CS4278/CS5478 Intelligent Robots: Algorithms and Systems

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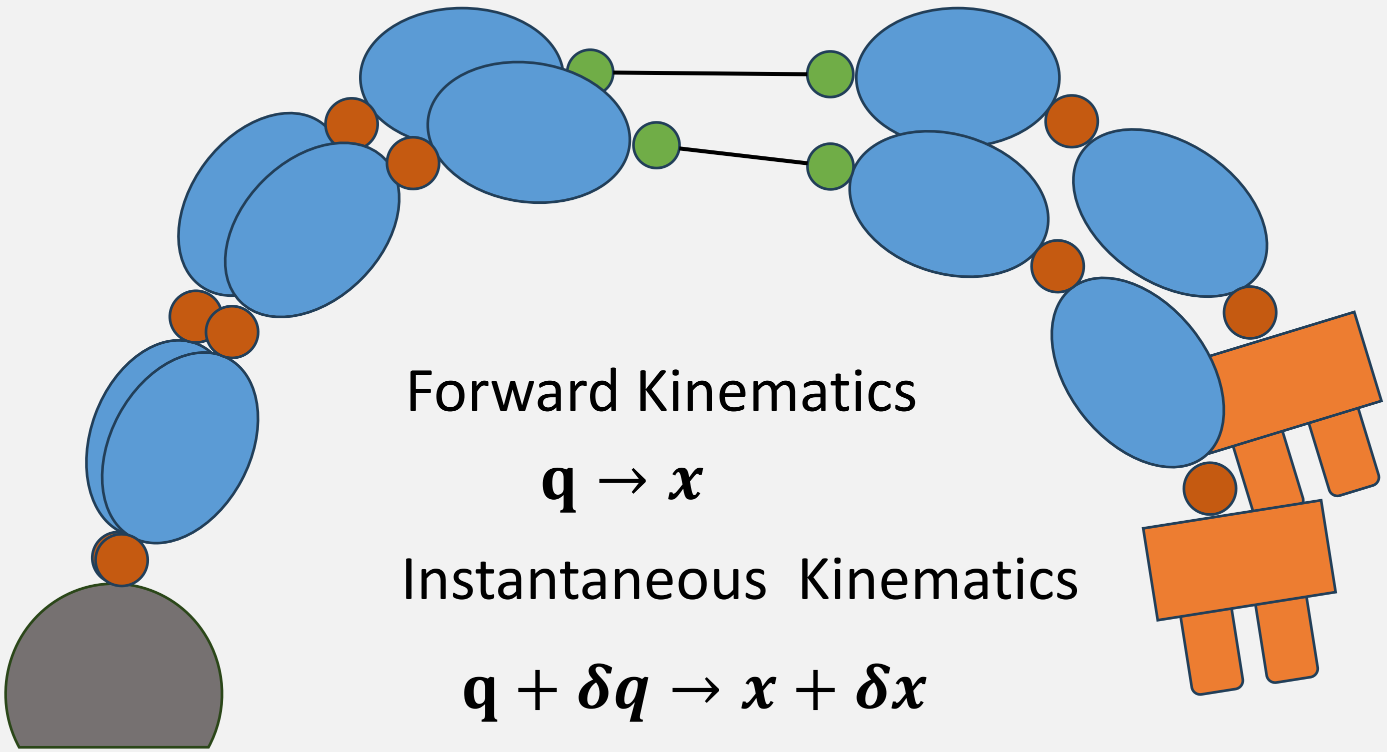
NUS

Today's Plan

- Jacobian
  - Direct Differentiation
  - Linear & Angular Motion
  - Velocity Propagation
  - Explicit Form
  - Static Forces

Instantaneous Kinematics



Forward Kinematics

$$\mathbf{q} \rightarrow \boldsymbol{x}$$

Instantaneous Kinematics

$$\mathbf{q} + \boldsymbol{\delta q} \rightarrow \boldsymbol{x} + \boldsymbol{\delta x}$$

Relationship: $\boldsymbol{\delta q} \leftrightarrow \boldsymbol{\delta x}$

$\dot{\boldsymbol{q}} \leftrightarrow \dot{\boldsymbol{x}}$ Linear Velocity

Angular Velocity

Joint Coordinates

Coordinate i: $\begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$

Joint coordinate: $q_i = \varepsilon_i \theta_i + \bar{\varepsilon}_i d_i$

$$\varepsilon_i : \begin{cases} 1 & \text{revolute} \\ 0 & \text{prismatic} \end{cases}$$

$$\bar{\varepsilon}_i = 1 - \varepsilon_i$$

Joint Coordinate Vector: $q = (q_1 q_2 q_3 \cdots q_n)^T$

Jacobian: Direct Differentiation

$$x = f(q) \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} f_1(q) \\ f_2(q) \\ \vdots \\ f_m(q) \end{bmatrix}$$

$$\delta x_1 = \frac{\delta f_1}{\delta q_1}\delta q_1 + \frac{\delta f_1}{\delta q_2}\delta q_2 + \cdots + \frac{\delta f_1}{\delta q_n}\delta q_n$$

$$\vdots$$

$$\delta x_m = \frac{\delta f_m}{\delta q_1}\delta q_1 + \frac{\delta f_m}{\delta q_2}\delta q_2 + \cdots + \frac{\delta f_m}{\delta q_n}\delta q_n$$

$$\delta x = \begin{bmatrix} \frac{\delta f_1}{\delta q_1} & \cdots & \frac{\delta f_1}{\delta q_n} \\ \vdots & \ddots & \vdots \\ \frac{\delta f_m}{\delta q_1} & \cdots & \frac{\delta f_m}{\delta q_n} \end{bmatrix} \delta q$$

$$\delta x_{(m\times 1)} = J(q)_{(m\times n)}\, \delta q_{(n\times 1)}$$

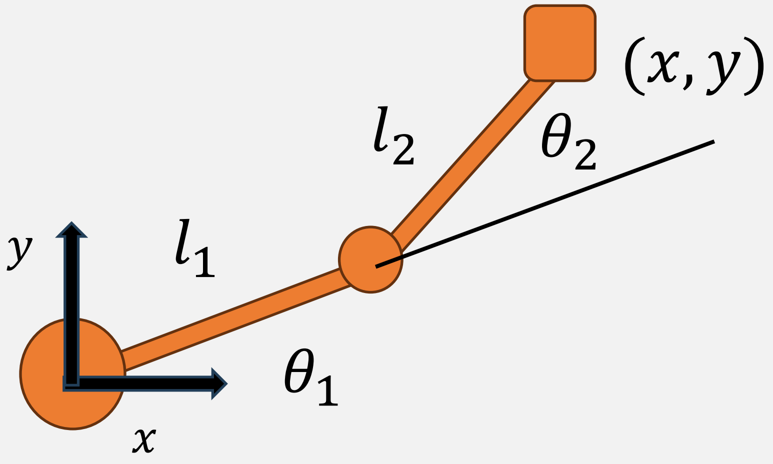
Jacobian

$$\delta x_{(m\times 1)} = J(q)_{(m\times n)}\, \delta q_{(n\times 1)}$$

$$\dot{x}_{(m\times 1)} = J(q)_{(m\times n)}\, \dot{q}_{(n\times 1)}$$

$$J(q)_{(ij)} = \frac{\delta f_i(q)}{\delta q_j}$$

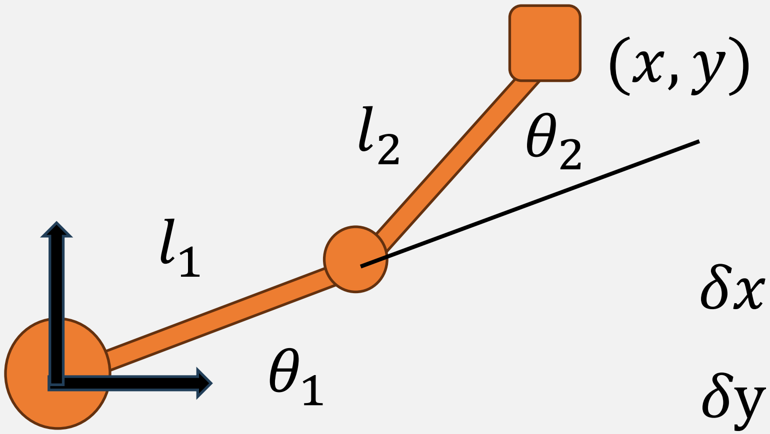
Example: RR Manipulator



$$x = l_1 c_1 + l_1 c_{12}$$

$$y = l_1 s_1 + l_2\, s_{12}$$

Example



$$x = l_1 c_1 + l_1 c_{12}$$

$$y = l_1 s_1 + l_2\, s_{12}$$

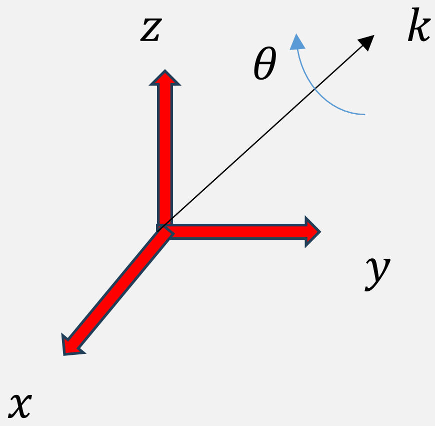
$$\delta x = -(l_1 s_1 + l_2 s_{12})\delta\theta_1 - l_2 s_{12}\delta\theta_2$$

$$\delta y = (l_1 c_1 + l_2\, c_{12})\, \delta\theta_1 + l_2 c_{12}\delta\theta_2$$

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ (l_1 c_1 + l_2\, c_{12}) & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \end{bmatrix}$$

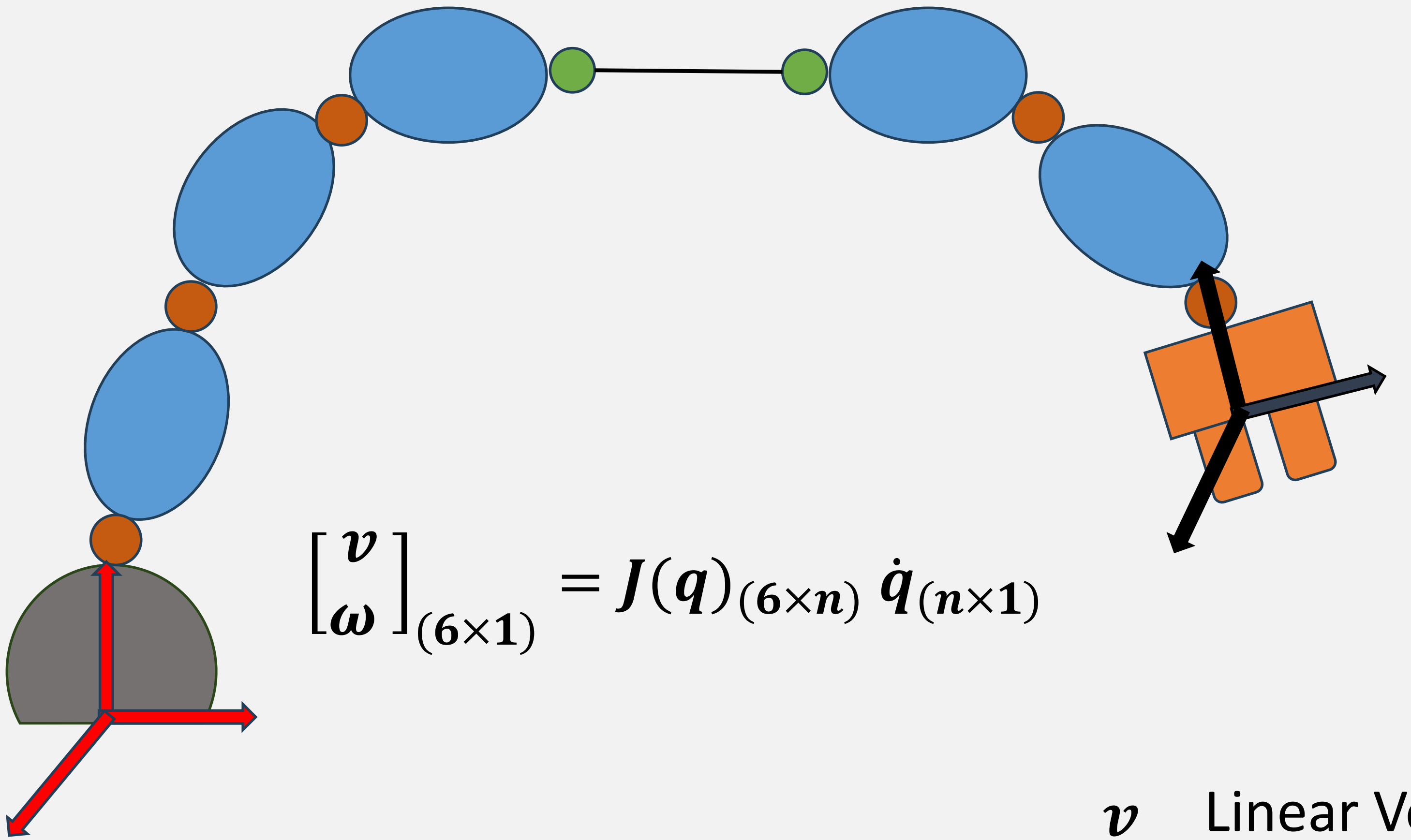
$$J = \begin{bmatrix} \frac{\delta x}{\delta\theta_1} & \frac{\delta x}{\delta\theta_2} \\ \frac{\delta y}{\delta\theta_1} & \frac{\delta y}{\delta\theta_2} \end{bmatrix} = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ (l_1 c_1 + l_2\, c_{12}) & l_2 c_{12} \end{bmatrix}$$

Angle-Axis Representation



$$\theta k = \theta \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} \theta k_x \\ \theta k_y \\ \theta k_z \end{bmatrix}$$

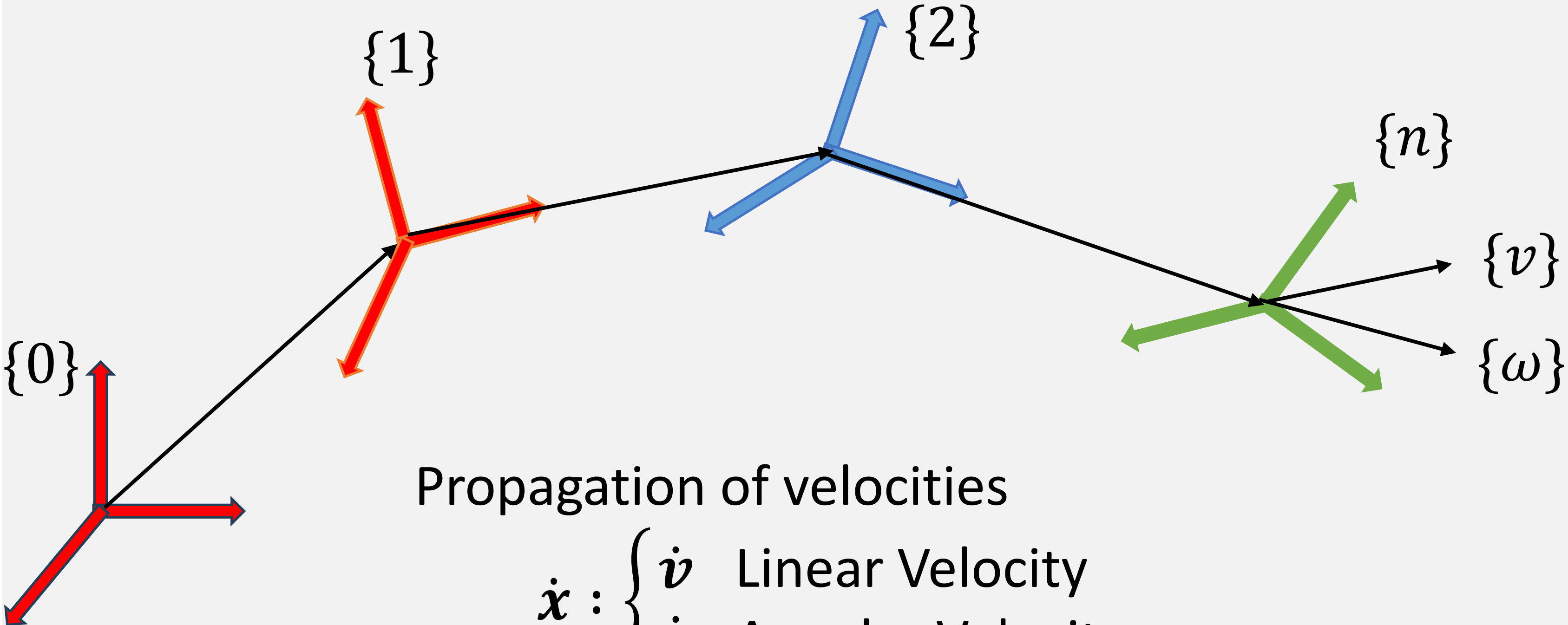
Jacobian



$$\begin{bmatrix} v \\ \omega \end{bmatrix}_{(6\times1)} = J(q)_{(6\times n)}\,\dot{q}_{(n\times1)}$$

v Linear Velocity
ω Angular Velocity

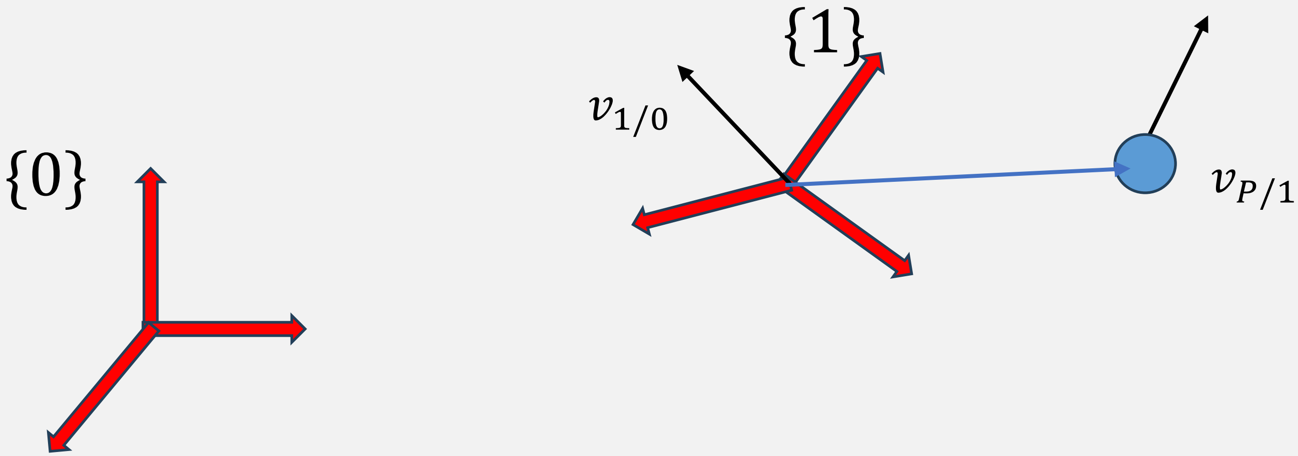
Spatial Mechanisms



Propagation of velocities

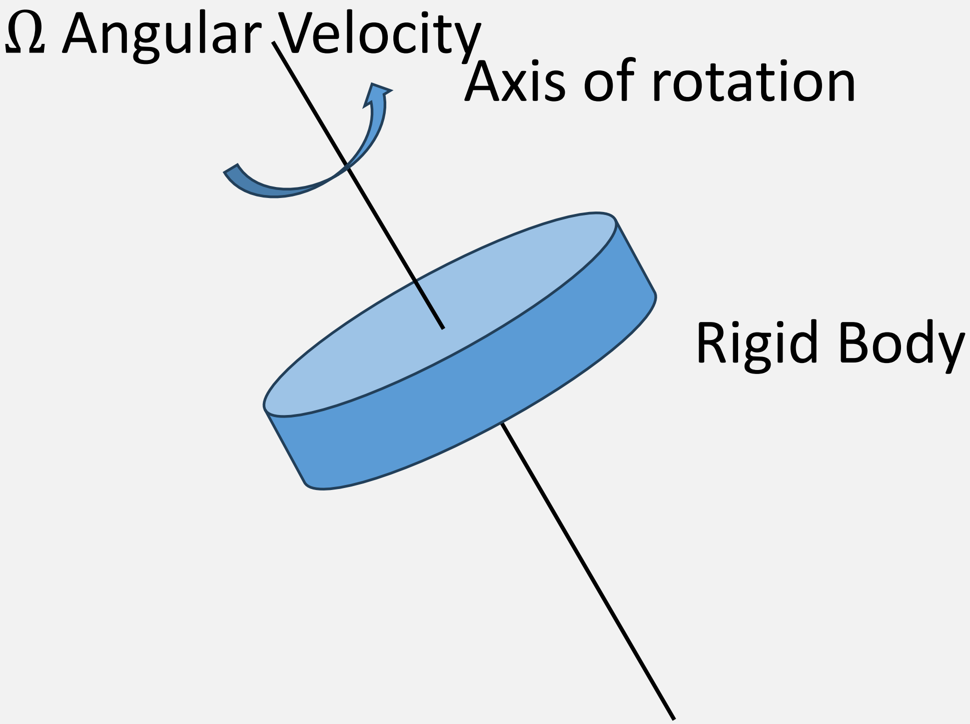
$$\dot{\boldsymbol{x}} : \begin{cases} \dot{\boldsymbol{v}} & \text{Linear Velocity} \\ \dot{\boldsymbol{\omega}} & \text{Angular Velocity} \end{cases}$$

Pure Translation

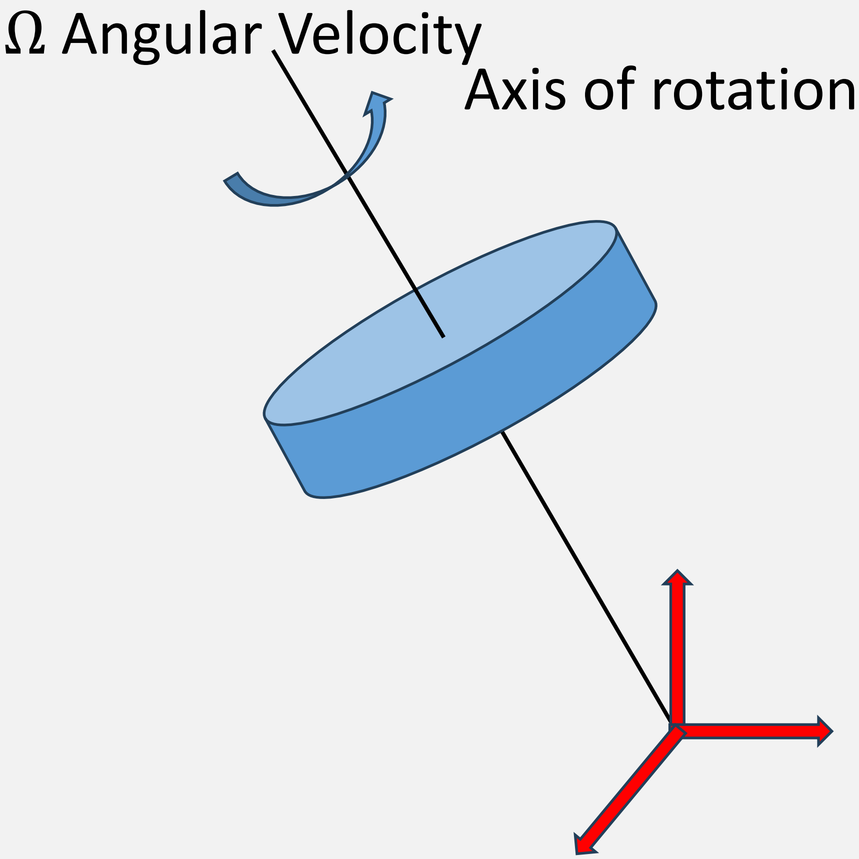


$$v_{P/0} = v_{1/0} + v_{P/1}$$

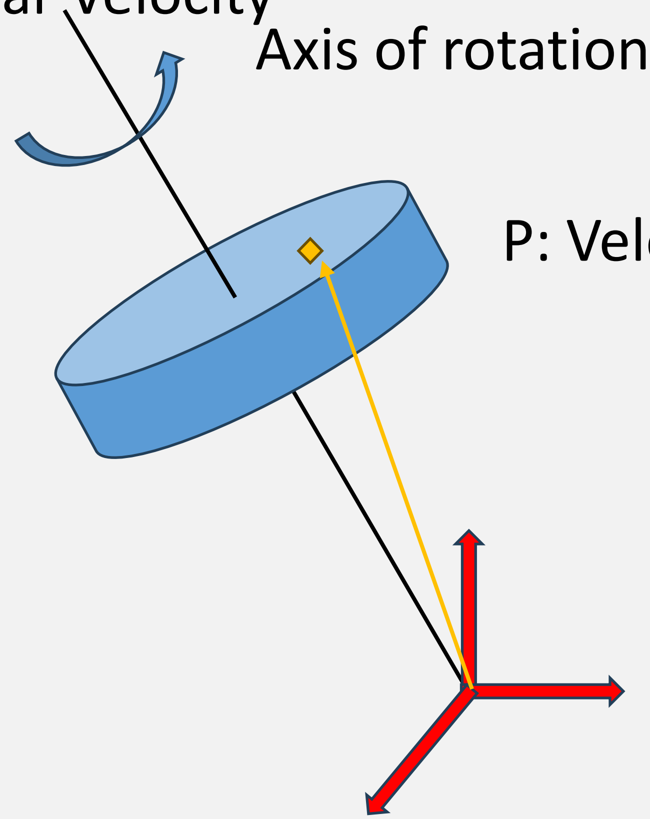
Rotational Motion



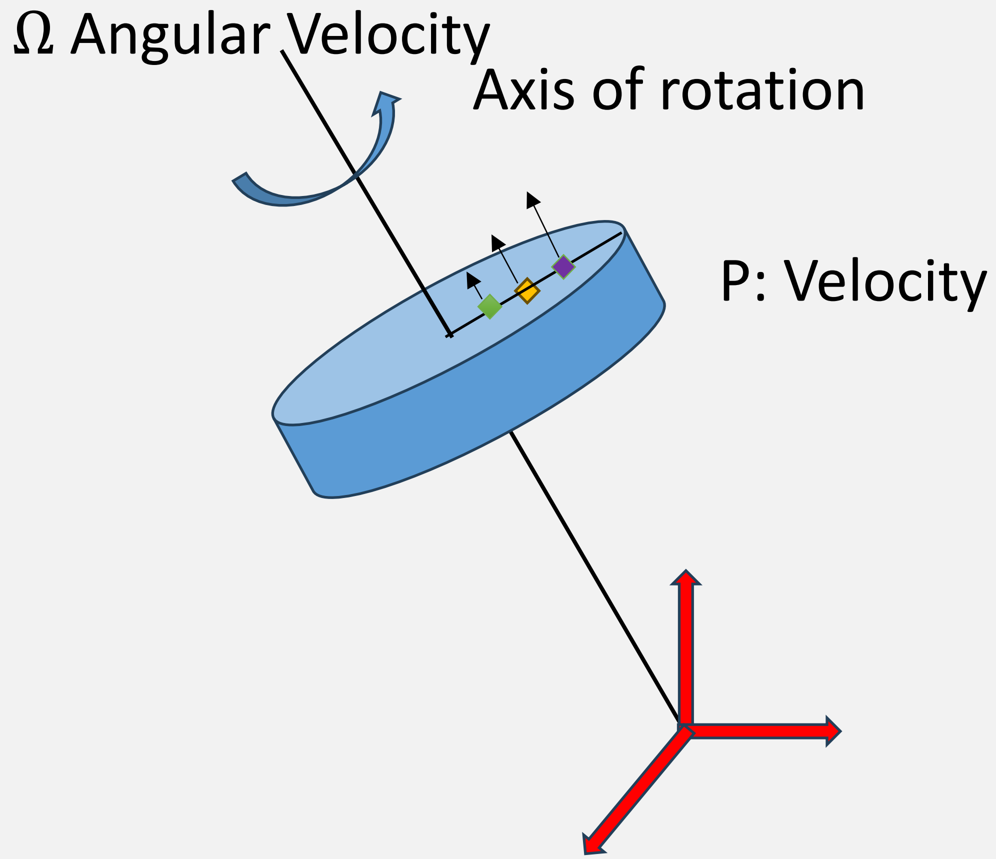
Rotational Motion



Rotational Motion



Rotational Motion

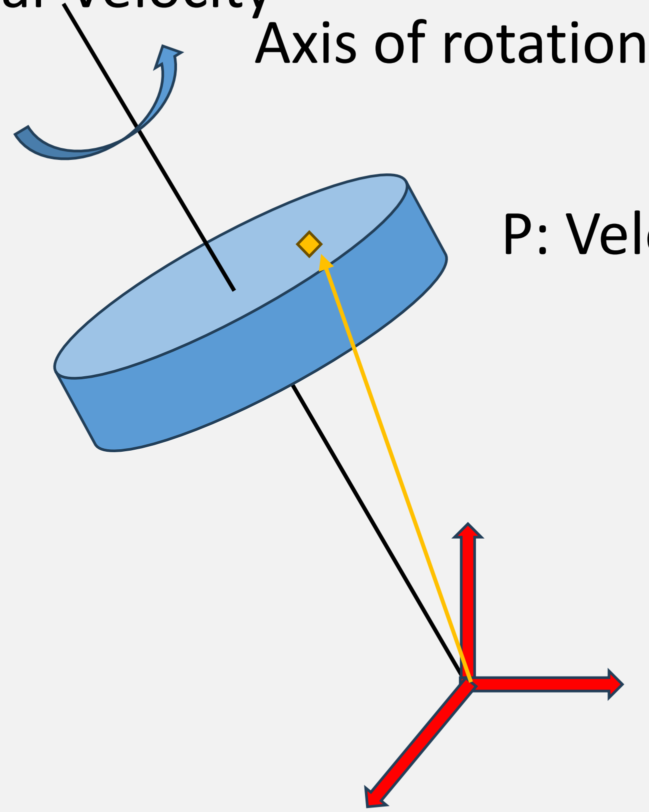


Rotational Motion

Ω Angular Velocity

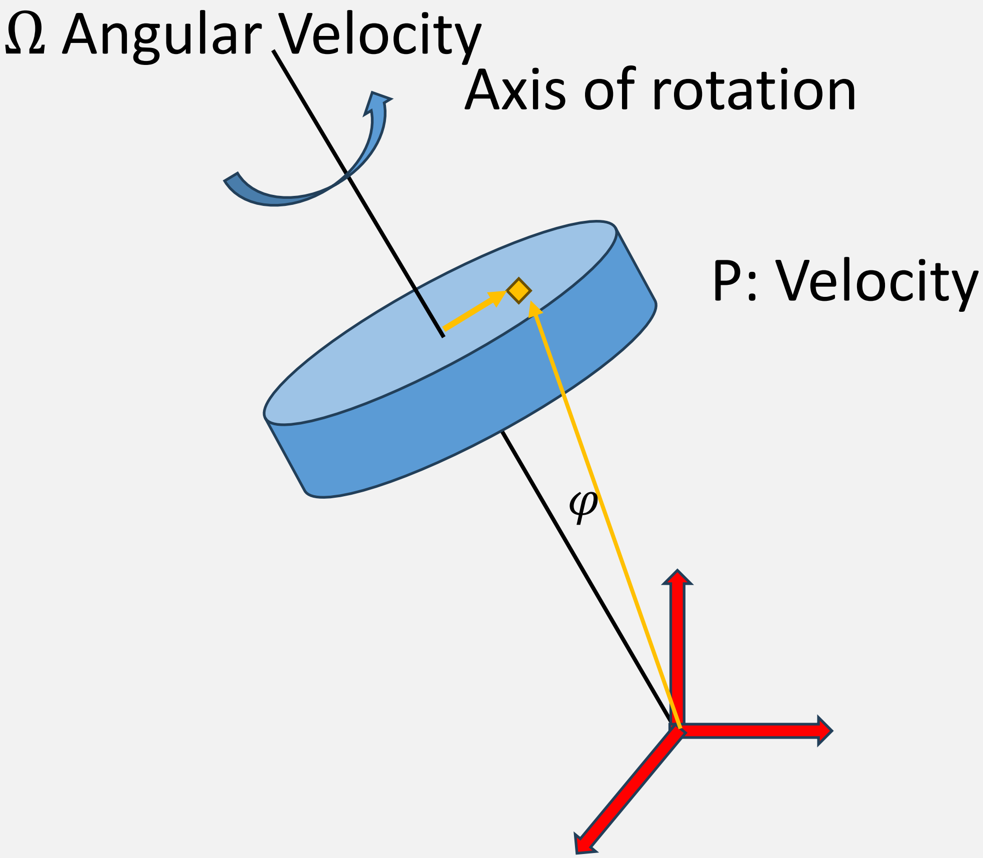
Axis of rotation

v_P?



P: Velocity

Rotational Motion



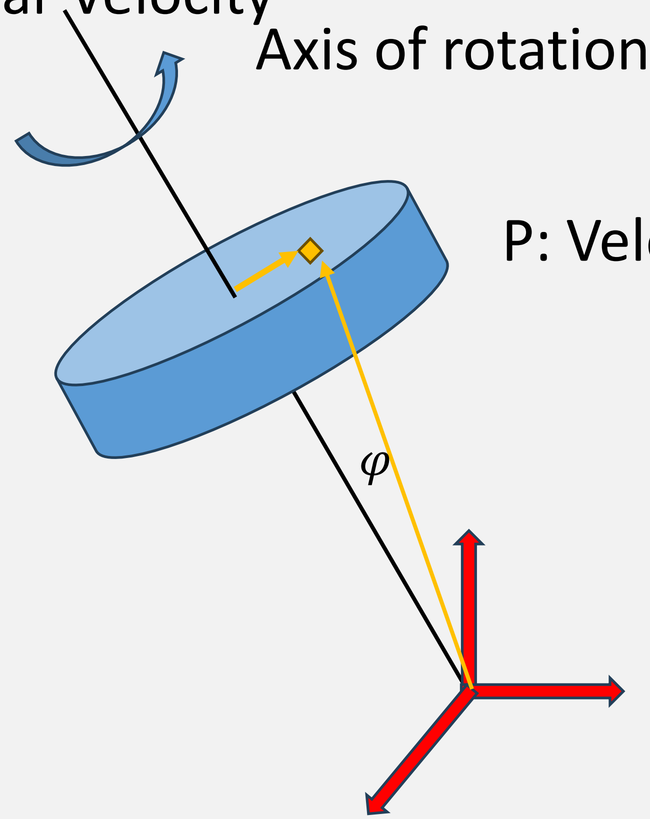
v_P?

$v_P \perp \Omega$

$v_P \perp P$

$\|v_P\| = \|\Omega\|\|Psin\varphi\|$

Rotational Motion



v_P?

$v_P \perp \Omega$

$v_P \perp P$

$$\|v_P\| = \|\Omega\|\|P sin\varphi\|$$

$$v_P = \Omega \times \mathrm{P}$$

Cross Product Operator

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \quad P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$v_P = \Omega \times \mathrm{P} \;\Rightarrow\; v_P = \widehat{\Omega}\mathrm{P}$$

Vectors

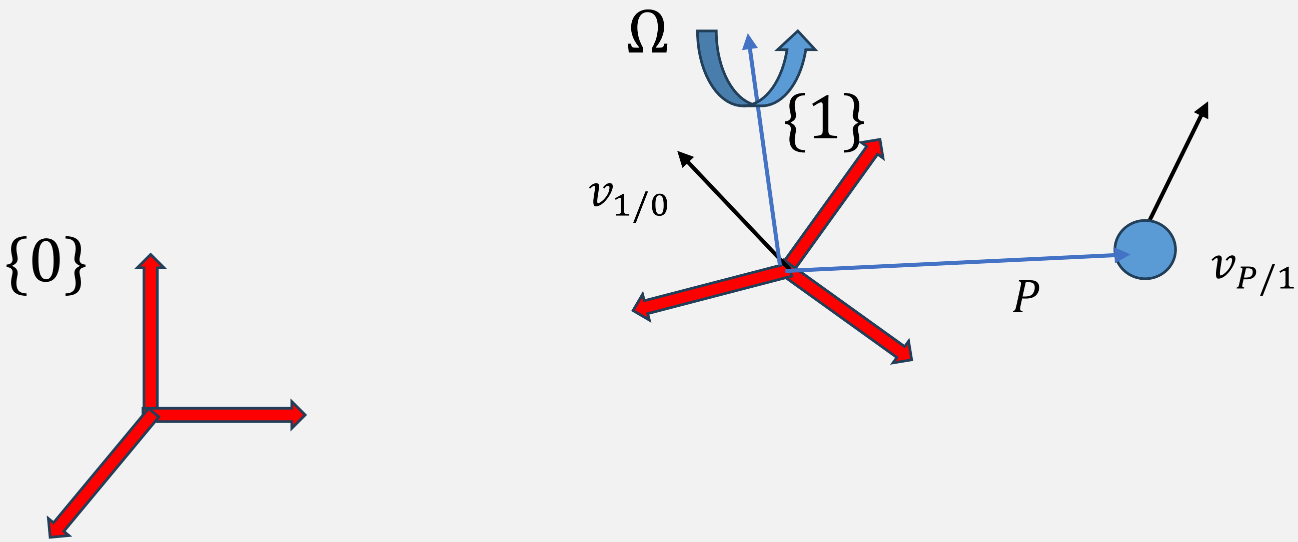
Matrices

$$\Omega \times \rightarrow \widehat{\Omega} \qquad \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

A skew-symmetric matrix

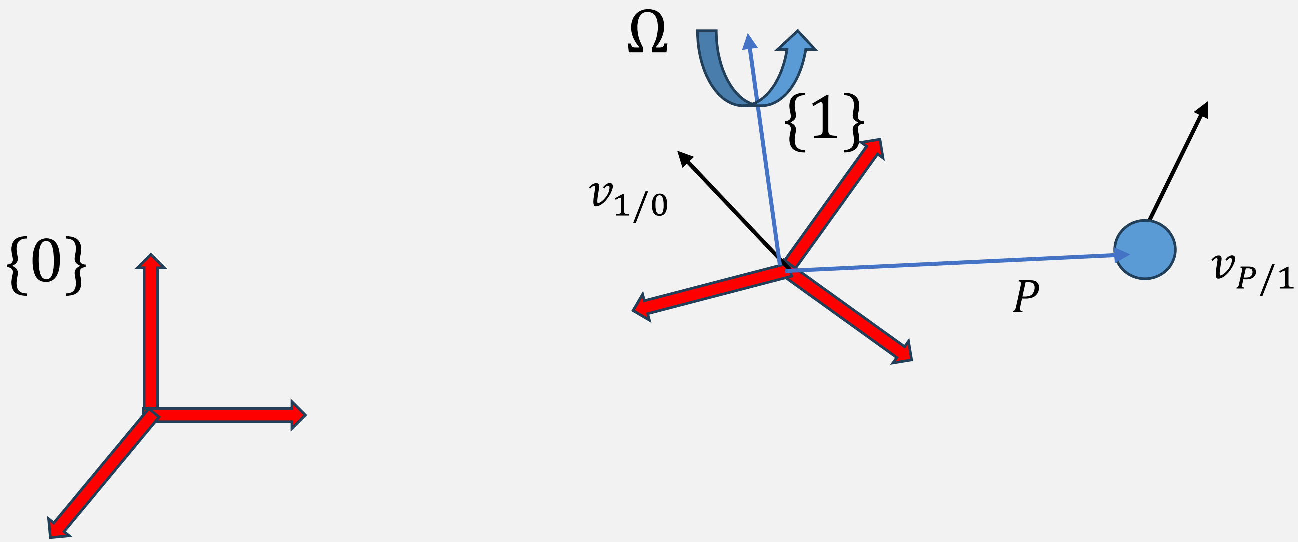
$$v_P = \widehat{\Omega}\mathrm{P} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

Linear and Angular Motion



$$v_{P/0} = v_{1/0} + v_{P/1} + \Omega \times P$$

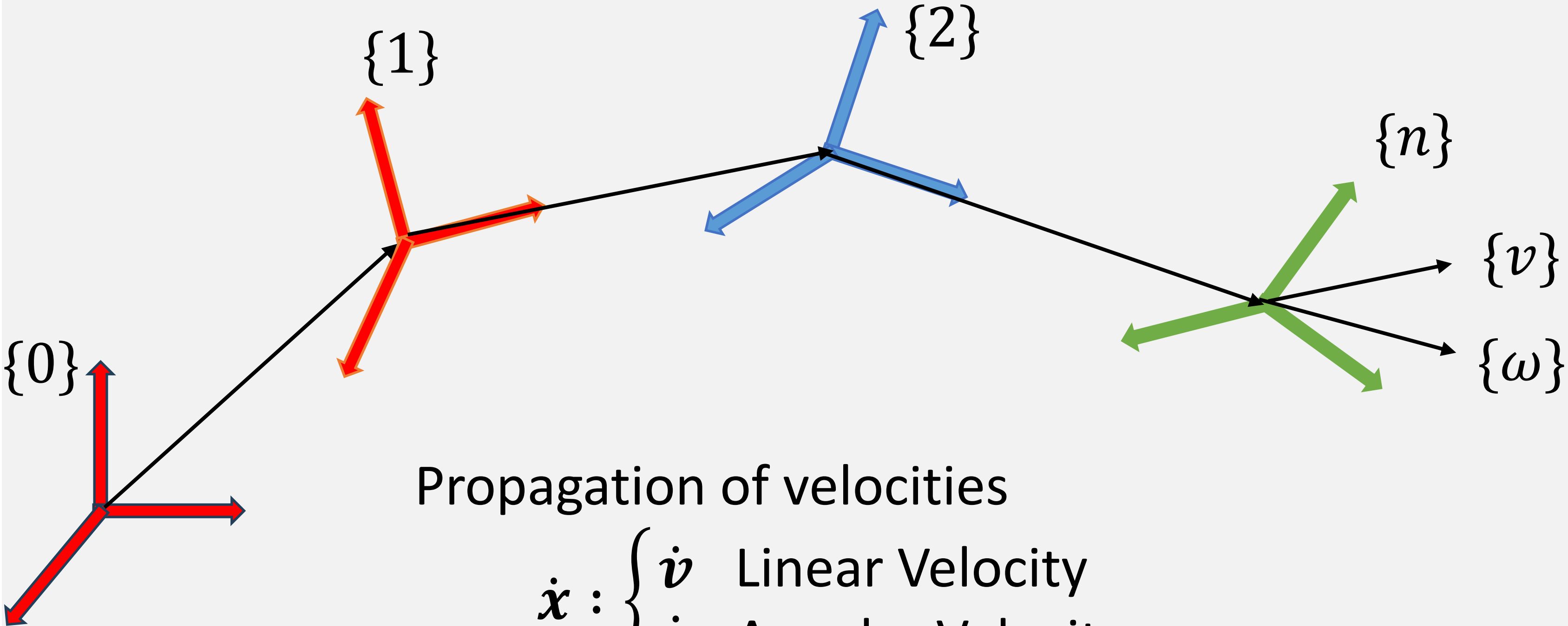
Linear and Angular Motion



$$v_{P/0} = v_{1/0} + v_{P/1} + \Omega \times P$$

$$^{0}v_{P/0} = \ ^{0}v_{1/0} + {}^{0}_{1}R\ ^{1}v_{P/1} + \ ^{0}\Omega \times {}^{0}_{1}R\ ^{1}P$$

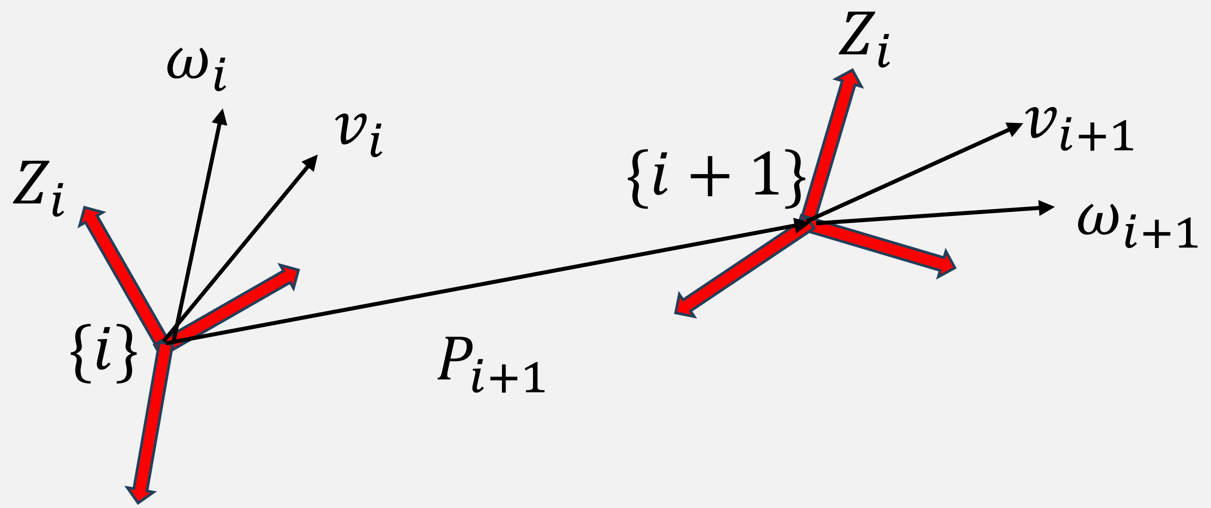
Spatial Mechanisms



Propagation of velocities

$$\dot{\boldsymbol{x}} : \begin{cases} \dot{\boldsymbol{v}} & \text{Linear Velocity} \\ \dot{\boldsymbol{\omega}} & \text{Angular Velocity} \end{cases}$$

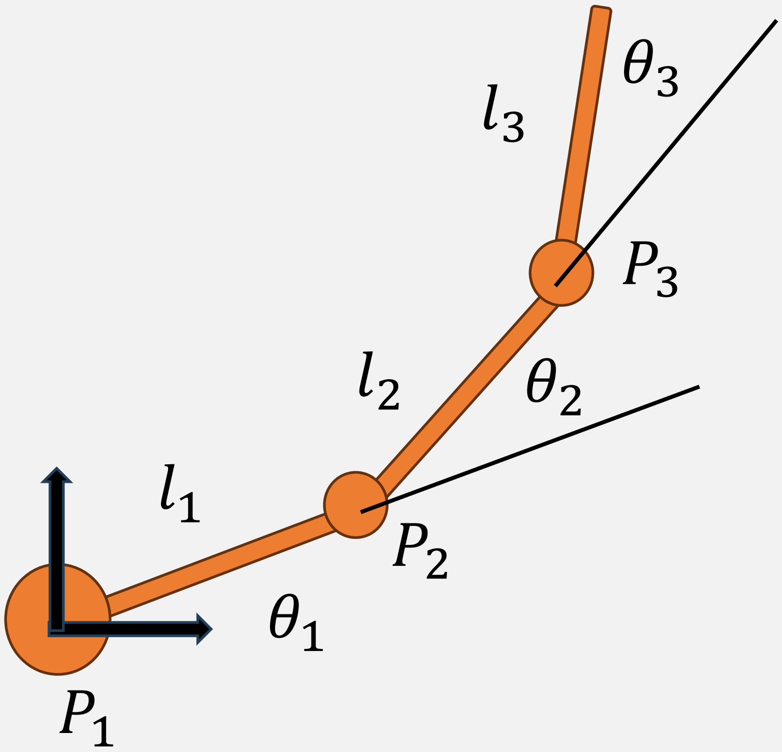
Velocity propagation



Linear $v_{i+1} = v_i + \omega_i \times P_{i+1} + (\dot{d}_{i+1}\, Z_{i+1}\ if\ prismatic)$

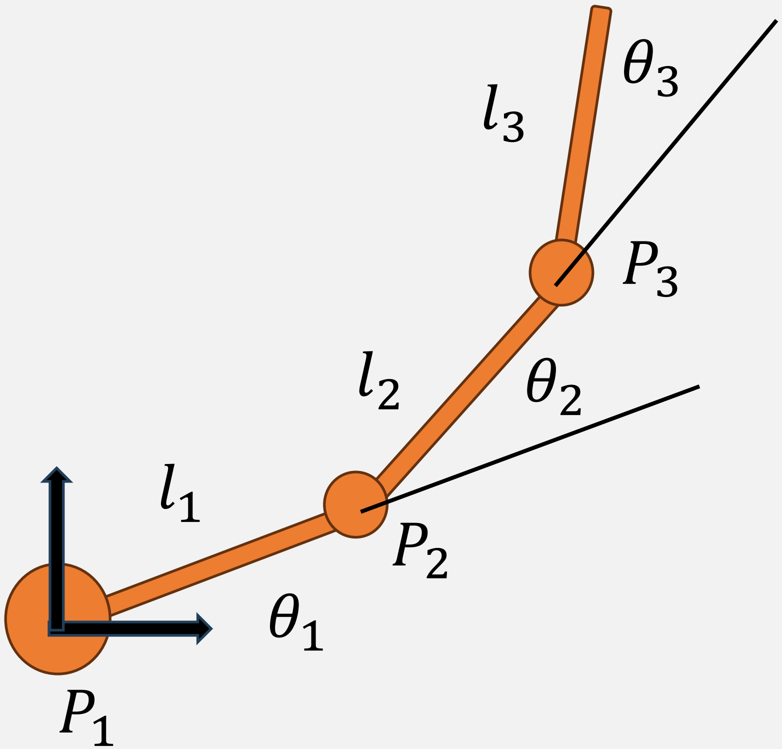
Angular $\omega_{i+1} = \omega_i + (\dot{\theta}_{i+1}\, Z_{i+1}\ if\ revolute)$

Example



$$v_{i+1} = v_i + \omega_i \times P_{i+1}$$

Example



$$v_{i+1} = v_i + \omega_i \times P_{i+1}$$

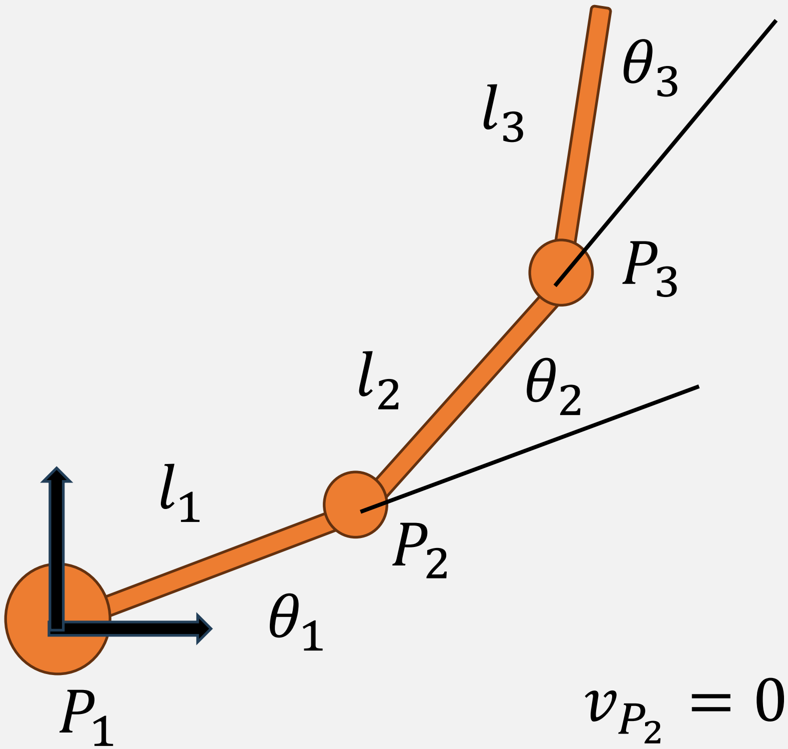
$$v_{P_1} = 0$$

$$v_{P_2} = v_{P_1} + {}^{0}\omega_1 \times P_2$$

$$v_{P_3} = v_{P_2} + {}^{0}\omega_2 \times P_3$$

$$^{0}\omega_2 = \dot{\theta}_1 \, Z_1 + \dot{\theta}_2 \, Z_2$$

Example



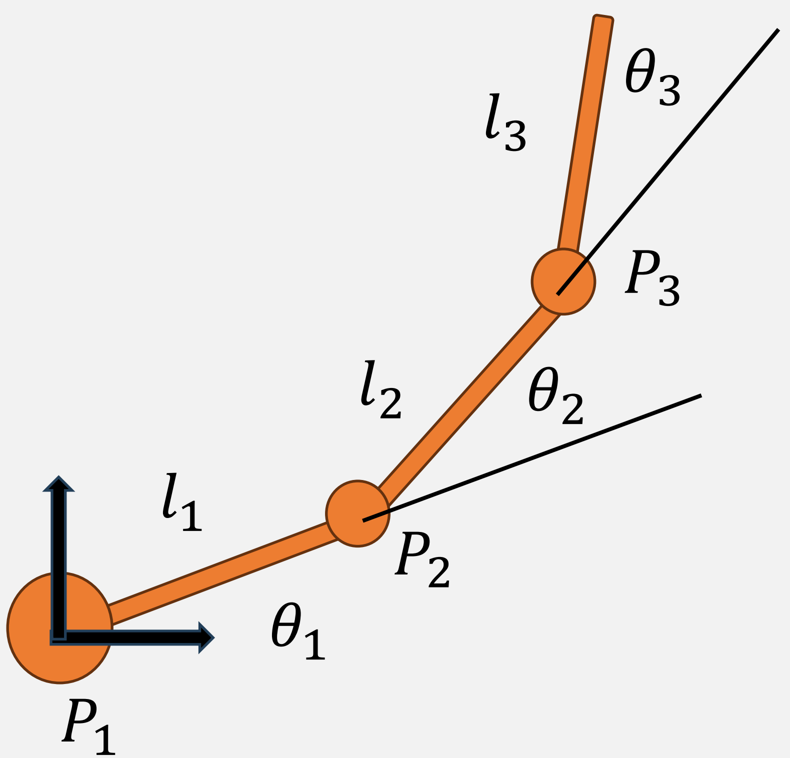
$$v_{i+1} = v_i + \omega_i \times P_{i+1}$$

$$v_{P_1} = 0 \qquad \omega_1 = \dot{\theta}_1 Z_1$$

$$v_{P_2} = v_{P_1} + {}^{0}\omega_1 \times P_2$$

$$v_{P_2} = 0 + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1 s_1 \\ l_1 c_1 \\ 0 \end{bmatrix} \dot{\theta}_1$$

Example



$$v_{i+1} = v_i + \omega_i \times P_{i+1}$$

$$v_{P_1} = 0 \qquad \omega_1 = \dot{\theta}_1 Z_1$$

$$v_{P_2} = v_{P_1} + {}^0\omega_1 \times P_2$$

$$v_{P_3} = v_{P_2} + {}^0\omega_2 \times P_3$$

$${}^0\omega_2 = (\dot{\theta}_1 + \dot{\theta}_2)$$

Example

$$v_{P_3} = v_{P_2} + \omega_2 \times P_3$$

$$v_{P_3} = \begin{bmatrix} -l_1 s_1 \\ l_1 c_1 \\ 0 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} 0 & -(\dot{\theta}_1 + \dot{\theta}_2) & 0 \\ (\dot{\theta}_1 + \dot{\theta}_2) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_2 c_{12} \\ l_2 s_{12} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 s_1 \\ l_1 c_1 \\ 0 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} -l_2 s_{12} \\ l_2 c_1 \\ 0 \end{bmatrix} (\dot{\theta}_1 + \dot{\theta}_2)$$

$$= \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} & 0 \\ (l_1 c_1 + l_2 c_{12}) & l_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = J_v \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Example

$$ {}^{0}\omega_3 = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) $$

$$ {}^{0}\omega_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = J_\omega \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} $$

Jacobian in a Different Frame

$$\begin{bmatrix} {}^{A}v_e \\ {}^{A}w_e \end{bmatrix} = {}^{A}J\dot{q}$$

$$\begin{bmatrix} {}^{B}v_e \\ {}^{B}w_e \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & 0 \\ 0 & {}^{B}_{A}R \end{bmatrix} \begin{bmatrix} {}^{A}v_e \\ {}^{A}w_e \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & 0 \\ 0 & {}^{B}_{A}R \end{bmatrix} {}^{A}J\dot{q} = {}^{B}J\dot{q}$$

$${}^{B}J = \begin{bmatrix} {}^{B}_{A}R & 0 \\ 0 & {}^{B}_{A}R \end{bmatrix} {}^{A}J$$

Jacobian

$$x = f(q) \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} f_1(q) \\ f_2(q) \\ \vdots \\ f_m(q) \end{bmatrix}$$

$$\delta x_1 = \frac{\delta f_1}{\delta q_1}\delta q_1 + \frac{\delta f_1}{\delta q_2}\delta q_2 + \cdots + \frac{\delta f_1}{\delta q_n}\delta q_n$$

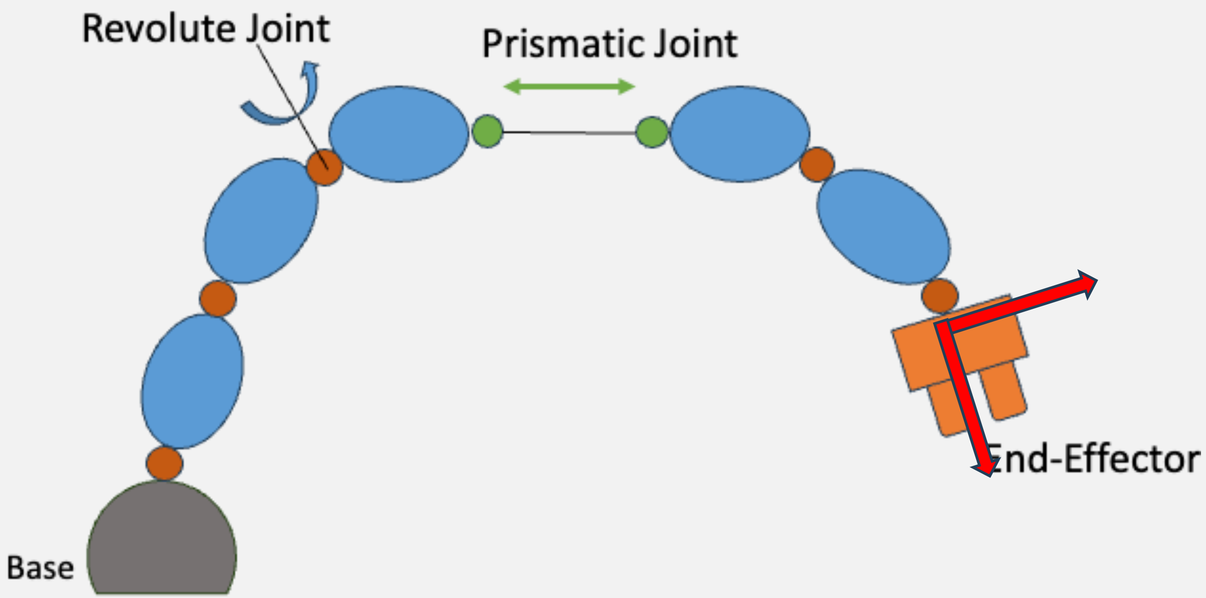
$$\vdots$$

$$\delta x_m = \frac{\delta f_m}{\delta q_1}\delta q_1 + \frac{\delta f_m}{\delta q_2}\delta q_2 + \cdots + \frac{\delta f_m}{\delta q_n}\delta q_n$$

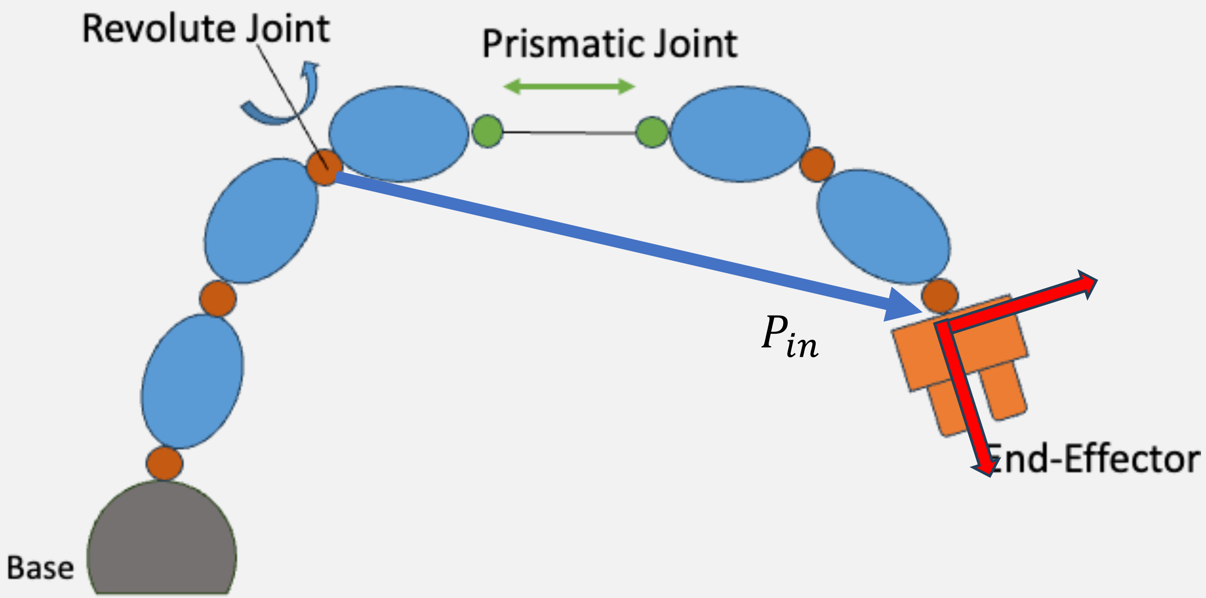
$$\delta x = \begin{bmatrix} \frac{\delta f_1}{\delta q_1} & \cdots & \frac{\delta f_1}{\delta q_n} \\ \vdots & \ddots & \vdots \\ \frac{\delta f_m}{\delta q_1} & \cdots & \frac{\delta f_m}{\delta q_n} \end{bmatrix} \delta q$$

$$\delta x_{(m\times 1)} = J(q)_{(m\times n)}\, \delta q_{(n\times 1)}$$

Jacobian (explicit form)



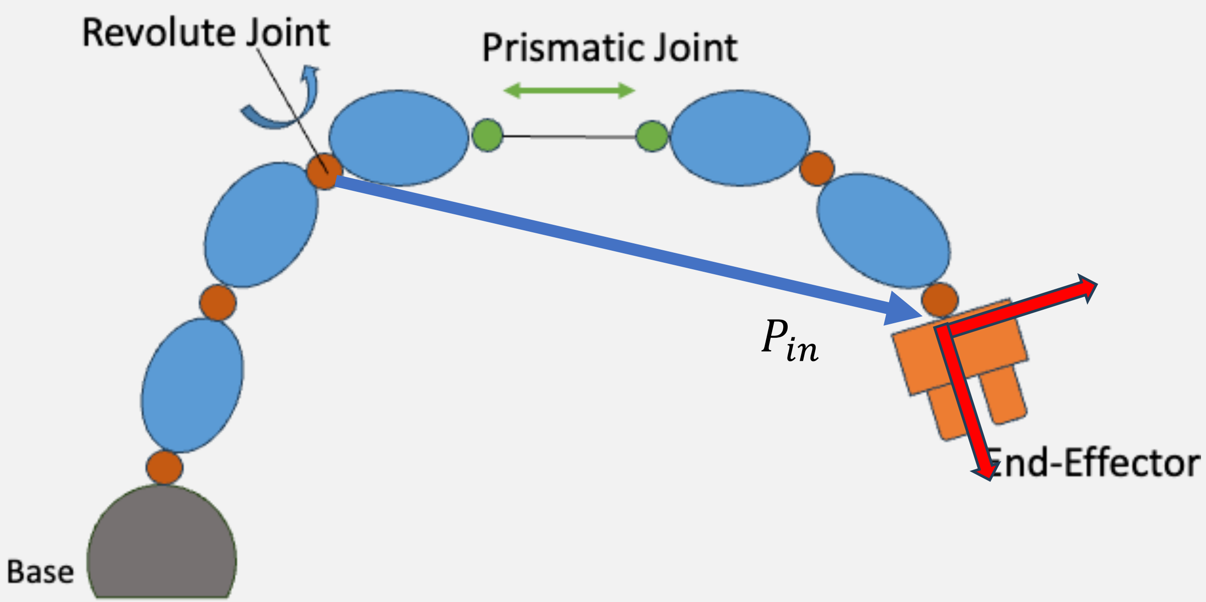
Jacobian (explicit form)



$$\Omega_i = Z_i \dot{q}_i$$

$$V_i = Z_i \dot{q}_i$$

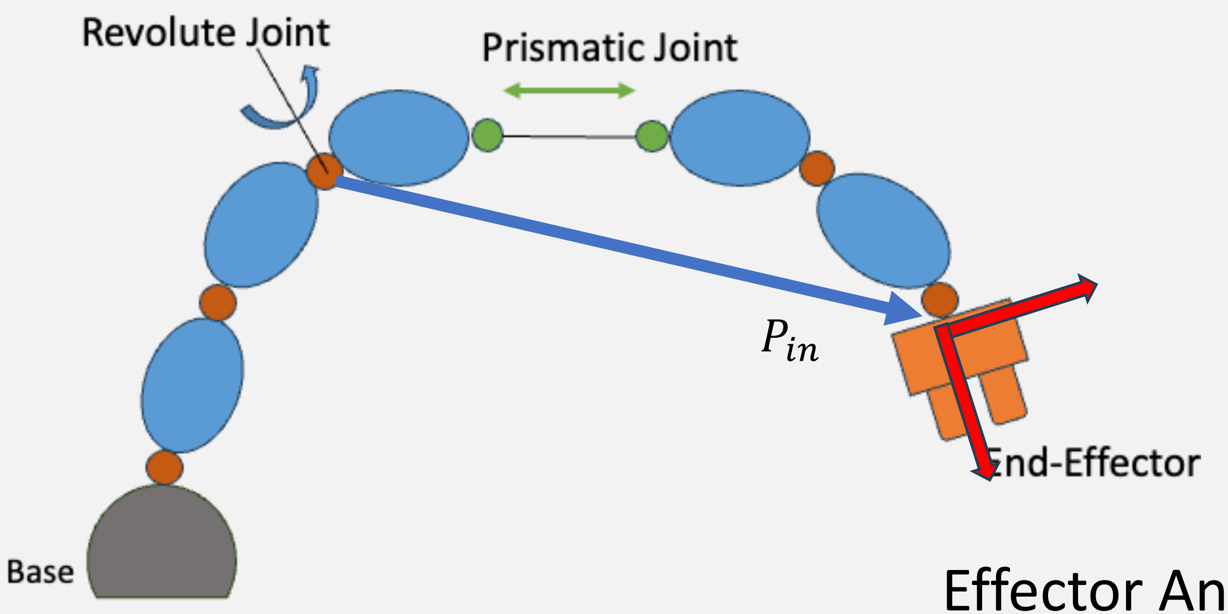
Jacobian (explicit form)



$$\Omega_i = Z_i \dot{q}_i$$
$$V_i = Z_i \dot{q}_i$$

Effector	Prismatic	Revolute
Angular Vel	None	Ω_i
Linear Vel	V_i	$\Omega_i \times P_{in}$

Jacobian (explicit form)



$$\Omega_i = Z_i \dot{q}_i$$

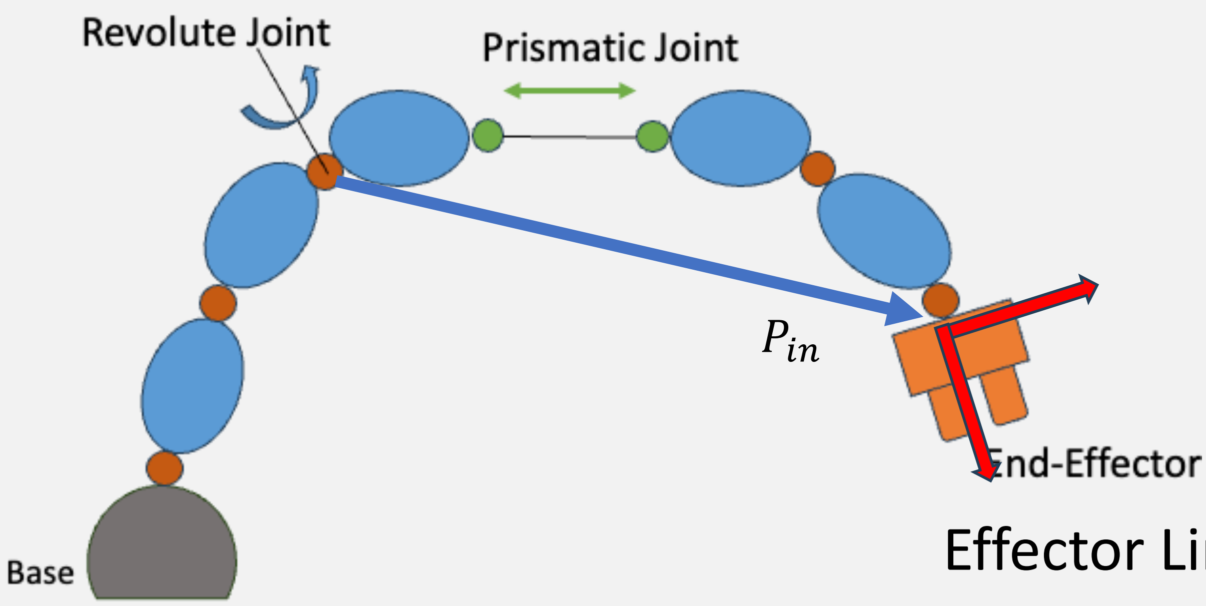
$$V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} \bar{\varepsilon}_i \Omega_i$$

Effector	Prismatic	Revolute
Angular Vel	None	Ω_i
Linear Vel	V_i	$\Omega_i \times P_{in}$

Jacobian (explicit form)



$$\Omega_i = Z_i \dot{q}_i$$

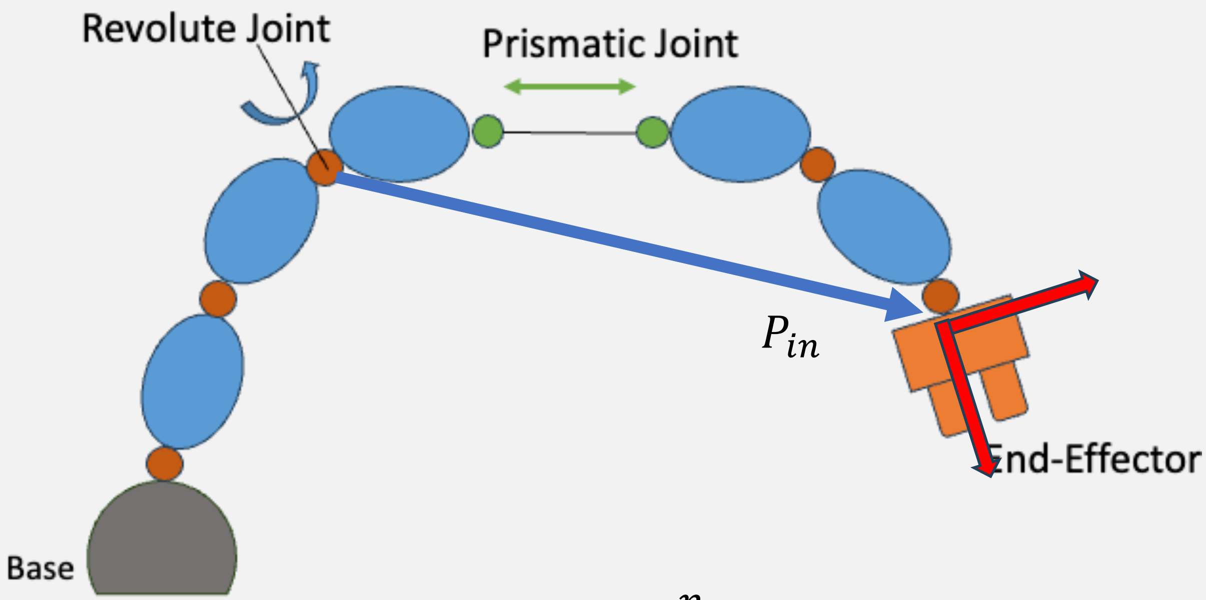
$$V_i = Z_i \dot{q}_i$$

Effector Linear Velocity

$$\mathrm{v} = \sum_{i=1}^{n} \varepsilon_i V_i + \bar{\varepsilon}_i (\Omega_i \times P_{in})$$

Effector	Prismatic	Revolute
Angular Vel	None	Ω_i
Linear Vel	V_i	$\Omega_i \times P_{in}$

Jacobian (explicit form)



Effector Linear Velocity

$$V_i = Z_i \dot{q}_i$$

$$\mathrm{v} = \sum_{i=1}^{n} \varepsilon_i V_i + \bar{\varepsilon}_i (\Omega_i \times P_{in}) = \sum_{i=1}^{n} [\varepsilon_i Z_i + \bar{\varepsilon}_i (Z_i \times P_{in})] \dot{q}_i$$

Effector Angular Velocity

$$\Omega_i = Z_i \dot{q}_i$$

$$\omega = \sum_{i=1}^{n} \bar{\varepsilon}_i \Omega_i = \sum_{i=1}^{n} (\bar{\varepsilon}_i Z_i) \dot{q}_i$$

Jacobian (explicit form)

$$v = \sum_{i=1}^{n} [\varepsilon_i Z_i + \bar{\varepsilon}_i (Z_i \times P_{in})] \dot{q}_i$$

$$\mathrm{v} = J_v \dot{q}$$

$$= \begin{bmatrix} \varepsilon_1 Z_1 + \bar{\varepsilon}_1 (Z_1 \times P_{1n}) & \varepsilon_2 Z_2 + \bar{\varepsilon}_2 (Z_2 \times P_{2n}) & \dots & \varepsilon_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_i \\ \dot{q}_i \\ \vdots \\ \dot{q}_i \end{bmatrix}$$

$$\omega = \sum_{i=1}^{n} (\bar{\varepsilon}_i Z_i) \dot{q}_i$$

$$\omega = J_\omega \dot{q}$$

$$= \begin{bmatrix} \bar{\varepsilon}_1 Z_1 & \bar{\varepsilon}_2 Z_2 & \dots & \bar{\varepsilon}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_i \\ \dot{q}_i \\ \vdots \\ \dot{q}_i \end{bmatrix}$$

Jacobian (explicit form)

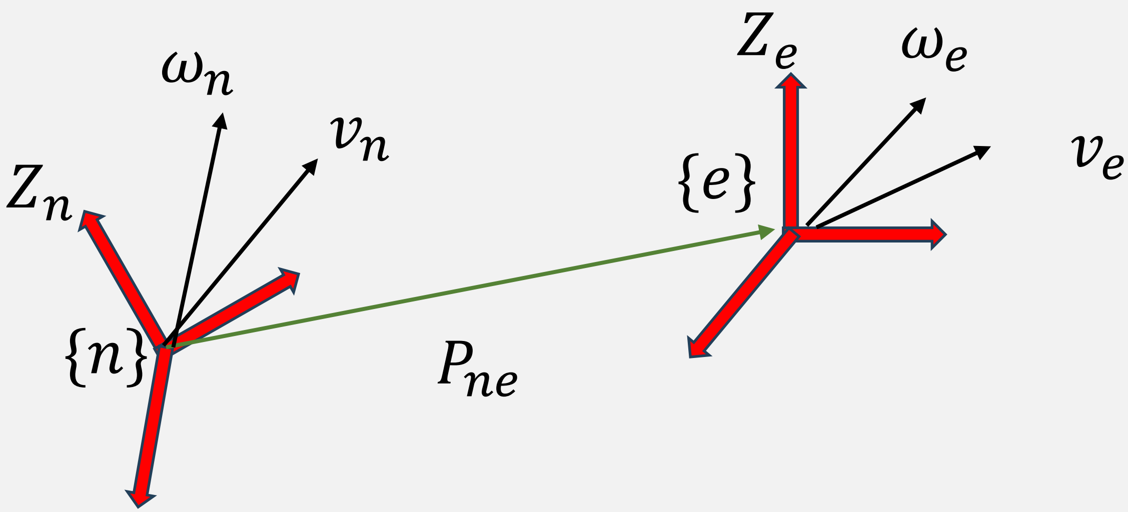
$$J = \begin{pmatrix} J_v \\ J_\omega \end{pmatrix}$$

J_v Linear Jacobian

J_ω Angular Jacobian

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{bmatrix} \varepsilon_1 Z_1 + \bar{\varepsilon}_1 (Z_1 \times P_{1n}) & \varepsilon_2 Z_2 + \bar{\varepsilon}_2 (Z_2 \times P_{2n}) & \dots & \varepsilon_n Z_n \\ \bar{\varepsilon}_1 Z_1 & \bar{\varepsilon}_2 Z_2 & \dots & \bar{\varepsilon}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_i \\ \dot{q}_i \\ \vdots \\ \dot{q}_i \end{bmatrix}$$

Jacobian at the End-Effector



$$v_e = v_n + \omega_n \times P_{ne}$$

$$\omega_e = \omega_n$$

$$v_e = v_n - P_{ne} \times \omega_n$$

Jacobian at the End-Effector

$$v_e = v_n - P_{ne} \times \omega_n$$

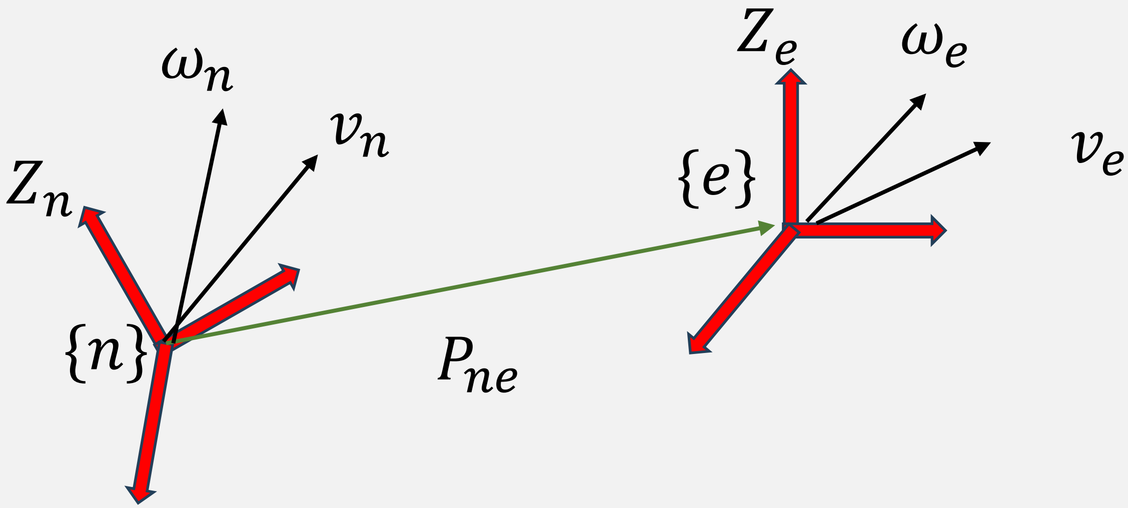
$$\omega_e = \omega_n$$

$$\begin{bmatrix} v_e \\ \omega_e \end{bmatrix} = \begin{bmatrix} I & -\hat{P}_{ne} \\ 0 & I \end{bmatrix} \begin{bmatrix} v_n \\ \omega_n \end{bmatrix}$$

$$J_e \dot{q} = \begin{bmatrix} I & -\hat{P}_{ne} \\ 0 & I \end{bmatrix} J_n \dot{q}$$

$$J_e = \begin{bmatrix} I & -\hat{P}_{ne} \\ 0 & I \end{bmatrix} J_n$$

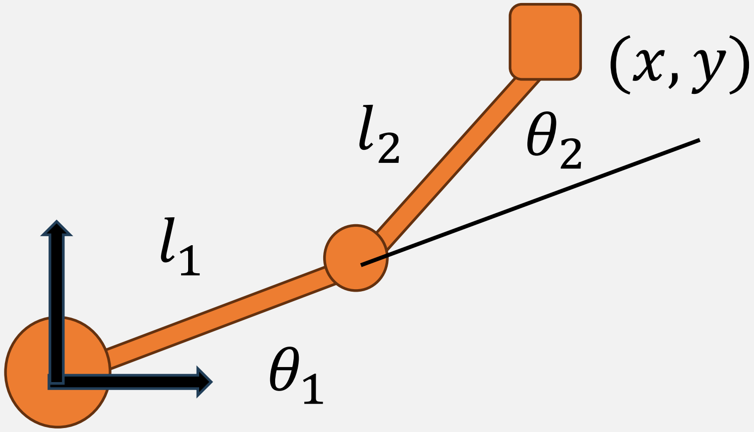
Cross Product Operator



$$J_e = \begin{bmatrix} I & -\hat{P}_{ne} \\ 0 & I \end{bmatrix} J_n$$

$$^{0}\hat{P} = {}^{0}_{n}R \quad {}^{n}\hat{P} \quad {}^{0}_{n}R^T$$

Example

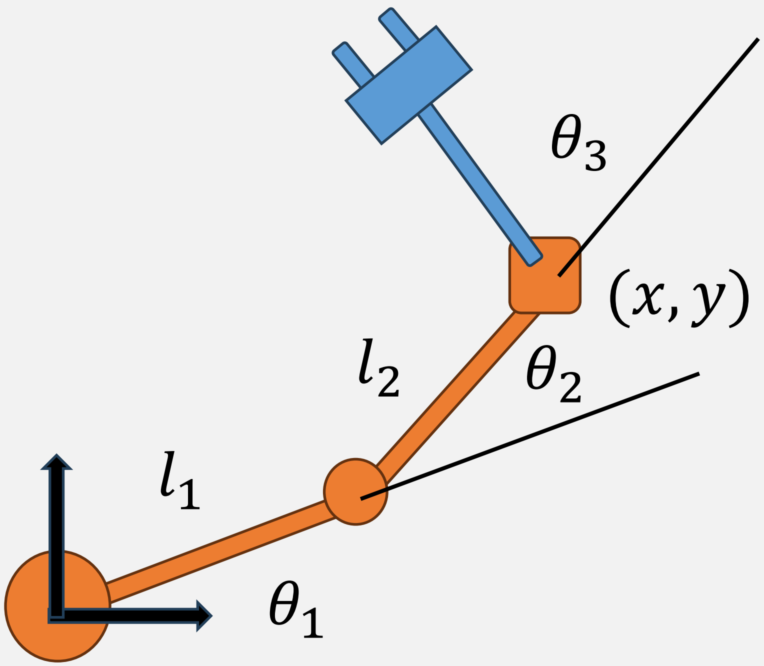


Wrist Point

$$x = l_1 c_1 + l_1 c_{12}$$

$$y = l_1 s_1 + l_2 \, s_{12}$$

Example



Wrist Point

$$x = l_1 c_1 + l_1 c_{12}$$

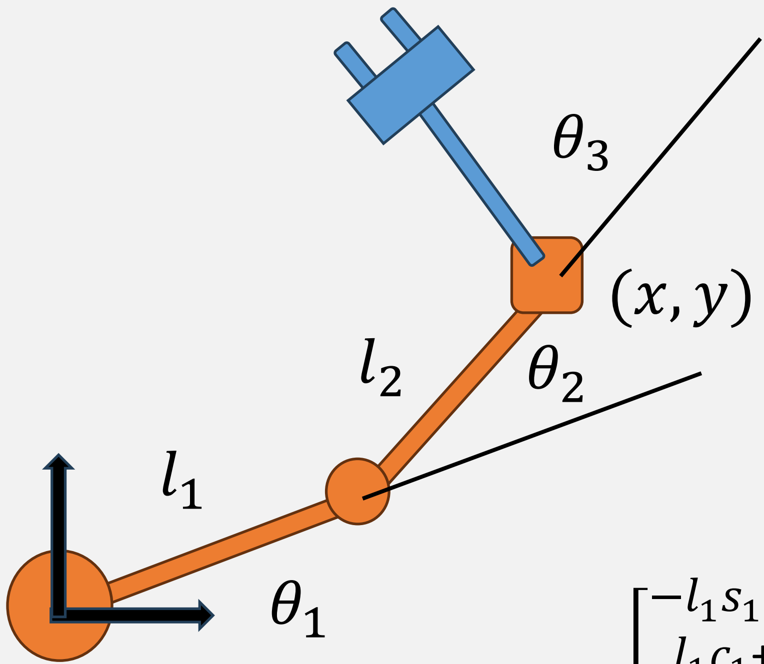
$$y = l_1 s_1 + l_2\, s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_1 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2\, s_{12} + l_3 s_{123}$$

Example



Wrist Point

$$x = l_1c_1 + l_1c_{12}$$

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End-Effector Point

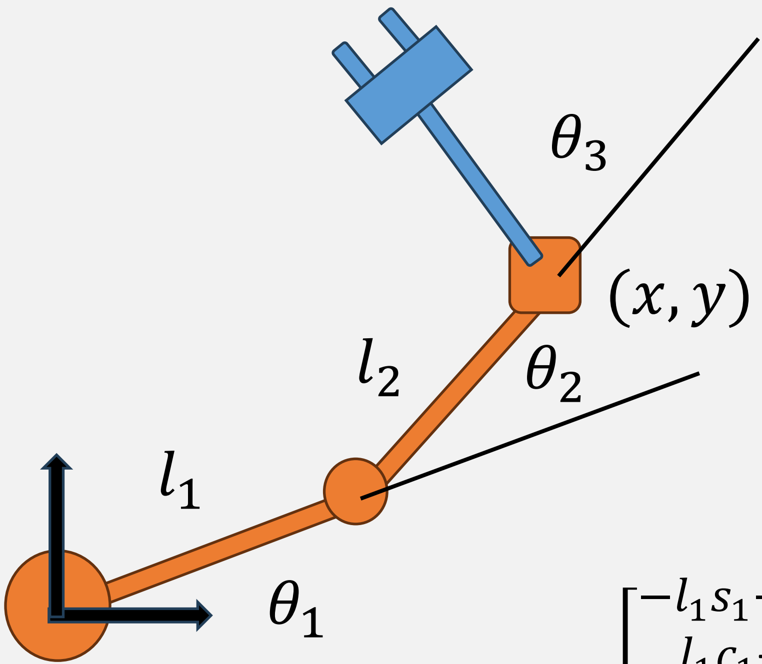
$$x = l_1c_1 + l_1c_{12} + l_3c_{123}$$

$$y = l_1s_1 + l_2\,s_{12} + l_3s_{123}$$

$$J_w = \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2\,s_{12} & 0 \\ l_1c_1 + l_2c_{12} & l_2\,c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$J_e = \begin{bmatrix} I & -\hat{P}_{we} \\ 0 & I \end{bmatrix} J_w$$

Example



Wrist Point

$$x = l_1 c_1 + l_1 c_{12}$$

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End-Effector Point

$$x = l_1 c_1 + l_1 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2\, s_{12} + l_3 s_{123}$$

$$J_e = \begin{bmatrix} -l_1 s_1 - l_2 s_{12}\ \ -l_3\, s_{123} & -l_2\, s_{12} - l_3\, s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2\, c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$J_e = \begin{bmatrix} I & -\hat{P}_{we} \\ 0 & I \end{bmatrix} J_w$$

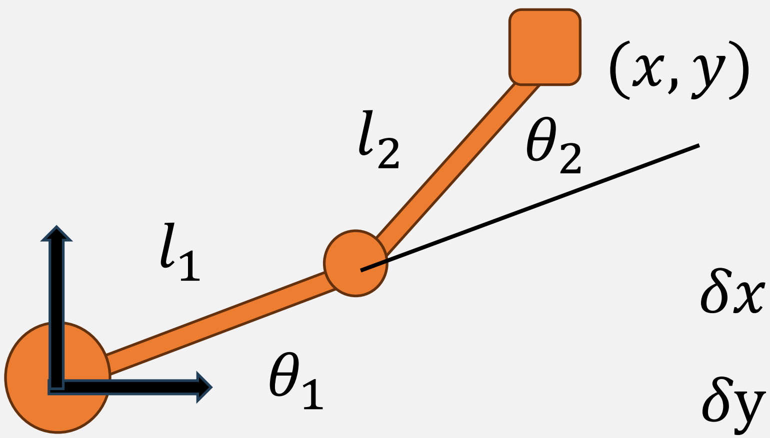
Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction

The direction is the singular direction

$$det[J(q)] = 0$$

Example



$$x = l_1 c_1 + l_1 c_{12}$$

$$\mathrm{y} = l_1 s_1 + l_2\, s_{12}$$

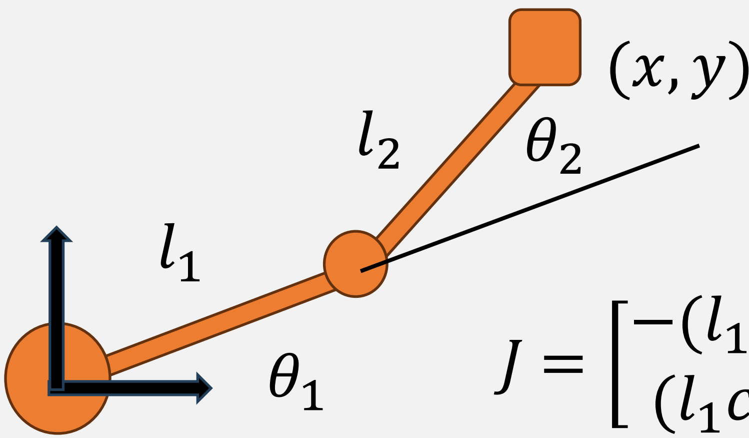
$$\delta x = -(l_1 s_1 + l_2 s_{12})\delta\theta_1 - l_2 s_{12}\delta\theta_2$$

$$\delta \mathrm{y} = (l_1 c_1 + l_2\, c_{12})\, \delta\theta_1 + l_2 c_{12}\delta\theta_2$$

$$\begin{bmatrix} \delta x \\ \delta \mathrm{y} \end{bmatrix} = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ (l_1 c_1 + l_2\, c_{12}) & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\delta x}{\delta\theta_1} & \frac{\delta x}{\delta\theta_2} \\ \frac{\delta y}{\delta\theta_1} & \frac{\delta y}{\delta\theta_2} \end{bmatrix} = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ (l_1 c_1 + l_2\, c_{12}) & l_2 c_{12} \end{bmatrix}$$

Example



$$x = l_1 c_1 + l_1 c_{12}$$

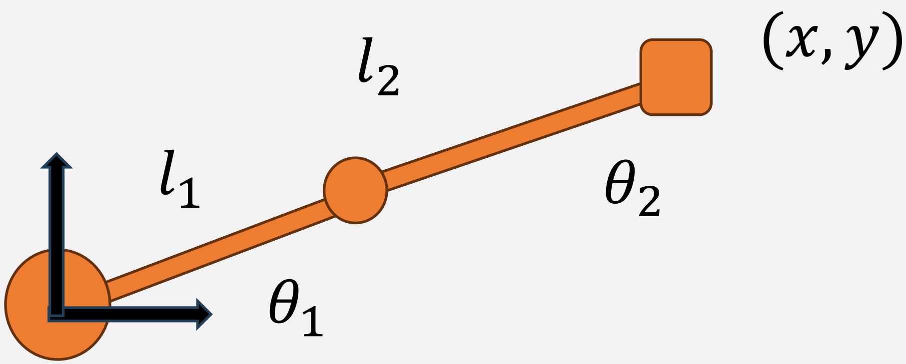
$$y = l_1 s_1 + l_2 \, s_{12}$$

$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ (l_1 c_1 + l_2 \, c_{12}) & l_2 c_{12} \end{bmatrix}$$

$$det[J] = l_1 l_2 s_2$$

Singularity at $\theta_2 \; = k\pi$

Example



$$x = l_1 c_1 + l_1 c_{12}$$

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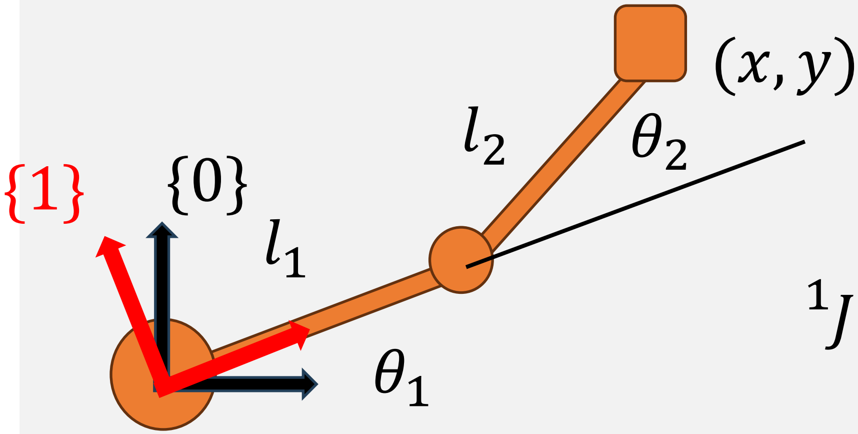
$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ (l_1 c_1 + l_2 \, c_{12}) & l_2 c_{12} \end{bmatrix}$$

$$det[J] = l_1 l_2 s_2$$

Singularity at $\theta_2 = k\pi$

Example



$$^1J = {}^1_0R\ {}^0J$$

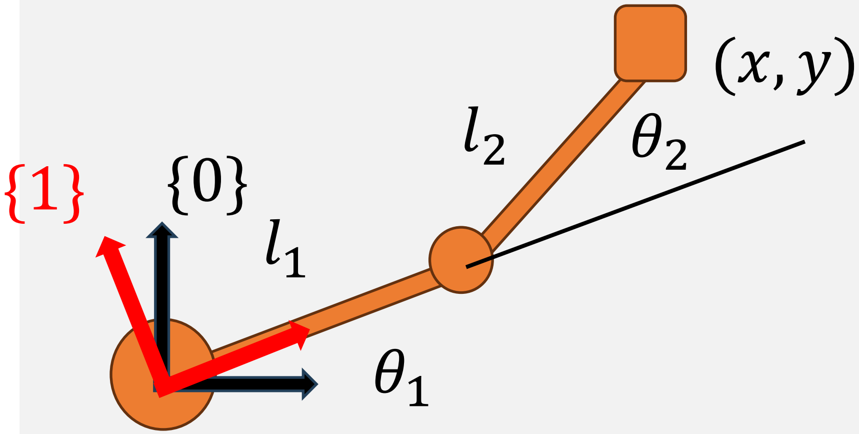
$$^1J = \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} -l_2 s_2 & -l_2 s_2 \\ (l_1 + l_2\, c_2) & l_2 c_2 \end{bmatrix}$$

Singularity at $\theta_2 = k\pi$ $\quad ^1J = \begin{bmatrix} 0 & 0 \\ (l_1 + l_2) & l_2 \end{bmatrix}$

$$^1\delta x = 0$$

$$^1\delta y = (l_1 + l_2)\delta\theta_1 + l_2\delta\theta_2$$

Small Displacement



$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

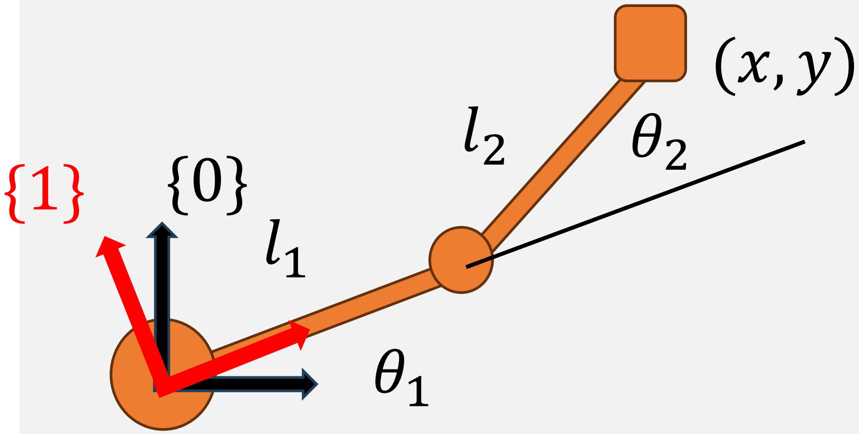
$$^{1}J = \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} -l_2 s_2 & -l_2 s_2 \\ (l_1 + l_2\, c_2) & l_2 c_2 \end{bmatrix}$$

$$\Delta q = J^{-1}\, \Delta X$$

$$J_{(1)}^{-1} \cong \begin{bmatrix} \frac{1}{l_2 \theta_2} & \frac{1}{l_1} \\ -\frac{l_1 + l_2}{l_1 l_2 \theta_2} & -\frac{1}{l_1} \end{bmatrix}$$

Singularity at $\theta_2 = k\pi$

Small Displacement



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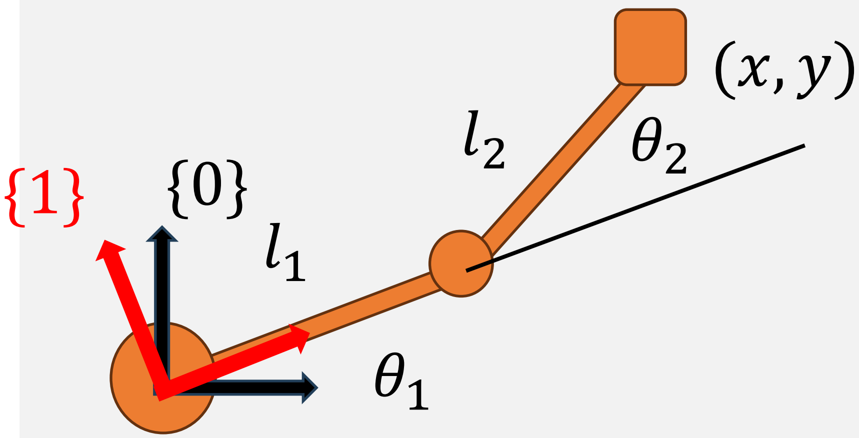
Small Displacement



$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

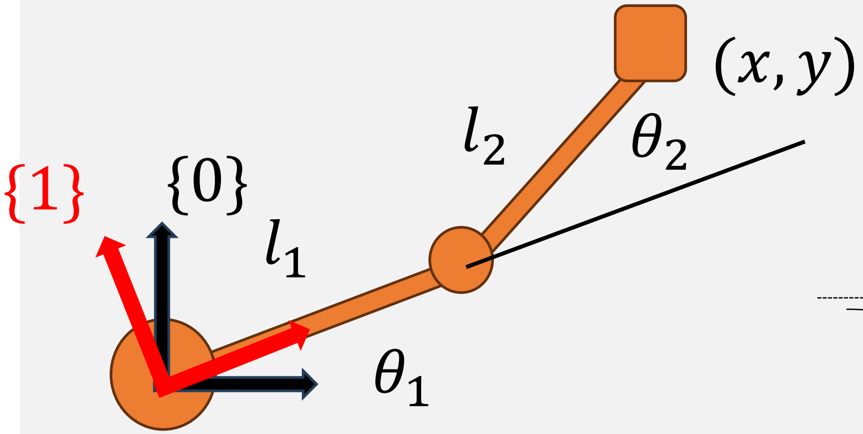
$$\Delta q = J^{-1}\,\Delta X$$

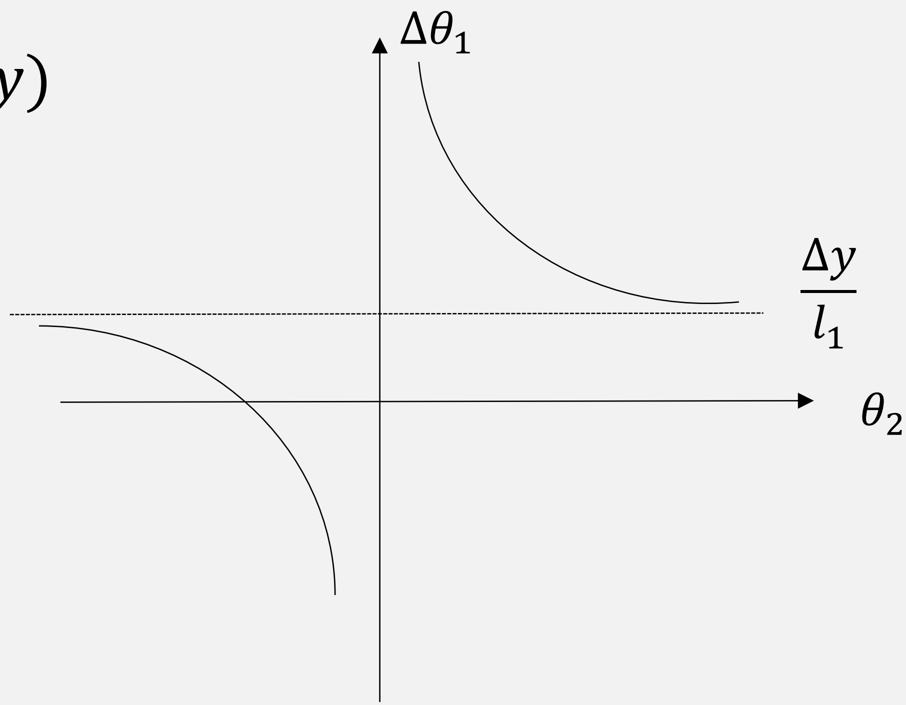
$$J^{-1}_{(1)} \cong \begin{bmatrix} \dfrac{1}{l_2\theta_2} & \dfrac{1}{l_1} \\ -\dfrac{l_1 + l_2}{l_1 l_2 \theta_2} & -\dfrac{1}{l_1} \end{bmatrix}$$

$$\Delta\theta_1 = \frac{\Delta x}{l_2\theta_2} + \frac{\Delta y}{l_1}$$

$$\Delta\theta_2 = -\frac{l_1 + l_2}{l_1 l_2 \theta_2}\Delta x - \frac{\Delta y}{l_1}$$

Small Displacement





$$\Delta\theta_1 = \frac{\Delta x}{l_2\theta_2} + \frac{\Delta y}{l_1}$$

$$\Delta\theta_2 = -\frac{l_1 + l_2}{l_1 l_2 \theta_2}\Delta x - \frac{\Delta y}{l_1}$$

Resolved Motion Rate Control

$$\delta x = J(q)\delta q$$

Outside singularities

$$\delta q = J(q)^{-1}\delta x$$

Arm at Configuration

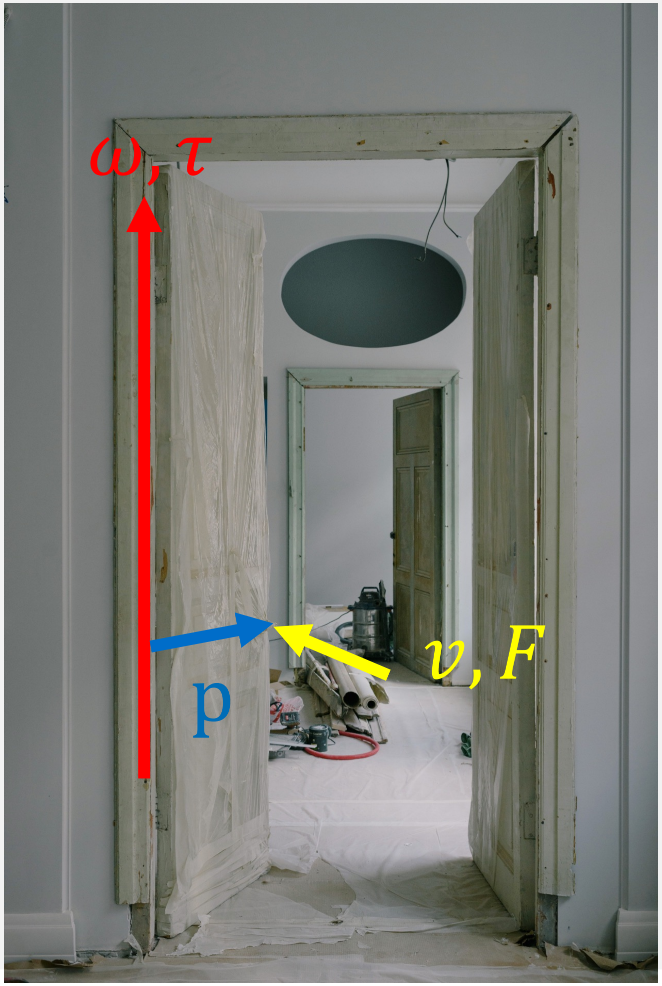
$$x = f(q)$$

$$\delta x = x_d - x$$

$$\delta q = J(q)^{-1}\delta x$$

$$q \leftarrow q + \delta q$$

Angular/Linear—Velocity/Force



$$v = \omega \times \mathrm{p}$$

$$\tau = \mathrm{p} \times F$$

Velocity/Force Duality

$$\dot{x} = J\dot{\theta}$$

$$\tau = J^T F$$

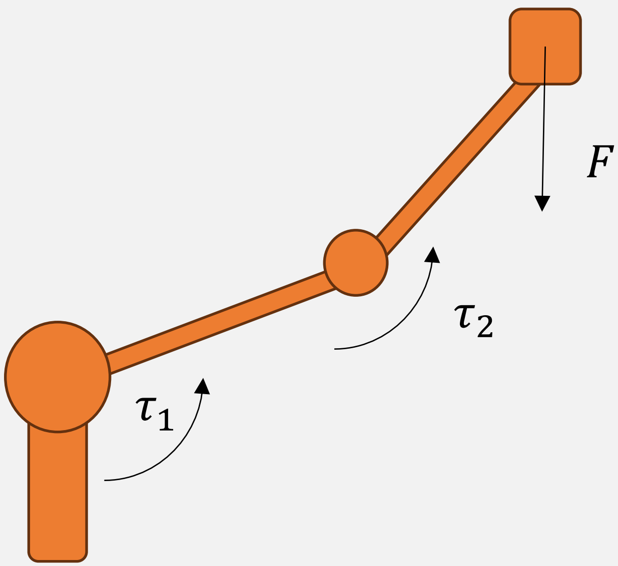
F Static forces and torques applied by the end-effector to the environment

τ The torques needed at the joints of the manipulator to produce F

Virtual Work Principal

Static Equilibrium $$\delta W = \sum_i f_i \delta x_i$$

At static equilibrium, the virtual work of all applied force is equal to zero



$$\tau^T \delta q + (-F^T)\, \delta x = 0$$

$$\tau^T \delta q = F^T \delta x \qquad \delta x = J \delta q$$

$$\tau^T \delta q = F^T J \delta q$$

$$\tau^T = F^T J$$

$$\tau = J^T F$$

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Static Equilibrium $$\delta W = \sum_i f_i \delta x_i$$

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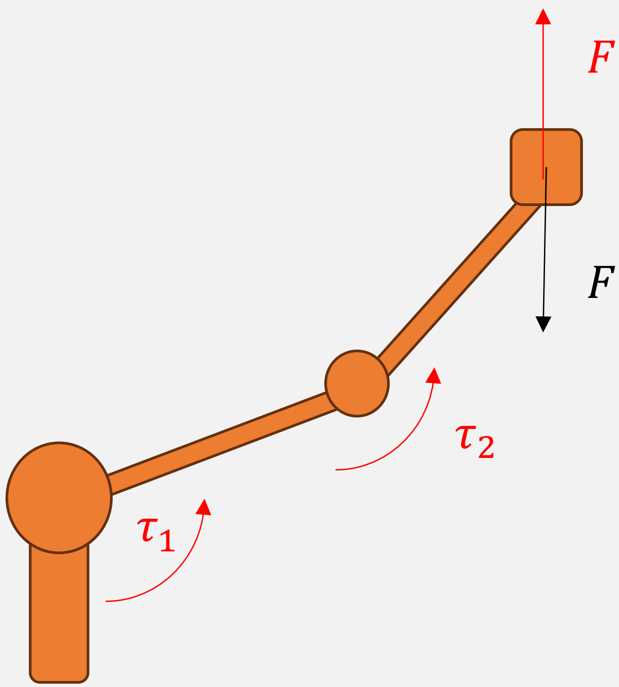

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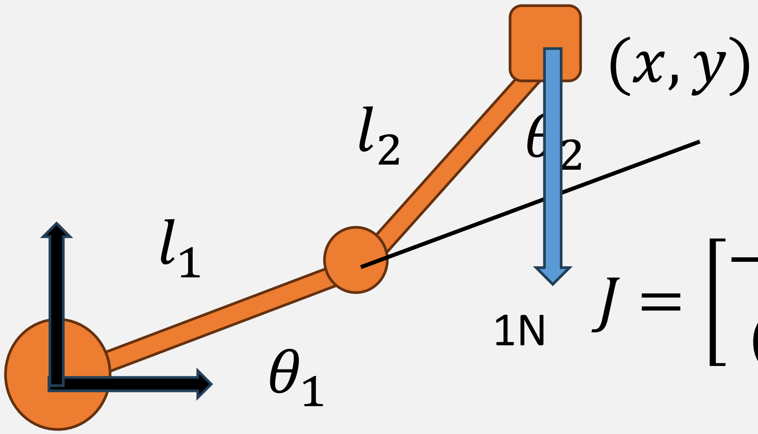
Example



$$x = l_1c_1 + l_1c_{12}$$

$$y = l_1s_1 + l_2\, s_{12}$$

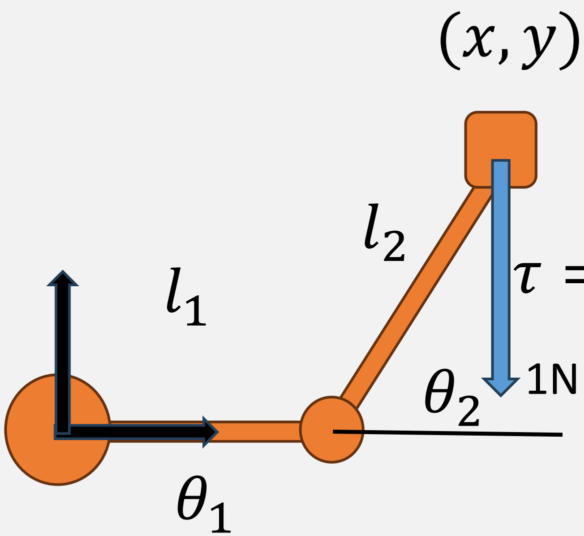
$$J = \begin{bmatrix} -(l_1s_1 + l_2s_{12}) & -l_2s_{12} \\ (l_1c_1 + l_2\, c_{12}) & l_2c_{12} \end{bmatrix}$$

$$J^T = \begin{bmatrix} -(l_1s_1 + l_2s_{12}) & (l_1c_1 + l_2\, c_{12}) \\ -l_2s_{12} & l_2c_{12} \end{bmatrix}$$

$l_1 = l_2 = 1 \quad \theta_1 = 0, \quad \theta_2 = 60,$

Example

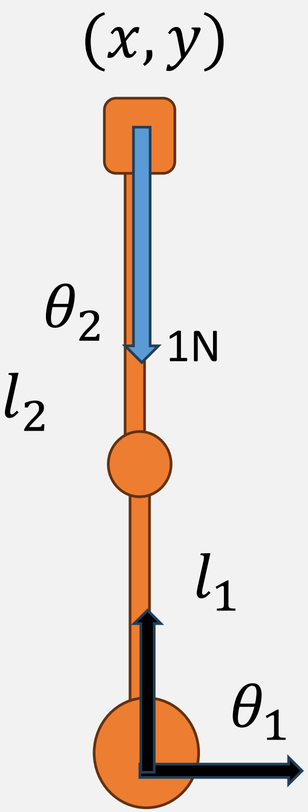
$$l_1 = l_2 = 1 \quad \theta_1 = 0, \quad \theta_2 = 60^{\circ},$$



$$\tau = J^T F = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & (l_1 c_1 + l_2\, c_{12}) \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -(l_1 c_1 + l_2\, c_{12}) \\ -l_2 c_{12} \end{bmatrix} = \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}$$

Example



$l_1 = l_2 = 1$

$\theta_1 = 90^{\circ},$

$\theta_2 = 0,$

$$\tau = J^T F = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & (l_1 c_1 + l_2\, c_{12}) \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -(l_1 c_1 + l_2\, c_{12}) \\ -l_2 c_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Velocity/Force Duality

$$\dot{x} = J\dot{\theta}$$

$$\tau = J^T F$$