Two Hours

UNIVERSITY OF MANCHESTER

GROUP THEORY

21 January 2020

09:45 - 11:45

Answer **THREE** of the four questions. If more than THREE questions are attempted, then credit will be given for the best THREE answers.

Electronic calculators may be used, provided that they cannot store text.

1.

(i) Let S be a non-empty subset of the group G.

(a) For $g, h \in G$, prove that $S^{(gh)} = (S^g)^h$.

(b) Prove that $N_G(S) = \{g \in G \mid S^g = S\}$ is a subgroup of G.

(ii) Suppose that $L = \{A \in GL(2,3) \mid \det(A) = 1\}$ (which you may assume is a subgroup of $GL(2,3)$). Prove that $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is the only element of L of order 2.

(iii) Suppose that $(H, *)$ and $(K, \odot)$ are groups. For $(h,k), (h',k') \in H \times K$ define

$$(h,k)(h',k') = (h * h', k \odot k').$$

Prove that this is a binary operation on $H \times K$ and that, with this binary operation, $H \times K$ is a group.

(iv) Prove that L (the group in part (ii)) is not isomorphic to $\mathbb{Z}_2 \times A_4$.

[20 MARKS]

2.

(i) (a) State the classification theorem for finitely generated abelian groups.

(b) Let G be the group $\{1, 2, 4, 7, 8, 11, 13, 14\}$ whose binary operation is multiplication mod 15. Determine the orders of each of the elements of G and hence show that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

(c) Determine the torsion coefficients of $\mathbb{Z}_{42} \times \mathbb{Z}_{12} \times \mathbb{Z}_{70} \times \mathbb{Z}_{120}$.

(ii) Suppose that G is a group with H and K subgroups of G such that $HK = KH$. Prove that HK is a subgroup of G.

(iii) Suppose that G is a group with H a subgroup of G and N a normal subgroup of G.

(a) Prove that NH is a subgroup of G.

(b) Prove that $C_G(N)$ is a normal subgroup of G (you may assume without proof that $C_G(N)$ is a subgroup of G).

[20 MARKS]

3.

Suppose that G is a finite group acting on a finite non-empty set Ω.

(i) Assume that $G = \langle g_1, g_2, g_3 \rangle \leq S_{19}$ where

$$\begin{aligned} g_1 &= (1,3)(2,4)(5,15)(6,12)(7,9)(11,13)(16,17), \\ g_2 &= (1,3,11,13)(2,6,4,8)(10,12)(14,18,16,17) \qquad \text{and} \\ g_3 &= (1,3,5)(7,11,13)(10,16,17)(12,14,18). \end{aligned}$$

Determine the G-orbits on $\Omega = \{1, \ldots, 19\}$.

(ii) Let $\alpha \in \Omega$. Prove that $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$ is a subgroup of G.

(iii) Let Δ be a G-orbit of Ω, and let $\alpha \in \Delta$. Prove that $[G : G_\alpha] = |\Delta|$.

(iv) If G has t orbits on Ω, prove that

$$t = \frac{1}{|G|} \sum_{g \in G} |\mathrm{fix}_\Omega(g)|.$$

(v) Assume that G acts transitively on Ω and that $|\Omega| > 1$. Show there exists $g \in G$ such that $\mathrm{fix}_\Omega(g) = \emptyset$.

[20 MARKS]

4.

(i) State Sylow's theorems.

(ii) Suppose G is a group with $|G| = pqr$ where p, q and r are distinct primes. Let n_p, n_q and n_r denote, respectively, the number of Sylow p-, q- and r-subgroups of G. Show that

$$|G| \geq 1 + n_p(p-1) + n_q(q-1) + n_r(r-1).$$

Hence prove that G is not a simple group.

(iii) Give an example, with reasons, of a Sylow 7-subgroup of S_{10}.

(iv) Prove that a group of order 882 cannot be a simple group.

[20 MARKS]

END OF EXAMINATION PAPER