School of Computing
National University of Singapore

CS4277/CS5477: 3D Computer Vision
Semester 2, AY 2022/23

Mid-Term Quiz – Solutions

Question 1

Given the image coordinates of three vertices of a square poster with unit length: $\mathbf{X}_1 = [0 \quad 0 \quad 0 \quad 1]^\top$, $\mathbf{X}_2 = [1 \quad 0 \quad 0 \quad 1]^\top$ and $\mathbf{X}_3 = [0 \quad 1 \quad 0 \quad 1]^\top$ are: $\mathbf{x}_1 = [0.0 \quad 0.0 \quad 1.0]^\top$, $\mathbf{x}_2 = [99.50 \quad 14.98 \quad 1.0]^\top$ and $\mathbf{x}_3 = [-9.98 \quad 149.25 \quad 1.0]^\top$.

a) Show that there exists a homography H that maps the vertices $\mathbf{X}_i$ of the square poster in the 3D space to the image space $\mathbf{x}_i$, i.e., H: $\mathbf{X}_i \mapsto \mathbf{x}_i$.

b) We further know that the principal points $p_x = p_y = 0$, the skew parameter $s = 0$, and the extrinsic $\mathrm{T}_{cw} \in \mathrm{SE}(3)$ (maps a point in the world to camera frame) consists of a rotation angle θ around the z-axis and a translation t_z along the z-axis. Find the homography H that maps the vertices of the square poster in the 3D space to the image space.

c) Using the homography H, find the focal lengths (f_x, f_y), rotation angle θ around the z-axis and the image of the absolute conic (IAC) ω.

d) Find the circular points on the image. Explain your answer.

Show all your workings clearly.

(*20 marks*)

Solution

a) The projection equation is: $\mathbf{x} = \mathrm{K}[\mathrm{R} \quad \mathbf{t}]\mathbf{X}$. Let us further write $\mathrm{R} = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3]$. We assign xy-plane of the world frame to be on the plane of the poster and the origin to be on the first vertex. Thus, the 3D points can be written as: $\mathbf{X} = [X \quad Y \quad 0 \quad 1]^\top$. Putting back into the project equation, we get:

$\mathbf{x} = \mathrm{K}[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3 \quad \mathbf{t}][X \quad Y \quad 0 \quad 1]^\top \Rightarrow \mathbf{x} = \mathrm{K}[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}][X \quad Y \quad 1]^\top$, where $\mathrm{H} = \mathrm{K}[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}]$ since we now have a $\mathbb{P}^2 \mapsto \mathbb{P}^2$ mapping.

(shown)

b) Since $p_x = p_y = s = 0$, the intrinsic is given by:

$$\mathrm{K} = \begin{bmatrix} f_x & s & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since there is only a rotation around the z-axis and a translation along the z-axis for T_{cw}, we have:

$$[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}] = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & t_z \end{bmatrix}.$$

Thus, the homography is given by:

$$\mathrm{H} = \mathrm{K}[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}] = \begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & t_z \end{bmatrix} = \begin{bmatrix} f_x r_{11} & f_x r_{12} & 0 \\ f_y r_{21} & f_y r_{22} & 0 \\ 0 & 0 & t_z \end{bmatrix}$$

, which we represent as:

$$\mathrm{H} = \begin{bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_{33} \end{bmatrix}.$$

We now have 4 equations and 4 unknowns:

$$\mathbf{x}_i = \mathrm{H}\mathbf{X}_i$$

$$[x \quad y \quad 1.0]^\top = \begin{bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_{33} \end{bmatrix} [X \quad Y \quad 1]^\top$$

$\Rightarrow x = Xh_{11} + Yh_{12}$ and $y = Xh_{21} + Yh_{22}$.

Using $\mathbf{x}_1 \leftrightarrow \mathbf{X}_1$, we have:

$h_{33} = 1.00$

Using $\mathbf{x}_2 \leftrightarrow \mathbf{X}_2$, we have:

$$99.50 = (1)h_{11} + (0)h_{12} \Longrightarrow h_{11} = 99.50$$
$$14.98 = (1)h_{21} + (0)h_{22} \Longrightarrow h_{21} = 14.98$$

Using $\mathbf{x}_3 \leftrightarrow \mathbf{X}_3$, we have:

$$-9.98 = (0)h_{11} + (1)h_{12} \Longrightarrow h_{12} = -9.98$$
$$149.25 = (0)h_{21} + (1)h_{22} \Longrightarrow h_{22} = 149.25$$

Therefore, we get:

$$\mathrm{H} = \begin{bmatrix} f_x c\theta & -f_x s\theta & 0 \\ f_y s\theta & f_y c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 99.50 & -9.98 & 0 \\ 14.98 & 149.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

c) $\mathrm{H} = \begin{bmatrix} f_x c\theta & -f_x s\theta & 0 \\ f_y s\theta & f_y c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 99.50 & -9.98 & 0 \\ 14.98 & 149.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we get:

$f_x c\theta = 99.50$ and $f_x s\theta = 9.98 \Longrightarrow \tan\theta = \frac{99.50}{9.98} \Longrightarrow \boldsymbol{\theta = 0.1\, rad}$.

$\boldsymbol{f_x} = \frac{\mathbf{99.50}}{\mathbf{\cos 0.1}} = \mathbf{100}$ and $\boldsymbol{f_y} = \frac{\mathbf{14.98}}{\mathbf{\sin 0.1}} = \mathbf{150.}$

Image of absolute conic $\omega = (\mathrm{K}\mathrm{K}^\top)^{-1} = \begin{bmatrix} \frac{1}{f_x^2} & 0 & 0 \\ 0 & \frac{1}{f_y^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{100^2} & 0 & 0 \\ 0 & \frac{1}{150^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

d) Circular points are: $[1 \quad \pm i \quad 0]^\top$. Since any plane intersects the plane at infinity at the circular points, the images of the circular points are given by:

$$\mathrm{H}[1 \quad \pm i \quad 0]^\top \Longrightarrow \mathbf{h}_1 \pm i\mathbf{h}_2 = \begin{bmatrix} 99.50 \\ 14.98 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} -9.98 \\ 149.25 \\ 0 \end{bmatrix}.$$

Question 2

a) Given four collinear points in $\mathbb{P}^3$:

$\mathbf{X}_1 = [2.00 \quad 1.00 \quad 2.00 \quad 1.00]^\top$, $\mathbf{X}_2 = [3.00 \quad 4.00 \quad 4.0000 \quad 1.00]^\top$,
$\mathbf{X}_3 = [5.00 \quad 10.00 \quad 8.00 \quad 1.00]^\top$, $\mathbf{X}_4 = [2.60 \quad 2.80 \quad 3.20 \quad 1.00]^\top$,

and the projections of $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ on an image in $\mathbb{P}^2$:

$\mathbf{x}_1 = [133.89 \;\; 124.25 \quad 1.00]^\top$, $\mathbf{x}_2 = [104.41 \;\; 136.56 \quad 1.00]^\top$,
$\mathbf{x}_3 = [86.63 \;\; 143.99 \quad 1.00]^\top$,

find the distance between the projection of $\mathbf{X}_4$ and $\mathbf{x}_1$.

b) A camera that has undergone a translation along the x-axis, and observes a point correspondence in two image frames:

$\mathbf{x_1} = [50 \quad 100 \quad 1]^\top \leftrightarrow \mathbf{x}'_1 = [100 \quad 150 \quad 1]^\top$,
$\mathbf{x_2} = [450 \quad 150 \quad 1]^\top \leftrightarrow \mathbf{x}'_2 = [600 \quad 300 \quad 1]^\top$.

We further note that the camera intrinsic has principal points $p_x = p_y = 0$ and skew parameter $s = 0$. Find the fundamental matrix F that relates that two image frames.

Show all your workings clearly.

(*20 marks*)

Solution

a)

Turn the collinear $\mathbb{P}^3$ points into $\mathbb{P}^1$:

$\mathbf{P}_1 = [\text{dist}(\mathbf{X}_1, \mathbf{X}_1), 1]^\top = [0, 1]^\top$, $\mathbf{P}_2 = [\text{dist}(\mathbf{X}_2, \mathbf{X}_1), 1]^\top = [3.74, 1]^\top$,

$\mathbf{P}_3 = [\text{dist}(\mathbf{X}_3, \mathbf{X}_1), 1]^\top = [11.23, 1]^\top$, $\mathbf{P}_4 = [\text{dist}(\mathbf{X}_4, \mathbf{X}_1), 1]^\top = [2.25, 1]^\top$.

Turn the collinear $\mathbb{P}^2$ points into $\mathbb{P}^1$:

$\mathbf{p}_1 = [\text{dist}(\mathbf{x}_1, \mathbf{x}_1), 1]^\top = [0, 1]^\top$, $\mathbf{p}_2 = [\text{dist}(\mathbf{x}_2, \mathbf{x}_1), 1]^\top = [31.94, 1]^\top$,

$\mathbf{p}_3 = [\text{dist}(\mathbf{x}_3, \mathbf{x}_1), 1]^\top = [51.21, 1]^\top$.

$$\text{CrossRatio} = \frac{\det[\mathbf{P_1} \quad \mathbf{P_2}]\det[\mathbf{P_3} \quad \mathbf{P_4}]}{\det[\mathbf{P_1} \quad \mathbf{P_3}]\det[\mathbf{P_2} \quad \mathbf{P_4}]} = \frac{(-3.7417)(8.9800)}{(-11.2250)(1.4967)} = 2.00$$

$\det[\mathbf{p_3} \quad \mathbf{p_4}] = p_3 - p_4 = 51.21 - p_4$

$\det[\mathbf{p_2} \quad \mathbf{p_4}] = p_2 - p_4 = 31.94 - p_4$

$$\text{CrossRatio} = \frac{\det[\mathbf{p_1} \quad \mathbf{p_2}]\det[\mathbf{p_3} \quad \mathbf{p_4}]}{\det[\mathbf{p_1} \quad \mathbf{p_3}]\det[\mathbf{p_2} \quad \mathbf{p_4}]} = \frac{(-31.94)(51.21 - p_4)}{(-51.21)(31.94 - p_4)} = 2.00$$

$\Longrightarrow (31.94)(51.21 - p_4) = (102.4200)(31.94 - p_4)$

$\Longrightarrow 1635.6 - 31.94p_4 = 3271.3 - 102.42p_4$

$\Longrightarrow 70.4800p_4 = 1635.7$

$\Longrightarrow p_4 = 23.21.$

The distance between the projection of $\mathbf{X}_4$ and $\mathbf{x}_1$ is $p_4 = 23.21$.

b)

$$\mathrm{K}^{-1} = \begin{bmatrix} 1/f_x & 0 & 0 \\ 0 & 1/f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix},$$

$$\Rightarrow [\mathbf{t}]_\times = \begin{bmatrix} 0 & 0 & t_y \\ 0 & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

$$\mathrm{F} = \mathrm{K}^{-\top}[\mathbf{t}]_\times \mathrm{R}\mathrm{K}^{-1} = \begin{bmatrix} 1/f_x & 0 & 0 \\ 0 & 1/f_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & t_y \\ 0 & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/f_x & 0 & 0 \\ 0 & 1/f_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & t_y/f_x \\ 0 & 0 & -t_x/f_y \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} 1/f_x & 0 & 0 \\ 0 & 1/f_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & t_y/f_x \\ 0 & 0 & -t_x/f_y \\ -t_y/f_x & t_x/f_y & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & -b \\ -a & b & 0 \end{bmatrix}.$$

Using the epipolar geometry, we have:

$$\mathbf{x}'^\top \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & -b \\ -a & b & 0 \end{bmatrix} \mathbf{x} = 0$$

$$[100 \quad 150 \quad 1] \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & -b \\ -a & b & 0 \end{bmatrix} \begin{bmatrix} 50 \\ 100 \\ 1 \end{bmatrix} = [-a \quad b \quad 100a - 150b] \begin{bmatrix} 50 \\ 100 \\ 1 \end{bmatrix}$$

$$= -50a + 100b + 100a - 150b = 50a - 50b = 0 \qquad \text{----- (1)}$$

$$[600 \quad 300 \quad 1] \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & -b \\ -a & b & 0 \end{bmatrix} \begin{bmatrix} 450 \\ 150 \\ 1 \end{bmatrix} = [-a \quad b \quad 600a - 300b] \begin{bmatrix} 450 \\ 150 \\ 1 \end{bmatrix}$$

$$= -450a + 150b + 600a - 300b = 150a - 150b = 0 \qquad \text{----- (2)}$$

From (1) and (2), we get $a = b$, which we can set to be $a = b = 1$ since F is up to a scale.

$$\Rightarrow \mathrm{F} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$