<u>TWO HOURS</u>

UNIVERSITY OF MANCHESTER

Advanced Quantum Mechanics

4th June 2021, 11.00 a.m. - 1.00 p.m.

Answer **<u>TWO</u>** questions

You MUST NOT confer with anyone in answering the questions on this assessment.

The numbers are given as a guide to the relative weights of the different parts of each question.

Solutions must be handwritten and scanned, or handwritten on a tablet, and uploaded to Blackboard **as a single pdf file**.

Order the pages so that the answers to different questions are sequential and make clear on every page which question part is being addressed. Number the pages and write your student ID on the first page.

Ensure the scan is clear. Do not use green or red pens.

One hour of the exam duration has been allowed for accessing the exam and uploading the answers. Multiple submissions are allowed and you should upload your first attempt 30 minutes before the deadline. Only the final submission will be marked.

Late penalties will apply to work submitted after the deadline.

If you are a DASS-registered student with extra time, write your own submission time, which will have been communicated to you in advance, on the first page of your solutions, and submit before that deadline.

You may assume the following formulae in any question *if proof is not explicitly requested*:

Special integrals

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2a)^n} \sqrt{\frac{\pi}{a}}, \qquad \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad a > 0.$$

4-vectors
Use notation $x^\mu = (x^0, \mathbf{r})$ for 4-position and $p^\mu = (E/c, \mathbf{p})$ for 4-momentum. In electromagnetism $A^\mu = (\Phi/c, \mathbf{A})$ for 4-potential.

Klein-Gordon equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[(\hat{\mathbf{p}} - \mathbf{V})^2 c^2 + (mc^2 + S)^2\right] \Psi = (i\hbar\partial_t - V_0)^2 \Psi.$$

Dirac equation in scalar S and vector $(V_0/c, \mathbf{V})$ potential field

$$\left[\boldsymbol{\alpha} \cdot (\hat{\mathbf{p}} - \mathbf{V})c + \beta(mc^2 + S)\right] \Psi = (i\hbar\partial_t - V_0)\Psi.$$

Standard spinor solutions to the free massive Dirac equation

$$u^{(s)}(E, \mathbf{p}) = \sqrt{E + mc^2} \begin{pmatrix} \phi^s \\ \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\phi^s \end{pmatrix}, \quad \phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

Standard matrices

$$\begin{aligned}
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \\
\alpha_i &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}; \\
\gamma_0 &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.
\end{aligned}$$

Space-time metric We use the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -, 1, -1)$.

Laplacian operator in terms of angular momentum operator $\hat{\mathbf{L}}$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2}.$$

Solution to Dirac equation in spherical vector potential $(V_0(r)/c, \vec{0})$

$$\begin{aligned}
\psi^\kappa_{jm}(\mathbf{r}) &= \begin{pmatrix} f^\kappa_j(r)\mathcal{Y}^\kappa_{jm}(\theta, \phi) \\ ig^\kappa_j(r)\mathcal{Y}^{-\kappa}_{jm}(\theta, \phi) \end{pmatrix} \quad \text{with } F = rf \text{ and } G = rg, \\
\hbar c \left[\frac{d}{dr} - \frac{\kappa}{r}\right] G^\kappa_j(r) &= -\left(E - V_0(r) - mc^2\right) F^\kappa_j(r), \\
\hbar c \left[\frac{d}{dr} + \frac{\kappa}{r}\right] F^\kappa_j(r) &= \left(E - V_0(r) + mc^2\right) G^\kappa_j(r).
\end{aligned}$$

1. a) A charged quantum rotor of angular-momentum quantum number $l = 1$ is placed in a uniform magnetic field along the z-axis with the Hamiltonian given by

$$\hat{H}_0 = \alpha \hat{L}_z,$$

where α is a positive constant. Initially the rotor is in the ground state of $\hat{H}_0$. At $t = 0$, a weak rotating magnetic field of angular frequency ω is switched on in the xy plane. The Hamiltonian of the rotor is now given by

$$\hat{H}(t) = \hat{H}_0 + \beta \left(\cos \omega t \, \hat{L}_x + \sin \omega t \, \hat{L}_y \right),$$

where β is a constant ($|\beta| \ll \alpha$). Use first-order perturbation theory to calculate the probability of transition to the first excited state as a function of time. Find the resonant frequency. Determine the transition probability at the resonant frequency and comment on the validity of your result in the long-time limit in this case.

You may use the following expressions for the two components of the angular momentum operator with quantum number $l = 1$:

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{L}_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

[11 marks]

b) i) Prove the following identity

$$\left(\boldsymbol{\sigma} \cdot \hat{\mathbf{O}} \right)^2 = (\hat{\mathbf{O}})^2 + i\boldsymbol{\sigma} \cdot (\hat{\mathbf{O}} \times \hat{\mathbf{O}}),$$

where $\hat{\mathbf{O}}$ is a general operator.

[3 marks]

ii) A spin-1/2 particle of mass m and charge q is in a field described by a time-independent 4-potential $A^\mu = (\Phi/c, \mathbf{A})$. Show that in the nonrelativistic limit, the Dirac equation of the particle reduces to the following Schrödinger eigen equation with eigenvalue ε,

$$\left[\frac{1}{2m} (\hat{\mathbf{p}} - q\mathbf{A})^2 - \frac{gq}{2m} \hat{\mathbf{S}} \cdot \mathbf{B} + q\Phi \right] \varphi = \varepsilon \varphi,$$

where $\hat{\mathbf{S}} = \hbar \boldsymbol{\sigma}/2$ and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. Find the value of the spin g factor.

[11 marks]

2. a) A sodium atom is placed in a uniform magnetic field $\mathbf{B}$ in the z direction. The corresponding weak-field Zeeman energy shift is given by

$$\Delta E = g\mu_B B M_J$$

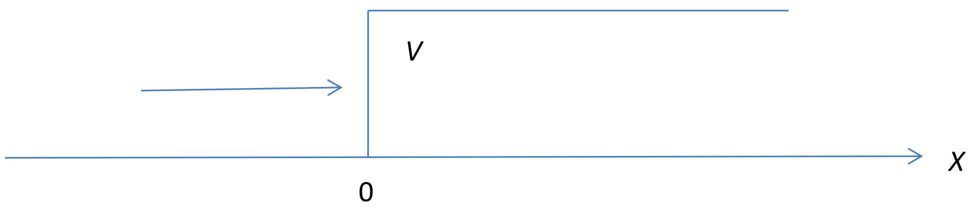
where μ_B is the Bohr magneton and the Landé g-factor is defined by

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)},$$

with the usual notation for angular momentum quantum numbers M_J, J, L, and S. Sketch weak-field Zeeman shifts for the three energy levels ${}^2S_{1/2}$, ${}^2P_{1/2}$ and ${}^2P_{3/2}$ of sodium. Mark all the transitions between these levels consistent with electric dipole selection rules.

[10 marks]

b) A beam of spinless particles with mass m and energy E is moving along the x-axis. It is incident from the left on an electrostatic barrier (step potential) of height $e\Phi = V(> 0)$ at $x = 0$ as shown in the following sketch:



i) Determine the reflection and transmission coefficients (probabilities) at the barrier.

[7 marks]

ii) Show that for $E - mc^2 < V < E + mc^2$, there is total reflection.

[3 marks]

iii) For the case $V > E+mc^2$, show that the wave-number for a wave propagating to the right inside the potential must be negative. Hence show that the magnitude of the reflection coefficient is greater than unity. Briefly state a resolution of this paradox.

[5 marks]

3. a) Write down the operator $\hat{U}(\alpha,\beta,\gamma)$ representing a rotation with Euler angles (α,β,γ) for a spin-1 particle with zero orbital angular momentum. Determine the corresponding Wigner matrix $d^{1}_{m_s m_s'}(\beta)$.

You may use the following expression for $\hat{S}_x$

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

[10 marks]

b) i) Show that the Dirac equation for a particle of mass m and charge q in the presence of the 4-potential $A^{\mu} = (\Phi/c, \mathbf{A})$ is invariant under the following charge conjugation transformation:

$$\Psi \to \Psi^{c} = -i\alpha_y \beta \Psi^{*}, \quad \text{and} \quad q \to -q.$$

You may use without proof the following identity:

$$(\alpha_y \beta)\boldsymbol{\alpha}^{*} = \boldsymbol{\alpha}(\alpha_y \beta).$$

[10 marks]

ii) Using the free particle spinor solution of the Dirac equation given in the formula sheet, write down the free antiparticle solution.

[5 marks]

END OF EXAMINATION PAPER