**Three hours**

# THE UNIVERSITY OF MANCHESTER

REPRESENTATIONS AND CHARACTERS OF GROUPS

22 January 2019

14.00 – 17.00

Answer FOUR of the FIVE questions. If more than four questions are attempted, then credit will be given for the best FOUR answers.

*Show all your work and justify your answers.*

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Electronic calculators may be used, provided that they cannot store text.

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**1.**

[20 marks]

(a) For a finite group $G$, define a $G$-space and a $G$-subspace.

(b) State and prove Maschke's Theorem (you may quote other results from the course without proof).

(c) Define a matrix representation of a finite group $G$ and state the condition for two matrix representations to be equivalent.

(d) If $\rho$ is a matrix representation of $G$, define $\sigma$ by $\sigma(g) = \overline{\rho(g^{-1})}^T$ for $g \in G$, where the $T$ denotes transpose and the bar denotes complex conjugation. Show that $\sigma$ is a matrix representation of $G$.

(e) Show that $\sigma$ and $\rho$ have the same character, i.e. $\chi_\sigma = \chi_\rho$, and deduce that there is an invertible matrix $M$ such that $M\rho(g)M^{-1} = \overline{\rho(g^{-1})}^T$ for all $g \in G$ (you may quote results from the course that you need).

**2.**

[20 marks]

(a) For a finite group, $G$, define a homomorphism of $G$-spaces and define what it means for a $G$-space to be irreducible.

(b) Show that the kernel and the image of a homomorphism of $G$-spaces are both $G$-subspaces.

(c) State and prove Schur's Lemma.

Let $\rho$ be an irreducible matrix representation of $G$ with character $\chi$, let $a \in G$ and let $a^G$ denote the conjugacy class of $a$ in $G$. Let $C_a = \sum_{g \in a^G} \rho(g)$.

(d) Prove that $C_a$ is a scalar multiple $\lambda$ of the identity matrix.

(e) Show that $\lambda = |a^G|\chi(a)/\chi(1)$.

Now suppose that $\rho(g)$ has integer entries for all $g \in G$.

(f) Show that $\lambda$ is an integer.

**3.**

[20 marks]

(a) Define a class function on a finite group $G$ and the inner product $(\phi, \mu)_G$ of two class functions $\phi$ and $\mu$.

(b) Prove that characters of $G$-spaces are class functions.

(c) If $H$ is a subgroup of $G$ and $\theta$ is a class function on $H$, write down the formula for the induced class function $\operatorname{Ind}_H^G \theta$ on $G$.

(d) State and prove the Frobenius Reciprocity Theorem for class functions.

Let $H$ be a subgroup of $G$ and let $V$ be an irreducible $G$-space. Let $U$ be an irreducible $H$-subspace of $\operatorname{Res}_H^G V$.

(e) Use Frobenius reciprocity to show that $V$ is isomorphic to a $G$-subspace of $\operatorname{Ind}_H^G U$ and deduce that $\dim V \leq |G : H| \dim U$.

(f) Deduce that $\dim V \leq |G : H|$ when $H$ is abelian.

(g) Returning to the case when $H$ is not necessarily abelian, let $W$ be an irreducible $H$-space and $X$ an irreducible $G$-subspace of $\operatorname{Ind}_H^G W$. Show that $W$ is isomorphic to an $H$-subspace of $\operatorname{Res}_H^G X$.

**4.**

[20 marks]

(a) Write down two formulas involving characters that express row orthogonality and column orthogonality.

(b) Write down representatives for the conjugacy classes of $S_4$, the symmetric group on 4 objects, and calculate the size of each class. How many irreducible characters does $S_4$ have?

(c) Find the two 1-dimensional characters of $S_4$.

From now on be careful to explain all the properties of characters that you use.

(d) Calculate the character of the natural permutation representation of $S_4$ and use it to produce a 3-dimensional irreducible character of $S_4$.

(e) Use standard properties of characters to complete the character table of $S_4$.

(f) Compute the square of the 2-dimensional irreducible character and express it as a sum of irreducible characters.

**5.**
Let $X$ be a $G$-set and $\mathbb{C}[X]$ the associated $G$-space. [20 marks]

(a) State and prove a formula for $\chi_{\mathbb{C}[\mathbb{X}]}(g)$ in terms of the action of $g$ on $X$.

(b) If $X$ is transitive, sketch briefly why the inner product $(\chi_{\mathbb{C}[X]}, 1)_G = 1$ (you may quote any other results from the course without proof).

(c) Deduce that even if $X$ is not transitive, $(\chi_{\mathbb{C}[X]}, 1)_G$ is equal to the number of orbits of $G$ on $X$.

(d) State and prove Burnside's Counting Theorem.

(e) Suppose that you have beads in $n$ different colours. How many different necklaces can be made consisting of 8 beads, up to rotating the necklace and turning it over?

**END OF EXAMINATION PAPER**