Representations and Applications of Poincaré Group

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1 Introduction

This article is devoted to explore the representation theory of Poincaré group and its application to physics, in particular, how is it related to Dirac equation. We will first introduce some preliminary mathematical knowledge useful to the development of the theory. Then we will use the method of induced representation to construct the representations of Poincaré group. The Dirac equation, a relativistic wave equation that describes the behavior of a spin-1/2 particle, is subsequently derived as a covariant projection to an irreducible subspace.

Poincaré group plays a vital role in describing the symmetries in physics. It is the group of all the isometries of Minkowski space. The group is generated by three basic symmetries: rotations, translations, and boosts. The rotations and translations preserve the distances between two points, while the boosts change the distances.

1.1 Lorentz Group

Before get down to the Poincaré group, first we need to look at the subgroup of the Poincaré group: Lorentz group, the set of all Lorentz transformations, which preserve the space-time isometries while leaving the origin of the reference frame fixed without translations.

Define a coordinate in $\mathbb{R}^4 : x = (x^0, x^1, x^2, x^3)$, where $x^0 \equiv ct^0$ is the time coordinate and (x^1, x^2, x^3) is the space coordinate.

Definition 1.1. A Lorentz metric is a bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle y, x \rangle = y^0x^0 - y^1x^1 - y^2x^2 - y^3x^3, \qquad \text{for all } x, y \in \mathbb{R}^4. \tag{1}$$

Written in a 4×4 matrix, it is a metric η such that

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

Such a vector space $\mathbb{R}^4$, endowed with a Lorentz metric, is known as the *Minkowski space.*

With summation convention, we can turn Eq.(1) into

$$\langle y, x \rangle = \eta_{\mu\nu} y^\mu x^\nu$$

Definition 1.2. A Lorentz transformation is a linear operator $\Lambda : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\langle \Lambda y, \Lambda x \rangle = \langle y, x \rangle \quad \text{ for all } x, y \in \mathbb{R}^4. \tag{2}$$

So Lorentz transformation preserves the Lorentz metric.

Eq.(2) can be written as

$$\Lambda^\mu{}_\nu \eta_{\mu\lambda} \Lambda^\lambda{}_\sigma = \eta_{\nu\sigma} \tag{3}$$

$$\text{where } \Lambda^\mu{}_\nu = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}.$$

This gives 10 independent quadratic equations by calculations, where Λ is a 4×4 matrix with 16 entries. Therefore, 6 parameters suffice to define Λ.

In particular, we have

$$\left(\Lambda^0{}_0\right)^2 - \left(\Lambda^1{}_0\right)^2 - \left(\Lambda^2{}_0\right)^2 - \left(\Lambda^3{}_0\right)^2 = 1.$$

Therefore, either

$$\Lambda^0{}_0 \geqslant 1 \text{ or } \Lambda^0{}_0 \leqslant -1. \tag{4}$$

Furthermore, the product of Lorentz transformation is still a Lorentz transformation and because $\det \Lambda = \pm 1$, they're invertible. Hence the set of all Lorentz transformations forms a group called the "*Lorentz group*", Denoted by $\mathcal{L} \equiv O(1,3)$.

Example 1.3. It's easy to check that rotations and boosts are Lorentz transformations if they satisfy the condition in Eq.(3). Rotations can be characterised by

$$\Lambda = \begin{pmatrix} 1 & 0^\top \\ 0 & \mathbf{R} \end{pmatrix}, \text{ where } \mathbf{R} \in SO(3). \tag{5}$$

Set of all such rotations is a subgroup of $\mathcal{L}$. For boosts with velocity $\boldsymbol{v}$,

$$\Lambda = \begin{pmatrix} \gamma(v) & \gamma(v)\boldsymbol{v}^\top \\ \gamma(v)\boldsymbol{v} & \mathbf{1}_3 + \frac{\gamma(v)-1}{v^2}\boldsymbol{v}\boldsymbol{v}^\top \end{pmatrix}, \text{ where } \gamma(v) = \frac{1}{\sqrt{1-v^2}}. \tag{6}$$

Note that the set of all boosts is *not* a subgroup as the product of two boosts in different directions is no longer a boost.

Remark 1.4. Because Lorentz group $\mathcal{L}$ is a closed subgroup of $GL(n,\mathbb{R})$ by Eq.(3), it is a Lie group with dimension 6 as only six variables are required to specify a Lorentz transformation. Intuitively this is because a Lorentz transformation can be generated by a product of a boost and a rotation, both of which have three degrees of freedom.

The group manifold of Lorentz group is not connected as seen from Eq.(4). It has four connected components, categorised by the sign of $\Lambda^0{}_0$ and its determinant:

$$\begin{aligned}
\mathcal{L}_+^\uparrow &\equiv \left\{\Lambda \in \mathcal{L} \mid \Lambda^0{}_0 \geq 1, \det\Lambda = +1\right\}, \\
\mathcal{L}_-^\uparrow &\equiv \left\{\Lambda \in \mathcal{L} \mid \Lambda^0{}_0 \geq 1, \det\Lambda = -1\right\} = P\mathcal{L}_+^\uparrow, \\
\mathcal{L}_-^\downarrow &\equiv \left\{\Lambda \in \mathcal{L} \mid \Lambda^0{}_0 \leq -1, \det\Lambda = -1\right\} = T\mathcal{L}_+^\uparrow, \\
\mathcal{L}_+^\downarrow &\equiv \left\{\Lambda \in \mathcal{L} \mid \Lambda^0{}_0 \leq -1, \det\Lambda = +1\right\} = PT\mathcal{L}_+^\uparrow.
\end{aligned} \tag{7}$$

$\Lambda^0{}_0 \geq 1$ indicates that the transformation Λ preserves the direction of time, denoted as $\mathcal{L}^\uparrow \equiv \{\mathcal{L}_+^\uparrow, \mathcal{L}_-^\uparrow\}$ with name *orthochronous* Lorentz subgroup. While $\det\Lambda = +1$ means that it preserves the orientation, denoted as $\mathcal{L}_+ \equiv \{\mathcal{L}_+^\uparrow, \mathcal{L}_+^\downarrow\}$ with name *proper* Lorentz subgroup. $\mathcal{L}_+^\uparrow$ has a special name: *restricted* Lorentz group.

In the above we have introduced two *discrete* Lorentz transformation, space inversion P and time reversal T, defined as

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -P, \quad PT = -\mathbf{1}_4. \tag{8}$$

$\mathcal{D} = \{\mathbf{1}, P, T, PT\}$ forms a discrete subgroup of Lorentz group. It is isomorphic to the Klein four-group V_4. Hence the quotient group $\mathcal{L}/\mathcal{L}_+^\uparrow \approx V_4$ (isomorphic).

1.2 Semidirect Product

Here we introduce the concept of semidirect product for the development of theory later.

Let H and G be two arbitrary groups. First we need to specify an *action* of G on H. This action is a homomorphism $\phi : G \to \mathrm{Aut}(H)$, where $\mathrm{Aut}(H)$ is the group of automorphisms of H. This means that for every $h \in H$ and $g \in G$, the automorphism $\phi(g)$ maps h to some element $\phi(g)(h) \in H$. This action must satisfy the following conditions:

- For every $g_1, g_2 \in G$ and $h \in H$, we have $\phi(g_1 g_2)(h) = \phi(g_1)\big(\phi(g_2)(h)\big)$.
- For every $g \in G$ and $h_1, h_2 \in H$, we have $\phi(g)(h_1 h_2) = \phi(g)(h_1)\ \phi(g)(h_2)$.

These conditions ensure that the action of G on H is a group homomorphism.

Definition 1.5. The semidirect product of H and G, denoted by $H \rtimes G$, in terms of the action ϕ, is a group defined by the set $H \times G$ with the group operation defined as

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot \phi(g_1)(h_2), g_1 g_2), \tag{9}$$

where $\phi : G \to \mathrm{Aut}(H)$ is a homomorphism from G to the group of automorphisms of H. The group operation is well-defined and satisfies the axioms of a group.

Remark 1.6. The semidirect product can be thought of as a "twisted" version of the direct product $H \times G$. In the semidirect product, the action of G on H is a crucial part of the structure of the group, whereas in the direct product, the action of one group on the other is trivial.

The semidirect product is a useful concept in algebra, as it allows us to construct new groups from known groups. This can be used to classify groups and to study the symmetries of geometric objects.

Example 1.7. The discrete Lorentz subgroup $\mathcal{D}$ we have defined in Eq.(8) can be used to express the full Lorentz group $\mathcal{L}$. It is a semidirect product of $\mathcal{D}$ and the restricted Lorentz group $\mathcal{L}_+^\uparrow$, namely, $\mathcal{L} = \mathcal{D} \ltimes \mathcal{L}_+^\uparrow$.

Proof. To show that the full Lorentz group is a semidirect product we need to prove that $\mathcal{L}$ satisfies the defining properties of a semidirect product.

The first property of a semidirect product is that the new group must be the direct product of the two groups, i.e., every elements of the new group can be written uniquely as pairs of elements from the two groups, In this case, each element in $\mathcal{L}$ can be written as pairs of elements from $\mathcal{D}$ and $\mathcal{L}_+^\uparrow$. This is obvious by examining Eq.(7).

The second property is that the operation of the first group on the second group must be non-trivial. This is true here because the operation of $\mathcal{D}$ on the restricted Lorentz group is not the same as the operation of the restricted Lorentz group on itself. □

We shall see later that Poincaré group is also a semidirect product, which is vital in finding its representations.

1.3 Poincaré Group

Different from the Lorentz transformation, an additional symmetry of translation is added to the *Poincaré transformation*.

Definition 1.8. A Poincaré transformation Π is described by two objects: $\Pi = (a, \Lambda)$, where $a \in \mathbb{R}^4$ and $\Lambda \in \mathcal{L}$. It is a map $\Pi : \mathbb{R}^4 \to \mathbb{R}^4$ with the rule:

$$\Pi(x) = \Lambda x + a, \quad x \in \mathbb{R}^4. \tag{10}$$

The set of all Poincaré transformation forms a group $\mathcal{P}$ with the law of composition:

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2). \tag{11}$$

From this composition law, we know that this group can be expressed as a semidirect product.

Definition 1.9. A Poincaré group $\mathcal{P}$ is a semidirect product of the Lorentz group and the four-dimensional abelian group of translations of space and time, i.e.,

$$\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^4. \tag{12}$$

$\mathcal{L}$ acts non-trivially on $\mathbb{R}^4$. With the introduction of translations, the Poincaré group is often known as the *inhomogeneous* Lorentz group.

1.4 Covering Group

Definition 1.10. A covering group is a group that is a "covering" of another group, in the sense that it contains multiple copies of the other group. Formally, a covering group is a group $\tilde{\mathcal{G}}$ and a homomorphism $\pi : \tilde{\mathcal{G}} \to G$ from $\tilde{\mathcal{G}}$ to another group $\mathcal{G}$, such that the map π is surjective and every element of $\mathcal{G}$ has a unique preimage under π except for a discrete set of elements.

If $\mathcal{G}$ is simply connected, and $\tilde{\mathcal{G}}$ is the smallest covering group, then $\tilde{\mathcal{G}}$ is said to be the *universal covering group* of $\mathcal{G}$.

Example 1.11. The special linear group $SL(2, \mathbb{C})$ is a universal cover of the restricted Lorentz group $\mathcal{L}_{+}^{\uparrow}$.

However, $SL(2, \mathbb{C})$ does not cover the full Lorentz group due to the space inversion P. We can solve this by introducing a linear representation $A \to \mathbf{L}_A$ as follows,

$$\mathbf{L}_A = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}, \quad \text{where } A \in SL(2). \tag{13}$$

And space inversion is represented by

$$\mathbf{L}_P = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}. \tag{14}$$

This map $A \to \mathbf{L}_A$ is reducible and has two invariant subspaces related by $\mathbf{L}_P$.

So we obtain the irreducible covering representation of $\mathcal{L}^{\uparrow}$,

$$\tilde{\mathcal{L}}^{\uparrow} \equiv \{\mathbf{L}_A, \mathbf{L}_P\mathbf{L}_A \mid A \in SL(2)\}. \tag{15}$$

2 Representations

2.1 Induced Representations

given a subgroup H of a group G, one can construct a representation of G from a representation of H by inducing it from H to G. This is achieved by taking the space of functions on G that are invariant under H, and then defining an action of G on this space. This induced representation encodes the symmetry of the larger group G.

Definition 2.1. More formally, let ρ be a representation of H on a vector space V, and let G be a group containing H as a subgroup. The induced representation π of G on the space of functions $\phi : G \to V$ is defined as follows:

$$\pi(g)\phi(g') = \phi(g^{-1}g') \tag{16}$$
$$\rho(h)\phi(g) = \phi(gh^{-1}) \tag{17}$$

for all $g, g' \in G$ and $h \in H$.

Remark 2.2. The set of functions ϕ on G with values in V can form a Hilbert space $\mathfrak{M}$, in which the elements are called *Mackey states*. It can be shown that if ρ is unitary irreducible, so is the induced representation π.

To determine the induced representationz of Poincaré group $\tilde{\mathcal{P}}^{\uparrow}$, we use the structure of its semidirect product, $\tilde{\mathcal{P}}^{\uparrow} = \tilde{\mathcal{L}}^{\uparrow} \ltimes \mathbb{R}^4$. Choose a closed subgroup $\mathcal{K}$ of $\tilde{\mathcal{P}}^{\uparrow}$,

$$\mathcal{K} = \mathcal{H} \ltimes \mathbb{R}^4 \tag{18}$$

where $\mathcal{H}$ is a subgroup of $\tilde{\mathcal{L}}^{\uparrow}$. Hence $\tilde{\mathcal{P}}^{\uparrow}/\mathcal{K} = \tilde{\mathcal{L}}^{\uparrow}/\mathcal{H}$.

Given a unitary representation τ of $\mathcal{K}$, we can define the representations of $\mathbb{R}^4$ and $\mathcal{H}$ by restrictions,

$$\begin{aligned} \alpha(a) &\equiv \tau(a, \mathbf{1}), \quad a \in \mathbb{R}^4, \\ \sigma(\mathbf{L}) &\equiv \tau(0, \mathbf{L}), \quad \mathbf{L} \in \mathcal{H}. \end{aligned} \tag{19}$$

By Eq.(9)

$$(a, \mathbf{L}) = (a, \mathbf{1})(0, \mathbf{L}) = (0, \mathbf{L})\left(\Lambda_{\mathbf{L}}^{-1} a, \mathbf{1}\right), \tag{20}$$

where $\Lambda_{\mathbf{L}}$ is an automorphism $\Lambda_{\mathbf{L}} : \mathbb{R}^4 \to \mathbb{R}^4$ for some $\mathbf{L}$ in $\mathcal{H}$.

Hence

$$\tau(a, \mathbf{L}) = \alpha(a)\sigma(\mathbf{L}) \tag{21}$$

$$\sigma(\mathbf{L})^{-1}\alpha(a)\sigma(\mathbf{L}) = \alpha\left(\Lambda_{\mathbf{L}}^{-1} a\right) \tag{22}$$

If we choose the subgroup $\mathcal{H}$ such that

$$\alpha\left(\Lambda_{\mathbf{L}}^{-1} a\right) = \alpha(a) \quad \text{for all } \mathbf{L} \in \mathcal{H} \text{ and } a \in \mathbb{R}^4, \tag{23}$$

then the product of the representations τ simplifies to

$$\tau(a, \mathbf{L}_1)\,\tau(b, \mathbf{L}_2) = \alpha(a+b)\sigma(\mathbf{L}_1\mathbf{L}_2) \tag{24}$$

2.2 Orbits and Dual Action

Definition 2.3. A one dimensional unitary representation of an abelian group $\mathcal{G}$ is called a *character*. If both ξ and χ are characters of $\mathcal{G}$, then multiplication $\xi\chi$ defined as $\xi\chi(g) = \xi(g)\chi(g)$ is again a character. The set of characters with this multiplication law is a group called the *dual group*.

Example 2.4. Define a map χ_p for some $p \in \mathbb{R}^4$

$$\chi_p(a) = e^{\mathrm{i}\langle p, a\rangle}, \quad \forall a \in \mathbb{R}^4. \tag{25}$$

Obviously, this is a character of $\mathbb{R}^4$. Its dual group $N \equiv \{\chi_p \mid p \in \mathbb{R}^4\}$ with multiplication

$$\chi_p\chi_q(a) = \chi_p(a)\chi_q(a) = \chi_{p+q}(a), \quad \forall a \in \mathbb{R}^4, \tag{26}$$

naturally defines an isomorphism $\chi : \mathbb{R}^4 \to N$, which maps p to χ_p.

We will choose the representation τ of the Poincaré subgroup such that for representation α,

$$\alpha(a) = \chi_q(a), \tag{27}$$

where q is fixed.

For any group $\mathcal{G}$, there is an an automorphism $I_h : g \to hgh^{-1}$. For Poincaré group, there is an action from the element $(b, \mathbf{L})$ on the element $(a, \mathbf{1})$ of its abelian normal subgroup $\mathbb{R}^4$ by $I_{(b,\mathbf{L})}$,

$$I_{(b,\mathbf{L})}(a, \mathbf{1}) = (b, \mathbf{L})(a, \mathbf{1})(b, \mathbf{L})^{-1} = (\Lambda_{\mathbf{L}} a, \mathbf{1}) \tag{28}$$

This action takes $a \to \Lambda_{\mathbf{L}} a$. We can similarly construct a *dual action* $\hat{I}_{(b,\mathbf{L})}$ from the Poincaré group on the element $\chi_p(a)$ of its *dual group* N of $\mathbb{R}^4$,

$$\hat{I}_{(b,\mathbf{L})}\left(\chi_p(a)\right) \equiv \chi_p\left(\Lambda_{\mathbf{L}}^{-1} a\right) \equiv \chi_{\Lambda_{\mathbf{L}} p}(a) \tag{29}$$

Notice that Eq.(22) is in fact a form of dual action by Eq.(27).

Now combine Eqs.(23),(27),(29), we get,

$$\chi_{\Lambda_{\mathbf{L}} q}(a) = \chi_q(a), \text{ for some fixed } q. \tag{30}$$

This suggests that the character is invariant under the dual action and this gives rise to the *isotropy group.*

Let group $\mathcal{G}$ acts on a set M, $q \in M$.

Definition 2.5. The subgroup $\mathcal{G}_q$ is a isotropy (*or stabilizer*) group of q under the given action Λ of $\mathcal{G}$ on M defined as

$$\mathcal{G}_q = \{g \in \mathcal{G} | \Lambda_g(q) = q\}. \tag{31}$$

Definition 2.6. The *orbit* of q under the action is the subset of M with

$$O_q \equiv \{\Lambda_h(q) \mid h \in \mathcal{G}\}. \tag{32}$$

Remark 2.7. The set M is a disjoint union of its orbits. If two elements q', q are both in the same orbit, then their isotropy groups are isomorphic.

2.3 Orbits of the Poincaré Group

By definition, the orbit of χ_q for a fixed $q \in \mathbb{R}^4$ under the dual action of Poincaré covering group $\tilde{\mathcal{P}}^{\uparrow}$ is the set

$$\hat{O}_q \equiv \left\{\hat{I}_h \chi_q \mid h = (a, \mathbf{L}) \in \tilde{\mathcal{P}}^{\uparrow}\right\} \tag{33}$$

which has a one-to-one correspondence to the set O_q by the isomorphism χ

$$O_q \equiv \left\{\Lambda_{\mathbf{L}} q \mid \mathbf{L} \in \tilde{\mathcal{L}}^{\uparrow}\right\} \tag{34}$$

The isotropy group of q is

$$\tilde{\mathcal{P}}_q \equiv \left\{g \in \tilde{\mathcal{P}}^{\uparrow} \mid \hat{I}_g \chi_q = \chi_q\right\} = \left\{(a, \mathbf{L}) \in \tilde{\mathcal{P}}^{\uparrow} \mid \Lambda_{\mathbf{L}} q = q\right\}. \tag{35}$$

We can write $\tilde{\mathcal{P}}_q$ as a semidirect product

$$\tilde{\mathcal{P}}_q = \mathbb{R}^4 \ltimes \tilde{\mathcal{L}}_q, \quad \text{where } \tilde{\mathcal{L}}_q = \left\{ \mathbf{L} \in \tilde{\mathcal{L}} \mid \Lambda_{\mathbf{L}} q = q \right\} \tag{36}$$

$\tilde{\mathcal{L}}_q$ is usually called the *little group* of q. We can then calculate the orbits of the dual group $N \approx \mathbb{R}^4$. There're six disjoint orbits, the union of which are N:

$$\begin{aligned}
&1) q = (\mu, 0, 0, 0)^\top, \mu > 0 && :O_q = \left\{ p \in \mathbb{R}^4 \mid \langle p, p \rangle = \mu^2, p_0 > 0 \right\} \\
&2) q = (-\mu, 0, 0, 0)^\top, \mu > 0 && :O_q = \left\{ p \in \mathbb{R}^4 \mid \langle p, p \rangle = \mu^2, p_0 < 0 \right\} \\
&3) q = (0, \mu, 0, 0)^\top, \mu > 0 && :O_q = \left\{ p \in \mathbb{R}^4 \mid \langle p, p \rangle = -\mu^2 \right\} \\
&4) q = (1, 1, 0, 0)^\top && :O_q = \left\{ p \in \mathbb{R}^4 \mid \langle p, p \rangle = 0, p_0 > 0 \right\} \\
&5) q = (-1, 1, 0, 0)^\top && :O_q = \left\{ p \in \mathbb{R}^4 \mid \langle p, p \rangle = 0, p_0 < 0 \right\} \\
&6) q = (0, 0, 0, 0)^\top && :O_q = \{0\}
\end{aligned}$$

where q is interpreted as the 4-momentum and $\mu \equiv mc$. The first and second orbit describe the massive particle with the positive and negative energy time-like solution respectively. Orbits 4 and 5 are the orbits of massless particles. The orbit 6 has only one element, meaning that the translations acts trivially, so its little group is the Lorentz group.

For the first orbit, the little group of q is given by

$$\tilde{\mathcal{L}}_q = \{ \mathbf{L}_U, \mathbf{L}_P \mathbf{L}_U \mid U \in SU(2) \}, \quad q = (\pm\mu, 0, 0, 0)^\top. \tag{37}$$

This representation is reducible as the matrix $\beta \equiv \mathbf{L}_P$ commutes with $\tilde{\mathcal{L}}_q$. Matrices

$$Q^\pm = \frac{1}{2}(1 \pm \beta) \tag{38}$$

are the two irreducible projections.

2.4 Induced Representations of the Poincaré group

From the construction of irreducible representation in Section 2.1, we apply that to the Poincaré group $\mathcal{G} = \tilde{\mathcal{P}}^\uparrow$. Choose the closed subgroup as $\mathcal{K} = \tilde{\mathcal{P}}_q$, where the $\mathcal{H}$ in Eq.(18) is $\tilde{\mathcal{L}}_q$. In what follows, the symbols will be consistent to that in Section 2.1.

The representations τ of $\tilde{\mathcal{P}}_q$ are irreducible if and only if σ is a irreducible representation of $\tilde{\mathcal{L}}_q$. We can get the induced representation of $\tilde{\mathcal{P}}^\uparrow$ for each orbit O_q and each σ. The Hilbert Space $\mathfrak{M}_q$ for the representation consists of functions on $\tilde{\mathcal{P}}^\uparrow$ with values in $\mathbb{R}^4$ and satisfies

$$\phi\left((a, \mathbf{L}) (b, \mathbf{L}')^{-1} \right) = e^{\mathrm{i}\langle q, b\rangle} \sigma(\mathbf{L}') \phi(a, \mathbf{L}), \quad \text{for all } (a, \mathbf{L}) \in \tilde{\mathcal{P}}^\uparrow \text{ and } (b, \mathbf{L}') \in \tilde{\mathcal{P}}_q. \tag{39}$$

The scalar product is given by

$$(\phi, \eta) = \int_{O_q} \frac{d^3 p}{|p_0|} (\phi(a, \mathbf{L}), \eta(a, \mathbf{L})). \tag{40}$$

2.5 Covariant States

Given separable Lie group $\mathcal{G}$, closed subgroup $\mathcal{K}$, unitary representation τ of $\mathcal{K}$ in the Hilbert space $\mathfrak{X}$. Assuming that the unitary representation τ is a restriction of the presentation of the group $\mathcal{G}$. We can define a new function out of ϕ in Section 2.1 as

$$\tilde{\psi}(g) \equiv \tilde{\tau}(g)\phi(g). \tag{41}$$

Calculations show that

$$\tilde{\psi}\left(gk^{-1}\right) = \tilde{\psi}(g) \tag{42}$$

meaning that $\tilde{\psi}$ depends on the coset $g\mathcal{K}$ only. Also

$$\tilde{\psi}(h) = \tilde{\psi}(s) = \tilde{\tau}(s)\zeta(s) \quad \forall\ h \in g\mathcal{K}, s = s(g\mathcal{K}) \tag{43}$$

Definition 2.8. s and ζ are used to define *Wigner states.* $s : \mathcal{G}/\mathcal{K} \to \mathcal{G}$ is a map such that $s(g\mathcal{K}) \in g\mathcal{K}$ and the image $S = s(\mathcal{G}/\mathcal{K}) \subset \mathcal{G}$ is a measurable set. By restricting a function ϕ to the set S, we define a function ζ of s such that for all $g \in \mathcal{G}$ and $s = s(g\mathcal{K})$.

$$\zeta(s) \equiv \phi(s), \quad \phi(g) = \phi\left(sk^{-1}\right) = \tau(k)\phi(s) = \tau(k)\zeta(s) \tag{44}$$

A Hilbert space $\mathfrak{C}$ of functions $\tilde{\psi}(h)$ can be defined on S with the scalar product

$$\left(\tilde{\psi}_1, \tilde{\psi}_2\right) \equiv \int_S \left(\tilde{\tau}(s)^{-1}\tilde{\psi}_1(s), \tilde{\tau}(s)^{-1}\tilde{\psi}_2(s)\right)_{\mathfrak{X}} d\mu(s) \tag{45}$$

Definition 2.9. A representation π^c of $\mathcal{G}$ is defined as,

$$\left(\pi^{\mathrm{c}}(h)\tilde{\psi}\right)(s) = \tilde{\tau}(h)\tilde{\psi}\left(\Lambda_h^{-1}s\right). \tag{46}$$

$\pi^{\mathrm{c}}(h) : \mathfrak{C} \to \mathfrak{C}$ is a unitary operator. π^c is the *covariant representation* and it acts on *covariant states* in $\mathfrak{C}$.

Now Let's turn to Poincaré group and find its covariant representations in the orbits $q_\pm = (\pm\mu, 0, 0, 0)^\top, \mu > 0$. Recall that the unitary representation τ in the Hilbert space $\mathfrak{X} = \mathbb{C}^4$ is

$$\tau(a, \mathbf{L}) = e^{\mathrm{i}\langle q,a\rangle}\mathbf{L} \quad \text{for all } (a, \mathbf{L}) \in \tilde{\mathcal{P}}_q \tag{47}$$

It is the restriction of

$$\tilde{\tau}(a, \mathbf{L}) = e^{\mathrm{i}\langle q,a\rangle}\mathbf{L} \quad \text{for all } (a, \mathbf{L}) \in \tilde{\mathcal{P}}_\uparrow. \tag{48}$$

Because τ is unitary, we can define two representations of $\tilde{\mathcal{P}}_\uparrow$ in Hilbert spaces $\mathfrak{W}_{q_+}$ and $\mathfrak{W}_{q_-}$. The covariant states in Hilbert space $\mathfrak{C}_q$ are

$$\tilde{\psi}(p) \equiv \tilde{\tau}\left(0, \mathbf{L}_p\right)\zeta(p) = \mathbf{L}_p\zeta(p), \quad \text{all } \zeta \in \mathfrak{W}_q \tag{49}$$

and the covariant representations are

$$\pi^{\mathrm{c}}(a, \mathbf{L})\tilde{\psi}(p) = e^{i\langle p,a\rangle}\mathbf{L}\tilde{\psi}\left(\Lambda_{\mathbf{L}}^{-1}p\right), \quad \forall\ (a, \mathbf{L}) \in \tilde{\mathcal{P}}^\uparrow,\ \tilde{\psi} \in \mathfrak{C}_q \tag{50}$$

2.6 Invariant Subspaces

With Eq.(38) we know that the subspaces $Q^{\pm}\mathfrak{W}_q$ are invariant and irreducible. In these invariant subspaces, Wigner states ζ satisfy $\zeta(p) = Q^{\pm}\zeta(p)$ and for the covariant states $\tilde{\psi} = \mathbf{L}_p\zeta$ in $\mathfrak{C}_q$,

$$\tilde{\psi}(p) = \mathbf{L}_p \left(Q^{\pm}\zeta\right)(p) = \left(\mathbf{L}_p Q^{\pm}\mathbf{L}_p^{-1}\right)\mathbf{L}_p\zeta(p) = \left(\mathbf{L}_p Q^{\pm}\mathbf{L}_p^{-1}\right)\tilde{\psi}(p) \tag{51}$$

So $\mathfrak{C}_q$ is also splited into two invariant subspaces

$$\mathfrak{C}_q^{\pm} = \left(\mathbf{L}_p Q^{\pm}\mathbf{L}_p^{-1}\right)\mathfrak{C}_q = \mathbf{L}_p Q^{\pm}\mathfrak{W}_q. \tag{52}$$

And

$$Q^{\pm}(p) \equiv \mathbf{L}_p Q^{\pm}\mathbf{L}_p^{-1} = \frac{1}{2}\left(1 \pm \mathbf{L}_p\beta\mathbf{L}_p^{-1}\right) = \frac{1}{2q_0}\left(q_0\mathbf{1} \pm \langle\gamma, p\rangle\right), \tag{53}$$

where $\langle\gamma, p\rangle = \mathbf{L}_p\langle\gamma, q\rangle\mathbf{L}_p^{-1} = q_0\mathbf{L}_p\gamma^0\mathbf{L}_p^{-1}$. Then following are then equivalent for all $\tilde{\psi} \in \mathfrak{C}_q^{\pm}$:

$$\begin{aligned} Q^{\pm}(p)\tilde{\psi}(p) =& \tilde{\psi}(p) \\ \langle\gamma, p\rangle\tilde{\psi}(p) =& \pm\, q_0\tilde{\psi}(p). \end{aligned} \tag{54}$$

2.7 Covariant Dirac Equation

From Eq.(50), we have

$$\pi^{\mathrm{c}}(a, 1)\tilde{\psi}(p) = e^{\mathrm{i}\langle p, a\rangle}\tilde{\psi}(p), \tag{55}$$

which shows that multiplication by p_0 is the self-adjoint generators of time translations, $p_k, k = 1, 2, 3$. Multiply Eq.(54) by $\gamma_0 = \beta$ and use $\alpha_k = \beta\gamma_k$, we get

$$p_0\tilde{\psi}(p) = (\alpha_k p_k \pm \beta q_0)\tilde{\psi}(p) \quad \text{for all } \tilde{\psi} \in \mathfrak{C}_q^{\pm} \tag{56}$$

The term $\alpha_k p_k \pm \beta q_0$ coincides with the Dirac operator in momentum space derived in the ordinary way: $H_0/c = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc$. ($q_0 = \pm\mu$ depending on which invariant subspace it is in). Hence

$$\langle\gamma, p\rangle\tilde{\psi}(p) - mc\tilde{\psi}(p) = 0, \quad \text{for } \tilde{\psi} \in \mathfrak{C}_{q+}^{+} \text{ or } \mathfrak{C}_{q-}^{-}. \tag{57}$$

This is the Dirac equation in covariant form. We have passed from the covariant projection of an induced representation to an irreducible subspace, to obtain the Dirac equation.

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