is to study the spherical electromagnetic field in vacuum. Firstly, we should write down the Maxwell's equations in Lorentz–Heaviside units:

$$\nabla \cdot \vec{E}(\vec{r},t) = 0, \tag{1}$$
$$\nabla \cdot \vec{B}(\vec{r},t) = 0, \tag{2}$$
$$\nabla \times \vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial \vec{B}(\vec{r},t)}{\partial t}, \tag{3}$$
$$\nabla \times \vec{B}(\vec{r},t) = \frac{1}{c}\frac{\partial \vec{E}(\vec{r},t)}{\partial t}. \tag{4}$$

If we consider the electromagnetic oscillation at a constant frequency ω ($\omega = 0$ corresponds to the electrostatic and magnetostatics), the time variable among the Maxwell's equations can be removed:

$$\nabla \cdot \vec{E}(\vec{r}) = 0, \tag{5}$$
$$\nabla \cdot \vec{B}(\vec{r}) = 0, \tag{6}$$
$$\nabla \times \vec{E}(\vec{r}) = ik\vec{B}(\vec{r}), \tag{7}$$
$$\nabla \times \vec{B}(\vec{r}) = -ik\vec{E}(\vec{r}). \tag{8}$$

Among the above equations, $\vec{E}(\vec{r},t) = \vec{E}(\vec{r})e^{-i\omega t}$, $\vec{B}(\vec{r},t) = \vec{B}(\vec{r})e^{-i\omega t}$, and $k = \omega/c$. $\vec{E}(\vec{r})$ and $\vec{B}(\vec{r})$ are the amplitude field. However, for convenience we still call them as electric field and magnetic field. Through the among first-order partial differential equations, we can obtain the decoupled second-order equation:

$$(\nabla^2 + k^2)\vec{E}(\vec{r}) = 0, \tag{9}$$
$$(\nabla^2 + k^2)\vec{B}(\vec{r}) = 0. \tag{10}$$

Furthermore, it can be proved that [1]

$$(\nabla^2 + k^2)\vec{r} \cdot \vec{E}(\vec{r}) = 0, \tag{11}$$
$$(\nabla^2 + k^2)\vec{r} \cdot \vec{B}(\vec{r}) = 0. \tag{12}$$

To solve these partial differential equations in spherical coordinate, we expand the field by vector spherical harmonic functions:

$$\vec{U}_{lm}(\theta,\phi) = |\vec{r}|\nabla Y_{lm}(\theta,\phi), \tag{13}$$
$$\vec{V}_{lm}(\theta,\phi) = \vec{r} \times \nabla Y_{lm}(\theta,\phi), \tag{14}$$
$$\vec{Y}_{lm}(\theta,\phi) = \frac{\vec{r}}{|\vec{r}|} Y_{lm}(\theta,\phi). \tag{15}$$

Vector spherical harmonic functions are complete bases for the vector field in spherical coordinate[2, 3]. Note that $\vec{U}_{lm}$ and $\vec{V}_{lm}$ are tangent vector fields on a sphere while $\vec{Y}_{lm}$ is along the radial direction. The complete spherical expansion for any vector field $\vec{\Psi}(\vec{r})$ is

$$\vec{\Psi}(\vec{r}) = \sum_{lm} A_{lm}(r)\vec{U}_{lm}(\theta,\phi) + B_{lm}(r)\vec{V}_{lm}(\theta,\phi) + C_{lm}(r)\vec{Y}_{lm}(\theta,\phi). \tag{16}$$

The divergence and curl of $\vec{\Psi}(\vec{r})$ can be calculated as

$$\nabla\cdot\vec{\Psi}(\vec{r}) = \sum_{lm}[\frac{dC_{lm}(r)}{dr}+\frac{2}{r}C_{lm}(r)-\frac{l(l+1)}{r}A_{lm}(r)]Y(\theta,\phi), \tag{17}$$

$$\nabla\times\vec{\Psi}(\vec{r}) = \sum_{lm}-[\frac{dB_{lm}(r)}{dr}+\frac{1}{r}B_{lm}(r)]\vec{U}_{lm}(\theta,\phi)+[\frac{dA_{lm}(r)}{dr}+\frac{1}{r}A_{lm}(r)-\frac{1}{r}C_{lm}(r)]\vec{V}_{lm}(\theta,\phi) \\ -\frac{l(l+1)}{r}B_{lm}(r)\vec{Y}_{lm}(\theta,\phi). \tag{18}$$

If we consider TE modes which means $\vec{r}\cdot\vec{E}(\vec{r})=0$, $\vec{E}(\vec{r})$ can be simplified as:

$$\vec{E}(\vec{r}) = \sum_{lm}E_{lm}(r)\vec{V}_{lm}(\theta,\phi). \tag{19}$$

Eq. (12) gives

$$\vec{r}\cdot\vec{B}(\vec{r}) = \sum_{lm}ie_{lm}\frac{l(l+1)h_l^{(1)}(kr)}{k}Y_{lm}(\theta,\phi), \tag{20}$$

where $h_l^{(1)}(kr)$ is the first type spherical Hankel function and e_{lm} is the corresponding coefficient. Since $\vec{B}$ is related with $\vec{E}$ through Maxwell's equations, we have

$$\vec{E} = \sum_{lm}e_{lm}h_l^{(1)}(kr)\vec{V}_{lm}(\theta,\phi) \\ = (\vec{r}\times\nabla)\Phi(\vec{r}), \tag{21}$$

$$\Phi(\vec{r}) = \sum_{lm}e_{lm}h_l^{(1)}(kr)Y_{lm}(\theta,\phi), \tag{22}$$

$$\vec{B} = -\frac{i}{k}\nabla\times\vec{E} \\ = -\frac{i}{k}\nabla\times(\vec{r}\times\nabla\Phi(\vec{r})) \\ = \frac{i}{k}\sum_{lm}e_{lm}\{(\frac{h_l^{(1)}(kr)}{r}+\frac{dh_l^{(1)}(kr)}{dr})\vec{U}_{lm}(\theta,\phi)+\frac{l(l+1)}{r}h_l^{(1)}(kr)\vec{Y}_{lm}(\theta,\phi)\}. \tag{23}$$

$\Phi(\vec{r})$ obeys the scalar Helmholtz equation.

Then we concern the TM modes. Similarly, we have

$$\vec{B} = \sum_{lm}b_{lm}h_l^{(1)}(kr)\vec{V}_{lm}(\theta,\phi) \\ = (\vec{r}\times\nabla)\Psi(\vec{r}), \tag{24}$$

$$\Psi(\vec{r}) = \sum_{lm}b_{lm}h_l^{(1)}(kr)Y_{lm}(\theta,\phi), \tag{25}$$

$$\vec{E} = \frac{i}{k}\nabla\times\vec{B} \\ = \frac{i}{k}\nabla\times(\vec{r}\times\nabla\Psi(\vec{r})) \\ = -\frac{i}{k}\sum_{lm}b_{lm}\{(\frac{h_l^{(1)}(kr)}{r}+\frac{dh_l^{(1)}(kr)}{dr})\vec{U}_{lm}(\theta,\phi)+\frac{l(l+1)}{r}h_l^{(1)}(kr)\vec{Y}_{lm}(\theta,\phi)\}. \tag{26}$$

Overall, the general spherical electromagnetic field should be combined by TE and TM modes:

$$\vec{E}(\vec{r}) = (\vec{r}\times\nabla)\Phi(\vec{r})+\frac{i}{k}\nabla\times(\vec{r}\times\nabla\Psi(\vec{r})), \tag{27}$$

$$\vec{B}(\vec{r}) = (\vec{r}\times\nabla)\Psi(\vec{r})-\frac{i}{k}\nabla\times(\vec{r}\times\nabla\Phi(\vec{r})), \tag{28}$$

where $\Phi(\vec{r})$ and $\Psi(\vec{r})$ are all the solutions of scalar Helmholtz equation. In fact, the above expressions are actually originated from the Poloidal–toroidal decomposition theorem.

[1] J. D. Jackson, *Classical electrodynamics* (Wiley, 1999), 3rd ed., ISBN 9780471309321.
[2] R. G. Barrera, G. A. Estevez, and J. Giraldo, European Journal of Physics **6**, 287 (1985).
[3] J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory (4th Edition)* (Addison-Wesley Publishing Company, USA, 2008), 4th ed., ISBN 0321581741.