Two hours

THE UNIVERSITY OF MANCHESTER

TOPOLOGY

21 May 2019

9:45-11:45

Answer **ALL** SIX questions

University approved calculators may be used.

1.

(a) Define what is meant by a *topology* on a set X.

(b) Define what is meant by the *usual* topology on a subset $X \subset \mathbb{R}^n$.

(c) Define what is meant by saying that a function $f\colon X \to Y$ between topological spaces is *continuous*. Define what is meant by saying that f is a *homeomorphism*.

(d) Consider the set $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Show that τ defines a topology on X. Find all continuous maps from $X \to \{1, 2\}$ with the usual topology on $\{1, 2\}$.

[15 marks]

2.

(a) Define what is meant by saying that a topological space X is *path-connected*.

(b) What is meant by saying that path-connectedness is a *topological property*?

(c) Define what is meant by the set $\pi_0(X)$ of *path-components* of a topological space X.

(d) Prove that a continuous map of topological spaces $f\colon X \to Y$ induces a map $f_*\colon \pi_0(X) \to \pi_0(Y)$ between the sets of path-components, taking care to prove that your function is well-defined. Prove that if f is a homeomorphism then f_* is a bijection.

(e) Use (d) to show that path-connectedness is a topological property.

(f) Consider the topological space (X, τ) from Question 1. (d), what are the path-components of this space? Justify your answer.

[15 marks]

3.

(a) Suppose that $q\colon X \to Y$ is a surjection from a topological space X to a set Y. Define the *quotient topology* on Y determined by q. State the *universal property* of the quotient topology.

(b) Suppose that $f\colon X \to Z$ is a continuous surjection from a compact topological space X to a Hausdorff topological space Z. Define an equivalence relation $\sim$ on X so that f induces a bijection $F\colon X/{\sim} \to Z$ from the identification space $X/{\sim}$ of this equivalence relation to Z. Prove that F is a homeomorphism. [*State clearly any general results which you use.*]

(c) Prove that the quotient space $(S^1 \times [0, 1])/(S^1 \times \{1\})$ is homeomorphic to the closed unit disc D^2 (where S^1 and D^2 have the usual topology).

[15 marks]

4.

(a) Define what is meant by saying that a topological space is *Hausdorff*.

(b) Suppose that X and Y are topological spaces. Define the *product topology* on the Cartesian product $X \times Y$. [*It is not necessary to prove that this is a topology.*]

(c) Consider the Sierpinski topology $\tau = \{\emptyset, \{a\}, \{a, b\}\}$ on $X = \{a, b\}$. Write down the product topology on $X \times X$.

(d) Prove that if X and Y are Hausdorff spaces, then the product space $X \times Y$ is Hausdorff, as well.

(e) Prove that, if a topological space X is Hausdorff, then all singleton subsets $\{x\}$ of X (where $x \in X$) are closed. Give an example to show that the converse of this statement is false; that is, if a topological space has all singleton subsets closed, it is not necessarily Hausdorff.

[15 marks]

5.

(a) Define what is meant by a *compact subset* of a topological space.

(b) Prove by using only the definition of compactness, that $\{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is not compact.

(c) Prove that every compact subset of a Hausdorff topological space is closed.

(d) Give an example to show that the Hausdorff condition in (c) cannot be omitted.

[15 marks]

6.

(a) Suppose that X_1 is a subspace of a topological space X. Define what is meant by saying that X_1 is a *retract* of X.

(b) Show that $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ is a retract of the punctured plane $\mathbb{R} \setminus \{0\}$.

(c) Explain how a continuous function of topological spaces $f\colon X \to Y$ induces a homomorphism of fundamental groups $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ for $x_0 \in X$. You should indicate why f_* is a well-defined homomorphism.

(d) Use the functorial properties of the fundamental group to prove that, if X_1 is a retract of X, then, for any $x_0 \in X_1$, the homomorphism induced by the inclusion map

$$i_*\colon \pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is injective.

(e) Hence prove that S^1 is not a retract of the closed disc D^2 .

[*You may quote any calculation of fundamental groups that you need, without proof.*]

[15 marks]

END OF EXAMINATION PAPER