NUS | Computing

Classical Planning

CS4246/CS5446

AI Planning and Decision Making

This lecture will be recorded!

Solving Planning Problems

- Planning Problem or Model
  - Appropriate abstraction of states, actions, effects, and goals
- Planning Algorithm
  - Input: a problem
  - Output: a solution in the form of an action sequence (a plan)
- Planning Solution
  - A plan or path from the initial state(s) to the goal state(s)
    - Any path
  - A goal state that satisfies certain properties

Classical Planning

Real-world example: Scheduling classes in NUS

- Definition:
  - Find a sequence of actions to achieve a goal in an environment that is:
    - Discrete
    - Deterministic
    - Static
    - Fully observable
- Main challenges:
  - How to represent the planning problem? – PDDL from STRIPS
  - How to search for a solution (plan)? – Search and satisfiability
  - How to use heuristics to solve for a solution (plan) more efficiently?

A Bit of History

- Classical planning – still very useful today!
  - Also called STRIPS planning
- What is STRIPS?
  - STanford Research Institute Problem Solver
  - R. Fikes and N. Nilsson (1971). STRIPS: a new approach to the application of theorem proving to problem solving. Artificial Intelligence, 2:189-208
  - Included planning language for automated planning problems
- Video:
  - https://media.ed.ac.uk/media/1_uhxvxo4a

Planning Model

PDDL and STRIPS Planning

Planning Domain Definition Language

- PDDL
  - Derived from the original STRIPS planning language
  - Logic-based, lifted propositional representation (to restricted FOL)
  - Basic version for classical planning, extensions in active research
- Planning problem representation

Usually, variables are letters (or words) in lower case; constants are in upper case

	Examples
State	$At(P_1, SFO) \wedge At(P_2, SIN)$
Action	$Action(Fly(P_1, SFO, SIN)$ PRECOND: $At(P_1, SFO) \wedge Plane(P_1) \wedge Airport(SFO) \wedge Airport\ (SIN)$ EFFECT: $\neg At(P_1, SFO) \wedge At(P_1, SIN))$
Goal	$At(p, SIN) \wedge Plane(p)$

Main Characteristics

- Factored representation – Why?
  - Represent state of the world as a set of state variables or fluents
  - Each fluent depicts an aspect of the world that changes with time
- Domain independent heuristics
  - Avoid need for domain specific heuristics to work well in planning
- On-demand state computation
  - "Implicit" transition function
    - Just specify the current state
    - Define actions that can compute state-transitions
    - Use actions to generate the other states as needed

State Representation

A set of fluents

- State
  - A conjunction of ground atomic fluents
    - Ground atomic – there is a single predicate, with only constant arguments; function-free
  - Examples:
    - $Hungry \wedge Sleepy$
    - $New(Plane_1) \wedge Safe(Plane_1)$
    - $At(Plane_1, SIN) \wedge At(Plane_2, SFO)$
- Database semantics:
  - Closed-world assumption – Any fluents that are not mentioned are False
    - e.g., $Fierce(CS4246\_Lecturer)$
  - Unique names assumption – Any two objects with different names are different
    - e.g., $Plane_1$ and $Plane_2$

Goal Representation

- Goal State:
  - Goal g is a partially specified state, represented as a conjunction of literals
  - A state s satisfies a goal g if s contains all the literals in g
  - Examples:
    - $Hungry \wedge Sleepy \wedge Bored$ satisfies the goal $Hungry \wedge Bored$
    - $At(Cargo_1, SFO)$ satisfies the goal $At(c, SFO)$ with substitution $\{c/Cargo_1\}$ where c is a variable for any cargo

Follow general definition of Goal in PDDL on Slide 21

Action Schema

- Defines a family of ground actions
  - All variables assumed to be universally quantified
  - Lifts propositional logic to restricted subset of first-order logic
  - Pre-Conditions:
    - Conjunctions of literals (positive and negative atomic sentences) defining the applicable states in which the action can be executed
  - Effects:
    - Conjunctions of literals defining the result of executing the action

$Action(Fly(p, from, to)$
PRECOND: $At(p, from) \wedge Plane(p) \wedge Airport(from) \wedge Airport\ (to)$
EFFECT: $\neg At(p, from) \wedge At(p, to))$

Grounded Actions

$Action(Fly(p, from, to)$
Precond: $At(p, from) \wedge Plane(p) \wedge Airport(from) \wedge Airport\ (to)$
Effect: $\neg At(p, from) \wedge At(p, to))$

- Grounded actions
  - When an action schema contains variables, it may have multiple applicable instantiations or grounded actions

$Action(Fly(P_1, SFO, SIN)$
PRECOND: $At(P_1, SFO) \wedge Plane(P_1) \wedge Airport(SFO) \wedge Airport\ (SIN)$
EFFECT: $\neg At(P_1, SFO) \wedge At(P_1, SIN))$

$Action(Fly(P_2, SIN, SFO)$
PRECOND: $At(P_2, SIN) \wedge Plane(P_2) \wedge Airport(SIN) \wedge Airport\ (SFO)$
EFFECT: $\neg At(P_2, SIN) \wedge At(P_2, SFO))$

Action Execution

- Current state and available action:
  $At(P_1, SFO) \wedge Plane(P_1) \wedge Airport(SFO) \wedge Airport\ (SIN)$
  $Action(Fly(P_1, SFO, SIN)$
  PRECOND: $At(P_1, SFO) \wedge Plane(P_1) \wedge Airport(SFO) \wedge Airport\ (SIN)$
  EFFECT: $\neg At(P_1, SFO) \wedge At(P_1, SIN))$
- Execute action
  $Fly(P_1, SFO, SIN)$
- Remove from database of states
  $At(P_1, SFO)$
- Add to database of states
  $At(P_1, SIN)$
- Updated state:
  $At(P_1, SIN) \wedge Plane(P_1) \wedge Airport(SFO) \wedge Airport\ (SIN)$

$s' = (s - DEL(a)) \cup ADD(a)$

Action Effects

- Applicable State
  - A ground action a is applicable in state s if s entails the precondition of a
- Execution Result
  - Result of executing applicable action a in state s is a state s' formed by:
    - Removing fluents in action effects that are negative literals - delete list $DEL(a)$, and
    - Adding fluents in action effects that are positive literals– add list $ADD(a)$

$$s' = (s - DEL(a)) \cup ADD(a)$$

  - New states are generated on demand with the $ADD(a)$ and $DEL(a)$ functions as a result of execution an action

The Frame Problem

	Examples
State	$At(P_1, SFO) \wedge At(P_2, SIN)$
Action	$Action(Fly(P_1, SFO, SIN)$ PRECOND: $At(P_1, SFO) \wedge Plane(P_1) \wedge Airport(SFO) \wedge Airport\ (SIN)$ EFFECT: $\neg At(P_1, SFO) \wedge At(P_1, SIN))$
Goal	$At(p, SIN) \wedge Plane(p)$

- Definition
  - What changes and what stays as the result of the action
  - Needs to be addressed by any real-world planning models
- Action representation in PDDL:
  - Times and states are implicit in action schemas:
  - Preconditions always refers to time t and the effect to time t + 1
  - Specifies result of an action in terms of changes; everything that stays the same unmentioned
- State representation in PDDL:
  - Uses closed world assumption that anything not mentioned is FALSE
  - The fluents do not explicitly refer to time

Planning Problem and Solution

- Initial state
  - A state – a conjunction of ground atoms
  - Example: $A \wedge t(P_1, SFO) \wedge Plane(P_1) \wedge Airport(SFO) \wedge Airport(SIN) \wedge \cdots$
- Goal
  - A conjunction of literals that may contain variables
  - Any variables are treated as existentially quantified
  - Example: $At(p, SIN) \wedge Plane(p)$.
- Problem solution
  - A sequence of actions that end in a state s that entails the goal
  - Example: Any plan that reaches state:

$$Plane(P_1) \wedge At(P_1, SIN)\ entails\ goal\ \ At(p, SIN) \wedge Plane(p)$$

STRIPS planning restricts Goal to positive literals with no variables

Example: Air Cargo Transport Planning

$Init(At(C_1, SFO) \wedge At(C_2, SIN) \wedge At(P_1, SFO) \wedge At(P_2, SIN)$
$\wedge Cargo(C_1) \wedge Cargo(C_2) \wedge Plane(P_1) \wedge Plane(P_2)$
$\wedge Airport\ (SIN) \wedge Airport(SFO))$
$Goal(At(C_1, SIN) \wedge At(C_2, SFO))$

$Action(Load(c, p, a),$
PRECOND: $At(c, a) \wedge At(p, a) \wedge Cargo(c) \wedge Plane(p) \wedge Airport(a)$
EFFECT: $\neg At(c, a) \wedge In(c, p))$
$Action(Unload(c, p, a),$
PRECOND: $In(c, p) \wedge At(p, a) \wedge Cargo(c) \wedge Plane(p) \wedge Airport(a)$
EFFECT: $At(c, a) \wedge \neg In(c, p))$
$Action(Fly(p, from, to),$
PRECOND: $At(p, from) \wedge Plane(p) \wedge Airport(from) \wedge Airport(to)$
EFFECT: $\neg At(p, from) \wedge At(p, to))$

Planning Domain

Planning Problem

Adapted from source: RN Figure 11.1

Algorithms for Classical Planning

- Possible approaches
  - Forward search through state space from initial state to goal
  - Backward search through state space from goal to initial state
  - Translate problem description into set of logic sentences and apply logical inference algorithm to find solution (SATPlan)

Planning as State-Space Search

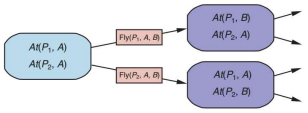
Forward (progression) search

Backward (regression) search

Planning as State-Space Search

- Map planning problem into search problem in state-space
  - States are ground states with only binary fluents
  - Goal state has all the positive fluents in problem's goal
  - Applicable actions in a state are ground actions
  - Different planning procedures have different search spaces
- Planning search tree :
  - Each node is a state, including initial state and goal state
  - Each branch is an action that allows a transition from one state to another
  - A plan is a path from the initial state to the goal state in search tree
- Algorithms:
  - Forward (progression) state-space search
  - Backward (regression) relevant-states search

Planning by Forward Search

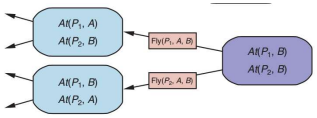


Start at a state s

Repeat:

1. Stop if goal g is satisfied
   - Check if the goal is a subset of the state s description: $g \subseteq s$
2. Compute applicable actions
   - Check if pre-conditions of an action a is a subset of the state s description: $Precond(a) \subseteq s$
3. Compute successor states
   - Add the list of positive effects of an action a to the state s description; delete the list of negative effects: $s' = (s - DEL(a)) \; U \; ADD(a)$
4. Pick a successor state s' as current state

Planning by Backward Search



Start at goal g

Repeat:

1. Stop if initial state s is satisfied
   - Check if the initial state s is a subset of the goal state g description: $s \subseteq g$
2. Compute relevant actions
   - Check if effects of an action a is a subset of the goal state g description: $Effects(a) \subseteq g$
3. Compute predecessor states
   - Add the preconditions and remove the list of positive and negative effects of an action a to the current goal state g description

$$POS(g') = (POS(g) - ADD(a)) \; U \; POS(Precond(a))$$
$$NEG(g') = (NEG(g) - DEL(a)) \; U \; NEG(Precond(a))$$

4. Pick a predecessor state g' to be current goal state

State Descriptions in Regression

- Notes
  - A set of relevant states to consider at each step (cf. forward search)
  - Regression from a state description to a predecessor state description
- State vs Description
  - In a state, every variable is assigned a value.
  - For n ground fluents, there are 2^n ground states
  - For n ground fluents, there are 3^n descriptions
    - Each fluent can be positive, negative, or not mentioned.
    - E.g. for goal: $\neg Hungry \wedge Sleepy$ describes states where $Hungry$ is False and $Sleepy$ is True, but other unmentioned fluents can have any value.
  - A description represent a set of states.

Relevant Actions in Regression

- Relevant actions
  - Those that can be the last step in a plan leading up to current goal state
  - At least one of its effects must unify with an element of the goal;
  - Must not have any effect that negates the elements of the goal
  - Substitute most general unifier to keep branching factor down but not rule out any solution
- Regression
  - Given a goal g and an action a, regression from g over a gives a state description g' whose positive and negative literals are given by:

$$Pos(g') = (Pos(g) - Add(a) \cup Pos(Precond(a))$$
$$Neg(g') = (Neg(g) - Del(a) \cup Neg(Precond(a))$$

Relevant Actions in Regression

- More formally:
  - Assume a goal description g that contains a goal literal g_i and an action schema A
  - If A has an effect literal e'_j where $Unify(g_i, e'_j) = \theta$ and where we define $A' = Subst(\theta, A)$ and if there is no effect A' that is the negation of a literal in g, then A' is a relevant action towards g

Example

- Goal
  - $At(C_2, SFO)$
- Relevant action schema
  - $Action(Unload(c, p, a)$
  - Precond: $In(c, p) \wedge At(p, a) \wedge Cargo(c) \wedge Plane(p) \wedge Airport(a)$
  - Effect: $At(c, a) \wedge \neg In(c, p)$
- New action schema after substitution
  - $Action(Unload(C_2, p', SFO)$
  - Precond: $In(C_2, p') \wedge At(p', SFO) \wedge Cargo(C_2) \wedge Plane(p') \wedge . Airport(SFO)$
  - Effect: $At(C_2, SFO) \wedge \neg In(C_2, p')$
- Regressed state (new goal)
  - $g' = In(C_2, p') \wedge At(p', SFO) \wedge Cargo(C_2) \wedge Plane(p') \wedge Airport(SFO)$

Substitution
$\theta = \{c/C_2, a/SFO\}$

POS(g') = (POS(g) – ADD(a)) U POS(Precond(a))
NEG(g') = (NEG(g) – DEL(a)) U NEG(Precond(a))

Forward and Backward Search in Large Domains

- Extended air cargo transportation problem
  - Air-cargo from airport A to airport B; 10 airports, each airport with 5 planes and 20 cargos
  - How to plan?
- Forward Search
  - Prone to exploring irrelevant actions
  - Need to traverse large state-spaces
  - Average branching factor is huge!
  - Need domain-specific or domain-independent heuristics that can be derived automatically
- Backward search
  - Keep branching factor lower than forward search
  - Using state sets for searching makes it harder to come up with heuristics

Planning as Boolean satisfiability

SATPlan

Planning as Boolean Satisfiability

- Transform classical planning problem into a Boolean satisfiability (SAT) problem
  - Translate planning problem into propositional formula in Conjunctive Normal Form (CNF)
  - Then determine whether propositional formula is satisfiable
  - Example:

$$(P \vee Q) \wedge (\neg Q \vee R \vee S) \wedge (\neg R \vee \neg P)$$

    Does there exist a model i.e., an assignment of truth values to the proposition that makes the formula true?
- SAT is a NP-complete problem
  - SAT solvers are effective in practice; can solve problems with millions of variables and constraints
  - Use domain independent heuristics to search for a solution (See RN 7.7.4)

The SATPlan Algorithm

function SATPLAN($init$, $transition$, $goal$, $T_{\max}$) **returns** solution or *failure*
  inputs: $init$, $transition$, $goal$, constitute a description of the problem
    $T_{\max}$, an upper limit for plan length

  for $t = 0$ **to** $T_{\max}$ **do**
    $cnf \leftarrow$ TRANSLATE-TO-SAT($init$, $transition$, $goal$, t)
    $model \leftarrow$ SAT-SOLVER(cnf)
    if $model$ is not null **then**
      return EXTRACT-SOLUTION($model$)
  return *failure*

Assignment of values to variables

Source: RN Fig 7.22

Example: Eat a Cake!

$Init(\neg Have(Cake))$

$Goal(\neg Have(Cake) \wedge Eaten(Cake))$

$Action(Eat(Cake)$
Precond: $Have(Cake)$
Effect: $\neg Have(Cake) \wedge Eaten(Cake))$

$Action(Bake(Cake)$
Precond: $\neg Have(Cake)$
Effect: $Have(Cake))$

Example: Eat a Cake!

Bounded planning problem with possible solution at n = 2

- Initial and goal states:

(Init) ¬Have(Cake, 0)
(Goal) ¬Have (Cake, 2) ∧ Eaten(Cake, 2)

- The actions:

(Eat) Eat(cake, 0) ➔ Have(Cake, 0) ∧¬Have(Cake, 1) ∧ Eaten(Cake, 1)
Eat(cake, 1) ➔ Have(Cake, 1) ∧¬Have(Cake, 2) ∧ Eaten(Cake, 2)

(Bake) Bake(cake, 0) ➔ ¬Have(Cake, 0) ∧Have(Cake, 1)
Bake(cake, 1) ➔ ¬Have(Cake, 1) ∧Have(Cake, 2)

Example: Eat a Cake! (cont.)

- Successor state axioms : (How fluents can change)

¬Have(Cake, 0) ∧Have(Cake, 1) ➔ Bake(Cake, 0)
¬Have(Cake, 1) ∧Have(Cake, 2) ➔ Bake(Cake, 1)
Have (Cake, 0) ∧ ¬Have(Cake, 1) ➔ Eat(Cake, 0)
Have (Cake, 1) ∧ ¬Have(Cake, 2) ➔ Eat(Cake, 1)
¬Eaten (Cake, 0) ∧ Eaten(Cake, 1) ➔ Eat(Cake, 0)
¬Eaten (Cake, 1) ∧ Eaten(Cake, 2) ➔ Eat(Cake, 1) …

- Action exclusion axioms: (Assume not eat and bake at same time!)

¬Eat (Cake, 0) ∨¬Bake(Cake, 0)
¬Eat (Cake, 1) ∨¬Bake(Cake, 1)

Planning Problem Translation

1. Propositionalize the actions
   - For each action schema, form ground propositions by substituting constants for each of the variables
2. Add action exclusion axioms
   - Assert that no two actions can occur at the same time
3. Add precondition axioms
   - For each ground action A^t, add the axiom $A^t \Rightarrow PRE(A)^t$, i.e., if an action is taken at time t, then the preconditions must have been true

Planning Problem Translation

4. Define the initial state
   - Assert F^0 for every fluent F in the problem initial state
   - Assert $\neg F^0$ for every fluent not mentioned in the initial state
5. Propositionalize the goal
   - Goal becomes a disjunction over all its ground instances; variables are replaced by constants
6. Add successor state axioms
   - For each fluent F, add an axiom of the form
     $F^{t+1} \Leftrightarrow ActionCausesF^t \vee (F^t \wedge \neg ActionCause\quad\quad^t)$, where
     $ActionCauses\quad^t$ stands for a disjunction of all the ground actions that add F and
     $ActionCausesNotF^t$ stands for a disjunction of all the ground actions that delete F

The Frame Problem

- The Frame Problem:
  - Most actions leave most fluents unchanged.
  - For each action, assert an axiom for each fluent the action leaves unchanged.
- Successor-state axioms:
  - A clever encoding that handles the frame problem well.
  - For m actions and n fluents, need $O(mn)$ such axioms
  - In contrast, with successor-state axioms, only need $O(n)$ axioms: each axiom is longer, but only involves actions that have an effect on the fluent
- SATPlan
  - Find possible reachable models specifying future action sequences
  - Guaranteed to find shortest path if one exists
  - Works only for fully observable or sensorless environments
  - Can find models with impossible action; memory requirement is still combinatorially large

Characteristics of Planning

Planning Algorithm Properties

- A planning algorithm or planner is sound
  - For any plan generated by the planning algorithm, this plan is guaranteed to be a solution
- A planning algorithm or planner is complete
  - If a solution exists then the planning algorithm will return a solution
- Question: How "good" are the solutions?
  - For any plan generated by the planning algorithm, can the plan be guaranteed to be an optimal solution?
  - Yes if admissible heuristics are used in the search for the solution

Complexity of Planning

- Two decision problems:
  - PlanSAT –whether there exists any plan that solves a planning problem
  - Bounded PlanSAT –whether there is a solution of length k or less
    - For classical planning, both problems are decidable as number of states is finite
- In general:
  - Both problems are in PSPACE: solvable with polynomial amount of space
  - In many domains, Bounded PlanSAT is NP-complete; PlanSAT is in P
    - Optimal planning is usually hard, but sub-optimal planning is sometimes easy
- In real world planning:
  - Agents usually asked to find plans in specific domains, not in worst-case instances
  - To do well, need good heuristics or alternate planning models

NUS | Computing

Real World Planning and Acting

CS4246/CS5446

AI Planning and Decision Making

This lecture will be recorded!

Please join:
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Solving Planning Problems

Heuristics: for classical planning
HTN: New planning model

- Planning Problem or Model
  - Appropriate abstraction of states, actions, effects, and goals (and costs and values)
- Planning Algorithm
  - Input: a problem
  - Output: a solution in the form of an action sequence
- Planning Solution
  - A plan or path from the initial state(s) to the goal state(s)
    - Any path; OR
    - An optimal path wrt to costs or values
  - A goal state that satisfies certain properties

Heuristics for Planning

Improving Efficiency

Heuristics for Planning

- Heuristic function
  - $h(s)$ estimates distance from a state s to the goal g
- Main idea
  - Find admissible heuristic for distance
  - A* or other heuristic search can then find optimal solutions

Heuristics for Planning

- Heuristic function
  - $h(s)$ estimates distance from a state s to the goal g
- Main idea
  - Find admissible heuristic for distance
  - A* or other heuristic search can then find optimal solutions
- Approach
  - Define a relaxed problem that is easier to solve
  - Exact cost of solution to easier problem is heuristic for original problem

Types of Heuristics

Real-world problems use a combination of domain-independent and domain-specific heuristics

- Definition and application
  - Facilitated by factored representation for states and action schemas
- Search problem
  - A graph with states as nodes and actions as edges
  - Find a path connecting initial state to goal state
- Domain independent heuristics –Why?
  - Two ways to relax the problem (make it strictly easier):
    - Add more **edges** – easier to find path
    - Group multiple **nodes** together – abstract with fewer states and easier to search
  - Prune away irrelevant branches of search tree

Heuristics of Adding Edges

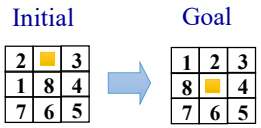
- Ignore preconditions heuristic
  - Drop all preconditions from actions
- Ignore selected preconditions heuristic
  - Derive simpler measures of distance from goal
- Ignore delete list heuristic
  - Allow monotonic progress toward goal
- Caveats:
  - Finding solution to relaxed problem is still NP-hard
  - Trade-off in optimality or admissibility sometimes required
  - Expensive to calculate heuristics
  - Do not reduce state-space size

How to Define an Edge-Based Heuristic?

- Approach:
  - Relax actions by removing all preconditions and all effects except goal literals
  - Count minimum no. of actions required for union effects to satisfy the goal
- Note:
  - An instance of the set cover problem – NP-hard
- Potential solution:
  - Simple greedy algorithm guaranteed to return a set covering whose size is within a factor of $\log(n)$ of the true minimum covering
    - where n is the number of literals in the goal
  - Loses guarantee of admissibility

Example: 8-Puzzle as Planning

- Planning as path search

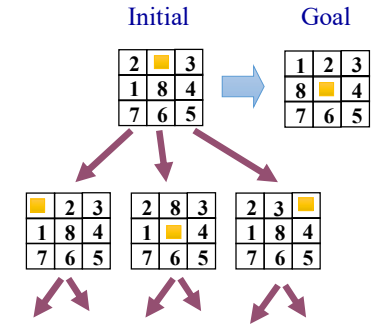


- Game objective:
  - To rearrange a given initial configuration (state) of eight numbered tiles arranged on a 3×3 board into given final or goal configuration (state)

$Action(Slide(t, s_1, s_2)$
Precond: $On(t, s_1) \wedge Tile(t) \wedge Blank(s_2) \wedge Adjacent(s_1, s_2)$
Effect: $On(t, s_2) \wedge Blank(s_1) \wedge \neg On(t, s_1) \wedge \neg Blank(s_2)$)

Example: 8-Puzzle as Planning

- States
  - A state description specifies the location of each of the eight tiles in one of the nine squares
- Actions or operators
  - Tile slides left, right, up, or down
- Goal test
  - State matches the goal configuration
- Path cost
  - Each step cost 1, so path cost is just the length of the path



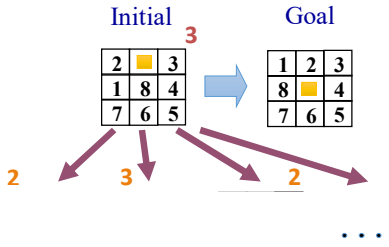
$Action(Slide(t, s_1, s_2)$
Precond: $On(t, s_1) \wedge Tile(t) \wedge Blank(s_2) \wedge Adjacent(s_1, s_2)$
Effect: $On(t, s_2) \wedge Blank(s_1) \wedge \neg On(t, s_1) \wedge \neg Blank(s_2)$)

Ignore Preconditions Heuristic

- Main idea:
  - All actions become applicable in all states
  - Any single goal fluent can be achieved in one step (if there is an applicable action)
  - No. of steps required to solve relaxed problem ≈ no. of unsatisfied goals
    1. Some actions may achieve multiple goals
    2. Some actions may undo the effects of others
  - Possible accurate heuristic – consider 1 and ignore 2

Ignore Selected Preconditions Heuristic

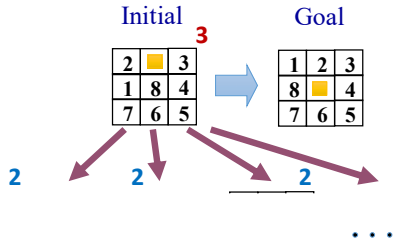
- Remove Preconditions
  - $Blank(s_2) \wedge Adjacent(s_1, s_2)$
- What does the heuristic do?
  - No. of misplaced tiles heuristic
- What are the estimates?



$Action(Slide(t, s_1, s_2)$
Precond: $On(t, s_1) \wedge Tile(t) \wedge Blank(s_2) \wedge Adjacent(s_1, s_2)$
Effect: $On(t, s_2) \wedge Blank(s_1) \wedge \neg On(t, s_1) \wedge \neg Blank(s_2))$

Ignore Selected Preconditions Heuristic

- Remove Preconditions
  - $Blank(s_2)$
- What does the heuristic do?
  - Manhattan-distance heuristic
- What are the estimates?



- Caveat:
  - Unclear which preconditions can be selectively ignored in general

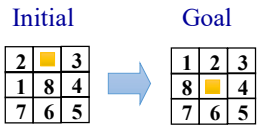
$Action(Slide(t, s_1, s_2)$
Precond: $On(t, s_1) \wedge Tile(t) \wedge Blank(s_2) \wedge Adjacent(s_1, s_2)$
Effect: $On(t, s_2) \wedge Blank(s_1) \wedge \neg On(t, s_1) \wedge \neg Blank(s_2))$

Ignore Delete List Heuristic

- Main idea:
  - Assume only positive literals in all plans and actions
  - Remove delete list from all actions (all negative literals from effects) to make monotonic progression toward goal
  - Create a relaxed version of original problem that is easier to solve, where solution length will serve as good heuristics
  - Approximate solution can be found in polynomial time by hill-climbing

Ignore Delete Lists Heuristic

- Ignore delete lists
  - How does the problem become easier when ignore delete lists heuristic is applied to this schema?
  - Goals are never undone
  - Blanks always increase



$Action(Slide(t, s_1, s_2)$
Precond: $On(t, s_1) \wedge Tile(t) \wedge Blank(s_2) \wedge Adjacent(s_1, s_2)$
Effect: $On(t, s_2) \wedge Blank(s_1) \wedge \neg On(t, s_1) \wedge \neg Blank(s_2)$

Domain-independent Pruning

- Symmetry reduction
  - Prune all symmetric branches of the search tree except for one
  - For many domains, efficiently solve intractable problems
- Forward pruning
  - Accept risk of pruning away an optimal solution, in order to focus search on promising branches
- Rule-out negative interactions
  - A problem has serializable subgoals if there exists an order of subgoals such that the planner can achieve them in that order without having to undo any of the previously achieved subgoals

How do you find preferred action(s)?

Serializable Subgoals

- Planning examples in the real world:
  - Build a tower of blocks on Table, in any order
  - Switch on all n lights with independent switches, in any order
  - Remote Agent Planner that commands NASA's Deep Space One spacecraft (1998) – serializable by design
    - able to command spacecraft in real-time

Heuristics of State Abstraction

- State abstraction:
  - Many-to-one mapping from states in the ground representation of the problem to the abstract representation
- Main approach:
  - Ignore some fluents
  - Solution in abstract state space will be shorter than a solution in the original space (assumes admissible heuristic)
  - Abstract solution extensible to solution for original problem

Example: Air Cargo Transportation

- Original problem:
  - 10 airports, 50 planes and 200 cargos
  - Total no. of states =
- Assumption:
  - All cargos are just in 5 airports, instead of 10, and all cargos in the same airport have the same destination.
- Reformulation:
  - Drop all the irrelevant At fluents (What are the relevant ones?)
  - Total no. of states =
- Solution:
  - Shorter than that for original problem (admissible heuristic)
  - Extensible by adding relevant actions

Decomposition

- Main idea:
  - Divide problem into parts, solving each part independently, and then combining the parts - Key idea in defining heuristics
- How to choose the right abstraction to reduce total cost?
  - Costs involved: Defining abstraction, doing abstract search, mapping abstraction back to original problem
  - Can the cost be less than original planning cost?
  - Example:
    - Pattern databases – cost of creation amortized over many problem instances

Planning Cost with Abstraction

- Problem definition:
  - Suppose the goal is a set of fluents G, divided into disjoint subsets $G_1, \ldots, G_n$.
  - Find plans $P_1, \ldots, P_n$ that solve the respective subgoals
  - What is the estimated cost of the plan for achieving all of G?
- Heuristic estimation:
  - Think of each $Cost(P_i)$ as a heuristic estimate
  - If each subproblem uses an admissible heuristic, taking Max is admissible
- Assuming subgoal independence:
  - Sum the cost of solving each independent subgoal; if admissible, Sum is better than Max – Why?
  - Solution optimistic when there are negative interactions between subplans for each subgoal; e.g., action in one subplan deletes goals in another subproblem.
  - Solution pessimistic (not admissible) when there are positive interactions; e.g., actions in one subplan achieves goals in another subproblem

Example: FF- FastForward

- Characteristics:
  - Forward state-space searcher making use of effective heuristics
  - Ignore-delete-list heuristic with graph plan for heuristic estimation
  - Hill-climbing search (modified to keep track of plan) with heuristic to find a solution
    - Non-standard hill-climbing algorithm: avoids local maxima by running a breadth-first search from the current state until a better one is found
    - If this fails, FF switches to greedy best-first search instead
  - [Hoffmann, 2001]

Classical Planning Today

Hint! Hint! Think about the project ☺

- Classical Planning research: Examples:
  - Families of systems in use – Heuristic Search Planner (HSP), Fast Forward (FF), Fast Downward
  - https://planning.wiki/ref/planners/
- International Planning Competitions
  - https://www.icaps-conference.org/competitions/
  - FastForward (FF) [Hoffmann, 2001] was the winner in the 2002 International Planning Competition (IPC)
  - SATPlan was the winner in the 2004 and 2006 IPC
  - IPC 2014 was won by SymBA*, based on bidirectional search using heuristics and abstraction.
  - IPC 2018 was won by Delfi, using deep learning to select a planner from a collection of planners including Fast Downward (uses forward search) with various heuristics and SymBA*
  - IPC 2020 – Hierarchical planning!
- Classical Planning in practice: Examples
  - Commercial video games [Neufeld, X., et al., 2019]
  - AI planning for enterprise [Sohrabi,S, 2019]
  - Space exploration [Estlin, T., et al. 07] [Rabideau, G., et al., 2020]

Hierarchical Planning and Acting

Manage complexity

Examples

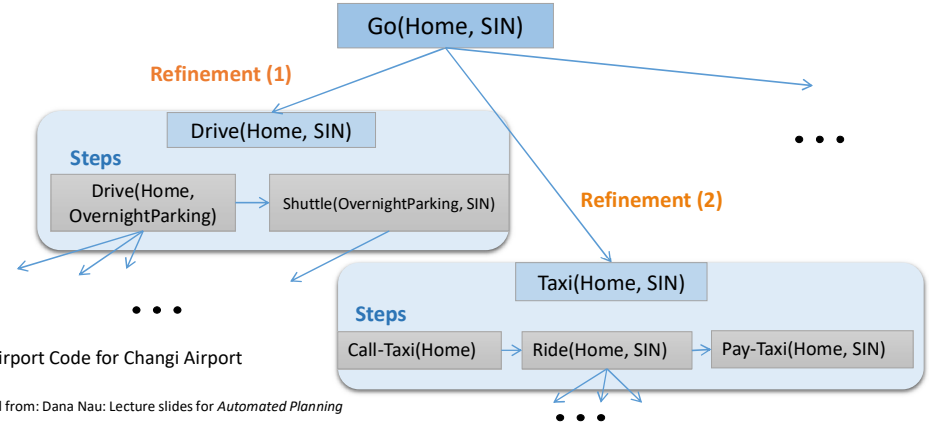
- Example 1:

  How to go to CSxx46 lecture from home?
  - Solution: Go to COM2 from home, find lecture hall
  - Solution: take MRT, change to internal bus, get off, find LT16.
  - Solution: Switch on laptop, launch Zoom, join lecture
- Example 2:

  How to land on the Jezero Crater of Mars from X space station?
- Example 3:

  How to get from home to Changi airport?

Example: Going to Changi Airport



SIN: Airport Code for Changi Airport

Adapted from: Dana Nau: Lecture slides for *Automated Planning*

Managing Complexity in Planning

- Hierarchical decomposition
  - Division of tasks into different subtasks at next level
  - At each level focus only a small number of tasks
- Deferred planning
  - Planning can occur before and during plan execution
  - Particular action can remain at an abstract level prior to the execution phase

Hierarchical Decomposition

- Key benefits
  - At each level of hierarchy, a task is reduced to a small number of subtasks or activities at the next lower level
  - Computational cost of finding the correct way to arrange activities for current problem is small
- Examples
  - Software components, subroutines
  - Military, government, and corporations

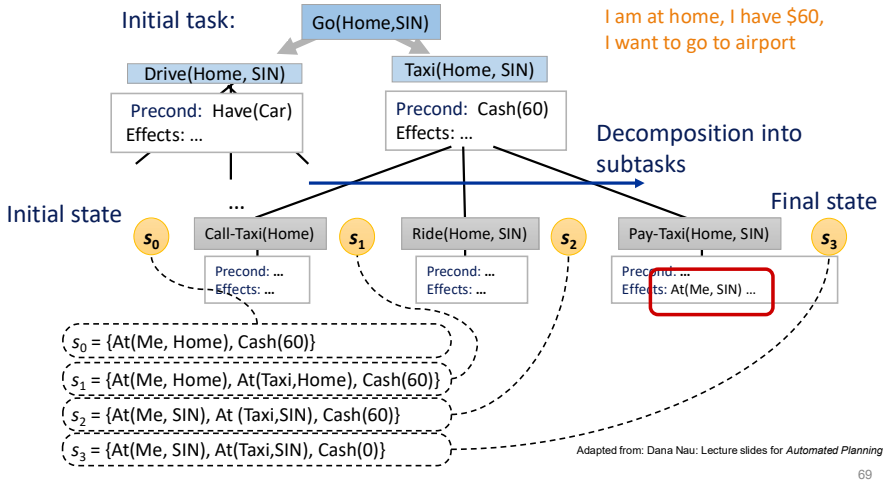
Hierarchical Task Networks

- Hierarchical task networks (HTNs)
  - A formalism to help understand hierarchical decomposition
  - A planning model that manages complexity through task abstractions
- Key concept
  - High-level actions (HLAs)
- Assumptions
  - Full observability
  - Deterministic
  - Availability of primitive actions with standard precondition-effect schemas
  - Main ideas are general in problem solving and planning and decision making

HTN Planning

- Planning Problem or Model
  - HLAs, action schemas, initial state, goal state, task list
- Planning Algorithm
  - Input: a problem
  - Output: a solution in the form of an action sequence
- Planning Solution
  - Any executable plan generated by recursively applying:
    - HLA to nonprimitive tasks
    - Actions to primitive tasks
  - A goal state that satisfies certain properties

Example: Going to Changi Airport



Adapted from: Dana Nau: Lecture slides for *Automated Planning*

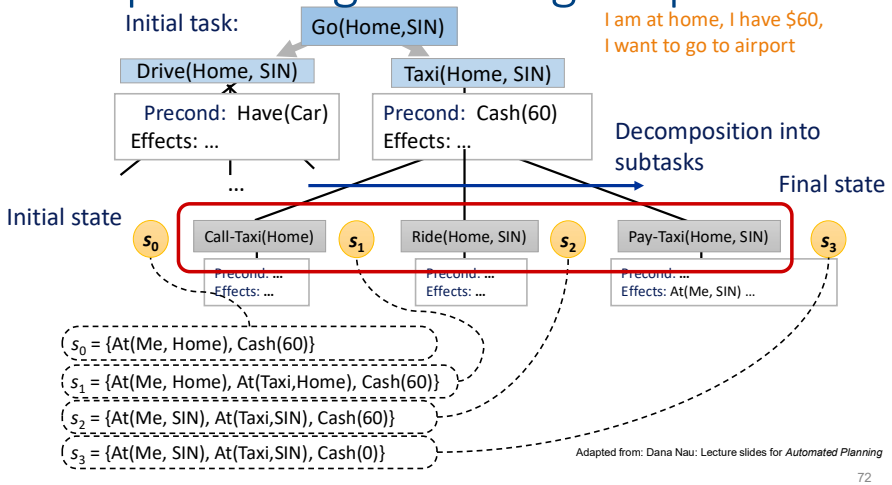
High-Level Actions (HLAs)

- Definition
  - Each HLA has one or more refinements into a sequence of actions
  - Each (refined) action can be an HLA or a primitive action
  - Recursive refinement may be needed
- Meaning
  - HLAs and their refinements embody knowledge about how to do things
    e.g., Go(Home, SIN) – drive or take a taxi

Implementation

- HLA implementation
  - An HLA refinement that contains only primitive actions
- High-level plan implementation
  - High-level plan – a sequence of HLAs
  - Concatenate implementations of each HLA in the sequence
- Observation
  - Given the precondition-effect definitions of each primitive action, can directly determine whether any given implementation of a high-level plan achieves the goal.

Example: Going to Changi Airport



Adapted from: Dana Nau: Lecture slides for *Automated Planning*

Planning with HLAs

- Definition
  - Achieves the goal from a given state if at least one of its implementations achieves the goal from that state
- Note
  - Not all implementations need to achieve the goal
  - The agent decides which implementation to execute
- Question:
  - How is this different from nondeterministic planning?

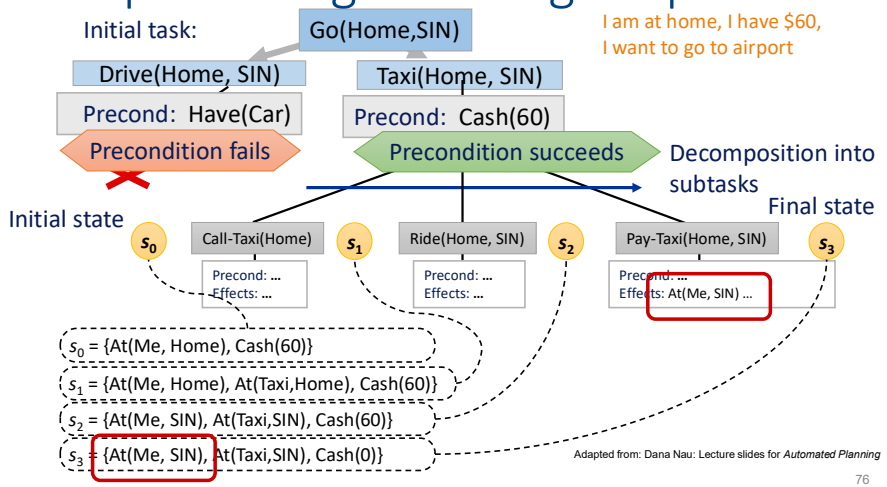
Planning with HLAs

- With one HLA implementation
  - Compute preconditions and effects of HLA from those of the implementation
  - Treat HLA exactly as if it were a primitive action
- Observation
  - Right collection of HLAs can reduce time complexity of (blind) search from exponential to linear in solution depth
  - Devising an appropriate collection of HLAs is HARD!
- With multiple HLA implementations
  - Search among implementations for one that works; OR
  - Reason directly about the HLAs - enables derivation of provably correct abstract plans, without having to consider their implementations

Hierarchical Planning as Search

Searching for Primitive Solutions

Example: Going to Changi Airport



Adapted from: Dana Nau: Lecture slides for *Automated Planning*

Searching for Primitive Solutions

- HTN Planning
  - Start with top level action Act
  - Find an implementation of Act that achieves the goal
- Hierarchical planning algorithm
  - Repeatedly choose an HLA in current plan and replace with refinement
  - Until the plan achieves the goal
- Example:
  - Breadth-first search tree
  - Plans are considered in order of depth of nesting of the refinements, rather than number of primitive steps
  - Can use graph-search, depth-first, and iterative deepening

Generic Planning Framework

- Classical planning definition:
  - For each primitive action a_i:
  - Provide one refinement of Act with steps – $[ai, Act]$
  - Create recursive definition of Act to add actions
  - Final refinement:
    - steps – empty, precondition – goal, effect – null
- Algorithm:
  - Repeatedly choose an HLA in the current plan
  - Replace it with one of its refinements
  - Until the plan achieves the goal

Hierarchical Search

function HIERARCHICAL-SEARCH(*problem*, *hierarchy*) **returns** a solution or *failure*

```
frontier ← a FIFO queue with [Act] as the only element
while true do
    if IS-EMPTY(frontier) then return failure
    plan ← POP(frontier)          // chooses the shallowest plan in frontier
    hla ← the first HLA in plan, or null if none
    prefix,suffix ← the action subsequences before and after hla in plan
    outcome ← RESULT(problem.INITIAL, prefix)
    if hla is null then           // so plan is primitive and outcome is its result
        if problem.IS-GOAL(outcome) then return plan
    else for each sequence in REFINEMENTS(hla, outcome, hierarchy) do
        add APPEND(prefix, sequence, suffix) to frontier
```

Source: RN Figure 11.8

Hierarchical Search

- Main idea:
  - Explore space of sequences that conform to knowledge in the HLA library about how things are to be done
  - Knowledge encoded in action sequences in each refinement and in the preconditions of the refinements
- Practical impact:
  - Can generate huge plans with little search
    - e.g., O-PLAN to develop production plans for HITACHI (Bell and Tate 1995)
  - Hierarchically structured – easier for human to understand
  - Find out more about recent applications in use!

Complexity Analysis

- Assumption
  - A planning problem has a solution with d primitive actions.
- For non-hierarchical, forward state-space planner
  - With b allowable actions at each state, cost is $O(b^d)$
- For HTN planner
  - Suppose each nonprimitive action has r possible refinements, each into k actions at the next lower level
  - So $r^{(d-1)(k-1)}$ possible regular decomposition trees could be constructed (see details in RN 11.4.2)
- Observation
  - Small r and large k - library of HLAs with small number of refinements each yielding a long action sequence - May be hard to construct!

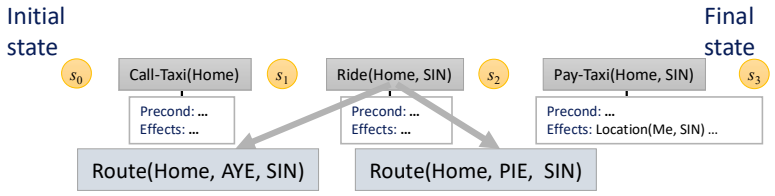
Proving Plan Properties

Searching for Abstract Solutions

Motivation

- Example:
  - We should determine if a high-level plan can get one to the airport, without going through all the specific details like precise route or alighting terminal

  [Call-Taxi(Home), Ride(Home, SIN), Pay-Taxi(SIN)]



Searching for Abstract Solutions

- Approach
  - Write precondition-effect description of the HLAs
  - Prove that the high-level plan achieves the goal
  - Work in small search space of high-level actions
  - Refine committed plan to achieve exponential reduction

Searching for Abstract Solutions

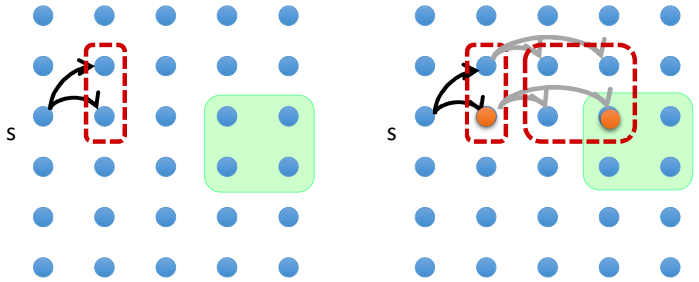
- Downward refinement property (of HLA descriptions)
  - Through description of the steps:
  - Every high-level plan that "claims" to achieve the goal achieves the goal
  - At least one implementation achieves the goal
- Main challenges
  - How to write HLAs with downward refinement property?
  - How to write HLAs with multiple implementations?
  - How to describe effects of an action that can be implemented in many different ways?
- Key idea
  - Determine if reachable sets of a sequence of HLAs in the plan overlap with goals

Reachable Set

- Reachable set of an HLA
  - Given a state s and an HLA h: $REACH(s, h)$ is the set of states reachable by any of the HLA's implementations
- Reachable set of a sequence of HLAs
  - Reachable set of a sequence of HLAs $[h_1, h_2]$ is the union of all the reachable sets obtained by applying h_2 in each state in the reachable set of h_1:

$$REACH(s, [h_1, h_2]) = \cup_{s' \in REACH(s, h_1)} REACH\ (s', h_2)$$

Example



$REACH(s, [h_1])$ $REACH(s, [h_1, h_2])$

Source: RN: Fig 11.6

High-Level Planning

- Practical implications
  - Agent can choose element of the reachable set it ends up in when it executes the HLA
  - HLA with multiple refinements is more "powerful" than the same with fewer refinements
- High-level plan
  - A sequence of HLAs
  - Achieves goal if its reachable set intersects set of goal states
  - Otherwise, the plan does not work
- Search algorithm
  - Search among high-level plans
  - Look for one whose reachable set intersects goal
  - Once that happens, commit to that abstract plan
  - Focus on refining the plan further

Representing HLA Effects

- Effects as reachable sets
  - As reachable set for each possible initial state
  - Represent changes made to each fluent or state variable
- Recall: Primitive action
  - Can add or delete a fluent or variable or leave it unchanged
- HLA
  - Can also control variable value, depending on implementation chosen
  - Description derivable, in principle, from descriptions of its refinements, such that the downward refinement property holds

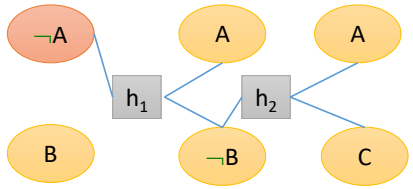
Representing Reachable Set

- Notations:
  - $\sim$ means possibly, if the agent chooses
  - E.g., $\tilde{+}A$ means "possibly add A", i.e., either leave A unchanged or make it True
- Questions:
  - What do $\tilde{=}A$ and $\tilde{\pm}A$ mean?
- Example

  $Go(Home, SIN)$ with two refinements
  - $Drive(Home, SIN)$ and $Taxi(Home, SIN)$
  - Possibly delete $Cash$ (if agent decides to take a taxi)
  - So should have effect $\tilde{=}Cash$

Example

- Consider:
  - Schemas for HLAs h_1 and h_2:
  - Action (h_1, Precond: $\neg$A, Effect: A $\wedge \tilde{=}$ B)
  - Action (h_2, Precond: $\neg$B, Effect: $\tilde{+}$A $\wedge \tilde{\pm}$C)
- Meaning:
  - h_1 adds A and possibly delete B
  - h_2 possibly adds A and has full control over C
- Exercise:
  - If only B is true in the initial state and goal is A $\wedge$ C
  - Q: What sequence of HLAs will achieve the goal?



NUS | Computing

Rational Decision Making

CS4246/CS5446

AI Planning and Decision Making

This lecture will be recorded!

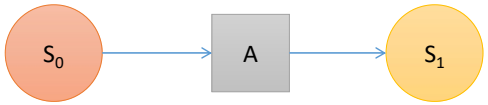
Please join:
pollev.com/anarayan

Nondeterminism

Partially observable, nondeterministic, and uncertain environments

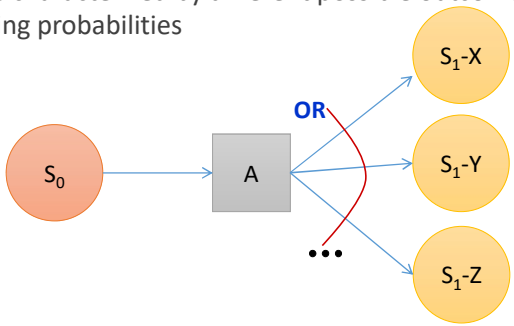
Definition

- Deterministic environment
  - Next state of the environment is completely determined by current state and action executed



Definition

- Non-deterministic environment
  - Actions are characterized by different possible outcomes, without any attaching probabilities



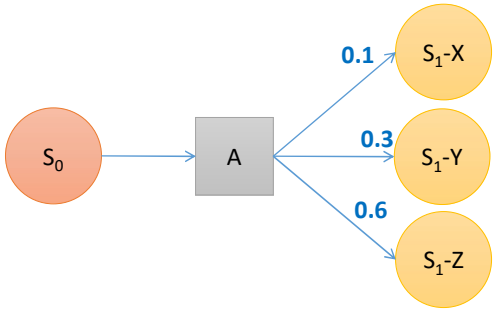
Non-Determinism

- Non-deterministic effects
  - Action may have different possible outcomes
- Controllable (angelic) nondeterminism
  - When an agent itself makes the choices
  - E.g., in hierarchical planning – still deterministic!
- Uncontrollable (demonic) nondeterminism
  - When an adversary or nature makes the choices
  - Planning in nondeterministic or uncertain environments

RN: 11.4.3

Definition

- Stochastic or uncertain environment
  - Outcomes not fully observable or not deterministic
  - Uncertainty usually characterized in terms of probabilities



Some Terminology

- State variables (or fluents)
  - Factored representations of states whose truth values may change over time
- Belief states
  - Sets of physical states the agent might be in
  - Represent agent's current belief about the possible physical states it might be in
  - Entire belief-state space contains every possible set of physical states.
  - If problem P has N fluents, then there are up to 2^N states in the belief space
- Percepts
  - Percepts supplied by sensors in acting, but a model of sensors is needed in planning
  - Need to reason about percepts obtained when plan is executed in partially observable environment

Example:
What are the belief states for binary fluents:
Have_Cake ∧ Eaten_Cake?

Decision Making under Uncertainty

Uncertain states and action effects

Rational Decision Making

- Main ideas:
  - Assume episodic, non-deterministic, partially observable environments
  - Make decisions based on combining beliefs and desires
  - Choose a strategy for experimentation and action that is logically consistent (cf. right) with:
    - basic judgments about the unknown states or events
    - basic preferences for consequences
- Rationality
  - Assume limited resources
  - Make decisions that will maximize profit/gain (or minimize cost/pain)
  - Rationality as a product (ends) or a process (means)

Types of Decision Theory

- Normative decision theory
  - Describes how ideal, rational agents should behave
- Descriptive decision theory
  - Describes how actual agents (humans) really behave
- Prescriptive decision theory
  - Prescribes guidelines for agents to behave rationally

Decision Theoretic Agents

- Recall: Goal-based agent
  - Has binary distinction between good (goal) and bad (non-goal) states
- Decision-theoretic agent
  - Based on combining probability theory and utility theory
  - Makes rational decisions based on beliefs and desires
  - Operates in contexts with uncertainty and possibly conflicting goals
  - Has continuous measure of outcome quality
- (Normative) Decision Theory
  - Choosing among actions based on desirability of immediate outcomes
  - Assuming episodic, non-deterministic, partially observable environments

Solving Decision Problems

- Decision (Planning) Problem or Model
  - Appropriate abstraction of states, actions, uncertain effects, and goals (wrt costs and values or preferences)
- Decision Algorithm
  - Input: a problem
  - Output: a solution in the form of an optimal action sequence
    - Optimal action at each decision or choice point
- Decision Solution
  - An action sequence or solution from an initial state to the goal state(s)
    - An optimal solution or action sequence; OR
    - An optimal policy that specifies "best" action in each state wrt to costs or values or preferences
  - (Optional) A goal state that satisfies certain properties

Decision Making under Uncertainty

- Decision (Planning) Model:
  - Actions: $a \in A$
  - Uncertain current state: $s \in S$ with probability of reaching: $P(s)$
  - Transition model of uncertain action outcome or effects:
    $P(s'|s,a)$ – probability that action a in state s reaches state s'
  - Outcome of applying action a:
    Result(a) – random variable whose values are outcome states
  - Probability of outcome state s', conditioning on that action a is executed:
    $P(\text{Result}(a) = s') = \sum_s P(s)P(s'|s,a)$
  - Preferences captured by a utility function:
    $U(s)$ – assigns a single number to express the desirability of a state s

Axioms of Utility

Constraints on rational preferences

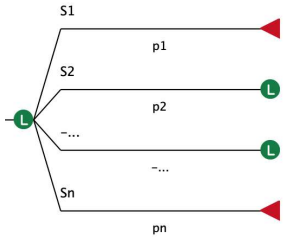
Modeling Preferences

- Preference
  - For any two acts or events A and B, $A \succ B$ (A is preferred to B) or $A \sim B$ (A and B are indifferent)
  - Agent is indifferent between two alternatives only if agent:
    - has considered both alternatives; and
    - is completely willing to trade one for the other
- Utility
  - Classify every item in an individual's preference ordering
    - Applying the conditions of rationality
  - Numeric ranking shows relative importance of an indifference class
    - Relative importance or utility of items in the class
- Utility function or Utility Scale $U(s)$:
  - Assigns a single number to express desirability of a state
  - Numbering items in a preference ordering

Rational Preferences

- Assumptions:
  - $A \succ B$: Agent prefers A to B
  - $A \sim B$: Agent is indifferent between A and B
  - $A \succcurlyeq B$: Agent prefers A over B or is indifferent between them
- Lottery:
  - Action as ticket to lottery L:
    - with set of outcomes S_1,..., S_n
    - with probabilities p_1,..., p_n

    $L = [p_1, S_1; p_2, S_2, \ldots, p_n, S_n]$
  - Each outcome S_i of a lottery can be either an atomic state or another lottery



Axioms of Utility Theory

- Utility Theory:
  - Understand how preferences between complex lotteries are related to preferences between the underlying states in the lotteries
- 6 axioms of utility
  - Constraints on preferences
  - An agent that violates any axiom will exhibit irrational behavior in some situations

6 Axioms of Utility

- A1: Orderability
- A2: Transitivity
- A3: Continuity
- A4: Substitutability
- A5: Monotonicity
- A6: Decomposibility

A1: Orderability

- Definition
  - Given any two lotteries, a rational agent must either:
    - prefer one to the other; OR
    - Rate the two as equally preferable
  - Exactly one of $(A \succ B)$, $(B \succ A)$, or $(A \sim B)$ holds
- Example:
  - A student will prefer
    - taking CS4246 to CS5340
    - taking CS5340 to CS4246
  - Or is indifferent between the two modules

A2: Transitivity

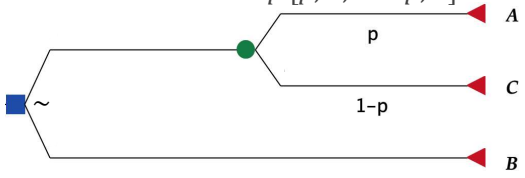
- Definition
  - Given any three lotteries, if an agent prefers A to B and prefers B to C, then the agent must prefer A to C

$$(A \succ B) \wedge (B \succ C) \implies (A \succ C)$$

A3: Continuity

- Definition
  - If some lottery B is between A and C in preference, then there is some probability p for which the rational agent will be indifferent between:
    - Betting B for sure; AND
    - The lottery that yields A with probability p and C with probability $1 - p$

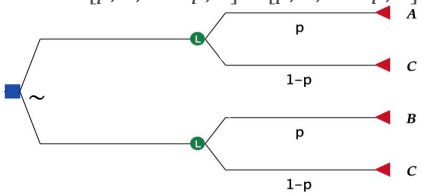
$$A \succ B \succ C \implies \exists p\,[p, A;\ 1 - p, C] \sim B$$



A4: Substitutability

- Definition
  - If an agent is indifferent between two lotteries A and B, then the agent is indifferent between two more complex lotteries that are the same except that B is substituted for A in one of them
  - This holds regardless of probabilities and other outcome(s) in lotteries
  - This also holds if $\succ$ is substituted for $\sim$ in the axiom

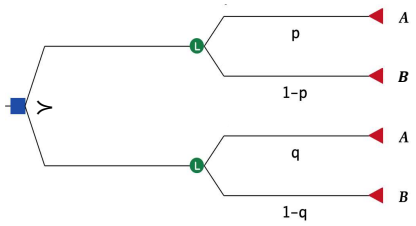
$$A \sim B \implies [p, A;\ 1-p, C] \sim [p, B;\ 1-p, C]$$



A5: Monotonicity

- Definition
  - Suppose two lotteries have the same two possible outcomes, A and B.
  - If an agent prefers A to B, then the agent must prefer the lottery that has a higher probability for A (and vice versa)

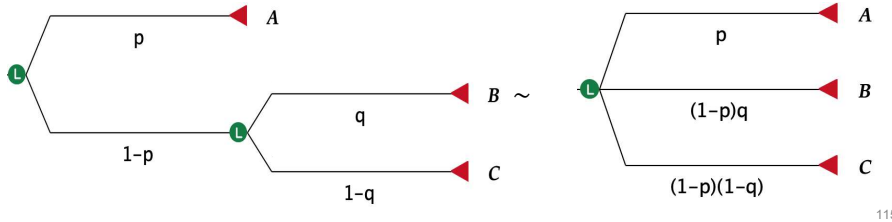
$$A \succ B \implies (p > q \iff [p, A;\ 1-p, B] \succ [q, A;\ 1-q, B])$$



A6: Decomposibility

- Definition
  - Compound lotteries can be reduced to simpler ones
  - No fun in gambling rule – two consecutive lotteries can be compressed into a single equivalent lottery

$$[p, A;\ 1-p, [q, B;\ 1-q, C]] \sim [p, A;\ (1-p)q, B;\ (1-p)(1-q), C]$$

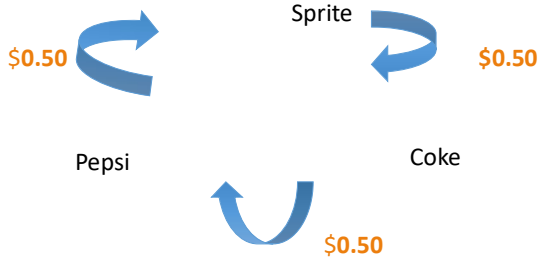


Exploiting Irrationality

- E.g., Coke≻Sprite≻Pepsi≻Coke

Non-transitive preferences

  - How to make money from someone without the above preference structure?



Rational Preferences and Utility

- Note:
  - Axioms of utility are really axioms about preferences
  - Nothing is mentioned about a utility function
  - Need to derive consequences from axioms of utility
- How should a rational agent behave?
  - If an individual can satisfy the conditions of rationality, then, a utility function U can be constructed according to:
    - Von Neumann and Morgenstern (1944) Expected Utility Theorem

Expected Utility Theorem

Von Neumann and Morgenstern (1944)

Existence of Utility Function

- If an agent's preferences obey the axioms of utility; then there exists a function U such that:
  - $U(A) > U(B)$ if and only if A is preferred to B, and
  - $U(A) = U(B)$ if and only if the agent is indifferent between A and B.

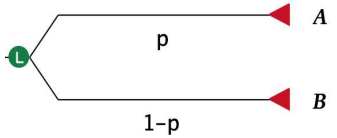
$$U(A) > U(B) \Leftrightarrow A \succ B \qquad \text{and} \qquad U(A) = U(B) \Leftrightarrow A \sim B$$

Expected Utility of A Lottery

- The utility of a lottery is the sum of the probability of each outcome times the utility of that outcome:

$$U([p_i, S_i; \ldots; p_n, S_n]) = \sum_i p_i U(S_i)$$

  - Once the probabilities and utilities of the possible outcome states are specified, the utility of a compound lottery involving those states is completely determined.



$$U(L) = pU(A) + (1 - p)U(B)$$

Computing Expected Utility

- Expected utility
  - Note that outcome of a nondeterministic action is a lottery
  - Expected utility of an action a is the average utility value of the outcomes, weighted by the probability that the outcome occurs

$$EU(a) = \sum_{s'} P(\text{Result}(a) = s')U(s') = \sum_{s'} \sum_{s} P(s)P(s'|s,a)U(s')$$

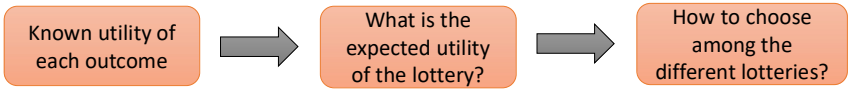
- Non-unique utility functions
  - Agent's behavior doesn't change if U is subjected to an affine transformation:

$$U'(s) = aU(s) + b \text{ with } a > 0$$

Maximum Expected Utility Principle

Basic principle of decision theory

Rational Decision Making



- An agent can act rationally – consistently with its preferences – only by choosing an action that maximizes expected utility according to:
- Maximum expected utility (MEU) principle
  - A rational agent should choose the action that maximizes its expected utility

$$action = \underset{a}{\text{argmax}}\, EU(a)$$

  - A prescription for intelligence behavior – "do the right thing" – a basis for AI

Rational Decision-Theoretic Agent

- Preference assumption
  - Assumes that agent's preferences satisfy all the axioms or conditions of rationality
- Rationality assumption
  - Assumes that agent always act rationally by choosing the most preferred option
- Ideal agent assumption
  - Agents of the theorem are ideal and hypothetical beings, but can be used as guides for our own problem solving or decision making
- Other assumptions:
  - Existence of a utility function that describes an agent's preference behavior does not necessarily mean that agent is explicitly maximizing that utility function itself
  - An observer can learn about the utility function that represents what the agent is actually trying to achieve through the agent's behavior

MEU as Performance Measures

- Given a set of environments, agents, and relevant perception histories:
  - If an agent acts so as to maximize a utility function that correctly reflects the performance measure, then the agent will achieve the highest possible performance score (averaged over all the possible environments).
- Main ideas:
  - Transition from external performance measure to internal utility function
  - Performance measure:
    - Gives a score for a history—a sequence of states
    - Applied retrospectively after an agent completes a sequence of actions
    - Utility function can be used to guide actions step by step

Computing MEU

- What are the major challenges?
  - Considering many actions a
  - Estimating $P(s)$ over possible states of the world in $P(\text{RESULT}(a) = s')$ requires perception, learning, knowledge representation, and inference
  - Computing $P(\text{RESULT}(a) = s')$ requires a causal model of the world
  - Computing outcome utilities $U(s')$ requires further searching or planning
  - Estimating uncertainty about U
- In summary, decision theory:
  - Provides a basic, general mathematical framework to define the AI problem
  - Cannot solve the AI problem!

Utility Functions

Basic Concepts and Assessment Methods

Encoding Preferences in Utility Functions

- Assess utility or value functions
  - To measure “desirability” of different outcomes and trade-off situations
- Possible scales:
  - Monetary cost
  - Revenue, profit
  - Life expectancy
  - Jobs saved
  - etc.

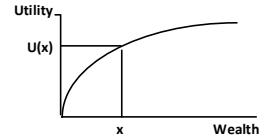
Measuring Utilities

- Provide summary scores that
  - aggregate different aspects of "goodness" measures
  - incorporate attitudes towards risks and "goodness" quantities
- Assessment methods
  - By direct assessment using:
    - probability equivalent
    - certainty equivalent
  - By consensus
  - From published reports
- Multi-attribute utility functions may be used

Utility Functions

- A utility function, U, represents a way to translate dollars or other "desirability" measures, x, into utility units $U(x)$.

Graphical Representation



Tabular Representation

Wealth (x)	Utility Value U(x)
2500	1.50
1500	1.24
1000	0.93
600	0.65
400	0.47
0	0.15

Mathematical Representation

$U(x) = \log(x)$

$U(x) = 1 - e^{-x/R}$

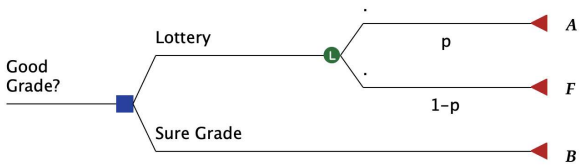
$U(x) = x^{0.5}$

Preference Elicitation: Method 1

- Using probability equivalents
  - Present choices to agents; define the utility function from observed responses
- Method:
  - Fix a scale; fix utilities of 2 outcomes to establish the scale
  - Worst: $u_\perp$; Best: $u_\top$
    - Normalized utilities: $u_\perp = 0; u_\top = 1$
  - Assess utility for prize S by asking the agent to choose between $U(S)$ and the lottery $[p, u_\top; (1-p), u_\perp]$
    - Adjust p until the agent is indifferent between S and lottery

Example: Getting Good Grade

- Demonstration:
  - Grade in CS4246/CS5446: $U(F) = 0; U(A) = 1$; sure grade: B, C, D
  - What value of p would you trade A for B?



- What about C, D?

Example: Pizza Party

- Demonstration:
  - Alice's preference: $Pasta \succ Pizza \succ Salad$
  - Bob's preference: $Salad \succ Pasta \succ Pizza$
  - Charlie's preference: $Pizza \succ Pasta \succ Salad$
- What is the group's preference?

UNSURE! Compromise and other considerations are needed

- So?
  - Group preference may be non-transitive
  - Studied in social choice theory

Certainty Equivalent

- Certainty Equivalent (CE)
  - Fixed amount of money equivalent to a given situation involving uncertainty
  - Value agent accepts in lieu of the lottery
- Example:
  - You are faced with the following lottery:
    - Win $2000 with probability 0.5
    - Lose $20 with probability 0.5

    EMV = $990
  - How much are you willing to sell the lottery for?
  - This amount, say $\$X$, is the least that you would accept for the lottery, then the lottery must be equivalent in your mind to a sure $\$X$

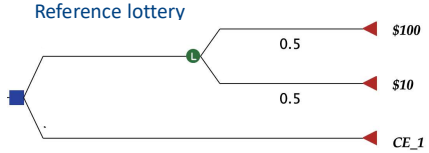
Preference Elicitation: Method 2

- Using certainty equivalents

1. Determine two initial points on curve
2. Arbitrarily assign utility values to the points
3. Create reference lottery to determine CE
4. Find third point on curve by formula:
   - $U(CE_1) = EU(Lottery)$
5. Create additional reference lotteries
6. Proceed as before until enough points are available to plot a curve

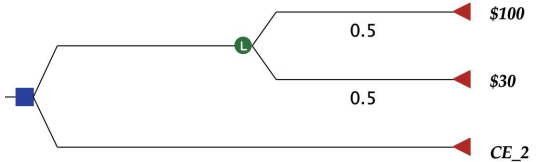
- Example:

1. Assume: worst: \$10, best: \$100
2. Set $U_\perp(10) = 0; U_\top(100) = 1$
3. Reference lottery



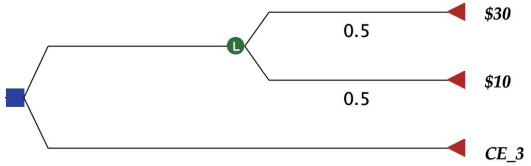
4. Assume: $CE_1 = 30$
5. $U(CE_1) = U(30)$
   $= 0.5U(100) + 0.5U(10)$
   $= 0.5$

Continue ...



CE_2

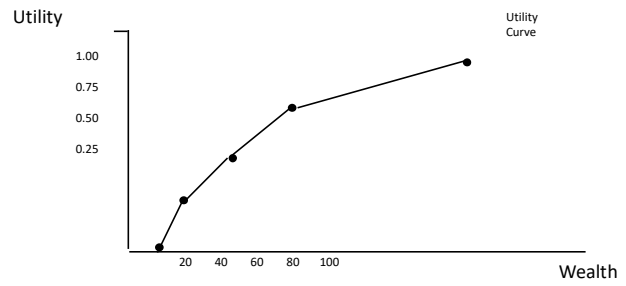
$$U(50) = 0.5U(100) + 0.5U(30) = 0.75$$



CE_3

$$U(18) = 0.5U(30) + 0.5U(10) = 0.25$$

Plotting the Utility Function



Example: Value of Statistical Life

- Value of statistical life:
  - How to put a value on human life?
  - Used by agencies of U.S. government, including the Environmental Protection Agency, the Food and Drug Administration, and the Department of Transportation
  - To determine costs and benefits of regulations and interventions
  - How much would it be?
    - Typical value in 2020 (in US) ~ $7.5 million.

Example: Micromort

value that people place on their own lives

- Micromort:
  - A one in a million chance of death, often used as a unit of risk
  - People appear to be willing to pay about $50 per micromort, e.g., $10,000 for a safer car that halves the risk of death (from 400 to 200 micromort).
  - Only for small risks, most people won't kill themselves for $50 million.
  - See video: https://www.youtube.com/watch?v=G3wolqD-acQ

Example: Quality Adjusted Life Year

- QALY:
  - A scale commonly used in health care literature
  - One year in perfect health is 1 QALY
  - One year bedridden would be less preferred, e.g. 0.5 QALY.
  - Death is 0 QALY
  - See video: https://www.youtube.com/watch?v=3tDXwKVkn68
  - Whose utility values are these?

Utility of Money and Risk Attitudes

Expected Monetary Value

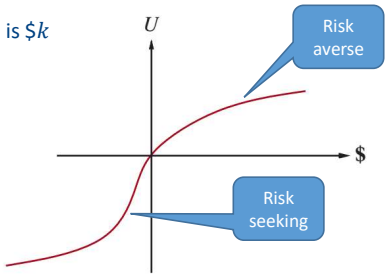
- Assumption:
  - Agent prefers more money than less – monotonic preference

Are people rational?

- Example:
  - You have won $1 mil so far
  - Now, flip a coin – Lose all the money ($0) if Head and gain $2.5mil if Tail
  - Will you play?
- What is the expected monetary value (EMV) of the lottery?
  - $EMV = [0.5, 0; 0.5, 2.5\ mil] = \$1.25\ mil$
- Is accepting the game the best decision?
  - Yes according to MEU Principle – EMV is higher than $1 mil won so far!

Utility of Money

- Let S_n denote the state of having $\$n$; current wealth is $\$k$
  - $EU(Accept) = \frac{1}{2}U(S_k) + \frac{1}{2}U(S_{k+2.5mil})$
  - $EU(Decline) = U(S_{k+1mil})$
- To decide:
  - Assign utility values to the outcome states: $S_k, S_{k+2.5mil}, S_{k+1mil}$
  - Will you accept the lottery if: $S_k = 5, S_{k+2.5mil} = 9, S_{k+1mi} = 8$?
  - $EU(Accept) = 0.5(5) + 0.5(9) = 7$
  - $EU(Decline) = 8$
- Note:
  - Utility of money is usually proportional to log of the amount



A typical utility of money function

Source: RN16.2 b

Example: A Choice of Games

- Consider the following two games: EMV

Game 1	Win $30 with prob 0.5
	Lose $1 with prob 0.5
Game 2	Win $2000 with prob 0.5
	Lose $1900 with prob 0.5

- Which game would you play? Why?
- What if you are to play a game 10 times?

Limitations of the EMV criterion

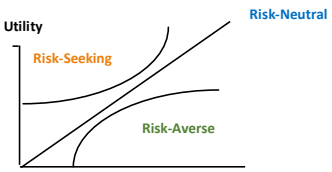
- Using money as decision objective:
  - Intuitive only if objective can be measured in terms of monetary value
  - Considers only the average or expected value; ignores the range of possible values
  - Does not take into account risk attitudes

Risk Attitudes

- A patient is forced to play the following game:
  - Live for another 30 years with probability 0.5
  - Die immediately with probability 0.5
  - Will he choose to live just for 5 years to get out of the game?
- Risk behaviors:
  - If he would trade a lottery for a sure amount that is less than the expected value, he is risk-averse
  - If he would pay an amount more than the expected value to enter the lottery, then he is risk-seeking
  - Otherwise, he is risk-neutral

Risk or Insurance Premium

- Risk attitudes and utility functions

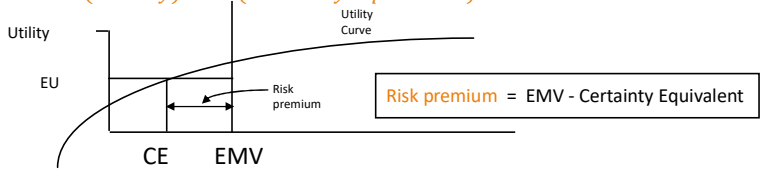


- Risk Premium
  - Risk premium = EMV - Certainty Equivalent
  - Premium paid, in the sense of a lost opportunity, to avoid the risk
- Risk premium and behavior
  - For risk-averse individual, risk premium is positive
  - For risk-seeking individual, risk premium is negative
  - For risk-neutral individual, risk premium is zero

Expected Utility and Certainty Equivalent

- Expected utility of a lottery is equal to the utility of its certainty equivalent:

$$EU(Lottery) = U(Certainty\ Equivalent)$$



- Note
  - If two alternatives have the same CE, then they must have the same EU; the decision maker would be indifferent to a choice between the two
  - Ranking alternatives by their CEs is the same as ranking them by their expected utilities

Example: Paying to Avoid Risk

- Recall the following gamble:
  - Win $2000 with probability 0.5
  - Lose $20 with probability 0.5
- What is the EMV?
  - EMV = $990
- How much will you be willing to accept to avoid the lottery?
  - Let CE = $300
- What is the risk premium?
  - Risk premium = $990 − $300 = $690
- What does this mean?
  - Trading lottery for $300: willing to give up $690 in expected value to avoid risk

Calculating Risk Premium

- Recall:
  - Risk premium = EMV - Certainty Equivalent
  - EU(Lottery) = U(Certainty Equivalent)
- How to calculate risk premium given a lottery and a utility function?
  1) Find EU for the Lottery
  2) Find CE, amount with utility value given by EU
  3) Calculate EMV for the lottery
  4) Risk premium = EMV – CE
- Other methods: Utility functions for non-monetary attributes
  - Utility functions derived for attributes other than money
  - Set up arbitrary scale with extremes (highest and lowest utilities)
  - Use either CE or PE assessment methods

Some Caveats

- Utilities do not add up
  - $U(a+b) \neq U(a) + U(b)$
  - Utility functions are non-linear
  - Calculate net payoffs at end-points of decision tree before transforming to utility values
- Utility differences do not express strengths of preferences
  - Utility provides numerical scale for ordering preferences, not a measure of their strengths
- Utility function
  - A subjective personal statement of an individual's preferences
  - Provides no basis for comparing utilities among individuals
- Certainty equivalent vs expected utility
  - Certainty equivalent is measured in $ (or any basic unit)
  - Expected utility is measured on the utility scale

Other Real-World Challenges

- What about:
  - Multi-attribute utility functions?
  - Unknown preferences?
    - Uncertainties about agent's own preferences
    - Uncertainties about human agent's preferences
- Areas of active research and innovation

NUS | Computing

Decision Analysis

CS4246/CS5446

AI Planning and Decision Making

This lecture will be recorded!

Please join:
pollev.com/anarayan

Types of Decision Theory

- Normative decision theory
  - Describes how ideal, rational agents should behave
- Descriptive decision theory
  - Describes how actual agents (humans) really behave
- Prescriptive decision theory
  - Prescribes guidelines for agents to behave rationally

Recall: Solving Decision Problems

- Decision Problem or Model
  - Appropriate abstraction of states, actions, uncertain effects, and goals (wrt costs and values or preferences)
- Decision Algorithm
  - Input: a problem
  - Output: a solution in the form of an action sequence
    - Optimal action at each decision or choice point
- Decision Solution
  - An action sequence or solution from an initial state to the goal state(s)
    - An optional solution or action sequence; OR
    - An optimal policy that specifies "best" action in each state wrt to costs or values or preferences
  - (Optional) A goal state that satisfies certain properties

Recall: Decision Making under Uncertainty

- Decision Model:
  - Actions: $a \in A$
  - Uncertain current state: $s \in S$ with probability of reaching: $P(s)$
  - Transition model of uncertain action outcome or effects:
    $P(s'|s,a)$ – probability that action a in state s reaches state s'
  - Outcome of applying action a:
    Result(a) – random variable whose values are outcome states
  - Probability of outcome state s', conditioning on that action a is executed:
    $P(\text{Result}(a) = s') = \sum_s P(s)P(s'|s,a)$
  - Preferences captured by a utility function:
    $U(s)$ – assigns a single number to express the desirability of a state s

Recall: Fundamentals of Decision Theory

- Decision theory
  - Choosing among actions based on desirability of outcomes
- In non-deterministic, partially observable, episodic environments:
  - An agent can act rationally – consistently with its preferences – only by choosing an action that maximizes expected utility according to MEU principle:
  - A rational agent should choose the action that maximizes its expected utility

$$action = \underset{a}{\operatorname{argmax}}\, EU(a)$$

  - Expected utility of an action a is the average utility value of the outcomes, weighted by the probability that the outcome occurs

$$EU(a) = \sum_{s'} P(\text{Result}(a) = s')U(s') = \sum_{s'}\sum_{s} P(s)P(s'|s,a)U(s')$$

Decision Analysis

A prescriptive framework for decision making

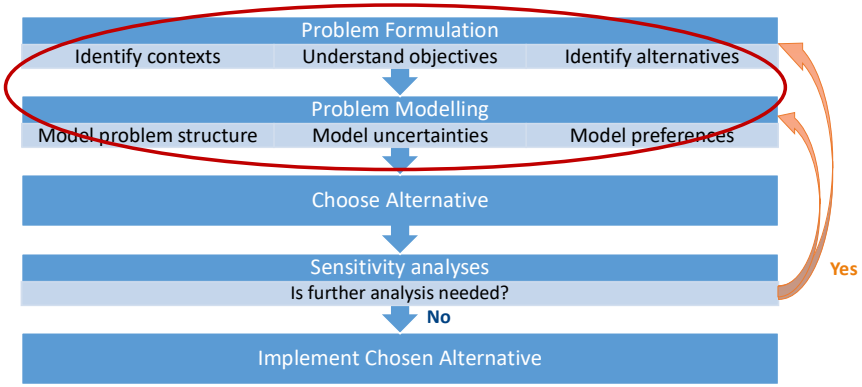
What is Decision Analysis?

- Emerged in the 1960s from operations research and game theory
  - (Howard, 1966)
  - (Raiffa and Abbas, 2016)
- A prescriptive framework
  - Vs. Normative decision theory - Assumes decision makers as ideally rational agents
  - Vs. Descriptive decision theory - Describes how people actually make decisions
- Provides structure and guidance for thinking systematically and recommends alternatives about hard decisions
  - Can help make better decisions, but not improve luck nor guarantee good outcomes!
- Why does it matter for AI Planning and Decision Making?

How to Improve Decision Making?

- Complexity Management
  - Effective methods for problem organization
  - Vocabulary on elements of decision structure
  - Graphical structuring tools
  - Solution and analysis procedures
- Uncertainty Reduction
  - Identification of sources of uncertainty
  - Quantitative representation of uncertainty
- Multiple Objectives Specification
  - A framework and specific tools for dealing with multiple objectives
- Multiple Perspectives Resolution
  - A framework and specific tools to sort through and resolve differences

The Decision Analysis Process



Decision Model: Decision Basis Formulation

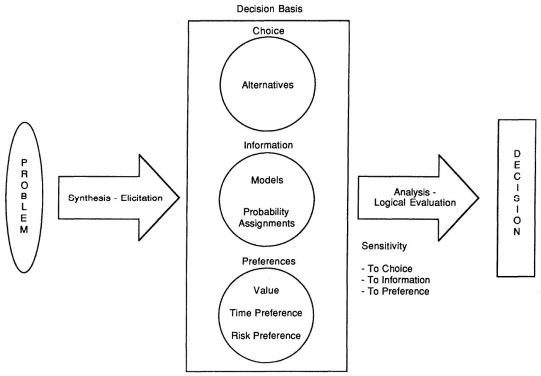


FIGURE 2. Elicitation and Evaluation of the Decision Basis.

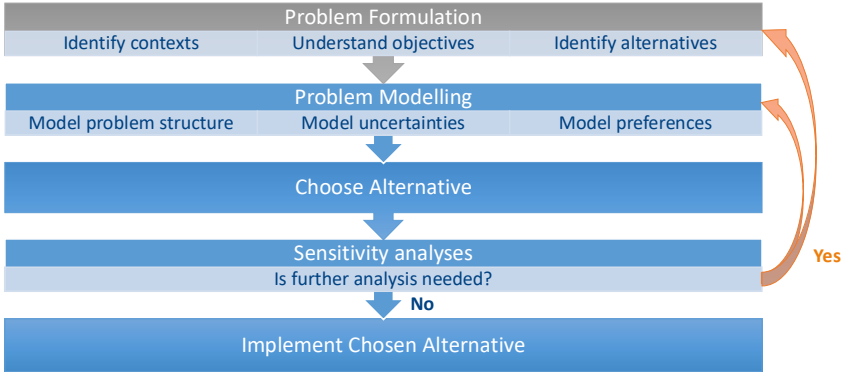
Source: Howard, R. A. 1988

Howard, R. A., Decision Analysis: Practice and Promise, *Management Science*, Vol. 34, No. 6 (Jun., 1988) , pp. 679-695

Example: Cryptocurrency Investment Problem

- Decision:
  - Whether or not to invest in a new cryptocurrency – KittyCoin
- Objective:
- Considerations:
  - Technology and exchange platforms have great credentials
  - Proposed project is more risky than most other cryptocurrencies
- Possible consequences:
  - If invest and value of KittyCoin rises, there will be high monetary returns
  - If not, capital may be put in stock market or other investment options

Problem Formulation



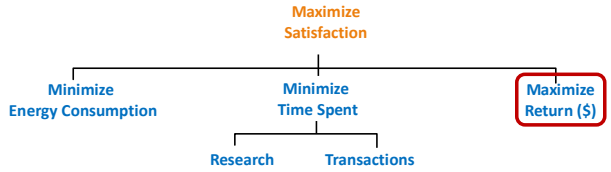
Problem Formulation

- Identify the decision context
  - Define precise problem to solve
  - Identify the decision maker and perspective
- Identify objectives
  - What are the goals of the decision?
- Identify alternatives or actions
  - Careful inspection of all aspects of a problem can lead to discovery of new alternatives
- Using:
  - Fundamental objectives
  - Other objectives (e.g., means objectives)

Organizing Objectives

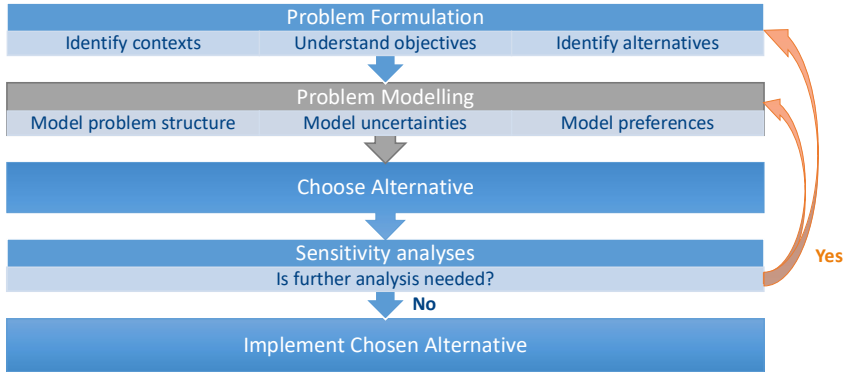
- Structure or organize objectives to
  - reveal all relevant details to be achieved
  - allow direct incorporation into problem definition
- Fundamental Objectives
  - Reflect things that the problem solver needs to achieve
  - Organized into hierarchies
- Means Objectives
  - Help achieve other objectives
  - Organized into networks

Example: Cryptocurrency Investment Problem



- How to construct the fundamental objectives hierarchy?
  - Ultimately, for any objective: Why is it important?
  - To move downward: What do you mean by that?
  - To move upward: Of what more general objective is this an aspect?
- The lowest level form the basis to measure consequences
- Identify alternatives to achieve objectives

Problem Formulation

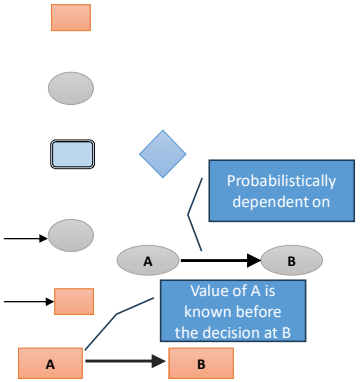


Decision Model Formulation

- Use "divide and conquer" method
  - To understand its structure
  - To measure the uncertainty, and
  - To specify the preferences
- Modeling Structure
  - Construct graphical, mathematical model to denote structure of decision problem
  - Influence Diagram
  - Decision Tree

Basic Elements of a Decision Model

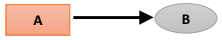
- Decision nodes
  - Decision points with several choices or alternatives
- Chance nodes
  - Uncertain or chance events with several outcomes
- Value or utility nodes
  - Deterministic monetary value or utility functions
  - Measure desirability of final objectives
- Probabilistic dependencies
  - Represent conditional dependence of chance events
- Informational dependencies
  - Represent information available at decision points



Relevance and Sequence

- Relevance Arcs
  - Predecessor is relevant for assessing the chances associated with the uncertain event or the values of the value (consequence) node

Outcomes of event B are probabilistically dependent on outcomes of event A



Outcomes of event B are probabilistically dependent on choices of decision A

- Sequence Arcs
  - Decision is made knowing the outcome of the predecessor node

The decision maker knows the outcome of event A when making decision B

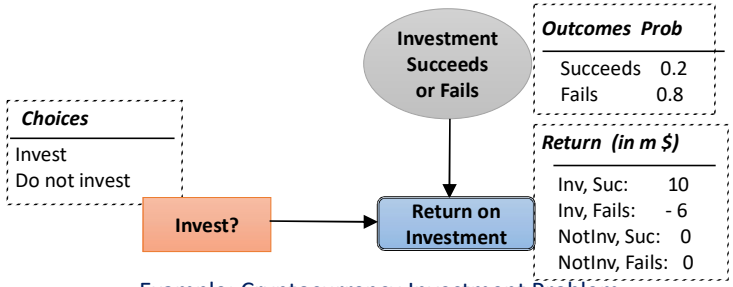
Decision A is made before decision B

Decision Networks

Influence Diagrams

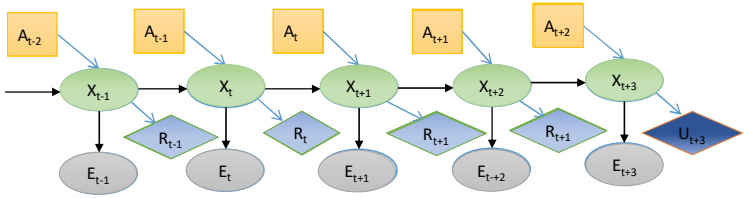
The Basic Risky Decision

- No information is available when decision is made
  - This example assumes objective is to maximize return ($)



Example: Cryptocurrency Investment Problem

Sequential Decisions



Sequential decision problem as dynamic decision network

- Sequential structure explicitly shown
- No cycles allowed
- "No-forgetting" arcs implied but not shown
- Final value is a function of individual values over all stages

Influence Diagram

- Characteristics:
  - Hierarchical representation captures current state of knowledge of decision maker
  - Arcs indicate probabilistic dependence or relevance if leading into chance nodes or value nodes
  - Arcs indicate informational dependence if leading into decision nodes
  - A node at the beginning of an arc is a predecessor
  - A node at the end of an arc is a successor
  - A proper influence diagram has no cycles
  - Factorized utility functions represented by chance or deterministic nodes leading into final unity node

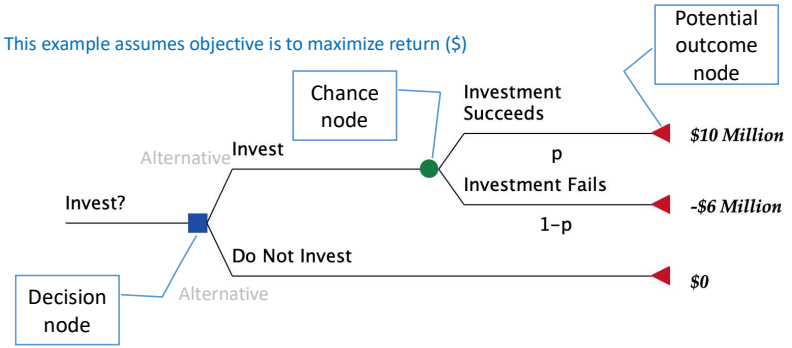
Building Influence Diagrams

- Hierarchical representation to facilitate communication
  - Captures current state of knowledge of decision maker
  - Details hidden beneath the structure to favor simplicity
- No best strategy for building influence diagrams; best approach is:
  - Put together a simple version of diagram
  - Add details as necessary until all relevant aspects are included
- No fixed order of "reasoning"
- Some Common Mistakes:
  - Influence diagrams (like BNs) are not flowcharts
    - An influence diagram is a snapshot of the decision situation
    - An arrow from a chance node to a decision node means that the chance event outcome is known at the time of decision
  - No cycles are allowed in an influence diagram

Decision Trees

The Basic Risky Decision

- This example assumes objective is to maximize return ($)



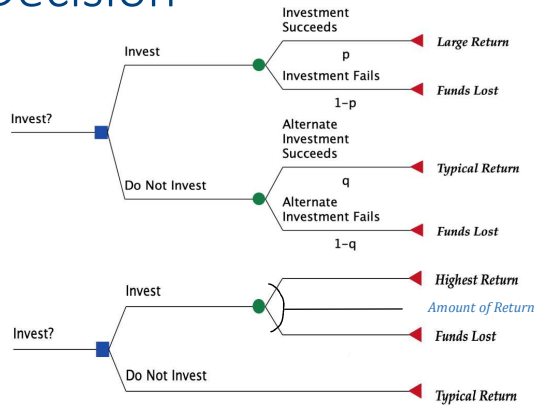
Example: Cryptocurrency Investment Problem

DT and DN are isomorphic

DT can potentially give immediate insights

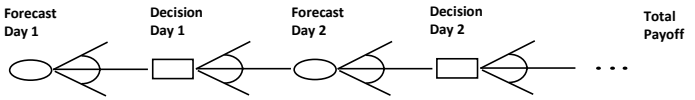
The Basic Risky Decision

- Double Risky Decision
  - e.g., if the investor decides not to invest, she may lose the money in the stock market or the less risky investment
- Range of Risk Decision
  - e.g., Return of the original risky decision may take a range of values



Sequential Decisions

- Number of branches increase exponentially as number of decisions and chance events increase
- May use "skeleton" version of decision tree to represent sequential decisions if:
  1) the sequential problem repeats itself; and
  2) the decision tree is symmetric at each stage



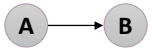
Decision Tree

- Characteristics:
  - Arcs denote decision alternatives or chance outcomes
  - Only one option can be chosen at each decision node
  - Each chance node has a set of mutually exclusive and collectively exhaustive outcomes
  - Represents all possible paths through time
  - Decisions and chance events are most naturally placed in a time order from left to right
  - Implicit probabilistic and information dependencies
  - Utility (terminal) node represents conditional utility associated with path of action-alternative-chance-outcome combinations
  - Collapse multidimensional objective description into a single score for final consequence

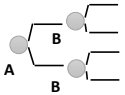
Building Decision Trees

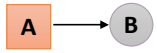
- Start with the first (leftmost) node:
  - For decision node, draw a branch for each alternative
  - For chance node, draw a branch for each possible outcome
- Label each branch emanating from a chance node with:
  - a unique outcome
  - the corresponding path dependent probability, and
  - the path dependent value for that branch
- Label each branch emanating from a decision node with:
  - a unique alternative, and
  - the value for that alternative
- Place final values on terminal nodes at the end of each path

Correspondence between ID and DT

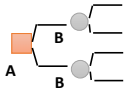


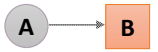
Outcomes of event B are probabilistically dependent on outcomes of event A



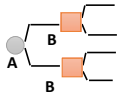


Outcomes of event B are probabilistically dependent on choices of decision A

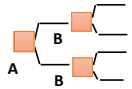




The decision maker knows the outcome of event A when making decision B



Decision A is made before decision B



Modeling Uncertainty

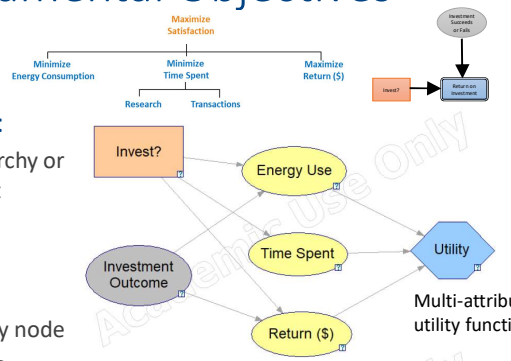
- Apply probability theory to represent uncertainty
- Approaches:
  - Using subjective judgment in assessing probabilities – difficult!
  - Using theoretical probability models
  - Using data

Modeling Preferences

- Use utility functions
- Assess utility or value functions to measure "desirability" of different outcomes and trade-off situations
- Possible scales:
  - Monetary cost
  - Revenue, profit
  - Life expectancy
  - etc.

Representing Fundamental Objectives

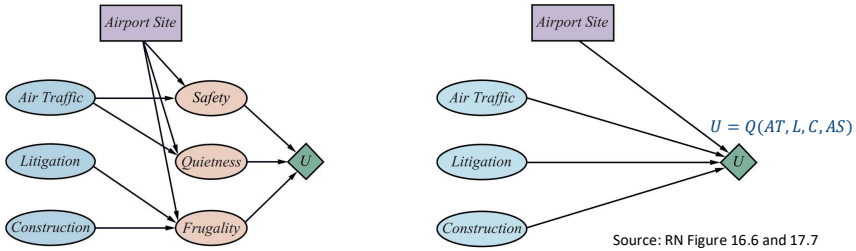
- In decision networks:
- To represent multiple attributes:
  - Use fundamental objectives hierarchy or utility node structure to represent multiple attributes
- To capture trade-offs:
  - Aggregate function for individual attribute utilities in "overall" utility node
  - Incorporate appropriate trade-offs among the different objectives



Multi-attribute utility function

Example: Multi-attribute Utility Function

- Utility node:
  - Represents the utility function; parents are all the variables (attributes) that directly affect utility
  - A special kind of deterministic nodes - outcome is certain given values of the parents
  - Simplified form: Utility node represents expected utility associated with each action: the action-utility function or Q-function

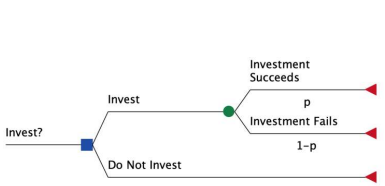


Source: RN Figure 16.6 and 17.7

Representing Fundamental Objectives

In decision trees:

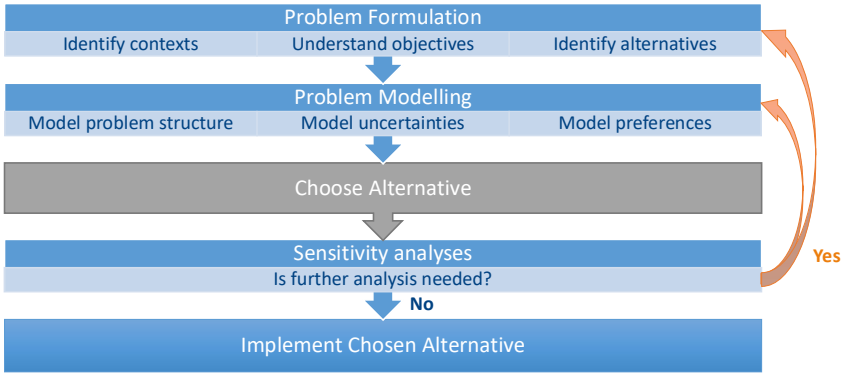
- Normally assign only one final value to each terminal node outcome or branch in consequence matrix
- Collapse multidimensional objective description into single score for final consequence
- Require assessment and application of appropriate trade-off weight
  - e.g., High return from KIttyCoin takes a long time; typical return from stock market takes a shorter time



Overall Utility		
Energy	Time	Return($)

Consequence Matrix

Model Solution



Solving a Decision Model

$$action = \text{argmax}\, EU(a)$$

$$EU(a) = \sum_{s'} P(\text{Result}(a) = s')U(s') = \sum_{s'}^{a} \sum_{s} P(s)P(s'|s,a)U(s')$$

- Solution algorithms often find optimal decisions by:
  - Chance values expectation
  - Decision value maximization
- Common solution methods for decision trees:
  - Rollback algorithm
  - Monte Carlo simulation
- Common solution methods for influence diagrams:
  - Reduction algorithm
  - Sampling
  - Inference in Bayesian networks

Chance Values Expectation

- Determine expected value for a chance node
  - e.g. Consider a chance event "Investment Outcome" with possible outcomes "Gain," "Unchanged," "Loss"

Investment Outcome

Outcome	Probability	Value (utility)
Gain	0.2	1.0
Unchanged	0.3	0.5
Loss	0.5	0

- EU [Investment outcome] =

Decision Values Maximization

- Determine maximum value for a decision node
  - e.g. Consider a decision point "Invest?" with possible alternatives "Invest" and "Not Invest"

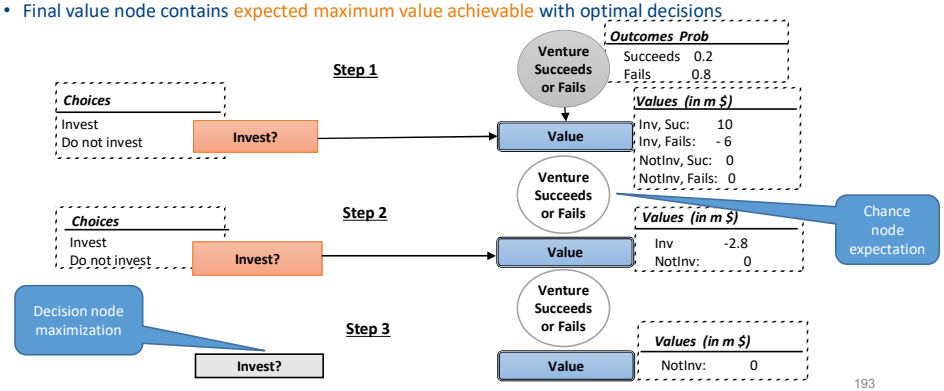
Invest?

Alternative	Value ($ in M)
Invest	6
Not Invest	1

- Max[Invest?] =

Solving an Influence Diagram

- Solution obtained by reducing diagram, while recording optimal choices, into single value node
- Final value node contains expected maximum value achievable with optimal decisions

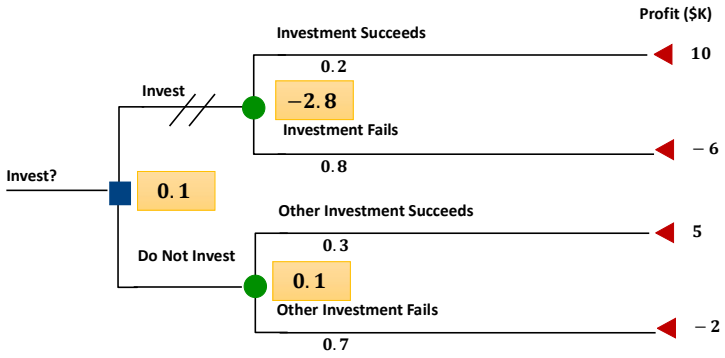


Solving an Influence Diagram: Common Approach

- Set evidence variable to the observed state
- For each possible value of the decision node:
  - Set the decision node to that value
  - Calculate the posterior probability of the parents of the utility using probabilistic inference algorithm
  - Calculate the resulting utility for the action
- Return the action with the highest utility

Use efficient Bayesian Network inference algorithms

Solving a Decision Tree (Rollback)



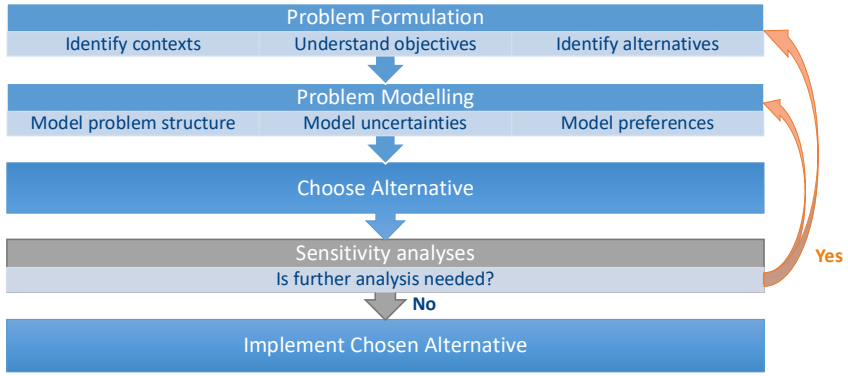
Solving a Decision Tree

- Choose among risky alternatives
  - Pick alternative with the highest EMV or utility
- Folding or rolling back the tree:
  - Begin at end points of branches on far right-hand side and move to left
  - Chance value expectation
    - Calculate expected values when encountering a chance node; OR
  - Decision value maximization
    - Choose the branch with the highest value or expected value when encountering a decision node
  - Record the value accumulated and alternative selected for each decision node along the optimal path

Influence Diagrams vs. Decision Trees

Feature \ Model	Influence Diagram	Decision Tree
Compact Representation	Yes	No
Explicit informational and probabilistic dependencies	Yes	No
Explicit value function structure	Yes	No
Explicit labels and bindings	No	Yes
Explicit representation of asymmetric situations	No	Yes
Straightforward solution algorithm(s)	No	Yes
Good for:	Communication; sequential decisions	Sensitivity analysis

Model Analysis

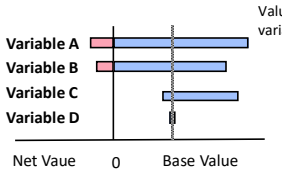


Sensitivity Analysis

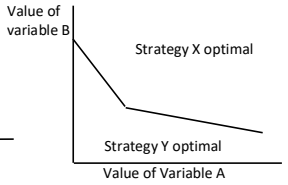
- Answers the question:
  - "What if … is changed in this decision"?
- Determines what matters and what does not throughout the modeling process
- Provides guidance to develop requisite decision model
- Integral part of decision analysis, may lead to:
  - Problem re-definition (what if something changed?)
  - New objectives, alternatives, structure, uncertainty, preference
- No optimal procedures, useful techniques facilitate :
  - identifying and structuring problems
  - detecting dominance among alternatives
  - assessing probabilities and utilities

Sensitivity Analysis Methods

- Common Methods:
  - Structural analysis methods
  - The clarity test
  - Tornado diagrams
  - One, two, three-way analysis



A Tornado Diagram



Two-Way Sensitivity Analysis

- The Clarity Test:
  - Resolve ambiguities
- Definition:
  - Imagine a clairvoyant who has access to all future information
  - Would the clairvoyant be able to determine unequivocally what the outcome would be for every event in the decision model?
  - No interpretation or judgment should be required of the clairvoyant
- Example:
  - Does "Temperature" of "*high*, *medium*, or *low*" pass the clarity test?

Robust Decisions

- Robust or minimax decision – Extreme case
  - One that gives the best result in the worst case.
  - "Worst case" means worst with respect to all plausible variations in the parameter values of the model
  - Letting θ stand for all the parameters in the model, the robust decision is defined by:

$$a^* = \max_a \min_\theta EU(a;\theta)$$

- For parametric uncertainty
  - Bayesian decision theory: model with hyperparameters
  - Analyze performance of hyperparameters or ranges of parameters
- For structural uncertainty
  - Examine ensemble of models
  - Use domain knowledge to propose alternatives

Requisite Decision Model

- When no new intuition emerges about the problem
- When it contains everything essential for solving the problem
- The decision maker's
  - thoughts about the problem
  - beliefs regarding uncertainty and preferences

  are fully developed

Value of Information

Information and Decision Making

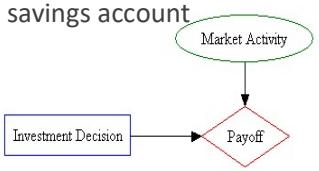
- Information is gathered to reduce uncertainty in decision making
  - Consult experts
  - Conduct surveys
  - Perform mathematical or statistical analyses
  - Do research
  - Read books, journals, newspapers
  - Learn from past data
- Relevant questions:
  - How to evaluate or measure the "value" or usefulness of the information?
  - What does it mean for a knowledge source to provide perfect information?
  - Shall we invest effort or pay $X for additional information to help problem solving?

Value of Information

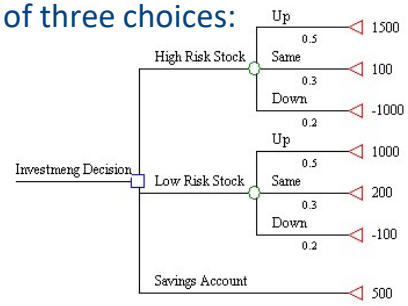
- Costly information gathered to reduce uncertainty
  - How to place value on information in a problem?
  - How to decide whether or not to gather more information?
- Main ideas:
  - Information has value to the extent that it is likely to cause a change of plan and to the extent that the new plan is significantly better than the old one
  - Use conditional probabilities and Bayes' Theorem to model the expected value of information

Example: Stock Investment

- An investor may invest in one of three choices:
  - A high-risk stock
  - A low-risk stock
  - A savings account



Influence diagram



Decision tree

Expected Value of Information

- Expected value of information (EVI)
  - Indicates if information is worth gathering
  - Lower bound: zero expected value
  - Upper bound: expected value of perfect information
- Information has:
  - no value or zero expected value if the same choice will be made before and after obtaining information
  - positive expected value if it leads to a different choice
  - maximum expected value if information is perfect
- EVI is defined in terms of the decision context
  - Different people in different situations may place different values on the same information

Stock Investment Example (cont.)

Should investor consult an expert?

- If the expert always provides perfect information:
  - P(Exp says "Up" | Market Up) = 1
  - P(Exp says "Down" | Market Up) = 0
  - P(Exp says "Up" | Market Down) = 0
- To show there is no uncertainty after hearing the expert, apply Bayes' theorem:

P(Market Up | Exp says "Up")

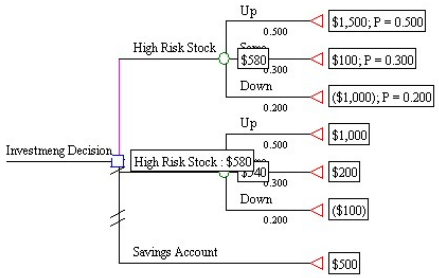
$$= \frac{P(\text{Exp says "Up"} \mid \text{Market Up})\, P(\text{Market Up})}{[P(\text{Exp says "Up"} \mid \text{Market Up})P(\text{Market Up}) + P(\text{Exp says "Up"} \mid \text{Market Down})P(\text{Market Down})]}$$

$= 1$

- Note:
  - The posterior probability is equal to 1 regardless of the prior probability
  - How do real problems differ from the above situation?

Stock Investment Example (cont.)

- The optimal choice is the high-risk stock with EMV $580
- Assumption: optimistic about market (Up, 0.5 prob)
- How much would the investor be willing to pay for information about the market activity?

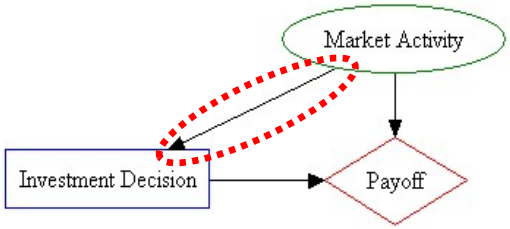


Expected Value of Perfect Information

- Max amount that the decision maker is willing to pay for perfect information
- To find expected value of perfect information (EVPI):
  - Modify the decision model to indicate perfect information
  - Solve the model and find the EMV ($1000)
  - EVPI = EMV (with perfect information) - EMV(original)
    = $1000 - $580 = $420

Modifying Influence Diagram for EVPI

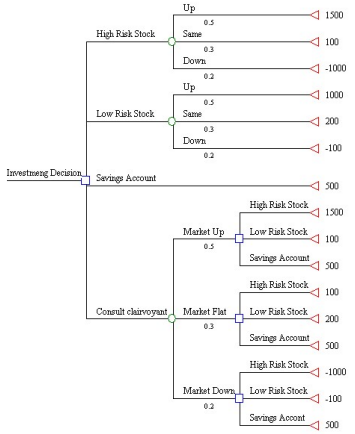
- Impose order on the decision and uncertain event nodes
- The uncertainty nodes for which perfect information is available comes before the decision node



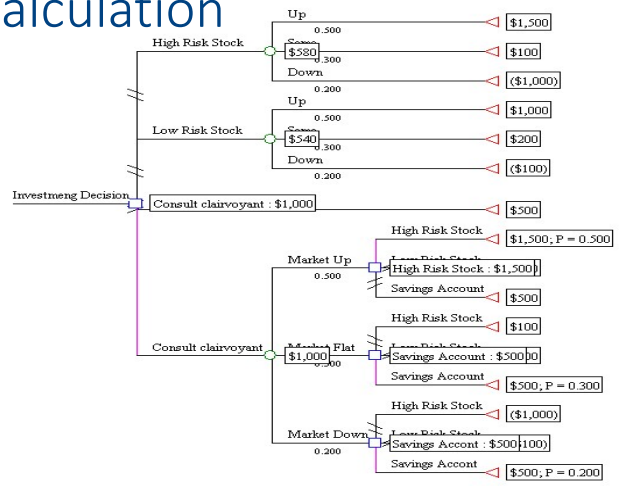
Influence diagram with perfect information

Modifying Decision Tree for EVPI

- Reorder the decision and the uncertain event nodes
- The uncertainty nodes for which perfect information is available comes before the decision node



EVPI Calculation



(Expected) Value of Perfect Information

Assume exact evidence about variable E_j; compute value of perfect information (VPI)

- Given expected utility with current best action α:

$$EU(\alpha) = \max_a \sum_{s'} P(Result(a) = s')U(s')$$

- Value of the best new action after $E_j = e_j$ is obtained

$$EU\left(\alpha_{e_j} \middle| e_j\right) = \max_a \sum_{s'} P\left(Result(a) = s' \middle| e_j\right)U(s')$$

- Variable E_j can take multiple values e_j, so on averaging:

$$VPI(E_j) = \sum_{e_j} P(E_j = e_j)EU\left(\alpha_{e_j} \middle| E_j = e_j\right) - EU(\alpha)$$

Properties of Value of information

- Expected value of information is always non-negative

$$\forall j \;\; VPI(E_j) \geq 0$$

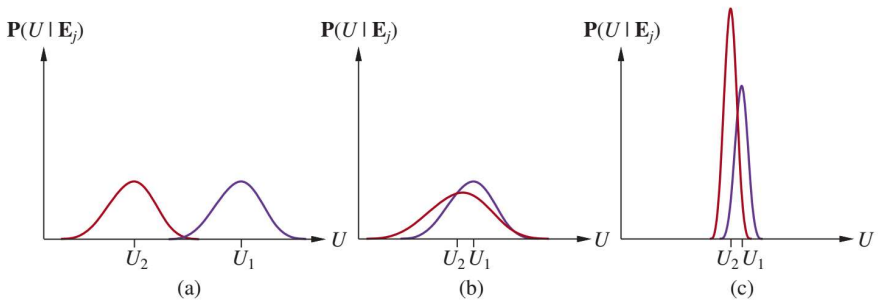
- VPI is not additive

$$VPI(E_j, E_k) \neq VPI(E_j) + VPI(E_k)$$

- VPI is order independent

$$VPI(E_j, E_k) = VPI(E_j) + VPI(E_k|E_j) = VPI(E_k) + VPI(E_j|E_k) \quad = VPI(E_k, E_j)$$

When to Gather More Information?



Source: RN Figure 15.8

Information Gathering, Decision-Theoretic Agent

- Agent should gather information before taking actions, if possible
  - Cost associated with getting the information, so how to choose which variable to get more information about?

function INFORMATION-GATHERING-AGENT(*percept*) **returns** an *action*
persistent: D, a decision network

integrate *percept* into D
$j \leftarrow$ the value that maximizes $VPI(E_j) \,/\, C(E_j)$
if $VPI(E_j) > C(E_j)$
  then return $Request(E_j)$
else return the best action from D

Unit gain per unit cost
OR
Information value

Source: RN Figure 15.9

NUS | Computing

Markov Decision Process

CS4246/CS5446

AI Planning and Decision Making

This Lecture Will Be Recorded!

Please join:
pollev.com/anarayan

Solving Sequential Decision Problems

- Decision (Planning) Problem or Model
  - Appropriate abstraction of states, actions, goals
    uncertain effects, and time horizon + observations (through sensing)
- Decision Algorithm
  - Input: a problem
  - Output: a solution in the form of an optimal action sequence or policy over time horizon
- Decision Solution
  - An action sequence or solution from an initial state to the goal state(s)
    - An optimal solution or action sequence; OR
    - An optimal policy that specifies "best" action in each state wrt to costs or values or preferences
  - (Optional) A goal state that satisfies certain properties

Recall: Decision Making under Uncertainty

- Decision Model:
  - Actions: $a \in A$
  - Uncertain current state: $s \in S$ with probability of reaching: $P(s)$
  - Transition model of uncertain action outcome or effects:
    $P(s'|s,a)$ – probability that action a in state s reaches state s'
  - Outcome of applying action a:
    Result(a) – random variable whose values are outcome states
  - Probability of outcome state s', conditioning on that action a is executed:
    $P(\text{Result}(a) = s') = \sum_s P(s)P(s'|s,a)$
  - Preferences captured by a utility function:
    $U(s)$ – assigns a single number to express the desirability of a state s

Sequential Decision Problems

- What are sequential decision problems?
  - An agent's utility depends on a sequence of decisions
  - Incorporate utilities, uncertainty, and sensing
  - Search and planning problems are special cases
  - Decision (Planning) Models:
    - Markov decision process (MDP)
    - Partially observable Markov decision process (POMDP)
    - Reinforcement learning: sequential decision making + learning

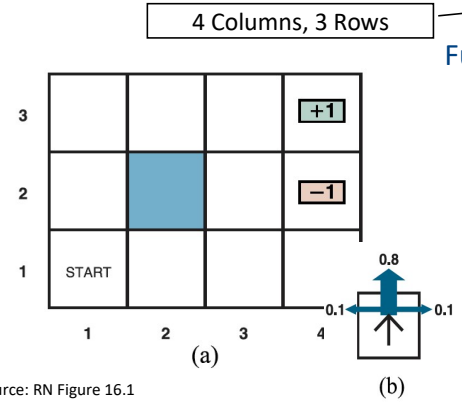
Why Study MDPs?

- Markov decision process (MDP)
  - A sequential decision problem for a fully observable stochastic environment with Markovian transition and additive rewards
- Advantages of using MDPs:
  - General, formal/principled framework for decision-theoretic planning
  - Model uncertainty in dynamics of environment (e.g., in actions)
  - Generate non-myopic action policies
  - Efficient algorithms for solving MDPs with performance guarantees
  - Many real-world applications spanning multiple disciplines:
    - Operations research and logistics (e.g., transportation systems), robotics (e.g., motion planning), computer games (e.g., path planning), multimedia (e.g., camera surveillance)

Model Formulation

Define the problem elements

Example: Navigation in Grid World



Source: RN Figure 16.1

Fully observable 4 X 3 environment

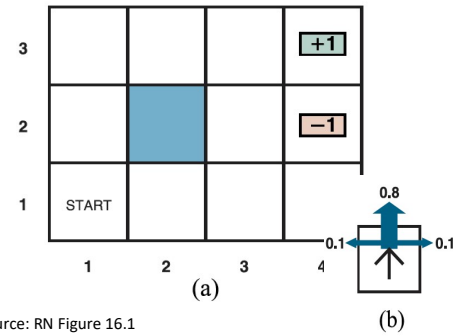
- Begin in START state s_0
- One action a per time step.
- Terminate when reaching goal states s_g
  - Example: States marked with +1 and –1

Uncertain action effects:

- Example: Up, Down, Left, Right:

0.8 - correct direction
0.1 - perpendicular to the left
0.1 - perpendicular to the right

Example: Navigation



Source: RN Figure 16.1

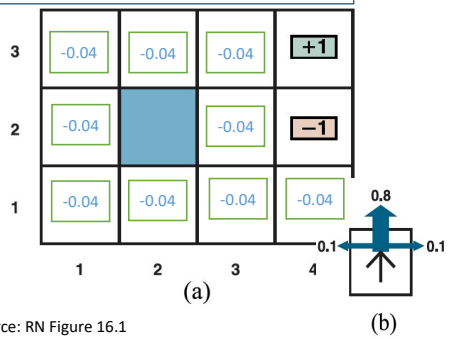
- Transition model:
  - Define (stochastic) outcome of each action a
  - $P(s'|s,a)$ – probability of reaching state s' if action a is done in state s
    - Also denoted as $T(s,a,s')$

Markovian assumption:

- Probability depends only on state s and not history of earlier states.

Example: Navigation

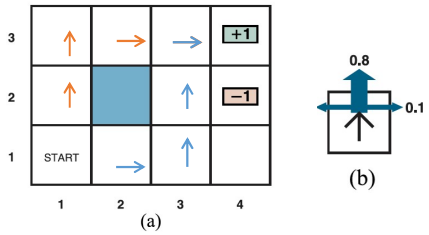
E.g., $R(s)$ is -0.04 for all states, except the terminal states, where $R(s)$ is $+1$ or -1



Source: RN Figure 16.1

- Reward model:
  - Define reward received for every transition
  - $R(s,a,s')$ – For every transition from s to s' via action a; AND/OR
  - $R(s)$ – For any transition into state s
  - Rewards may be positive or negative, bound by $\pm R_{max}$
- Utility function $U(s)$ depends on the sequence of states and actions – environment history – sum of rewards of the states in the sequence

Example: Navigation



Uncertain action effects:

Up, Down, Left, Right:

0.8 - correct direction

0.1 - perpendicular to the left

0.1 - perpendicular to the right

- If action effects are deterministic:
  - What is the optimal policy or action sequence?
- But agent's actions are stochastic
  - What is the chance of getting to the goal with: $[U,U,R,R,R]$?
  - How to derive optimal policy from the transition function directly?

Markov Decision Process (MDP)

- Formally:
  - An MDP $M \triangleq (S,A,T,R,\gamma)$ consists of:
  - A set S of states
  - A set A of actions
  - A transition function $T: S \times A \times S \rightarrow [0,1]$ that satisfies the Markov property such that:

$$\forall s \in S, \forall a \in A: \sum_{s' \in S} T(s,a,s') = \sum_{s' \in S} P(s'|s,a) = 1$$

  - A reward function $R: S \rightarrow \Re$ or $R: S \times A \times S \rightarrow \Re$
  - A discount factor $0 < \gamma < 1$
  - Solution is a policy – a function to recommend an action in each state: $\pi: S \rightarrow A$
    - Solution involves careful balancing of risk and reward

Transition Function

- Formally:
  - $T(s, a, s') = P(s'|s, a)$ is the probability of going from state s to state s' taking action a
  - Define $T(s, a, s')$ for all $s, s' \in S, a \in A$
- Markov Property
  - The next state is conditionally independent of the past states and actions given the current state s and action a, i.e.,

$$P(s_{t+1}|s_t, a_t, s_{t-1}, a_{t-1}, \dots, s_0, a_0) = P(s_{t+1}|s_t, a_t)$$
$$\text{for all } s_t, a_t, s_{t-1}, a_{t-1}, \dots, s_0, a_0$$

Markov Property

- Is the Markov Property applicable in real-world problems?
  - It is a simplifying assumption!
- Potential violations of Markov assumption
  - State variables and dynamics of environment not captured in model
  - Inaccuracies in the transition function
- Nevertheless:
  - Markov assumption helps reduce time complexity of algorithms
  - Most, if not all, stochastic processes can be modeled as Markov processes

Reward Function

- Formally:
  - Define $R(s, a, s')$ for all $s, s' \in S$ and for all $a \in A$.
- Alternate forms: (given $R(s, a, s')$ above)
  - $R(s, a)$ as $\sum_{s'} P(s'|s, a)R(s, a, s')$ independent of s'
  - $R(s)$ as $\sum_{s'} P(s'|s, a)R(s, a, s')$, independent of a and s'
- Challenges
  - It is hard to construct reward functions with multiple attributes
  - Balance risk vs reward

Other possible ways to define reward functions

Clarifications on Reward Functions

a) Representation of states and transitions:
- ***s*** always denotes the "current state", ***a*** is the action (to be) taken in state ***s*** (current state), and ***s'*** is the outcome or next state after taking action ***a*** in state ***s***.

b) The common reward definitions are as follows:

- On state:
  - b.1) ***R(s)*** - amount of reward (or cost) of "being" in state ***s***
  - When to record or "count" reward: 1) on arriving at ***s***, OR more commonly, 2) on exiting ***s*** (in the next transition), depending on context/implementation/choice in the problem model and/or solution
  - b.2) ***R(s, a)*** - amount of reward (or cost) of taking action ***a*** in state ***s***
  - When to record reward: - on exiting ***s*** upon taking action ***a***
- On transition:
  - b.3) ***R(s, a, s')*** - amount of reward (or cost) of the transition from ***s*** to ***s'*** given that the action ***a*** is taken
  - When to record reward: 1) on exiting ***s*** before reaching (the intended) ***s'***, OR 2) on arriving at ***s'***

Clarifications on Reward Functions

- Notes:
  1) The above formulations allow accumulation of rewards if agent/system remains in the same state for multiple time steps. But the actual time of "counting" may not matter that much in calculating expectations. It matters more in the simulation counts.
  2) In practice, there can be more than one set of rewards defined for each ***s***, ***(s, a)***, and/or ***(s, a, s')*** combinations, i.e., you can have ***R(s)*** and ***R(s, a, s')*** separately defined in the same model, e.g., in diagnostic test and therapy planning models in healthcare.
  3) For slide 15 in the lecture notes - the **descriptions** (they are not meant to be equations) in the **"Alternate Forms"** are meant to be interpreted as: **How to formulate *R(s, a)* and *R(s)* in terms of the main definition of *R(s, a, s')* given in the first line**.
  4) Remember, MDP is a modeling "language", these definitions are by "choice" of the designer and conventions commonly adopted - there are variations in different problem and solution

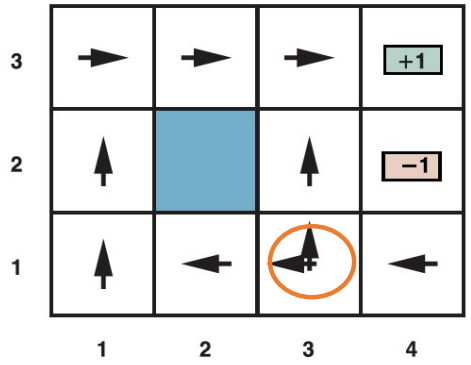
Solving MDPs

Deriving policies – actions to take at each state

Solving MDPs

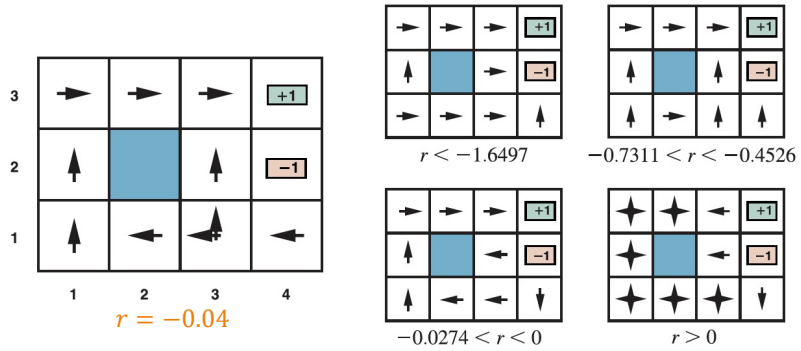
- What does a solution look like?
  - A policy $\pi(s): S \rightarrow A$ is a function from states to actions:
    - For every state s, outputs an appropriate action a.
  - Quality of policy – measured by expected utility of possible state sequence generated by the policy
  - Optimal policy $\pi^*$is a policy that generates highest expected utility
- An MDP agent:
  - Given optimal policy π^*: Agent decides what to do by consulting its current percept, which tells it the current state s, and then executing action $a^* = \pi^*(s)$
  - The (optimal) policy represents the agent function explicitly – how to behave!

Example: Illustration of π^*



Source: RN Figure 16.2

Example: Balancing Risk and Reward



Changes depending on the value of $r = R(s, a, s')$ for transitions between nonterminal states
There may be many optimal policies for various ranges of r.

Source: RN Figure 16.2

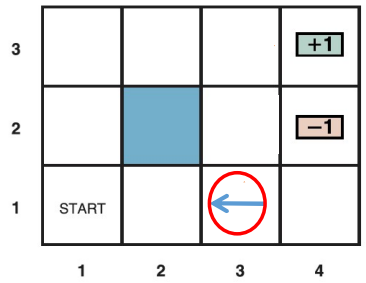
Finite and Infinite Horizon Problems

- Finite horizon: Fixed time N and terminate
  - Return is usually the addition of rewards over the sequence
    - $U_h([s_0, a_0, s_1, a_1, s_2, s_3 \dots s_{N+k}]) = U_h([s_0, a_0, s_1, a_1, s_2, s_3 \dots s_N])$
    - $U_h([s_0, a_0, s_1, a_1, s_2, s_3 \dots s_N]) = R(s_0, a_0, s_1) + R(s_1, a_1, s_2) + \cdots + R(s_{N-1}, a_{N-1}, s_N)$
  - Optimal action in a given state can change over time N, depending on remaining steps
  - Nonstationary optimal policy: π_t^*
- Infinite horizon
  - No fixed deadline
  - No reason to behave differently in the same state at different times
  - Stationary optimal policy: π^*

Example: Finite Horizon Problem

N = 3?

N = 100?



Assume the agent is in (3, 1)

Source: RN Figurer 16.2

Rewards in Infinite Horizon Problems

- Infinite horizon – comparing utilities are difficult
  - Undiscounted utilities can be infinite
- Additive discounted rewards

Allows preference independence assumption

$U_h([s_0, a_0, s_1, a_1, s_2, s_3 \dots]) = R(s_0, a_0, s_1) + \gamma R(s_1, a_1, s_2) + \gamma^2 R(s_2, a_2, s_3) + \cdots$
where $\gamma \in [0,1]$ is the discount factor

- Discounted rewards with $\gamma < 1$ and rewards bounded by $\pm R_{max}$, utility is always finite

$$U_h([s_0, a_0, s_1, \dots]) = \sum_{t=0}^{\infty} \gamma^t R(s_t, a_t, s_{t+1}) \leq \sum_{t=0}^{\infty} \gamma^t R_{max} = \frac{R_{max}}{1-\gamma}$$

Rewards in Infinite Horizon Problems

- Infinite rewards
  - Environment does not have terminal state; or
  - Agent never reaches terminal state; and
  - Additive, undiscounted rewards
- Why additive discounted rewards?
  - Preference independence assumption
    - Read: If you prefer one future to another starting tomorrow, then you should still prefer that future it were to start today instead.
  - Empirical – humans and animas appear to value near term rewards more
  - Economic – Early rewards can be invested to produce returns
  - Uncertainty about the true rewards – Rewards may never arrive
  - Discounted rewards make some nasty infinities go away

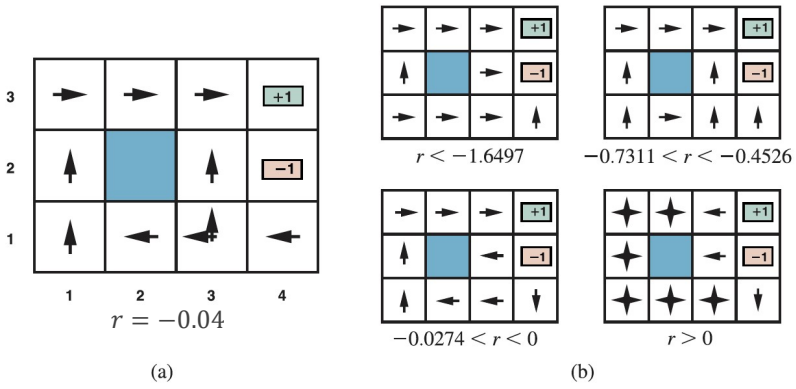
Preference Independence Assumption

- Assumptions
  - Each transition $s_t \xrightarrow{a_t} s_{t+1}$ regarded as an attribute of the history $[s_0, a_0, s_1, a_1, s_2, s_3 \ldots]$
  - Preference independence assumption – preferences between state sequences are stationary
- Stationary preferences
  - If two histories $[s_0, a_0, s_1, a_1, s_2, \ldots]$ and $[s'_0, a'_0, s'_1, a'_1, s'_2, \ldots]$ begin with the same transition (i.e., $s_0 = s'_0, a_0 = a'_0$, and $s_1 = s'_1$)
  - Then the two histories should be preference-ordered the same way as the histories $[s_1, a_1, s_2, \ldots]$ and $[s'_1, a'_1, s'_2, \ldots]$

How to Achieve Finite Rewards?

- 3 possible solutions to achieve finite rewards
  1. With discounted rewards, utility of infinite sequence is finite, if $\gamma < 1$, and rewards bounded by $\pm R_{max}$, then $U_h([s0, a0, s1, a1, s2, \ldots]) = R_{max}/(1 - \gamma)$
  2. Environment has terminal states and if there is a proper policy, i.e., agent is guaranteed to get to a terminal state, can even use $\gamma = 1$ in additive rewards. (See counter examples)
  3. Compared in terms of average reward obtained per time step
     - Harder to compute and to analyze

Example: Proper and Improper Policies



Source: RN Figure 17.2

Utility of State and Optimal Policy

- Main ideas:
  - Utility of sequence: Sum of the discounted rewards obtained during that sequence
  - Comparing expected utilities obtained when executing policies
  - Utility function of state $U(s)$ allows selection of optimal action using MEU
- Assumptions:
  - Define random variable S_t : State reached at time t when executing a particular policy π; $S_0 = s$
  - Probability distribution over state sequences $S_1, S_2, \ldots$ determined by: Initial state s, policy π, and transition model T for the environment
- Expected utility of executing π starting from s:

$$U^{\pi}(s) = E[\sum_{t=0}^{\infty} \gamma^t R(S_t, \pi(S_t), S_{t+1})]$$

  - With $S_0 = s$, expectation wrt distribution of state sequences determined by π and s.

Utility of State and Optimal Policy

- Utility of state (or value of state) is the value of optimal policy

$$U(s) = U^{\pi^*}(s) = \max_{a \in A(s)} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')] = V(s)$$

  - Expected sum of discounted rewards if an optimal policy is executed
    - $R(s) = \sum_{s'} P(s'|s,a) R(s,a,s')$ is the "short term" reward for being in s
    - $U(s) = V(s)$ is the "long term" total expected reward from s onward
- Optimal action selected through maximizing utility of state $U(s)$ based on MEU:

$$\pi^*(s) = \text{argmax}_{a \in A(s)} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')]$$

  - Optimal policy is independent of the starting state in infinite horizon problems with discounted utilities

Example: Utilities of States

$R(s) = -0.04$

Numbers are $U(s)$

3	0.8516	0.9078	0.9578	+1
2	0.8016		0.7003	−1
1	0.7453	0.6953	0.6514	0.4279
	1	2	3	4

How to compute the numbers?

Using $\gamma = 1$

Source: RN Figure 16.3

Bellman Equation

- Principle of optimality: (Bellman, 1957)
  - An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision

Bellman Equation: Finite Horizon

- Definition for finite horizon problem:
  - The dynamic programming algorithm finds the utility or value functions (state utilities):
  - If the horizon is 0, $U_0(s) = R(s)$.
  - If horizon is k, following Bellman's principle of optimality (or optimal substructure)

$$U_k(s) = \max_{a \in A(s)} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U_{k-1}(s')] \quad \text{OR}$$

$$U_k(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) U_{k-1}(s'))$$

Bellman Equation: Infinite Horizon

- Definition for infinite horizon problem:

$$U(s) = \max_{a \in A(s)} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')] \quad \text{OR}$$

Utility of a state, s

$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) U(s')$$

Immediate reward | Optimal action | Expected utility of next states

  - $U(s) = U^{\pi^*}(s)$: The utility of a state is the expected reward for the next transition (or the current state) plus the discounted utility of the next state, assuming optimal action taken
- Solution
  - The utilities of the states as defined as the expected utility of subsequent state sequences are the unique solutions of the set of Bellman equations
  - Gives direct relationship between utility of a state and the utility of its neighbors.
  - There is 1 Bellman equation per state. So, $|S|$ nonlinear equations (due to max) with $|S|$ unknowns (utility of states).

Q-Function

- Action-Utility Function or Q-Function
  - $Q(s,a)$: Expected utility of taking a given action in a given state

$$U(s) = \max_a Q(s,a)$$

- Computing optimal policy:

$$\pi^*(s) = \text{argmax}_a Q(s,a)$$

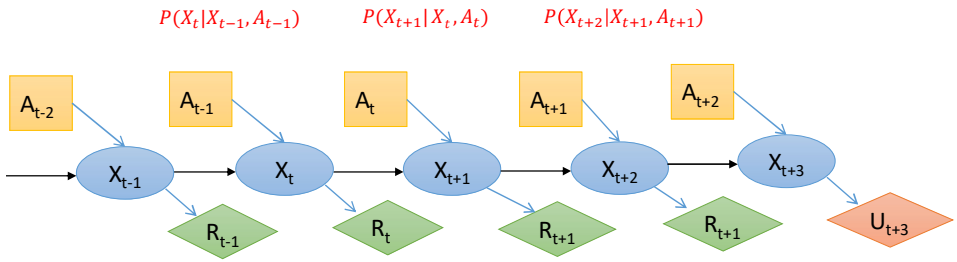
- Bellman Equation for Q-functions

$$Q(s,a) = \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')] = \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma \max_{a'} Q(s',a')]$$

- Q-Value Function
  - Inputs: mdp, s, a, U
  - Return: $\sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')]$

Used in the algorithms later

MDP as Dynamic Decision Network



What are X_t, A_t, R_t, U_t?

Value Iteration

Solution Method

Bellman Equation and Value Iteration

- Value iteration (A solution algorithm for MDP)
  - Bellman equation is basis of the value iteration algorithm for solving MDPs based on MEU
  - $|S|$ nonlinear equations (due to max) with $|S|$ unknowns (utility of states).
  - Iterative approach to solve Bellman equations
  - Propagating information through state space by means of local updates
  - Solutions are unique solutions

Value Iteration

- Repeatedly perform the Bellman Update

$$U_{i+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U_i(s')] \quad \text{OR}$$

$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) U_i(s')$$

- Simultaneous updates of all states:
  - $|S|$ nonlinear equations (due to max) with $|S|$ unknowns (utility of states).
- Guaranteed to reach an equilibrium through convergence
- Final utility values must be unique solutions to the Bellman equations
- Corresponding policy is optimal

$$\pi^*(s) = \text{argmax}_{a \in A(s)} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')]$$

Value Iteration Algorithm

function VALUE-ITERATION(mdp, ϵ) **returns** a utility function
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$,
    rewards $R(s, a, s')$, discount γ
    ϵ, the maximum error allowed in the utility of any state
  local variables: U, U', vectors of utilities for states in S, initially zero
    δ, the maximum relative change in the utility of any state

  repeat
    $U \leftarrow U'$; $\delta \leftarrow 0$
    for each state s **in** S **do**
      $U'[s] \leftarrow \max_{a \in A(s)}$ Q-VALUE(mdp, s, a, U) $= \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')]$
      if $|U'[s] - U[s]| > \delta$ **then** $\delta \leftarrow |U'[s] - U[s]|$
  until $\delta \leq \epsilon(1-\gamma)/\gamma$
  return U

Bellman update

Termination condition

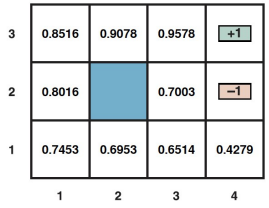
Source: RN Figure 17.6

Example: Solving Bellman Equations

$R(s) = -0.04$

Numbers are $U(s)$

How to compute the numbers?
What is the optimal policy?



Assume $\gamma = 1$

For U(1,1): $U_{i+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U_i(s')]$

$$\max\{\ [0.8(-0.04+\gamma U(1,2))+0.1(-0.04+\gamma U(2,1))+0.1(-0.04+\gamma U(1,1))], \quad \text{(U)}$$
$$[0.9(-0.04+\gamma U(1,1))+0.1(-0.04+\gamma U(1,2))], \quad \text{(L)}$$
$$[0.9(-0.04+\gamma U(1,1))+0.1(-0.04+\gamma U(2,1))], \quad \text{(D)}$$
$$[0.8(-0.04+\gamma U(2,1))+0.1(-0.04+\gamma U(1,2))+0.1(-0.04+\gamma U(1,1))]\} \quad \text{(R)}$$

$\pi^*((1,1)) = \text{Up}$

Source: RN Figure 17.3

Some Terminology

- Definitions:
  - A fixed point is any input unchanged by a function
  - A contraction is a function, when applied to two inputs, produces two outputs that are closer together
    - E.g., "divide by two" is a contraction
    - A contraction has only one fixed point
    - When contraction is applied to any argument, the value must get closer to the fixed point; repeated application reaches fixed point in the limit
  - Max norm or distance between two vectors is the maximum difference between any two corresponding elements
    - It measures "length" of a vector by the absolute value of its biggest component

Convergence

Bellman update operator

- Value iteration:
  - Converges to the (unique) utility function for discounted problems with $\gamma < 1$
- The Bellman update $U_{i+1} \leftarrow BU_i$, is a contraction by a factor of γ on the space of utility vectors:

$$\|BU_i - BU_i'\| \leq \gamma \|U_i - U_i'\|$$

  - where max norm $\|U\| = \max_s U(s)$; $\|U - U'\| = \max_s \|U(s) - U'(s)\|$
- For Bellman equations, the utility or value function U is a fixed point

$$U = BU$$

Convergence

- Repeated application of a contraction reaches a unique fixed point U

  $\|U_{i+1} - U\| = \|BU_i - BU\| \leq \gamma \|U_i - U\| \leq \gamma^i \|U_0 - U\|$ for any initial U_0
- Reaching fixed point:
  - $U = BU$ is the fixed point utility function
  - Error $\|U_i - U\|$ reduced by a factor of at least γ on each iteration; converges exponentially fast
  - Utilities of all states bounded by $\pm R_{max}/(1-\gamma)$:

$$\|U_0 - U\| \leq 2R_{max}/(1-\gamma)$$

Convergence

- Factors influencing convergence:
  - N iterations to reach error of at most ϵ:

$$\|U_N - U\| \le \gamma^N \cdot 2R_{max}/(1-\gamma) \le \epsilon$$

$$N = \left\lceil \frac{\log\left(\frac{2R_{max}}{\epsilon(1-\gamma)}\right)}{\log\left(\frac{1}{\gamma}\right)} \right\rceil$$

- Termination condition:
  - If the update is small (i.e., no state utility changes by much), then the error, compared with the true utility function, also is small

$$\text{if } \|U_{i+1} - U_i\| < \frac{\epsilon(1-\gamma)}{\gamma} \text{ then } \|U_{i+1} - U\| < \epsilon$$

Convergence: Calculations

- If we run N iterations, we get:

Error

$$\|U_N - U\| \le \frac{\gamma^N R_{max}}{1-\gamma}$$

- To get error at most ϵ, we have:

$$\frac{\gamma^N R_{max}}{1-\gamma} \le \epsilon$$
$$\gamma^N R_{max} \le \epsilon(1-\gamma)$$
$$\frac{R_{max}}{\epsilon(1-\gamma)} \le \frac{1}{\gamma^N} = \left(\frac{1}{\gamma}\right)^N$$

- Take log:

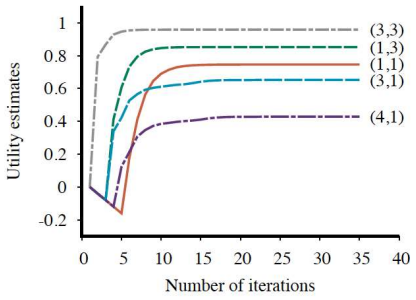
$$N \log\left(\frac{1}{\gamma}\right) \ge \log\left(\frac{R_{max}}{\epsilon(1-\gamma)}\right)$$

$$N = \left\lceil \frac{\log\left(\frac{2R_{max}}{\epsilon(1-\gamma)}\right)}{\log\left(\frac{1}{\gamma}\right)} \right\rceil$$

- Terminating condition:
  - $\|U_{t+1} - U_t\| \le \epsilon(1-\gamma)/\gamma$
  - $\Rightarrow \|U_{t+1} - U\| \le \epsilon$

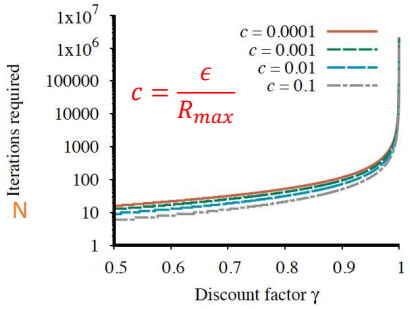
Remember: Values at each state bounded by $\pm \frac{R_{max}}{1-\gamma}$
The 2 in the numerator reflects the bounds

Example: Convergence



(a)

Evolution of utilities of selected states using VI



(b)

Num. iterations of VI required to guarantee an error of ϵ, as a function of γ

Source: RN Figure 17.7

Computing Optimal Policy

Would an MEU policy based on the estimated utilities behave optimally?

- Policy Loss
  - Let π_i be the MEU policy wrt U_i
  - Recall: $U^{\pi_i}(s)$is utility obtained if π_i is executed starting in state s
  - Policy loss $\|U^{\pi_i} - U\|$ is the max loss by following π_i instead of π^*.
- Connecting utility error and policy loss

$$\text{if } \|U_i - U\| < \epsilon \quad \text{then } \|U^{\pi_i} - U\| < 2\epsilon$$

- In practice:
  - π_i often becomes optimal long before convergence of U_i

Example: Computing Optimal Policy

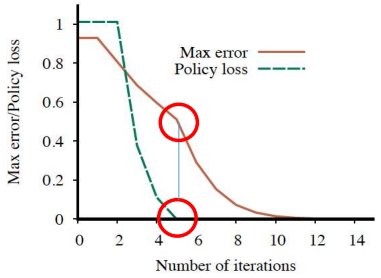


Figure 17.8 The maximum error $\|U_i - U\|$ of the utility estimates and the policy loss $\|U^{\pi_i} - U\|$, as a function of the number of iterations of value iteration on the 4×3 world.

Source: RN Figure 17.8

Policy Iteration

Solution Method

Motivation for Policy Iteration

- Value iteration:
  - Exact algorithm
  - Scales poorly
- Observations:
  - If one action is clearly better than all others, then exact magnitude of utilities on the states need not be precise
  - Utility function estimate can be inaccurate to derive optimal policy
  - There is another way to find optimal policies: Policy Iteration

Policy Iteration

- Begin with Initial policy: π_0
- Alternate between:
  - Policy evaluation:
    - Given a policy π_i, calculate $U_i = U^{\pi_i}$, utility of each state if π_i is executed
      $$U_i(s) = \sum_{s'} P(s'|s, \pi_i(s))[R(s, \pi_i(s), s') + \gamma U_i(s')]$$
  - Policy improvement:
    - Calculate new MEU policy π_{i+1} using one-step look-ahead based on U_i.
- Terminate when there is no change in utilities for policy improvement step
  - Reaching fixed point of the Bellman update
  - Finite number of policies, hence algorithm must terminate
- Complexity:
  - Assuming $|S| = n$ linear equations with n unknows – can be solved in $O(n^3)$ time.

Policy Iteration Algorithm

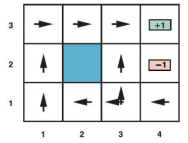
function POLICY-ITERATION(mdp) **returns** a policy
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$
  local variables: U, a vector of utilities for states in S, initially zero
    π, a policy vector indexed by state, initially random

  repeat
    $U \leftarrow$ POLICY-EVALUATION(π, U, mdp)
    $unchanged? \leftarrow$ true
    for each state s **in** S **do**
      $a^* \leftarrow \underset{a \in A(s)}{\text{argmax}}$ Q-VALUE(mdp, s, a, U) $= \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')]$
      if Q-VALUE(mdp, s, a^*, U) $>$ Q-VALUE($mdp, s, \pi[s], U$) **then**
        $\pi[s] \leftarrow a^*$; $unchanged? \leftarrow$ false
  until $unchanged?$
  return π

Policy Improvement

Source: RN Figure 17.9

Example: Implementing Policy Evaluation



- Policy evaluation

$$U_i(s) = \sum_{s'} P(s'|s,\pi_i(s))[R(s,\pi_i(s),s) + \gamma U_i(s')]$$
$$= R(s) + \gamma \sum_{s'} P(s'|s,\pi_i(s)) U_i(s')$$

No max operator

$$U_i(1,1) = 0.8[-0.04 + U_i(1,2)] + 0.1[-0.04 + U_i(2,1)] + 0.1[-0.04 + U_i(1,1)]$$
$$U_i(1,2) = 0.8[-0.04 + U_i(1,3)] + 0.2[-0.04 + U_i(1,2)]$$
$$\vdots$$

- Policy evaluation equations are linear equations

Similar, but simpler than Bellman equations

$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) U(s')$$

  - Simplified Bellman equation
  - For n states, can be solved in $O(n^3)$ time
  - For large state-spaces, use iterative method:

$$U_{t+1}(s) \leftarrow R(s) + \gamma \sum_{s'} P(s'|s,a) U_t(s')$$

Source: RN 17.2.2

Example: Implementing Policy Improvement

- Policy improvement:
  - For all the states, to find best policy:

$$\pi^*(s) = \underset{a \in A(s)}{\text{argmax}} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')]$$

  - In state (1, 1), compute:
  - If any action is better than $\pi_{old}(1,1)$, update

$$\pi_{new}(1,1) = \underset{a \in A(s)}{\text{argmax}} \sum_{s'} P((s'|(1,1),a)U(s')$$
$$\sum_{s'} P((s'|(1,1),U)U(s') = 0.8U(1,2) + 0.1U(1,1) + 0.1U(2,1) = \ldots$$
$$\sum_{s'} P((s'|(1,1),L)U(s') = \ldots$$
$$\sum_{s'} P((s'|(1,1),R)U(s') = \ldots$$
$$\sum_{s'} P((s'|(1,1),D)U(s') = \ldots$$

Assumed R(s): Needs updating wrt to eqn --->

Policy Iteration: Why Does It Work?

- Termination
  - Policy improvement step yields no change in utilities
  - Utility function U_i is a fixed point of Bellman update, and a solution to the Bellman equations
  - π_i must be an optimal policy
  - Only finitely many policies for a finite state space, each iteration can be shown to yield a better policy, hence policy iteration must terminate
- Correctness
  - Follows from Policy Improvement Theorem [SB 4.2]

Policy Improvement Theorem

- Let:
  - $Q^{\pi}(s,a) = \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U^{\pi}(s')$ be the action-utility function of π (one-step look-ahead using action a)
- Theorem: [Proof in SB 4.2]
  - Let π and π' be any pair of deterministic policies such that for all $s \in S$,
    $$Q^{\pi}(s,\pi'(s)) \geq U^{\pi}(s)$$
    $U(s) = \max_a Q(s,a)$
  - Then
    $$U^{\pi'}(s) \geq U^{\pi}(s) \text{ for all } s \in S$$
  - If the inequality of $Q^{\pi}(s,\pi'(s))$ is strict for any state s, then corresponding inequality for $U^{\pi'}(s)$ is strict for that s

Source: Sutton and Barto, 2018/2020 Section 4.2

Policy Improvement theorem – Proof

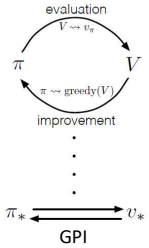
- Start with $Q^{\pi}(s,\pi'(s)) \geq U^{\pi}(s)$ and keep expanding the policy

$$U^{\pi}(s) \leq Q^{\pi}(s,\pi'(s))$$
$$= R(s) + \gamma \sum_{s'} P(s'|s,\pi'(s)) U^{\pi}(s)$$
$$= E_{\pi'}[R(S^t) + \gamma U^{\pi}(S^{t+1})|S^t = s]$$
$$\leq E_{\pi'}[R(S^t) + \gamma E_{\pi'}[R(S^{t+1}) + \gamma U^{\pi}(S^{t+2})|S^{t+1}]|S^t = s]$$
$$= E_{\pi'}[R(S^t) + \gamma R(S^{t+1}) + \gamma^2 U^{\pi}(S^{t+2})|S^t = s]$$
$$\leq E_{\pi'}[R(S^t) + \gamma R(S^{t+1}) + \gamma^2 R(S^{t+2}) + \gamma^3 U^{\pi}(S^{t+3})|S^t = s]$$
$$\vdots$$
$$\leq E_{\pi'}[R(S^t) + \gamma R(S^{t+1}) + \gamma^2 R(S^{t+2}) + \gamma^3 R(S^{t+3}) + \cdots |S^t = s]$$
$$= U^{\pi'}(s)$$

- If the first inequality is strict, we have $U^{\pi}(s) < U^{\pi'}(s)$ □

Source: Sutton and Barto, 2018/2020 Section 4.2

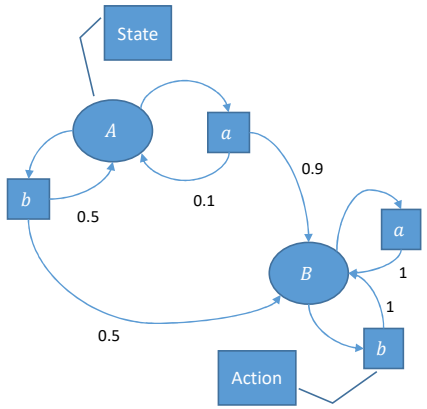
General Policy Iteration



- Modified policy iteration:
  - Do only k iterations of (simplified) value iteration instead of reaching convergence
  - Approximate evaluation
- Asynchronous policy iteration:
  - Pick only a subset of states for policy improvement or for updating in policy evaluation
  - Converges as long as continuously update all states
- Generalized policy iteration: (SB) 4.6
  - Update utility according to policy, and improve policy wrt the utility function.
  - Value iteration, policy iteration, asynchronous policy iteration are all special cases.
  - May interleave in different ways based on conditions

MDP Representation as State Transition Graph

- For a 2-state MDP (States: $\{A, B\}$), with actions $\{a, b\}$, and the following transition function:
  - State A:
    - a takes to B with 90% probability; remain at A with 10% probability
    - b takes to B with 50% probability; remain at A with 50% probability
  - State B:
    - Both actions will loop with 100% probability
- We can represent the MDP as a state transition graph as shown alongside

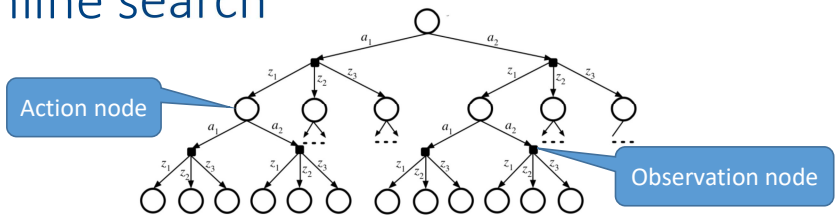


Online search

Curse of dimensionality

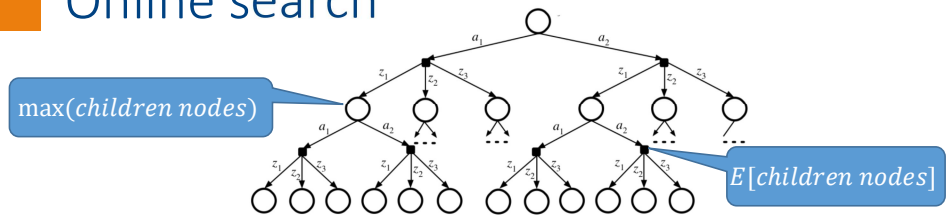
- State space grows exponential with the number of variables
- VI/PI iterates through all states; so exponential with the number of variables
- To handle curse of dimensionality
  - Use function approximation
    - Linear function of features
    - Deep neural networks
  - Do online search with sampling

Online search



- At every step, construct a search tree
  - Up to a fixed depth D
  - Root is the current state
  - $|A|$children of the root (and other action nodes)
  - $|S|$ children of observation nodes

Online search

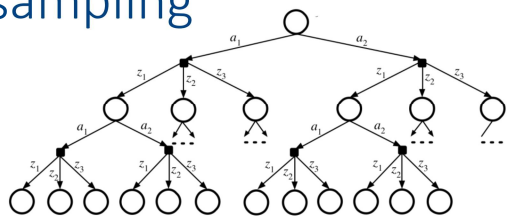


- To compute the value at the root:
  - Initialize leaf with value estimates (or zeros)
  - At observation node, compute the expected values of the children
  - At the action nodes, compute the max of the children

Q: Have we fixed the curse of dimensionality?

Sparse sampling

Don't search the entire tree!

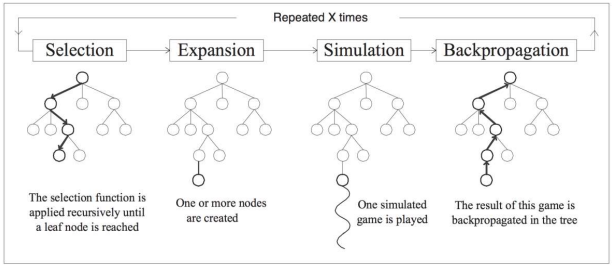


- Tree size: $|A|^D |S|^D$
- Sparse sampling[1]
  - Estimate by sampling k observations at observation nodes instead of using $|S|$ states as possible observations
    - Tree size now: $|A|^D k^D$
    - Curse of dimensionality is solved …
      … but still exponential with the search depth – curse of history

[1]Michael Kearns, Yishay Mansour, and Andrew Y Ng. "A sparse sampling algorithm for near-optimal planning in large Markov decision processes". In: Machine learning 49.2-3 (2002), pp. 193-208.

Monte Carlo Tree Search

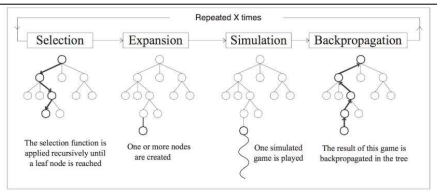
MCTS[2]



- Commonly used to solve MDP and games

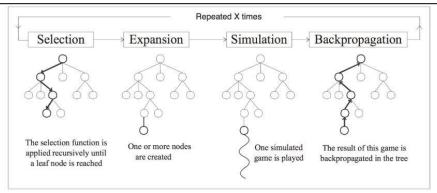
[2]Guillaume Chaslot et al. "Monte-Carlo Tree Search: A New Framework for Game AI". In: AIIDE. 2008.

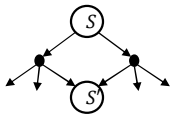
MCTS



- Repeatedly run trials from the root (current state in online search)
- Trial:
  - Repeatedly select the node to go to at the next level until
    - Target depth is reached, or
    - Selected node has not been discovered – create a new node; run a simulation using a default policy till required depth
  - Backup the outcomes all the way to the root
- Anytime policy: When time is up, use the action that looks best at the root at that time

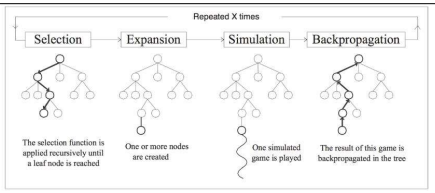
MCTS





- For MDP, a tree (actually DAG) node n is associated with a state s
- A node n' at the next level is selected by applying an action a to s
  - Sample next state s' (corresponding to n') according to $P(s'|s,a)$
- Action a is selected to balance exploration with exploitation

MCTS



Estimated

- Estimated value $\hat{V}(n)$ at node n is the average return of all trials at n
  - Return $r_t(n)$ of trial t starting from n with state s and next node n' is
$$r_t(n) = R(s) + \gamma r_t(n')$$
- Estimated Q-function at n, $\hat{Q}(n,a)$, is the average return of all trials at n that start with action a
  - $\hat{Q}(r,a)$ at the root r is used to select the action to take at the root node
- All these are updated in the backup operation to the root

Upper confidence tree (UCT)

UCT[3]

Used for tuning the performance

- Action selection is guided by this function:

Confidence interval

Being greedy

$$\pi_{UCT}(n) = \arg\max_a Q(n,a) + c\sqrt{\frac{\ln(N(n))}{N(n,a)}}$$

  - Where, $N(n)$ and $N(n,a)$ count the number of trials through n and (n,a) respectively and c is a constant
- UCT will eventually converge to the optimal policy with enough trials
  - Worst case can be very bad[4] $\Omega(\exp(\exp(\ldots \exp(1) \ldots)))$ ($D-1$ times)
  - Often works well in practice
    - E.g., PROST planner[5], which uses UCT, won the international probabilistic planning competition in 2011 & 2014

[4] Levente Kocsis and Csaba Szepesvari. "Bandit based monte-carlo planning". In: European conference on machine learning. Springer. 2006, pp. 282-293.

MCTS in practice – AlphaGo Zero

- AlphaGo Zero[6]
  - Uses MCTS + Approximate Policy Iteration
  - Play against Self to learn
  - Defeated AlphaGo (that beat Lee Sedol) 100-0!
- Go has a state space size of about 10^{170}
  - Need function approximation to represent value and policy functions
- AlphaGo Zero uses deep neural network with two heads (outputs)
  - Value head – outputs real value estimate of the value function (board position)
  - Policy head – outputs a vector of size 19×19 (maps board position to action) Incorrect in the video
    - Each item represents the probability that the policy will play that board position

[6]David Silver et al. "Mastering the game of Go without human Knowledge". In: Nature 550.7676 (2017), p. 354.

AlphaGo Zero – MCTS

- Variant of UCT that exploits the policy head output: $P(s,a)$

$$\pi_{UCT}(s) = \arg\max_a \hat{Q}(s,a) + c\, P(s,a) \sqrt{\frac{\sum_b N(s,b)}{1+N(s,a)}}$$

  - When leaf node is reached, the value head output is used to evaluate the state instead of doing a roll-out (simulation)
- Go is a zero-sum, turn taking game instead of MDP
  - Search alternates between:
    - selecting action that maximizes when it is first player's turn
    - selecting the action that minimizes for second player's turn
  - At termination: reward $+1$ for the first player win and -1 for second player win

AlphaGo Zero – Approximate PI

- Recollect: Policy iteration has 2 stages, policy evaluation and policy improvement
- AlphaGo Zero does both using supervised learning
- With the current value function, MCTS can be viewed as a policy improvement operator – gives improved policy values for the evaluated states
- Self-play with search gives the policy evaluation for the evaluated states
- Supervised learning is used to interpolate the values and the policy over the whole domain using data from a set of states

NUS | Computing

Reinforcement Learning

CS4246/CS5446

AI Planning and Decision Making

Please join:
pollev.com/anarayan

This lecture will be recorded!

Recall: Sequential Decision Problems

- What are sequential decision problems?
  - An agent's utility depends on a sequence of decisions
  - Incorporate utilities, uncertainty, and sensing
  - Search and planning problems are special cases
  - Decision (Planning) Models:
    - Markov decision process (MDP)
    - Partially observable Markov decision process (POMDP)
    - Reinforcement learning: sequential decision making + learning

Solving Sequential Decision Problems

- Decision (Planning) Problem or Model
  - Appropriate abstraction of states, actions, uncertain effects, goals (wrt costs and values or preferences), and time horizon + observations (through sensing)
- Decision Algorithm
  - Input: a problem
  - Output: a solution in the form of an optimal action sequence or policy over time horizon
- Decision Solution
  - An action sequence or solution from an initial state to the goal state(s)
    - An optional solution or action sequence; OR
    - An optimal policy that specifies "best" action in each state wrt to costs or values or preferences
  - (Optional) A goal state that satisfies certain properties

Recall: Decision Making under Uncertainty

- Decision Model:
  - Actions: $a \in A$
  - Uncertain current state: $s \in S$ with probability of reaching: $P(s)$
  - Transition model of uncertain action outcome or effects:
    $P(s'|s,a)$ – probability that action a in state s reaches state s'
  - Outcome of applying action a:
    Result(a) – random variable whose values are outcome states
  - Probability of outcome state s', conditioning on that action a is executed:
    $P(\text{Result}(a) = s') = \sum_s P(s)P(s'|s,a)$
  - Preferences captured by a utility function:
    $U(s)$ – assigns a single number to express the desirability of a state s

Recall: Markov Decision Process (MDP)

- Formally:
  - An MDP $M \triangleq (S, A, T, R)$ consists of:
  - A set S of states
  - A set A of actions
  - A transition function $T: S \times A \times S \to [0,1]$ such that:

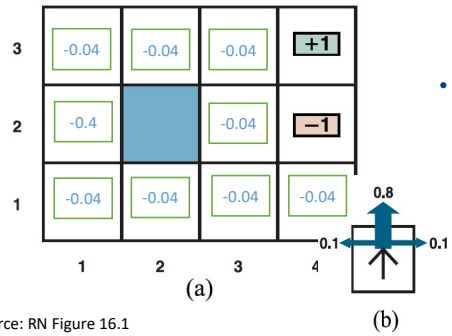
$$\forall s \in S, \forall a \in A: \sum_{s' \in S} T(s,a,s') = \sum_{s' \in S} P(s'|s,a) = 1$$

  - A reward function $R: S \to \Re$ or $R: S \times A \times S \to \Re$

Solution is a policy – a function to recommend an action in each state: $\pi: S \to A$

- Solution involves careful balancing of risk and reward

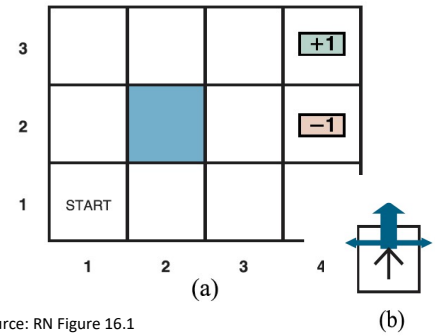
Example: Navigation in Grid World



Source: RN Figure 16.1

- Transition model:
  - Define (stochastic) outcome of each action
  - $P(s'|s,a)$ – probability of reaching state s' if action
- Reward model:
  - Define reward received for every transition
  - $R(s,a,s')$ – For every transition from s to s' via action a; AND/OR
  - $R(s)$ – For any transition into state s
  - Rewards may be positive or negative, but bound by $\pm R_{max}$
  - Utility function $U(s)$ depends on the sequence of states and actions – environment history – sum of rewards of the states in the sequence

Example: Navigation



Source: RN Figure 16.1

- Transition model:
  - Unknown
- Reward model:
  - Unknown

Reinforcement Learning

- Decision making in complex and uncertain environments
  - Learning to behave proficiently in unfamiliar environment, given only percepts and occasional rewards
  - E.g., playing chess, flying helicopter, exploring Mars, assisting elderly
- Assume:
  - Environment is a Markov Decision Process (MDP)
  - Optimal policy – policy that maximizes the expected total reward

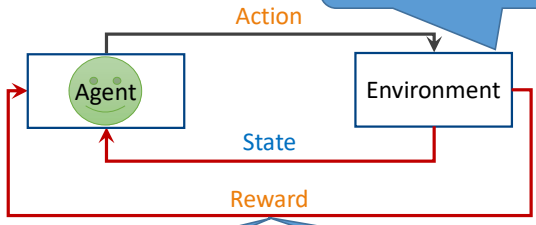
Rely on the Markov property

- Reinforcement learning
  - Use observed rewards to learn an optimal policy for the environment
  - Often with no knowledge of the environment model or reward function
    - E.g., don't know the game rules, just play until Win/Loss declared
    - Would this be a good approach for an AI tutor?
  - Influences from psychology, neuroscience, operations research, and optimal control theory.

Reinforcement Learning

- The agent – environment loop

Fully observable environment based on percepts
Unknown environment
Unknown probabilistic action outcomes



Reward or Reinforcement as feedback for learning – achieved along the way or at the end

"Hardwired" or given rewards, e.g., hunger and pain, pleasure and food

Part of the input perception; agent must differentiate & recognize it as a *reward*

Types of RL Agents

- Reinforcement learning
  - [PL] Passive Learning: Learning Utility functions
  - [AL] Active Learning : Learning to Plan or Decide
  - [MB] Model-based: Learn transition model to solve problem
  - [MF] Model- free: Do not (need to) learn transition model to solve problem
- Passive learning agent
  - Has a fixed policy that determines his behavior
  - sf policy evaluation in policy iteration
- Active learning agent
  - Must decide what actions to take
  - sf value iteration

Passive Learning

Predict utility function (value function) of state given a fixed policy

Passive Learning

Main idea:

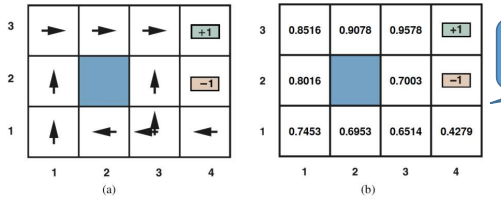
- Agent executes a set of trials in environment using policy π; starts in $(1,1)$ until reaches terminal state
- Percepts supply both current state and reward received in state
- Use reward information to learning expected utility $U^{\pi}(s)$associated with each non-terminal state s

$$U^{\pi}(s) = E[\sum_{t=0}^{\infty} \gamma^t R(S_t)]$$

where $R(s)$ is reward for a state s, S_t is a random variable indicating state reached at time t when executing policy π

  - sf policy evaluation (in policy iteration)
- Assume $\gamma = 1$ in examples that follow

Learning Utility Function

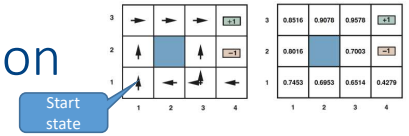


Source: RN Figure 23.1

- The policy π is fixed:
  - Agent always executes $\pi(s)$ in state s
- Goal: Learn utility function or value function $U^{\pi}(s)$ from observations
  - Unknown transition model, $P(s'|s,a)$
  - Unknown reward function, $R(s)$

} If known, simply do policy evaluation!

Learning Utility Function

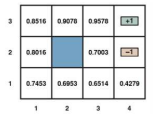


- Agent executes a series of trials using π, e.g.,
  - Trial 1: $(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{+1} (4,3)$
  - Trial 2: $(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{-.04} (3,2) \xrightarrow[Up]{-.04} (3,3) \xrightarrow[Right]{+1} (4,3)$
  - Trial 3: $(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{-.04} (3,2) \xrightarrow[Up]{-1} (4,2)$
- Utility or value of a state:
  - Expected total reward from that state onwards
  - Also called expected reward-to-go / expected return
- Utility/value of s under π is:

$$U^{\pi}(s) = E\left[\sum_{t=0}^{\infty} \gamma^t R(S_t) | S_0 = s\right]$$

Assume $\gamma = 1$

Monte Carlo Learning



$(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{+1} (4,3)$

$(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{-.04} (3,2) \xrightarrow[Up]{-.04} (3,3) \xrightarrow[Right]{+1} (4,3)$

$(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{-.04} (3,2) \xrightarrow[Up]{-1} (4,2)$

- Main idea:
  - Maintain a running average of expected return for each state in a table
  - Run infinitely many trials; value converges to the true expected value
  - Number of visits may be treated differently: e.g., first visit vs. every visit
    - E.g., consider only 0.8 for (1,2); 0.88 is ignored
    - E.g., consider both 0.8 and 0.88 for (1,2)
    - Both converge to the true expected value in the limit

Monte Carlo Learning

Also called Direct Utility Estimation:

An instance of supervised learning - Each example has state as input and observed return (unbiased estimate) as output

$$((x_1, y_1), u_1), ((x_2, y_2), u_2), \ldots, ((x_n, y_n), u_n)$$

where u_j is the measured utility of the j^{th} example

Very slow convergence:

Need to wait till the end of the episode before learning can begin

By ignoring constraints, searches much larger space, including utility functions that violate the Bellman equations for a fixed policy:

$$U^{\pi}(s) = \sum_{s'} P(s'|s, \pi(s))[R(s, \pi(s), s') + \gamma U^{\pi}(s')$$

Example: Adaptive Dynamic Programming

$(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{+1} (4,3)$

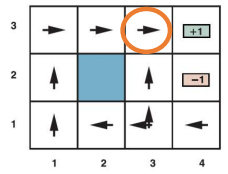
$(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{-.04} (3,2) \xrightarrow[Up]{-.04} (3,3) \xrightarrow[Right]{+1} (4,3)$

$(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{-.04} (3,2) \xrightarrow[Up]{-1} (4,2)$

What is the transition probability function from $s = (3,3)$?

To compute: Why are all the actions $Right$?

- $P(s' = (3,2) | s = (3,3), a = Right) = \frac{1}{3}$
- $P(s' = (4,3) | s = (3,3), a = Right) = \frac{2}{3}$



Adaptive Dynamic Programming (ADP)

- Learn transition function T empirically through experience
  - Count action outcomes for each s, a — Learn by counting and average!
  - Normalize to give estimates of transition probabilities $P(s'|s,a)$
- Learn reward function $R(s,a,s')$ upon entering state s'
- Solve MDP with learned T
  - Given π, perform policy evaluation for all s:
    $$U^{\pi}(s) = \sum_{s'} P(s'|s,\pi(s))[R(s,\pi(s),s') + \gamma U^{\pi}(s')]$$
  - Solve $|S|$ linear equations with $|S|$ unknowns in $O(|S|^3)$ time
- Notes:
  - Can learn $P(s'|s,a)$ using supervised learning
  - Exploits constraints among utilities of states

ADP-Learner

function PASSIVE-ADP-LEARNER($percept$) **returns** an action
  inputs: $percept$, a percept indicating the current state s' and reward signal r
  persistent: π, a fixed policy
    mdp, an MDP with model P, rewards R, actions A, discount γ
    U, a table of utilities for states, initially empty
    $N_{s'|s,a}$, a table of outcome count vectors indexed by state and action, initially zero
    s, a, the previous state and action, initially null

Using tabular representation

  if s' is new **then** $U[s'] \leftarrow 0$
  if s is not null **then**
    increment $N_{s'|s,a}[s,a][s']$
    $R[s,a,s'] \leftarrow r$
    add a to $A[s]$
    $\mathbf{P}(\cdot \mid s,a) \leftarrow$ NORMALIZE($N_{s'|s,a}[s,a]$)
  $U \leftarrow$ POLICYEVALUATION(π, U, mdp)
  $s,a \leftarrow s', \pi[s']$
  return a

Reward function R

Transition function T

MLE

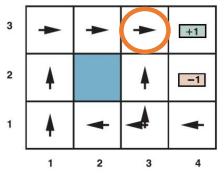
Source: RN Figure 23.2

Example: Temporal-Difference (TD) Learning

$$(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{+1} (4,3)$$
$$(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{-.04} (3,2) \xrightarrow[Up]{-.04} (3,3) \xrightarrow[Right]{+1} (4,3)$$
$$(1,1) \xrightarrow[Up]{-.04} (1,2) \xrightarrow[Up]{-.04} (1,3) \xrightarrow[Right]{-.04} (2,3) \xrightarrow[Right]{-.04} (3,3) \xrightarrow[Right]{-.04} (3,2) \xrightarrow[Up]{-1} (4,2)$$

After first trial:

- $U^{\pi}(1,3) = \frac{0.84+0.92}{2} = 0.88$
- $U^{\pi}(2,3) = 0.96$
- In the long run, must obey:
  - $U^{\pi}(1,3) = -0.04 + U^{\pi}(2,3)$
- Hence:
  - $U^{\pi}(1,3) = 0.92$ Current value of 0.88 needs updating!



Temporal Difference Learning (TD)

Update rule uses difference in utlities between successive states

- Temporal difference learning (TD)
  - For a transition from state s to s', TD learning does:
    $$U^{\pi}(s) \leftarrow U^{\pi}(s) + \alpha\big(R(s,\pi(s),s') + \gamma U^{\pi}(s') - U^{\pi}(s)\big)$$
  - where, α is the learning rate
- In TD learning:
  - TD target: $R(s,\pi(s),s') + \gamma U^{\pi}(s')$
  - TD term or error: $R(s,\pi(s),s') + \gamma U^{\pi}(s') - U^{\pi}(s)$
    - TD term is error signal, update is intended to reduce error
  - Increases $U^{\pi}(s)$ if $R(s,\pi(s),s') + \gamma U^{\pi}(s')$ is larger than $U^{\pi}(s)$; decreases otherwise
  - Converges if α decreases appropriately with the number of times the state has visited. E.g., $\alpha(n) = O\left(\frac{1}{n}\right)$

Temporal-Difference (TD) Learning

- Main idea:
  - Adjust utility estimates towards ideal (local) equilibrium with correct utility estimates
  - In passive learning equilibrium is given by the Bellman equation for a fixed policy:

$$U_i(s) = \sum_{s'} P(s'|s, \pi_i(s))[R(s, \pi_i(s), s') + \gamma U_i(s')]$$

- TD update:
  - Involves only observed successor s', average value of $U^{\pi}(s)$ will converge to correct value
  - If learning rate α is changed from fixed parameter to decreasing function of increasing number of visits to a state, then $U^{\pi}(s)$ converges
- TD vs MC and ADP:
  - Exploits more of the Bellman equation constraints than MC
  - Does not need to learn the model (here refers to transition function T), unlike ADP

TD-Learner (Model-Free RL)

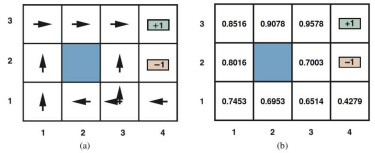
function PASSIVE-TD-LEARNER($percept$) **returns** an action
  inputs: $percept$, a percept indicating the current state s' and reward signal r
  persistent: π, a fixed policy
    s, the previous state, initially null
    U, a table of utilities for states, initially empty
    N_s, a table of frequencies for states, initially zero

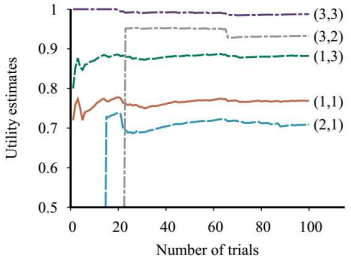
  if s' is new **then** $U[s'] \leftarrow 0$
  if s is not null **then**
    increment $N_s[s]$
    $U[s] \leftarrow U[s] + \alpha(N_s[s]) \times (r + \gamma\, U[s'] - U[s])$
  $s \leftarrow s'$
  return $\pi[s']$

- TD needs only observed transitions to do the updates; no transition model is needed!
- Learn from every experience – update each time we experience s, $\pi(s)$, and s'

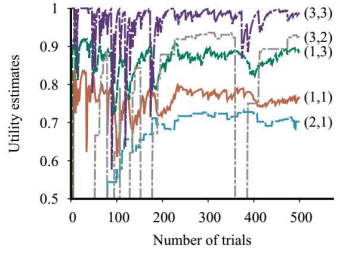
Source: RN Figure 23.4

Example: ADP vs TD





Adaptive Dynamic Programming



Temporal Difference Learning

Source: RN Figure 23.3 (a) and 23.5 (a)

ADP vs TD/MC

- ADP
  - Learns the model and then solves for the value
- TD/MC
  - Don't need a model
  - Can work with measurements from real-world or simulator
- ADP
  - More data efficient
  - Requires less data from the real-world
- TD
  - Doesn't need to compute expectation
  - Computationally more efficient

Active Learning

Choose to act optimally based on prediction of utility or value of states

Exploration vs exploitation, Q-learning, SARSA

Active Learning

- Main ideas:
  - T and R unknown; can choose any action a at each step
  - Goal: Learn optimal policy π^* that obey the Bellman equations:

$$U(s) = \max_{a \in A(s)} \sum_{s'} P(s'|s,a)[R(s,a,s') + \gamma U(s')]$$

- In summary:
  - Learner makes choices
  - Tradeoff between exploration vs. exploitation:
  - If learned $\hat{T}$ is no the true T, optimal policy obtained using $\hat{T}$ may be suboptimal wrt T, possibly incurring large policy loss

From Passive to Active ADP-Learner

function PASSIVE-ADP-LEARNER($percept$) **returns** an action
  inputs: $percept$, a percept indicating the current state s' and reward signal r
  persistent: π, a fixed policy ← π not fixed
    mdp, an MDP with model P, rewards R, actions A, discount γ
    U, a table of utilities for states, initially empty
    $N_{s'|s,a}$, a table of outcome count vectors indexed by state and action, initially zero
    s, a, the previous state and action, initially null

  if s' is new **then** $U[s'] \leftarrow 0$
  if s is not null **then**
    increment $N_{s'|s,a}[s,a][s']$
    $R[s,a,s'] \leftarrow r$
    add a to $A[s]$
    $\mathbf{P}(\cdot \mid s,a) \leftarrow$ NORMALIZE($N_{s'|s,a}[s,a]$)
    $U \leftarrow$ POLICYEVALUATION(π, U, mdp) ← $U \leftarrow \text{PolicyIteration}(\pi, U, mdp)$
  $s,a \leftarrow s', \pi[s']$
  return a

Reward function R

Transition function T

Question:

Is this a good reinforcement learning algorithm?

- Replace POLICY-EVALUATION step by $\pi \leftarrow$ POLICY-ITERATION(mdp) above
- π is no longer fixed; changes as transitions & rewards are learned

Source: RN Figure 23.2

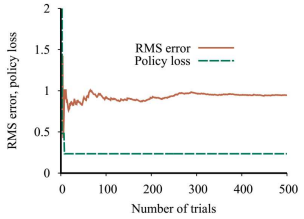
Active Adaptive Dynamic Programming

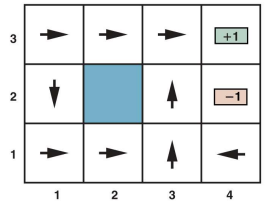
- For active-ADP:
  - Learn model T with outcome probabilities for all actions, not just for a fixed policy.
  - Learn utilities defined by optimal policy that obey the Bellman equation
  - Then, run VI or PI to determine optimal policy
  - Extract an optimal action by one-step look-ahead to maximize expected utility
- Note:
  - Learner has choice of actions
  - Actions don't just maximize rewards for given current learned model T, they also affect the percepts (states and rewards) received
  - Should learner then just execute action recommended by optimal policy? Why?

Greedy ADP Learner

RMS error in the utility estimates averaged over non-terminal states

Policy loss: $U^{\pi^*} - U^{\pi}$





- Resulting algorithm is greedy w.r.t the policy
- Policy resulting from executing action suggested by the learned model
  - Note, for e.g., at $(2, 1)$ it goes right instead of left
  - Policy reasonable, but not optimal
- Agent is greedy – sometimes doesn't try better policies sufficiently after finding a suboptimal one

Source: RN Figure 23.6

Exploration Vs Exploitation

Hope to learn something new about the problem!

- Actions in RL gain reward and help to learn a better model
  - By improving the model, better reward may potentially be obtained
- Need to trade off
  - Exploration: Choose action to learn the true model T (by affecting the states and rewards received) to receive greater rewards in future
  - Exploitation: Given current learned model T, choose action to maximize the reward (as reflected in current utility estimates)
- Main questions:
  - How to balance exploration and exploitation to maximize long-term expected rewards?
  - When to "stop" learning the model?
- With greater understanding, less exploration is necessary!

Greedy in the Limit of Infinite Exploration

- GLIE: A scheme for balancing exploration and exploitation
  - Try each action in each state an unbounded number of times to avoid a finite probability of missing an optimal action
  - Eventually become greedy so that it is optimal w.r.t the true model
- GLIE-based schemes for forcing exploration in RL:
  - ϵ-greedy exploration
    - Choose the greedy action with probability $(1 - \epsilon)$
    - Choose a random action with probability ϵ
  - GLIE based ϵ-greedy eventually converges to the optimal, but can be slow
    - One solution: lower ε over time (e.g., $\epsilon = \frac{1}{t}$)
    - Another solution: use exploration functions!

Optimism

- Alternatively, balance exploration and exploitation using greedy action selection w.r.t an optimistic estimate of the utility, $U^+(s)$
- One way to construct $U^+(s)$ is to use an exploration function $f(u, n)$ with the following update:

$$U^+(s) \leftarrow \max_a f\left(\sum_{s'} P(s'|s,a)[R(s,\mathrm{a},s') + \gamma U^+(s)], N(s,a)\right)$$

  - Where, $N(s, a)$ is the number of times action a has been tried in state s

Exploration Functions

- When to explore:
  - Random actions: explore a fixed amount
  - Better idea: explore areas where "badness" has not been established
- Exploration function: f
  - Takes a value estimate (u) and a count (n), and returns an optimistic utility $f(u,n)$
  - Determines how greed (preference for high values of u) is traded off against curiosity (preference for actions that have not been tried often with low n)
  - f should be increasing in u and decreasing in n (form is not important)

Example: Exploration Functions

- An exploration function $f(u,n)$ that is increasing with u and decreasing with n:

$$f(u,n) = \begin{cases} R^+ & \text{if } n < N_e \\ u & \text{otherwise} \end{cases}$$

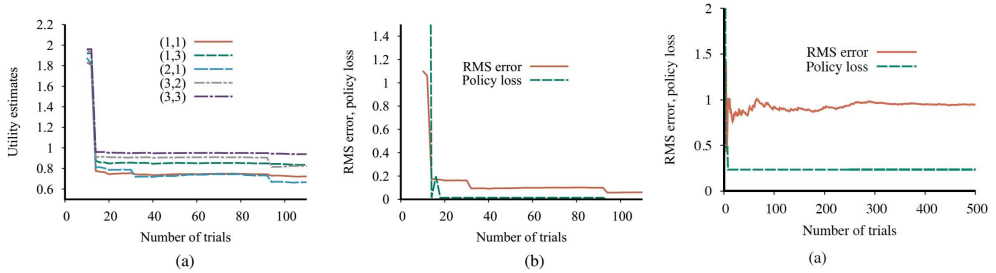
Where:
- R^+ is an optimistic estimate of the best possible reward
- N_e is the parameter

Agent tries each state-action pairs N_e times

- Example: RMAX [1]
  - Works well for small problems, has guarantees

[1] Brafman, R.I. and M. Tennenholtz, *R-max - a general polynomial time algorithm for near-optimal reinforcement learning.* J. Mach. Learn. Res., 2003. **3**(null): p. 213–231.

Example: Navigation in Grid World



Exploring ADP

Greedy ADP

- Exploring ADP using exploration function with: $R^+ = 2, N_e = 5$

Source: RN Figure 23.6 (a) 23.7 (a) and (b)

Q-Learning: Learning Action-Utility Functions

- Q-function – Action-utility function $Q(s,a)$
  - Expected total discounted reward if action a is taken in state s
  - Q-functions (also called Q-values) are related to utilities by: $U(s) = \max_a Q(s,a)$
- Q-Learning
  - Model-free learning: Agent can act optimally knowing Q-function; no need for look-ahead based on a transition model
  - The Bellman equation for Q-functions:

$$Q(s,a) = \sum_{s'} P(s'|s,a)\,[R(s,a,s') + \gamma \max_{a'} Q(s',a')]$$

  - Uses TD update to learn action-utility functions or Q-functions:

$$Q(s,a) \leftarrow Q(s,a) + \alpha[R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)]$$

TD Term or Error

Q-learning: Active TD Learning

function Q-LEARNING-AGENT($percept$) **returns** an action
  inputs: $percept$, a percept indicating the current state s' and reward signal r
  persistent: Q, a table of action values indexed by state and action, initially zero
    N_{sa}, a table of frequencies for state–action pairs, initially zero
    s, a, the previous state and action, initially null

  if s is not null **then**
    increment $N_{sa}[s, a]$
    $Q[s, a] \leftarrow Q[s, a] + \alpha(N_{sa}[s, a])(r + \gamma \max_{a'} Q[s', a'] - Q[s, a])$
  $s, a \leftarrow s', \text{argmax}_{a'} f(Q[s', a'], N_{sa}[s', a'])$
  return a

TD Update

Exploration function

TD Term or Error

- TD Q-learning does not need a transition model, $P(s'|s, a)$ for learning or action selection

Source: RN Figure 23.8

SARSA

- State-Action-Reward-State-Action (SARSA):
  - Uses TD for prediction (evaluation) and ϵ-greedy for action selection

$$Q(s, a) \leftarrow Q(s, a) + \alpha[R(s, a, s') + \gamma Q(s', a') - Q(s, a)]$$

  where a' is the action taken at state s'
- Note:
  - Rule is applied at the end of each quintuple (s, a, r, s', a')
  - With appropriately decreasing ϵ, SARSA converges to the optimal policy

Q-Learning vs SARSA

```
1: controller Q-learning(S, A, γ, α)
2:     Inputs
3:         S is a set of states
4:         A is a set of actions
5:         γ the discount
6:         α is the step size
7:     Local
8:         real array Q[S, A]
9:         states s, s'
10:        action a
11:    initialize Q[S, A] arbitrarily
12:    observe current state s
13:    repeat
14:        select an action a
15:        do(a)
16:        observe reward r and state s'
17:        Q[s,a] := Q[s,a] + α*(r + γ*max_a' Q[s',a'] − Q[s,a])
18:        s := s'
19:    until termination
```

Figure 12.3: Q-learning controller

```
1: controller SARSA(S, A, γ, α)
2:     Inputs
3:         S is a set of states
4:         A is a set of actions
5:         γ the discount
6:         α is the step size
7:     Local
8:         real array Q[S, A]
9:         state s, s'
10:        action a, a'
11:    initialize Q[S, A] arbitrarily
12:    observe current state s
13:    select an action a using a policy based on Q
14:    repeat
15:        do(a)
16:        observe reward r and state s'
17:        select an action a' using a policy based on Q
18:        Q[s,a] := Q[s,a] + α*(r + γ*Q[s',a'] − Q[s,a])
19:        s := s'
20:        a := a'
21:    until termination
```

Figure 12.5: SARSA: on–policy reinforcement learning

Source: Poole and Macwark, AI Foundations of Computational Agents, 2e, 2018

On-policy vs Off-policy Learning[1]

- On-policy learning – data used for learning is generated by the policy being learned
- Situations where data isn't generated by the policy currently being learned
  - Data collected using currently running policy
  - Explore using a different policy
- In off-policy methods:
  - Generate data using a behavior policy
  - Try to learn a target policy

What is the issue here?

[1]Material for this section are mostly from Sutton & Barto

Off-Policy and On-Policy Learning

- Q-learning update rule:

$$Q(s,a) \leftarrow Q(s,a) + \alpha\left(R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)\right)$$

- SARSA update rule:

$$Q(s,a) \leftarrow Q(s,a) + \alpha\left(R(s,a,s') + \gamma Q(s',a') - Q(s,a)\right)$$

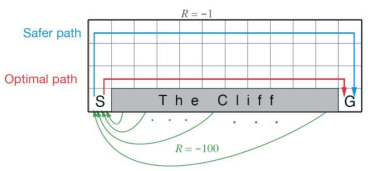
- Q-learning is off-policy – Target of Q-learning uses the max over possible actions a' at s'
  - Doesn't need the actual action in the next state in the update
  - Can converge to the optimal policy even when trained off-policy
    - Needs appropriately decaying α
    - Each action is taken in each state infinitely often
- SARSA is on-policy – Target uses Q-function from the policy that is running
  - Using Q-function from another state-action pair will give incorrect target value

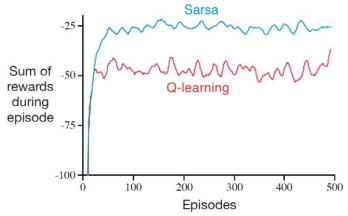
Q-Learning vs SARSA

- Similarities:
  - If agent is greedy and always takes action with the best Q-function: the two are identical
  - Both converge slowly as local updates do not enforce consistency among all the Q-functions via model
- Differences:
  - Q-learning backs up Q-function from the best action in s'
  - SARSA waits until an action is taken and backs up Q-function for that action
  - If exploration yields a negative reward: SARSA penalizes the action, Q-learning does not

Example: Cliff Walking

From Sutton & Barto



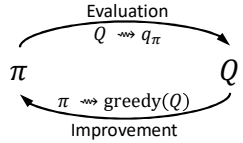


Source: SB: Example 6.6

- Actions: left, right, up, down
- Reward: -1 for each step; -100 if you fall off
- SARSA & Q-learning used with ϵ-greedy; $\epsilon = 0.1$
  - Why does SARSA outperform Q-learning?
  - What happens if GLIE exploration is used?

Generalized Policy Iteration

- How can MC and TD prediction methods be used to do control?
- Can use the idea of generalized policy iteration (GPI)



- Repeatedly:
  - Do some policy evaluation (using MC or TD), then use greedy action with respect to the current policy (policy improvement) to get more samples

NUS | Computing

"Real World" Reinforcement Learning

CS4246/CS5446

AI Planning and Decision Making

Please join:
pollev.com/anarayan

This lecture will be recorded!

Recall: Reinforcement Learning (RL)

- Based on:
  A Markov decision process (MDP) $M \triangleq (S, A, T, R)$ consisting of:
  - A set S of states
  - A set A of actions
  - Missing transition function T?
  - Missing reward function R?
- Learning (prediction):
  - Assume policy
  - Solution is an (optimal) utility (value) function: $U: S \to \Re$ or $V: S \to \Re$
- Planning (control)
  - Assume utility function or Q-function (action-utility function)
  - Solution is an (optimal) policy: $\pi: S \to A$

- Categorization of methods
- Monte Carlo (MC) Learning
  - aka Direct utility estimates
- Adaptive dynamic programming (ADP)
- Temporal difference (TD) learning:
  - Q-learning and SARSA

Scaling

- Number of states grow exponentially with no. of state variables (features)
- Tabular representation (for utility function and Q-function) scales to tens of thousands of states
  - e.g., Number of states in Backgammon & Chess are of the order of 10^{20} & 10^{40}
  - Still considered relatively small as compared to real-world problems!
  - Cannot visit all the states infinitely often
- Approaches to scaling up:
  - Function approximation – approximating utility (value) functions
  - Policy search – systematic search for good policies

Function Approximation

Linear:
- Approximate Monte Carlo Learning
- Approximate temporal difference (TD) learning

Non-linear:
- Deep neural networks

Function Approximation

- Function approximation constructs compact representation of true utility (value) function and Q-function
  - Example: Represent evaluation function for chess as a linear function of features (or basis functions)

$$\hat{U}_\theta(s) = \theta_1 f_1(s) + \theta_2 f_2(s) + \cdots + \theta_n f_n(s)$$

  - Function approximation uses the n no. of θ parameters to represent a function over a very large number of states
  - RL agent learns θ that best approximate the evaluation (utility) function

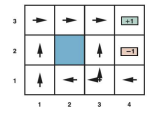
Function Approximation

- Function approximation allows generalization from small number of states observed in training data to entire state space
  - Example: Backgammon agent learned to play as well as the best human players by observing only $\approx 10^{12}$ states out of 10^{20} states
- Caveat:
  - If n (no. of parameters) is too small, may fail to achieve good approximation

Linear Function Approximation

Approximate Monte Carlo Learning
Approximate temporal difference (TD) learning

Example: Navigation in Grid World



Source: RN Figure 17.2

- Use:

Compact representation + generalization

$$\hat{U}_\theta(x, y) = \theta_0 + \theta_1 x + \theta_2 y$$

  - If $(\theta_0, \theta_1, \theta_2) = (0.5, 0.2, 0.1)$, then $\hat{U}(1,1) = 0.8$
- Given a collection of trials:
  - Obtain a set of sample values of $\hat{U}_\theta(x, y)$
  - Find the best fit, in the sense of minimizing the squared error, using standard linear regression

Approximating Monte Carlo Learning

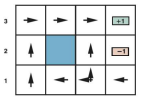
- For MC learning we get a set of training samples

$$((x_1, y_1), u_1), ((x_2, y_2), u_2), \ldots, ((x_n, y_n), u_n)$$

Where u_j is the measured utility of the j^{th} example - observed total reward from state s onward in the j^{th} trial

- This is a supervised learning problem
  - Standard linear regression problem with squared error and linear function
  - Minimize squared-error (loss) function – when partial derivatives wrt to coefficients of linear function are zero

Example: Navigation in Grid World



- Use:

$$\hat{U}_\theta(x, y) = \theta_0 + \theta_1 x + \theta_2 y$$

  - If $(\theta_0, \theta_1, \theta_2) = (0.5, 0.2, 0.1)$, then $\hat{U}(1,1) = 0.8$

- After a single trial:
  - Suppose we run a trial and the total reward obtained starting at $(1,1)$ is 0.4. This suggests that $\hat{U}(1,1)$ currently 0.8, is too large and must be reduced.
  - How?

Online Learning

- To minimize the squared error using online learning
  - Update the parameters after each trial.
- For the j^{th} example, take a step in the direction of the gradient of error function:

Half the squared difference of predicted total and actual total

$$\mathcal{E}_j(s) = \frac{\left(\hat{U}_\theta(s) - u_j(s)\right)^2}{2}.$$

Observed total reward from state s onward in the j^{th} trial

Widrow-Hoff Rule Or Delta rule for online least squares

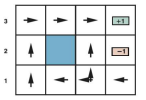
- For parameter θ_i:

$$\theta_i \leftarrow \theta_i - \alpha \frac{\partial \mathcal{E}_j(s)}{\partial \theta_i} = \theta_i + \alpha \left(u_j(s) - \hat{U}_\theta(s)\right) \frac{\partial \hat{U}_\theta(s)}{\partial \theta_i}$$

Rate of change of error w.r.t each parameter θ_i

- Notes:
  - Changing the parameters θ_i in response to an observed transition between two states also changes the values of $\hat{U}_\theta$ for every other state! $\hat{U}_\theta(x, y) = \theta_0 + \theta_1 x + \theta_2 y$
  - Function approximation allows a reinforcement learner to generalize from its experiences.

Example: Navigation in Grid World



- Use:

$$\hat{U}_\theta(x, y) = \theta_0 + \theta_1 x + \theta_2 y$$

  - If $(\theta_0, \theta_1, \theta_2) = (0.5, 0.2, 0.1)$, then $\hat{U}(1,1) = 0.8$
- After a single trial:
  - Suppose we run a trial and the total reward obtained starting at $(1,1)$ is 0.4.
  - Apply delta rule for online least squares to the example where $\hat{U}_\theta(x, y)$ is 0.8 and $u_j(1,1)$ is 0.4.
  - Parameters θ_0, θ_1, and θ_2 are all decreased by 0.4α, which reduces the error for $(1,1)$.
- Applying Delta Rule for linear function approximator:

$$\theta_i \leftarrow \theta_i - \alpha \frac{\partial \mathcal{E}_j(s)}{\partial \theta_i} = \theta_i + \alpha \left(u_j(s) - \hat{U}_\theta(s)\right) \frac{\partial \hat{U}_\theta(s)}{\partial \theta_i}$$

$$\theta_0 \leftarrow \theta_0 + \alpha \left(u_j(s) - \hat{U}_\theta(s)\right)$$

$$\theta_1 \leftarrow \theta_1 + \alpha \left(u_j(s) - \hat{U}_\theta(s)\right) x$$

$$\theta_2 \leftarrow \theta_2 + \alpha \left(u_j(s) - \hat{U}_\theta(s)\right) y$$

Approximating Temporal Difference Learning

- For TD learning the same idea of online learning can be applied
  - Adjust the parameters to reduce the temporal difference between successive states
  - For utilities:

$$\theta_i \leftarrow \theta_i + \alpha \left[R(s,a,s') + \gamma \hat{U}_\theta(s') - \hat{U}_\theta(s) \right] \frac{\partial \hat{U}_\theta(s)}{\partial \theta_i}$$

TD target

Update parameter to reduce temporal difference

  - For Q-learning:

$$\theta_i \leftarrow \theta_i + \alpha \left[R(s,a,s') + \gamma \max_a \hat{Q}_\theta(s',a') - \hat{Q}_\theta(s,a) \right] \frac{\partial \hat{Q}_\theta(s,a)}{\partial \theta_i}$$

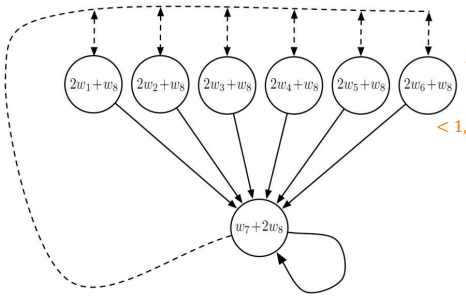
- Notes:
  - Also called semi-gradient as the target is not a true value, depends on θ
  - For passive TD learning, update rule converges for linear function when using on-policy

The Deadly Triad

- Challenges for active learning and non-linear functions:
  - Instability and divergence may arise when following 3 elements are combined:
- Function approximation:
  - Required when state space is large, e.g., using linear function approximation or deep neural nets
- Bootstrapping:
  - Using existing estimates as targets, e.g., in TD, rather than complete returns like in MC methods
- Off-policy training:
  - Training on transitions other than those produced by the target policy

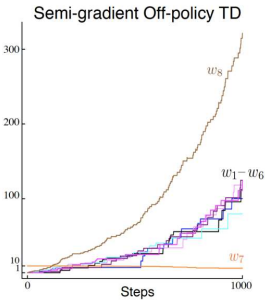
Example: Baird's Counterexample

- Off-policy TD with linear function approximation diverges



Source: SB Figure 11.1

$r(s,a,s') = 0$
$\hat{U}_\theta(s)$=0
$< 1,1,1,1,1,1,10,1 >^T$
$\pi(\text{solid}|\cdot) = 1$
$b(\text{dashed}|\cdot) = 6/7$
$b(\text{solid}|\cdot) = 1/7$
$\gamma = 0.99$



Source: SB Figure 11.2

Catastrophic Forgetting

- Problems with over-training
  - Forgotten about the "dangerous zones" of the learning regions
- Potential solution: Experience replay
  - Retain "relevant" training examples or trajectories from entire learning process
  - Replay those trajectories to ensure utility or value function is still accurate for parts of state space it no longer visits

Non-Linear Functional Approximation

Deep reinforcement learning

Deep Reinforcement Learning

- Deep neural network in function approximation
  - Discovers useful features by itself
  - "Transparent" to show selected features if last network layer is linear

$$\hat{U}_\theta(s) = \theta_1 f_1(s) + \theta_2 f_2(s) + \cdots + \theta_n f_n(s)$$

  - Parameters are all weights in all the layers of the network
  - Gradients required the same for supervised learning, computed by back propagation

Deep Reinforcement Learning

- Learning the parameters:
  - For utilities:

$$\theta_i \leftarrow \theta_i + \alpha\left[R(s,a,s') + \gamma \hat{U}_\theta(s') - \hat{U}_\theta(s)\right]\frac{\partial \hat{U}_\theta(s)}{\partial \theta_i}$$

  - For Q-learning:

$$\theta_i \leftarrow \theta_i + \alpha\left[R(s,a,s') + \gamma \max_a \hat{Q}_\theta(s',a') - \hat{Q}_\theta(s,a)\right]\frac{\partial \hat{Q}_\theta(s,a)}{\partial \theta_i}$$

Deep Q-Network (DQN)

- Uses deep neural networks with Q-learning to play 49 Atari games
  - Online Q-learning with non-linear function approximators is unstable and may diverge
- DQN uses experience replay with fixed Q-targets:
  - Take action a_t using ϵ-greedy policy
  - Store $(s_t, a_t, r_{t+1}, s_{t+1})$ in a large buffer D of most recent transitions
  - Sample a random mini-batch (s, a, r, s') from D
  - Set targets to $r + \gamma \max_{a'} Q(s', a', \theta^-)$
  - Do gradient step on the minibatch squared loss w.r.t θ – Optimize MSE btw Q-network and Q-learning targets:

$$\mathcal{L}_i(\theta_i) = E_{s,a,r,s' \sim D_i}\left[\left(r(s,a,s') + \gamma \max_{a'} Q(s',a';\theta_i^-) - Q(s,a;\theta_i)\right)^2\right]$$

  - Set θ^- to θ every C steps
- Experience replay and fixed target
  - Help reduce instability by making input less correlated

Deep Q-Network (DQN)

Algorithm 1: deep Q-learning with experience replay.

Initialize replay memory D to capacity N
Initialize action-value function Q with random weights θ
Initialize target action-value function $\hat{Q}$ with weights $\theta^- = \theta$
For episode = 1, M **do**
  Initialize sequence $s_1 = \{x_1\}$ and preprocessed sequence $\phi_1 = \phi(s_1)$
  For t = 1,T **do**
    With probability ε select a random action a_t
    otherwise select $a_t = \text{argmax}_a Q(\phi(s_t), a; \theta)$
    Execute action a_t in emulator and observe reward r_t and image x_{t+1}
    Set $s_{t+1} = s_t, a_t, x_{t+1}$ and preprocess $\phi_{t+1} = \phi(s_{t+1})$
    Store transition $(\phi_t, a_t, r_t, \phi_{t+1})$ in D
    Sample random minibatch of transitions $(\phi_j, a_j, r_j, \phi_{j+1})$ from D
    Set $y_j = \begin{cases} r_j & \text{if episode terminates at step j+1} \\ r_j + \gamma \max_{a'} \hat{Q}(\phi_{j+1}, a'; \theta^-) & \text{otherwise} \end{cases}$
    Perform a gradient descent step on $\left(y_j - Q(\phi_j, a_j; \theta)\right)^2$ with respect to the network parameters θ
    Every C steps reset $\hat{Q} = Q$
  End For
End For

Source: Mnih et al., Nature 2015

- Optimise MSE between Q-network and Q-learning targets

$$\mathcal{L}_i(w_i) = \mathbb{E}_{s,a,r,s' \sim \mathcal{D}_i}\left[\left(r + \gamma \max_{a'} Q(s', a'; w_i^-) - Q(s, a; w_i)\right)^2\right]$$

Source: Silver, D. Lecture 6 Notes on RL, 2015.

← Take action a_t using ϵ-greedy policy

← Store $(s_t, a_t, r_{t+1}, s_{t+1})$ in a large buffer D
← Sample a random mini-batch (s, a, r, s') from D

← Set targets to $r + \gamma \max_{a'} Q(s', a', \theta^-)$

← Do gradient step on minibatch squared loss w.r.t θ

← Set θ^- to θ every C steps

DQN in Atari

- End-to-end learning of values $Q(s, a)$ from pixels s
- Input state s is stack of raw pixels from last 4 frames
- Output is $Q(s, a)$ for 18 joystick positions
- Reward is change in score for that step

Network architecture and hyperparameters fixed across all games

Source: Silver, D. Lecture 6 Notes on RL, 2015, p 36.

AlphaGo[2]

- Go Game
  - Space size of about 10^{170} with branching factor that starts at 361
  - Difficult to define good evaluation function
  - Need function approximation to represent value and policy functions
- AlphaGo[2] used deep reinforcement learning to beat best human players
  - A Q-function with no look-ahead suffices for Atari games
  - Go requires substantial lookahead.
  - AlphaGo learned both a value function and a Q-function that guided its search by predicting which moves are worth exploring.
  - Q-function implemented as a convolutional neural network – accurate enough by itself to beat most amateur human players without search.

[2]Silver, D., et al., *Mastering the game of Go with deep neural networks and tree search.* Nature, 2016. **529**: p. 484+.

Deep Reinforcement Learning: Reality

"

Despite its impressive successes, deep RL still faces significant obstacles: it is often difficult to get good performance and the trained system may behave very unpredictably if the environment differs even a little from the training data.

Deep RL is rarely applied in commercial settings. It is, nonetheless, a very active area of research.

"

Russell, Stuart; Norvig, Peter. Artificial Intelligence (Pearson Series in Artificial Intelligence) (p. 858). Pearson Education. 4th Edition.

Policy Search

Policy gradient, actor critic, correlated sampling

Policy Search

- A policy $\pi: S \rightarrow A$ is a mapping from state to action
- Assume the policy is parametrized by some parameters θ
  - Dimensions of θ should be smaller than the number of states
- Often use: Q-function parameterized by θ to represent π

$$\pi(s) = \arg\max_a \hat{Q}_\theta(s, a)$$

What problem can this representation pose when trying to optimize the policy?

- Policy search adjusts θ to improve the policy
- Idea:
  - Keep *twiddling* the policy as long as its performance improves, then stop

Intuition

- Policy search tries to find a good policy, e.g., represented as Q-function
  - Results in process that learns Q-functions
  - Q-learning with function approximation: find a value of θ such that $\hat{Q}_\theta$ is close to Q^*, the optimal Q-function
  - Policy search: find a value of θ that results in good policy
- Difference between good Q-function and optimal Q-function
  - Approximate Q-function defined by $\hat{Q}_\theta = \frac{Q^*}{100}$ gives optimal performance, even though it is not at all close to Q^*

Stochastic Policy

- For policy representation of the form:

$$\pi(s) = \arg\max_a \hat{Q}_\theta(s, a)$$

- Problem:
  - When actions are discrete, policy is a discontinuous function of θ
  - This makes gradient-based search difficult
- Stochastic policy:
  - $\pi_\theta(s, a)$ specifies the probability of selecting an action a in state s
  - E.g., Softmax function, with $\beta > 0$ modulating softness of the softmax:

$$\pi_\theta(s, a) = \frac{e^{\beta \hat{Q}_\theta(s,a)}}{\sum_{a'} e^{\beta \hat{Q}_\theta(s,a')}}$$

Distribution of action: probability of selecting a in s

Differentiable function of θ

Normalizer

How to Improve Policy?

- Definition:
  - Let $\rho(\theta)$ be the policy value – expected reward-to-go when π_θ is executed.
- For deterministic policy and deterministic environment:
  - If $\rho(\theta)$ is differentiable:
    Take a step in the direction of the policy gradient vector $\nabla_\theta \rho(\theta)$ – Look for the local optimum
- For stochastic environment and/or policy $\pi_\theta(s,a)$:
  - Obtain an unbiased estimate of the gradient at θ, $\nabla_\theta \rho(\theta)$ directly from results of trials executed at θ

Use gradient ascent or stochastic gradient ascent

Policy Gradient

- Consider: single action from single state s_0

$$\nabla_\theta \rho(\theta) = \nabla_\theta \sum_a R(s_0,a,s_0)\pi_\theta(s_0,a) = \sum_a R(s_0,a,s_0)\,\nabla_\theta \pi_\theta(s,a)$$

- Approximate the summation using samples generated from $\pi_\theta(s_0,a)$:

$$\nabla_\theta \rho(\theta) = \sum_a \pi_\theta(s_0,a)\cdot \frac{R(s_0,a,s_0)\nabla_\theta \pi_\theta(s,a)}{\pi_\theta(s_0,a)} \approx \frac{1}{N}\sum_{j=1}^{N} \frac{R(s_0,a_j,s_0)\nabla_\theta \pi_\theta(s_0,a_j)}{\pi_\theta(s_0,a_j)}$$

- For sequential case, this generalizes to:

$$\nabla_\theta \rho(\theta) \approx \frac{1}{N}\sum_{j=1}^{N} \frac{u_j(s)\nabla_\theta \pi_\theta(s,a_j)}{\pi_\theta(s,a_j)}$$

Sample using policy

for each state s visited, where a_j is executed in s on the jth trial and $u_j(s)$ is the total reward received from state s onward in the jth trial.

Derivation: REINFORCE*[1]

- Alternately, for sequential case, policy gradient by sample approximation generalizes to

$$\nabla_\theta \rho(\theta) = \nabla_\theta \sum_\tau p_\theta(\tau)u(\tau)$$

We want to find the θ that maximizes the value of $\sum_\tau p_\theta(\tau)u(\tau)$

Where, τ is the trajectory generated by the policy and $u(\tau)$ is the sum of rewards from trajectory τ

- Using the policy gradient theorem[1] this can be written as:

$$\nabla_\theta \rho(\theta) \propto \sum_s p_{\pi_\theta}(s) \sum_a \nabla_\theta \pi_\theta(s,a)\hat{Q}_{\pi_\theta}(s,a)$$

States generated by policy

Q-function at that state

Gradient of policy

[1]Sutton & Barto Section 13.2

Derivation: REINFORCE*[2]

Sample using policy

- We can approximate the gradient using sampling

$$\nabla_\theta \rho(\theta) \propto \sum_s p_{\pi_\theta}(s) \sum_a \frac{\pi_\theta(s,a)\nabla_\theta \pi_\theta(s,a)\hat{Q}_{\pi_\theta}(s,a)}{\pi_\theta(s,a)}$$

$$\approx \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{n_i} \frac{\nabla_\theta \pi_\theta(s_{ij},a_{ij})u_i(s_{ij})}{\pi_\theta(s_{ij},a_{ij})}$$

i: trial, j: seq within trial

- Where, a_{ij} is executed in s_{ij} on the j^{th} step of the i^{th} trial and $u_i(s_{ij})$ is the total reward (return) received from the j^{th} step onward in the i^{th} trial

REINFORCE

- Using an online update, we get the REINFORCE algorithm:

$$\theta_{j+1} = \theta_j + \alpha u_j \frac{\nabla_\theta \pi_\theta(s, a_j)}{\pi_\theta(s, a_j)}$$

- Using the identity:

$$\nabla_\theta \ln \pi_\theta(s, a_j) = \frac{\nabla_\theta \pi_\theta(s, a_j)}{\pi_\theta(s, a_j)}$$

- Rewriting:

$$\theta_{j+1} = \theta_j + \alpha\, u_j\, \nabla_\theta \ln \pi_\theta(s, a_j)$$

Ref: SB Section 13.3
Original ref: Williams, R. 1992

Variance reduction using a Baseline

- We estimate

$$\nabla_\theta \rho(\theta) = \sum_s p_{\pi_\theta}(s) \sum_a \nabla_\theta \pi_\theta(s, a_j) \hat{Q}_{\pi_\theta}(s, a)$$

Lower variance

- This is the same as

$$\sum_s p_{\pi_\theta}(s) \sum_a \nabla_\theta \pi_\theta(s, a_j) \left(\hat{Q}_{\pi_\theta}(s, a) - B(s) \right)$$

Sum to 1

For any function $B(s)$ because

$$\sum_a \nabla_\theta \pi_\theta(s, a_j) B(s) = B(s) \nabla_\theta \sum_a \pi_\theta(s, a_j) = B(s) \nabla_\theta 1 = 0$$

Variance reduction using a Baseline

- Using a baseline function $B(s)$ can reduce variance
- Natural choice: estimated $\hat{U}_{\pi_\theta}(s)$
- The function $A_{\pi_\theta}(s, a) = \hat{Q}_{\pi_\theta}(s, a) - \hat{U}_{\pi_\theta}(s)$ is called the advantage function

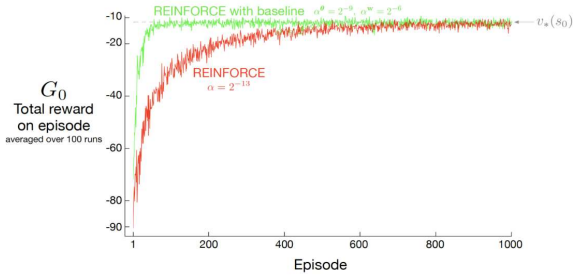


Figure 13.1 Sutton & Barto

Actor-Critic

- In policy search, a gradient step to update parameters w (to differentiate from the policy parameters) for the utility (value) function estimator is usually also done
  - To estimate both utility (value) and policy
- This is one form of actor-critic method
  - Learn a policy (actor) that takes action
  - Simultaneously, learn a utility (value) or Q-function that is used *only* for evaluation (critic)

Actor-Critic

- REINFORCE: Uses a Monte Carlo estimate of the advantage function, which has a higher variance
- For the TD method, the advantage function is:

$$\hat{Q}_{\pi_\theta}(s,a) - \hat{U}_{\pi_\theta}(s) = E\left[r + \gamma \hat{U}_{\pi_\theta}(s')\right] - \hat{U}_{\pi_\theta}(s)$$

- Using utility (value) function estimator $\hat{U}(s,w)$ with parameter w, TD-type update becomes:

$$\theta_{j+1} = \theta_j + \alpha \nabla_\theta \ln \pi_\theta(s_j, a_j)\left(r_j + \gamma \hat{U}(s_{j+1}, w) - \hat{U}(s_j, w)\right)$$

advantage

- It is common to use multiple steps of rewards instead of one step:

$$r_j + \gamma r_{j+1} + \gamma^2 r_{j+2} + \cdots + \gamma^k \hat{U}(s_{j+k} + w)$$

Correlated Sampling

- Improve performance of policy search
  - Given environment simulator with repeatable random-number sequences
  - Generate a number of experiences in advance, and check the policies with the same set of experiences
  - Eliminate errors due to actual experiences encountered
- Main idea:
  - No. of random sequences required to ensure value of every policy is well estimated depends only on complexity of policy space, and not on complexity of underlying domain
- Example:
  - PEGASUS: for stable autonomous helicopter flighted (Ng and Jordan 2000)

NUS National University of Singapore | Computing

Partially Observable Markov Decision Process

CS4246/CS5446

AI Planning and Decision Making

This lecture will be recorded!

Solving Sequential Decision Problems

- Decision (Planning) Problem or Model
  - Appropriate abstraction of states, actions, uncertain effects, goals (wrt costs and values or preferences), and time horizon + observations (through sensing)
- Decision Algorithm
  - Input: a problem
  - Output: a solution as an optimal action sequence or policy over time horizon
- Decision Solution
  - An action sequence or solution from an initial state to the goal state(s)
    - An optional solution or action sequence; OR
    - An optimal policy that specifies "best" action in each state wrt to costs or values or preferences
  - (Optional) A goal state that satisfies certain properties

Recall: Decision Making under Uncertainty

- Decision Model:
  - Actions: $a \in A$
  - Uncertain current state: $s \in S$ with probability of reaching: $P(s)$
  - Transition model of uncertain action outcome or effects:
    $P(s'|s,a)$ – probability that action a in state s reaches state s'
  - Outcome of applying action a:
    Result(a) – random variable whose values are outcome states
  - Probability of outcome state s', conditioning on that action a is executed:
    $P(\text{Result}(a) = s') = \sum_s P(s)P(s'|s,a)$
  - Preferences captured by a utility function:
    $U(s)$ – assigns a single number to express the desirability of a state s

Sequential Decision Problems

- What are sequential decision problems?
  - An agent's utility depends on a sequence of decisions
  - Incorporate utilities, uncertainty, and sensing
  - Search and planning problems are special cases
  - Decision (Planning) Models:
    - Markov decision process (MDP)
    - Partially observable Markov decision process (POMDP)
    - Reinforcement learning: sequential decision making + learning

Why Study POMDPs?

- Uncertainty in action outcomes
  - MDP: fully observable environment
  - Agent knows exactly which state it is in
- Uncertainty in observations
  - POMDP: partially observable environment
  - Agent does not know exactly which state it is in – cannot observe the state directly
  - Some noisy observations projected from a state
- Real-world challenges
  - Model sequential decision problem for an uncertain, dynamic, partially observable environment
  - If the state is not directly observable, how does an agent reason about its decisions?

Partially Observable Markov Decision Process (POMDP)

- An POMDP $M \triangleq (S, A, E, T, O, R)$ consists of
- A set S of states
- A set A of actions
- A set E of evidences or observations or percepts
- A transition function $T: S \times A \times S \rightarrow [0,1]$ such that:

$$\forall s \in S, \forall a \in A: \sum_{s' \in S} T(s,a,s') = \sum_{s' \in S} P(s'|s,a) = 1$$

- An observation function O: $S \times E \rightarrow [0,1]$ such that:

$$\forall s \in S, \forall e \in \mathrm{E}: \sum_{e \in \mathrm{E}} O(s,e) = \sum_{e \in E} P(e|s) = 1$$

- A reward function $R: S \rightarrow \Re$ or $R: S \times A \times S \rightarrow \Re$
- Solution: What is a policy in POMDP?

Observation Function and Sensor Model

- Observation function:
  - $O(s, e) = P(e|s)$ is the probability of observing (or perceiving evidence) e from state s
  - Define $O(s, e)$ for all $s \in S$ and $e \in E$
  - Assumption of Markov property
- Sensor model
  - The observation function defines the sensor model
  - An observation is also called a measurement or test

Evidence – Real-life Examples

Robotics: grasping an item in a clutter
evidence by pixels

Navigation: driving from A to B
evidence by road signs/buildings/etc

Medicine: detecting a diagnosis of a patient
evidence by symptoms

Belief State as Probability Distribution

- Belief State:
  - Actual state of the system is unknown, but we can track the probability distribution or belief state over the possible states
  - An action a changes the belief state b, not just the physical state s
  - $b(s)$ denotes probability assigned to actual state s by belief state b
  - If $b(s)$ is the current belief, the agent executes action a and receives evidence e', then the updated belief is given by filtering:

Sensor model: $P(e'|s')$ — Transition model: $P(s'|s, a)$

$$b'(s') = \alpha P(e'|s') \sum_s P(s'|s, a) b(s)$$

where α is the normalizing constant that makes the belief state sum to 1

- Filtering Function:

$$b' = \text{FORWARD}(b, a, e')$$

Exercise

$$b'(s') = \alpha P(e'|s') \sum_s P(s'|s, a) b(s)$$

- Consider a problem with two states s_1 and s_2:
  - Current belief $b(s_1) = 0.6$ and $b(s_2) = 0.4$.
- For action a:
  - Let the transition probabilities be: $P(s_1|s_1, a) = 0.2$ and $P(s_2|s_1, a) = 0.8$; $P(s_2|s_2, a) = 0.3$ and $P(s_1|s_2, a) = 0.7$
  - Let the observation probabilities be: $P(o_2|s_1) = 0.7$ and $P(o_2|s_2) = 0.2$
- Assume that evidence $e' = o_2$ is received. What is b'?

$$b'(s_1) = \alpha P(o_2|s_1)[P(s_1|s_1, a)b(s_1) + P(s_1|s_2, a)b(s_2)]$$
$$= \alpha 0.7[0.2 \times 0.6 + 0.7 \times 0.4] = \quad ?$$

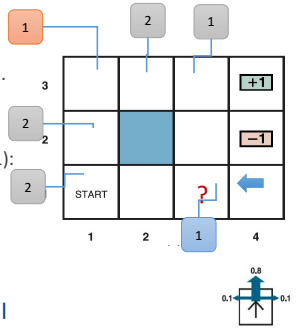
$$b'(s_2) = \alpha P(o_2|s_2)[P(s_2|s_1, a)b(s_1) + P(s_2|s_2, a)b(s_2)]$$
$$= \alpha 0.2[0.8 \times 0.6 + 0.3 \times 0.4] = \quad ?$$

Decision Making in POMDPs

- Main ideas:
  - Optimal action depends only on agent's current belief
  - Optimal policy can be described by a mapping $\pi^*(b)$ from belief to action
  - Optimal policy does not depend on the actual state agent is in!
  - Include value of information as a component in decision making
- Decision cycle of a POMDP agent:
  - Given current belief b, execute action $a = \pi^*(b)$
  - Receive percept e'
  - Set belief to $b' = \text{FORWARD}(b, a, e')$ and repeat

Example: Navigation in Grid World

- POMDP :
  - States S, actions A, transition T , reward R, and observation model O.
- Assume:
  - An observation function measures no. of adjacent walls in a state s
  - For all non-terminal states except those in column 3 (where value = 1):
  - $O(s,2) = 0.9, O(s,*) = 0.1$ (where * indicates a wrong value)
- Belief state:
  - $b = \langle 1/9, 1/9/, 1/9, 1/9, 1/9, 1/9, 1/9, 1/9, 1/9, 0, 0\rangle$
  - $b((1,1)) = 1/9, b((1,2)) = 1/9, \ldots, b((3,4)) = 0$
- Suppose agent moves Left and sensor reports 1 adjacent wall
  - It is quite likely that agent is now in $(3,1)$
- What is the exact probability values of the new belief state?



Source: RN Chapter 16

Calculating New Belief State

- What is the probability that an agent in belief state b reaches belief state b' after executing action a?
  - With current belief $b(s)$, agent executes action a and perceives e', updated belief is:

$$b'(s') = \alpha P(e'|s') \sum_s P(s'|s,a) b(s) = \text{FORWARD}(b, a, e')$$

    where α is normalizing constant for belief state to sum to 1
  - New belief b' is a conditional probability over actual state given sequence of percepts and actions so far
  - If action and subsequent percept were known, deterministic update to the belief using $b' = \text{FORWARD}(b, a, e')$
  - But subsequent percept is not yet known, so agent might arrive in one of several possible belief states b', depending on percept received

Calculating Probability of Percept

- Probability of percept
  - Probability of perceiving e', given that a was performed starting in belief state b, is given by summing over all the actual states s' that the agent might reach:

$$P(e'|a,b) = \sum_{s'} P(e'|a,s',b) P(s'|a,b)$$
$$= \sum_{s'} P(e'|s') P(s'|a,b)$$
$$= \sum_{s'} P(e'|s') \sum_s P(s'|s,a) b(s)$$

Sensor model — Transition model

Calculating Transition Model and Reward Model

- Probability of reaching b' from b, given action a:
  - Transition model for belief state space:

$$P(b'|a,b) = \sum_{e'} P(b'|e',a,b)P(e'|a,b)$$
$$= \sum_{e'} P(b'|e',a,b) \sum_{s'} \underbrace{P(e'|s')}_{\text{Sensor model}} \sum_{s} \underbrace{P(s'|s,a)}_{\text{Transition model}} b(s) .$$

where $P(b'|e',a,b)$ is 1 if $b' = \text{FORWARD}(b,a,e')$ and 0 otherwise

- Reward function of belief state space:
  - Expected reward if the agent does a in belief state b:

$$\rho(b,a) = \sum_{s} b(s) \sum_{s'} P(s'|s,a)\, R(s,a,s')$$

Reducing POMDP into an MDP

- Belief space MDP:
  - Transition model $P(b'|b,a)$ and reward model $\rho(b,\mathrm{a})$ define an observable MDP on the space of belief states!
  - Solving a POMDP on a physical state space – solving a continuous, usually high-dimensional MDP on the corresponding belief-state space
  - An optimal policy for this MDP, $\pi^*(b)$ is also an optimal policy for the original POMDP
- Remember:
  - The belief state is always observable to the agent, by definition

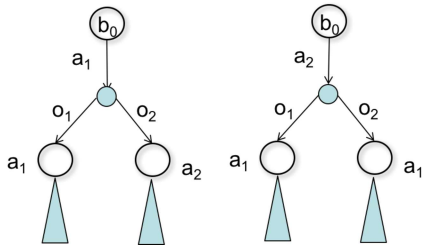
Value Iteration

Value Iteration for POMPDs

- Policy and conditional plan
  - Consider an optimal policy π^* and its application in a single belief state b
  - Policy generates an action, then for each subsequent observation or percept, belief state is updated and new action is generated, and so on
  - For b, the policy is equivalent to a conditional plan
- Main issue:
  - How does the expected utility of executing a fixed conditional plan varies with the initial belief state?

Conditional Plan

- A policy at a belief b_0 is a conditional plan

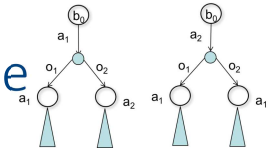


- Multiple conditional plans are possible

Utility Function for Belief State

- Note:
  - A belief state b is a probability distribution
  - Each utility value in a POMDP is a function of an entire probability distribution
- Problems:
  - Probability distributions are continuous
  - Huge complexity of belief spaces
- Solution:
  - For finite state, action, and observation spaces and planning horizon: utility functions represented by piecewise linear functions over belief space

Utility Function of Belief State

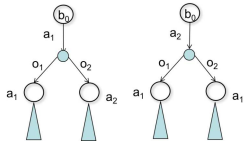


- Let the utility of executing a fixed conditional plan p starting in physical state s be $\alpha_p(s)$ – alpha vector

- Expected utility of executing p in belief state b is:

  $U_p(b) = \sum_s b(s)\alpha_p(s)$ or $b \cdot \alpha_p$ (inner product of vectors b, α_p)
  - A linear function of b, corresponding to a hyperplane in belief space

Utility Function of Belief State



- At any particular belief state b, optimal policy is to choose the conditional plan with highest utility:

$$U^{\pi^*}(b) = U(b) = \max_p b \cdot \alpha_p$$

- Utility or value function $U(b)$ on belief states, being the maximum of a collection of hyperplanes, will be piecewise linear and convex
- For finite depth, there are only a finite number of conditional plans
  - With $|A|$ actions, $|E|$ observations, there are $|A|^{O(|E|^{d-1})}$ distinct depth-d plans

Example: A Two-State World

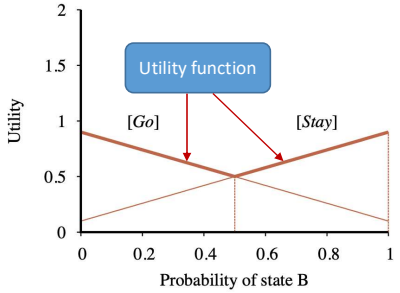
- Given:
  - Two states: A , B (or 0, 1 in RN 3e) [Belief space is 1-dimensional]
  - Rewards: $R(.,.,A) = 0$, $R(.,.,B) = 1$ [Any transition ending in A is 0, …]
  - Two actions:
    - Stay – Stays put with probability = 0.9
    - Go – Switches to the other state with probability = 0.9
  - Discount factor: $\gamma = 1$
  - Sensor: Reports correct state with probability = 0.6
- Obviously:
  - Agent should: Stay when it thinks it is in state B and Go when it thinks it is in state A
- What are the values for 1-step plans (α-vectors)?

Example: Solving a POMDP, $d = 1$

- Consider one-step plans: [Stay] and [Go]
  - Each receives reward for one transition as follows:

$$\alpha_{[Stay]}(A) = 0.9\,R(A,Stay,A) + 0.1\,R(A,Stay,B) = 0.1$$
$$\alpha_{[Stay]}(B) = 0.1\,R(B,Stay,A) + 0.9\,R(B,Stay,B) = 0.9$$
$$\alpha_{[Go]}(A) = 0.1\,R(A,Go,A) + 0.9\,R(A,Go,B) = 0.9$$
$$\alpha_{[Go]}(B) = 0.9\,R(B,Go,A) + 0.1\,R(B,Go,B). = 0.1$$

- Hyperplanes for $b \cdot \alpha_{[Stay]}$ and $b \cdot \alpha_{[Go]}$
  - Utility or value function: max of the two linear functions
- What is the one-step optimal policy?
  - Optimal policy is to Stay if $b(B) > 0.5$ and Go otherwise



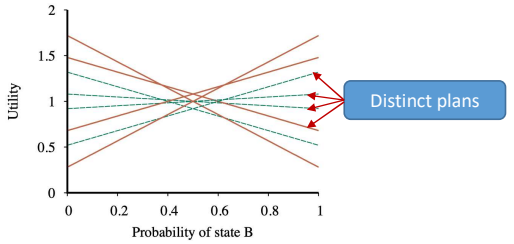
Utility of Belief-State $b(B)$: $d = 1$

Source: RN Figure 16.15 (a)

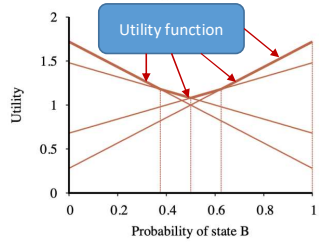
Example: Solving a POMDP, $d = 2$

- Deriving a solution:
  - Obtain utilities $\alpha_p(s)$ for all the conditional plans p of depth 1 in each physical state s
  - Compute utilities for conditional plans of depth 2 by considering:
    - each possible first action
    - each possible subsequent percept, and
    - each way of choosing a depth-1 plan to execute for each percept:
      [Stay; **if** Percept = A **then** Stay **else** Stay]
      [Stay; **if** Percept = A **then** Stay **else** Go]
      [Go; **if** Percept = A **then** Stay **else** Stay] …
  - There are 8 distinct depth-2 plans in all
    - 4 suboptimal and dominated plans – dominated plans are never optimal
    - 4 undominated plans, each optimal in a specific region
  - The regions partition the belief-state space

Example: Utility of Belief-State $b(B)$: $d = 2$



Utilities of 8 distinct 2-step plans

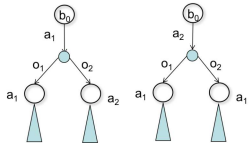


Utilities of 4 undominated 2-step plans

- Continuous belief space is divided into regions; the regions partition the belief-state space
  - Each region corresponds to a conditional plan that is optimal for that region
- $U(b)$ is piecewise linear and convex

Source: RN Figure 16.15 (b) and (c)

Value Iteration in POMDP



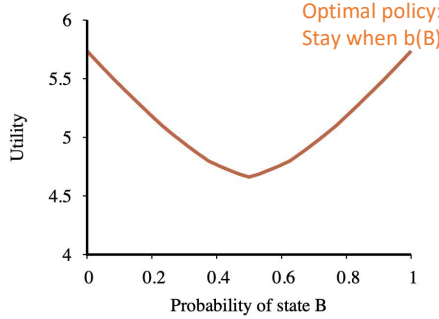
- In general:
  - Let p be a depth-d conditional plan with initial action a followed by depth $(d-1)$ subplans $p \cdot e'$ for percept e'

$$\alpha_p(s) = \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \sum_{e'} P(e'|s')\alpha_{p\cdot e'}(s')\right]$$

Transition model; Observation model

- Given utility function:
  - Executable policy extracted by:
    - looking at which hyperplane is optimal at any given belief state b
    - executing the first action of the corresponding plan
- POMDP Value Iteration (VI):
  - maintains a collection of undominated plans $\{p\}$ with their utility hyperplanes $\{alpha\ vectors\ \alpha_p\}$
  - Cf MDP VI computes one utility number, $U(s)$ for each state s

Example: Utility of Belief-State $b(B)$: $d = 8$



Utility function for optimal 8-step plans

Source: RN Figure 17.15 (d)

POMDP Value Iteration Algorithm

function POMDP-VALUE-ITERATION($pomdp, \epsilon$) **returns** a utility function
  inputs: $pomdp$, a POMDP with states S, actions $A(s)$, transition model $P(s'|s,a)$,
    sensor model $P(e|s)$, rewards $R(s)$, discount γ
    ϵ, the maximum error allowed in the utility of any state
  local variables: U, U', sets of plans p with associated utility vectors α_p

  $U' \leftarrow$ a set containing just the empty plan [], with $\alpha_{[]}(s) = R(s)$
  repeat
    $U \leftarrow U'$
    $U' \leftarrow$ the set of all plans consisting of an action and, for each possible next percept,
      a plan in U with utility vectors computed according to Equation (17.18)
    $U' \leftarrow$ REMOVE-DOMINATED-PLANS(U')
  until MAX-DIFFERENCE(U, U') $\leq \epsilon(1-\gamma)/\gamma$
  return U

Source: RN Figure 17.16

Notes on Value Iteration

- Algorithm's complexity depends primarily on how many plans get generated.
  - Given $|A|$ actions and $|E|$ observations, there are $|A|^{O(|E|^{d-1})}$ distinct depth-d plans.
  - $|A|n^{|E|}$ conditional plans generated at level $d+1$ before elimination, where n is the no. of conditional plans at level d
  - Elimination of dominated plans to reduce doubly exponential growth
  - P-SPACE hard – very inefficient in practice
  - Approximate methods necessary
- Intermediate belief states have lower value than state A and state B
  - In the intermediate states the agent lacks information needed to choose a good action.
  - Information has value and optimal policies in POMDPs often include information-gathering actions.

Online Methods

Approximate solutions

Scaling POMDP solvers

- POMDP solvers need to solve two problems
  - Belief tracking/filtering
    - Given the history observed, what is the current belief?
  - Planning
    - Given the current belief, what is the optimal action to take?
- To scale up, need to scale up for both problems

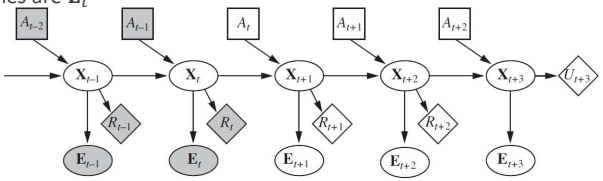
Online Agent for POMDP

- Main ideas:
  - Represent transition model and sensor model by a dynamic decision network (DDN)
  - Belief tracking – Inference in DDN – Exact inference is computationally intractable
    - No known polynomial time algorithm, as number of state variables grow
  - Approximate solution: Deploy a filtering algorithm (exact or particle filtering) to incorporate each new percept and action and to update belief state representation
  - Projecting forward possible action sequences and choosing the best one
- Note:
  - A DDN can actually be used as inputs for any POMDP algorithm, include those for value iteration and policy iteration

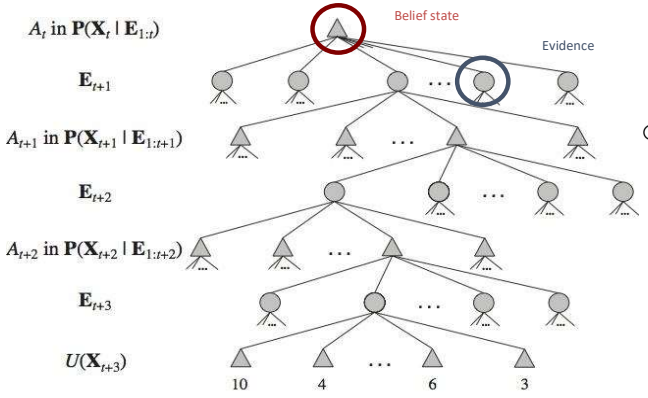
Dynamic Decision Network (DDN)

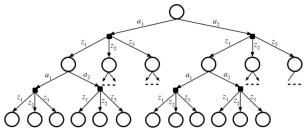
- Execution of POMDP over time can be represented as a DDN
  - Transition and sensor models represented by a Dynamic Bayesian Network (DBN)
  - Add decision and utility nodes to get DDN
- In DDN:
  - State S_t becomes set of variables $\mathbf{X}_t$
  - Evidence/observation variables are $\mathbf{E}_t$
  - Action at time t is A_t
  - Transition: $P(\mathbf{X}_{t+1}|\mathbf{X}_t, A_t)$
  - Sensor model: $P(\mathbf{E}_t|\mathbf{X}_t)$

Note:
- Variables with known values are shaded
- Current time is t and agent must decide what to do



Look-Ahead Solution of POMDP





Alternate form

Time complexity: $O(|A|^d|E|^d)$

Source: RN 3e Figure 16.11

Notes on Online Solution

- Main ideas:
  - Project action sequences forward from current belief state
  - Determine utility estimates at projected depth based on future rewards
  - Extract decision by backing up utilities from the leaves
    - Averaging chance nodes and maximizing/minimizing decision nodes
  - Non-leaf states may have rewards
  - Decision nodes correspond to belief states rather than actual states
- Complexity
  - Time complexity of exhaustive search to depth d is $O(|A|^d|E|^d)$ where $|A|$ is the no. of available actions and $|E|$ is the number of possible percepts

POMCP[1]

Partially Observable Monte Carlo Planning

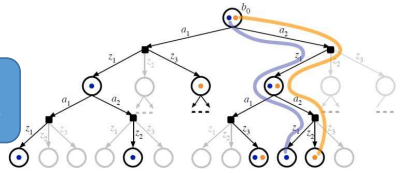
MCTS:
- Select
- Expand
- Simulate
- Backup

- To run UCT on POMDP, we need to represent beliefs in the nodes
  - Inference to construct beliefs is intractable in general
  - For MDP, states are easy to generate
  - Beliefs are difficult to generate, so POMCP uses action-observation history
  - History is equivalent to belief assuming same initial belief
- Instead of propagating beliefs forward in a trial
  - POMCP samples a state at the root from the initial belief.
  - Run the simulation using the state to generate action-observation history
  - Only need to sample from $P(s'|s,a)$ then $P(e'|s')$ to generate observation e' for the action-observation history
  - Avoid constructing beliefs

Silver, D. and J. Veness, *Monte-Carlo planning in large POMDPs*, in *Advances in Neural Information Processing Systems 23: 24th Annual Conference on Neural Information Processing Systems 2010.* . 2010: Vancouver, British Columbia, Canada. p. 2164-2172

DESPOT[2]

Determinized Sparse Partially Observable Tree



Source: Ye et al 2017

- Construct search tree differently! (make tree smaller)
  - Construct k different search trees
  - Each search tree, sample a state at the root to initialize
  - At every action node, try all actions on the (single) state at that node
  - At every observation node, sample a single observation from the (single) state at node
- Size of the tree:
  - Each of the k trees have size A^d. Combine together to get tree of size $O(|A|^d k)$
  - Exponentially smaller than $|A|^d|E|^d$
- Can show that searching sampled tree is sufficient if a small good policy exists
  - Use heuristics to do anytime search of this tree

[2]Ye, N., et al., *DESPOT: online POMDP planning with regularization.* J. Artif. Int. Res., 2017. **58**(1): p. 231–266.

NUS | Computing

Game Theory

CS4246/CS5446

AI Planning and Decision Making

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This lecture will be recorded!

Recall: Decision Theory

- Utility theory:
  - Quantifies degrees of preferences of alternatives
- Decision Model:
  - Explicates impact of uncertainty on these preferences
- Utility function:
  - Mapping from states of the world to real numbers, indicating agent's level of happiness with that state of the world
- Decision-theoretic rationality:
  - Takes actions to maximize expected utility.

Game Theory: Overview

- Multiagent decision making
  - Simultaneous moves and partial observability
  - Perfect information vs imperfect information
- Agent design
  - Analyze agent's decisions; compute expected utility for each decision for optimal agents
  - E.g., Two finger Morra
- Mechanism design
  - Define rules for the environment with multiple agents to maximize collective good
  - E.g., Protocols for internet traffic routers, intelligent multiagent systems
- Assumption:
  - All agents obey optimal decision making rules according to game theory

Elements of A Game

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- Rules of the Game: PAPI
  - Players
  - Actions
  - Payoffs
  - Information
- Objective
  - Describe a situation in terms of the rules of the game
  - Players will maximize payoffs
  - Players will devise strategies to pick actions depending on information available
  - Combination of strategies chosen by each player is the equilibrium
  - Given an equilibrium, predict outcome of the game

Playing Games

- Which of the following is a game?
  1. OPEC members choosing their annual output
  2. Company deciding to hire 10 senior programmers
  3. Apple purchasing OLED displays from Samsung

Source: Adapted from Rasmussen 2005, Chapter 1

Non-Cooperative Games

- What are non-cooperative games?
  - Mathematical study of interaction between rational, self-interested agents
- Why is it called non-cooperative?
  - While most interested in situations where agents' interests conflict, not restricted to these settings
  - Key is that the agent is the basic modeling unit (with beliefs, preferences, possible actions), and that agents pursue their own interests
- What is a self-interested agent?
  - Not that it wants to harm other agents
  - Not that it cares about only things that benefit it
  - Agent has a description of states of the world that it likes; its actions are motivated by this description

Single-Move Games

Normal form games
Pure strategy games

Single-Move Games

- Definition:
  - All players take actions "simultaneously"
  - Result of game is based on this single set of actions
- Examples:
  - Prisoner's Dilemma; Two finger Morra; etc.
  - Many real decision making situations in business, politics, defense, financial planning, etc.
- A single-move game is defined by 3 components
  1. Players or agents who make the decision
     - Focus on 2-player games; n-player games for $n > 2$ are also common
  2. Actions that players can choose
  3. Payoff function that gives the utility to each player for each combination of actions by all players
     - Can be represented as a payoff matrix – called the strategic form or normal form
     - Row Player: Player 1; Column Player: Player 2

Example: Prisoner's Dilemma

	$Al: Testify$	$A: Ref$
$Bo: Testify$	$B = -5; A = -5$	$B = 0; A = -10$
$Bo: Refuse$	$B = -10; A = 0$	$B = -1; A = -1$

- Game:
  - Two suspects: Bo and Al, are caught near a burglary scene and interrogated separately
  - Choices:
    - One Testifies (blames the other) and the other Refuses (to cooperate): Payoffs – [0, -10] or [-10, 0]
    - Both Refuse: Payoffs – [-1, -1]
    - Both Testify: Payoffs – [-5, -5]
  - Should each of them testify?

Example: Prisoner's Dilemma

- What should Al and Bo do?

	$Al: Testify$	$Al: Refuse$
$Bo: Testify$	$B = -5; A = -5$	$B = 0; A = -10$
$Bo: Refuse$	$B = -10; A = 0$	$B = -1; A = -1$

Both should testify!

General Form of Single-Move Games

- Prisoner's dilemma written as a matrix
  - "strategic form" or "normal form"

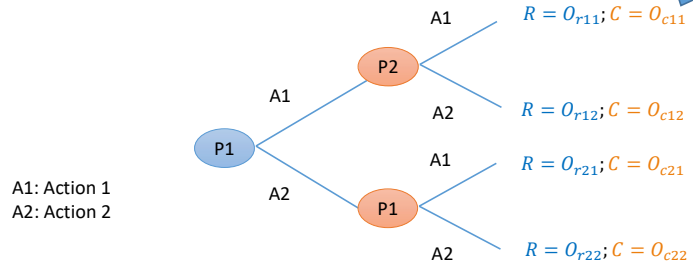
Subscripts in payoffs denote action numbers of P1 and P2

	Column Player (P2) Action 1	*Column Player (P2) Action 2*
Row Player (P1) Action 1	$R = O_{r11}; C = O_{c11}$	$R = O_{r12}; C = O_{c12}$
Row Player (P1) Action 2	$R = O_{r21}; C = O_{c21}$	$R = O_{r22}; C = O_{c22}$

General Form of Single-Move Games

- Prisoner's dilemma written as a game tree
  - "extensive form"

Subscripts in payoffs denote action numbers of P1 and P2



A1: Action 1
A2: Action 2

Game Strategies

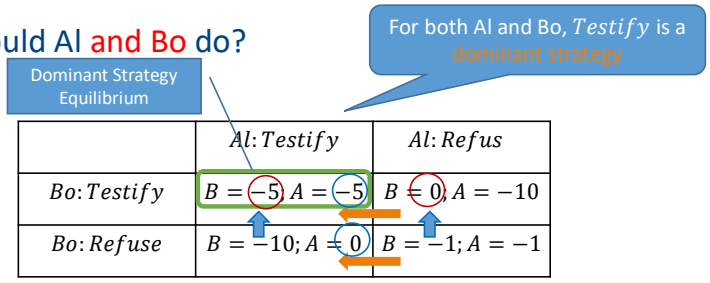
- Strategy
  - Each player must adopt and execute a strategy (policy)
- Pure strategy
  - Deterministic policy
  - A single action for a single-move game
- Mixed strategy
  - Randomized policy that selects actions according to a probability distribution
  - [p: a; (1 – p): b]
  - chooses action a with probability p and b otherwise
- Strategy profile
  - Assignment of a strategy to each player

Game Solutions

- Outcome
  - Given the strategy profile, the game's outcome is a numeric value for each player (number or expected utility)
- Solution
  - A strategy profile in which each player adopts a rational strategy
  - Examples: Pareto optima and Nash equilibrium
- What is "rational"?
  - Each agent chooses only part of the strategy profile that determines the outcome
  - Outcomes are actual results of playing a game
  - Solutions are theoretical constructs for analyzing a game

Example: Prisoner's Dilemma

- What should Al and Bo do?



Both should testify!

Domination

- Dominant strategy
  - A strategy that dominates all others
  - Irrational to play a dominated strategy
  - Irrational not to play a dominant strategy if one exists
- Strongly dominant strategy
  - Strategy s for player P strongly dominates strategy s' if the outcome for s is better for P than the outcome for s', for every choice of strategies by other player(s)
- Weakly dominant strategy
  - Strategy s weakly dominates s' if s is better than s' on at least one strategy profile and no worse on any other

Deriving Solution with Dominated Strategies

- One way to solve the game: iteratively eliminate dominated strategies
  - If we end up with only one strategy for each player, we have a solution!
- Al first reasons that Ref is dominated, so it can be eliminated
- Simplified game after eliminating Al's Ref strategy
- In the simplified game, Bo reasons that $Refuse$ is dominated
- On eliminating, we get the solution!

	$Al: Testify$	$Al: Refu$
$Bo: Testify$	$B = -5; A = -5$	$B = 0; A = -10$
$Bo: Refus$	$= -10; A = 0$	$B = -1; A = -1$

	$Al: Testi$
$Bo: Testify$	$B = -5; A = -5$
$Bo: Ref$	$= -10; A = 0$

	$Al: Testify$
$Bo: Testify$	$B = -5; A = -5$

Dominant Strategy Equilibrium

Deriving Solution with Dominated Strategies

- Strongly dominant strategy
  - If iterated elimination of strictly dominated strategies result in one strategy for each player, we have found a unique equilibrium
  - Order of the elimination does not matter
- Weakly dominant strategy
  - If we also eliminated weakly dominated strategies and end up with one strategy for each player, we have also found an equilibrium, but there may be other equilibriums that we did not find

Equilibrium

E.g., in Prisoner's Dilemma, both players testifying

- Dominant strategy equilibrium
  - When each player has a dominant strategy, combination of strategies is called a dominant strategy equilibrium
  - It is an "equilibrium" because no player has any incentive to deviate from their part of it: by definition, if they did so, they could not do better, and may do worse.
- Nash Equilibrium
  - A strategy profile forms an equilibrium if no other player can benefit by switching strategies, given that every other player sticks with the same strategy
  - Local optimum in the space of strategies
- Note:
  - Every game has at least one equilibrium (John Nash)
  - A dominant strategy equilibrium is a Nash equilibrium
  - Some games have Nash equilibrium but no dominant strategies

Pareto Optimality

- An outcome is Pareto dominated by another outcome if all players would prefer the other outcome
- An outcome is Pareto optimal if there is no other outcomes that all players would prefer, i.e., if there is no other outcome that would make one player better off without making someone else worse off
  - An outcome is Pareto-optimal if there is no other outcome that Pareto-dominates it.
  - A game can have more than one Pareto-optimal outcome
  - Every game has at least one Pareto-optimal outcome

Note: Pareto optimality is named for the Italian economist Vilfredo Pareto (1848–1923).

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Example: Prisoner's Dilemma

	$Al: Testify$	$Al: Refu$
$Bo: Testify$	$B = -5; A = -5$	$B = 0; A = -10$
$Bo: Refus$	$= -10; A = 0$	$B = -1; A = -1$

Equilibrium

Pareto optimal

Pareto optimal

Pareto dominating

- Dilemma:
  - Equilibrium outcome of both Al & Bo testifying is worse than the outcome both would get if they refuse
  - The $(Testi \quad, Testify)$ equilibrium is pareto dominated by $(Refuse, Ref \quad)$ strategy profile which is better for both players
  - Although$(Refuse, Refuse)$ combination is better, both players playing rationally end up in $(Testify, Testify)$ solution

Example: Prisoner's Dilemma

	$Al: Testify$	$Al: Ref$
$Bo: Testify$	$B = -5; A = -5$	$B = 0; A = -10$
$Bo: Refus$	$= -10; A = 0$	$B = -1; A = -1$

- By modifying the game, it may be possible to end up with $(Refuse, Refuse)$ solution.
- For example:
  - In a repeated game, the players will meet again and again
  - Agents may have moral beliefs that encourage cooperation and fairness, changing the payoff matrix

Example: Finding Pure Strategy Nash Equilibrium

	$Acme: bluray$	$Acme: dvd$
$Best: bluray$	$B = +9; A = +9$	$B = -1; A = -4$
$Best: dvd$	$B = -1; A = -3$	$B = +5; A = +5$

- Game: (Classic)
  - Two video game console manufacturers, Acme and Best, need to decide whether to use Blu-ray discs or DVD
  - Profits are positive if they both agree to use the same format, but negative otherwise
  - There is no dominant strategy, so iterative elimination of dominant strategies will fail to find a solution.
- How do we find the solution(s) in this case?

Example: Finding Pure Strategy Nash Equilibrium

	$Acme: bluray$	$Acme: dvd$
$Best: blura$	$= +9; A = +9$	$B = -1; A = -4$
$Best: dvd$	$B = -1; A = -3$	$B = +5; A = +5$

- There are 2 Nash equilibriums:
  - $(bluray, bluray)$ and (dvd, dvd)
- Choice in this case: $(bluray, bluray)$, a Pareto optimal solution
  - Every game has at least one Pareto-optimal solution, and may have multiple

Finding Pure Strategy Nash Equilibrium

- A simple algorithm to find pure Nash equilibrium:
  - For each column, mark cell if it has the maximum payoff for the row player
    - May be more than one
  - For each row, mark cell if it has the maximum payoff for the column player
    - May be more than one
  - Cells with both row and column marks are pure Nash equilibrium
- There are multiple acceptable solutions, but if agents disagree, both suffer
- There are various ways the agents can coordinate in choosing solutions
  - e.g., by communicating & negotiation
  - Games where agents need to communicate are called coordination games

Mixed Strategies Games

Probabilistic action selection with randomized policy

Example: Game of Chicken

	$Player\ A: continue$	$Player\ A: swerve$
$Player\ B: continue$	$B = -10; A = -10$	$B = 2; A = -2$
$Player\ B: swerve$	$B = -2; A = 2$	$B = 0; A = 0$

- Game: (Terrible!)
  - Two teenagers on opposite ends of a road drive towards each other
  - At the last possible moment, they must decide whether to swerve
  - If one continues while the other swerves, the one who swerves is chicken, while the other is brave
  - If both swerve, then both are OK
  - If both continue – *crash*

Example: Game of Chicken

	$Player\ A: continue$	$Player\ A: swerve$
$Player\ B: continue$	$B = -10; A = -10$	$B = 2; A = -2$
$Player\ B: swerve$	$B = -2; A = 2$	$B = 0; A = 0$

- There are two pure strategy Nash equilibriums
  - $(A: continue, B: swerve)$
  - $(A: swerve, B: continue)$
- But there is also a mixed strategy that we will compute

Example: Game of Chicken

[p; 1-p]

	Player A: continue	*Player A: swerve*
Player B: continue	$B=-10; A=-10$	$B=2; A=-2$
Player B: swerve	$B=-2; A=2$	$B=0; A=0$

- B plays $continue$:
  - Let B play $continue$ with probability p and swerve with probability $(1-p)$
  - If A plays $continue$ then the expected payoff for A is:
    $$-10p + 2(1-p) = -12p + 2$$
  - If A plays $swerve$ then the expected payoff for A is:
    $$-2p + 0(1-p) = -2p$$
  - It would not matter which action A plays if both the expected payoffs are the same; i.e.,
    $$-12p + 2 = -2p \implies p = 1/5$$

Example: Game of Chicken

[q; 1-q]

	Player A: continue	*Player A: swerve*
Player B: continue	$B=-10; A=-10$	$B=2; A=-2$
Player B: swerve	$B=-2; A=2$	$B=0; A=0$

- A plays $continue$:
  - Let A play $continue$ with probability q and swerve with probability $(1-q)$
  - If B plays $continue$ then the expected payoff for B is:
    $$-10q + 2(1-q) = -12q + 2$$
  - If B plays $swerve$ then the expected payoff for B is:
    $$-2q + 0(1-q) = -2q$$
  - It would not matter which action B plays if both the expected payoffs are the same; i.e.,
    $$-12q + 2 = -2q \implies q = 1/5$$

Example: Game of Chicken

	Player A: continue	*Player A: swerve*
Player B: continue	$B=-10; A=-10$ $pq = 1/25$	$B=2; A=-2$ $p(1-q) = 4/25$
Player B: swerv	$=-2; A=2$ $(1-p)q = 4/25$	$B=0; A=0$ $(1-p)(1-q) = 16/25$

$p = q = \frac{1}{5}$

Probabilities of the pairs of actions

Finding mixed strategy:

- For player B the expected utility is:
  - $\frac{1}{25} \times -10 + \frac{4}{25} \times 2 + \frac{4}{25} \times -2 + \frac{16}{25} \times 0 = -\frac{2}{5}$
- For player A the expected utility is:
  - $\frac{1}{25} \times -10 + \frac{4}{25} \times -2 + \frac{4}{25} \times 2 + \frac{16}{25} \times 0 = -\frac{2}{5}$

Expected payoff for each player for this strategy profile

Finding Mixed Strategies

- Algorithm finds a mixture for one player where the other player becomes indifferent to the choices; repeat this for both players
  - Does not always work! (If there is no mixture where the player becomes indifferent)
  - E.g., if one choice dominates the other choice, the player cannot be indifferent to the choices
- Questions:
  - Is there an efficient algorithm for computing a Nash equilibrium in general?

Interpreting Mixed Strategy Equilibria

- What does it mean to play a mixed strategy? (Different interpretations)
  - Randomize to confuse your opponent
    - E.g., Matching pennies
  - Players randomize when they are uncertain about the other's action
    - E.g., Battle of the sexes
  - Mixed strategies are a concise description of what might happen in repeated play
    - Count of pure strategies in the limit
  - Mixed strategies describe population dynamics
    - 2 agents randomly chosen from a population, all having deterministic strategies. Mixed strategy is the probability of getting an agent who will play one pure strategy or another.

Zero-Sum Games

Minimax Theorem

Example: Two-Finger Morra

	O: *one*	O: *two*
E: *one*	$E = +2; O = -2$	$E = -3; O = +3$
E: *two*	$E = -3; O = +3$	$E = +4; O = -4$

- Two players: E and O simultaneously display one or two fingers
- Let, total number of fingers be $f \in \{1, 2, 3, 4\}$:
  - If f is odd, O collects $\$f$ from E
  - If f is even, E collects $\$f$ from O
- What is the optimal mixed strategy against a rational player?
- What is the expected utility for each player?
- Special case of two player zero-sum games: sum of the payoffs is always zero
  - Only need to consider payoff of one player, since the sum is 0: Use the payoff of the even player, E
  - E will try to maximize the payoff $U_E(e, o)$; O will try to minimize the payoff

Efficient algorithm to compute the equilibrium is known

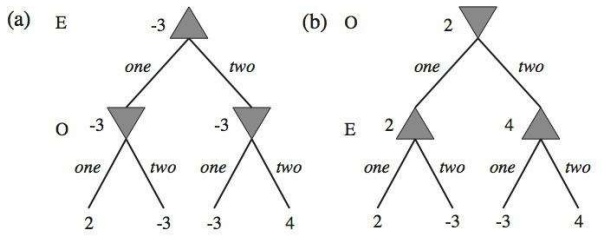
Two Finger Morra: Minimax Game Tree

	O: *one*	O: *two*
E: *one*	$E = +2; O = -2$	$E = -3; O = +3$
E: *two*	$E = -3; O = +3$	$E = +4; O = -4$

- Find range of true utility U of solution
  - Examining pure strategies:

$$U_{E,O} \leq U \leq U_{O,E} : -3 \leq U \leq 2$$

E picks action first & reveals it to O



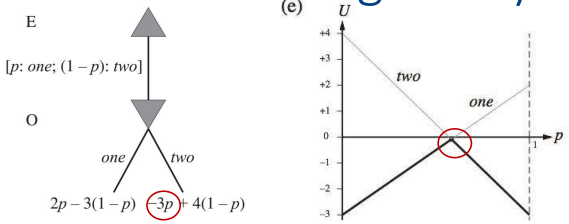
O picks action first & reveals it to E

Source: RN Figure 17.12 (a) and (b)

Two Finger Morra: Calculating Utility

- For E:

	one	*two*
One	2, −2	−3, +3
Two	−3, +3	+4, −4



  - If O chooses one: expected payoff to E: $2p - 3(1-p) = 5p - 3$
  - If O chooses two: expected payoff to E: $-3p + 4(1-p) = 4 - 7p$
- Draw straight lines in graph (e)
  - O, the minimizer, will always choose lower of the two lines
  - The best E can do at the root is to choose p at intersection pt:

$$5p - 3 = 4 - 7p \implies p = 7/12 \text{ and hence } U_{E,O} = -1/12$$

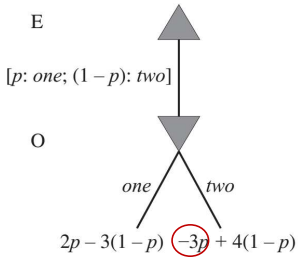
Source: RN Figure 18.12 (c) and (e)

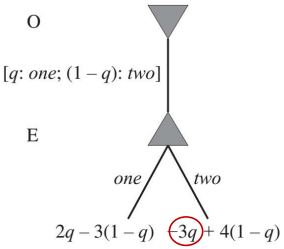
Two Finger Morra: Parameterized Game Tree

	O: one	*O: two*
E: one	$E = +2; O = -2$	$E = -3; O = +3$
E: two	$E = -3; O = +3$	$E = +4; O = -4$

- Determine true value of utility U of solution
  - Examining mixed strategies:

$$U_{E,O} = -1/12, U_{O,E} = -1/12\text{: } U = -1/12$$

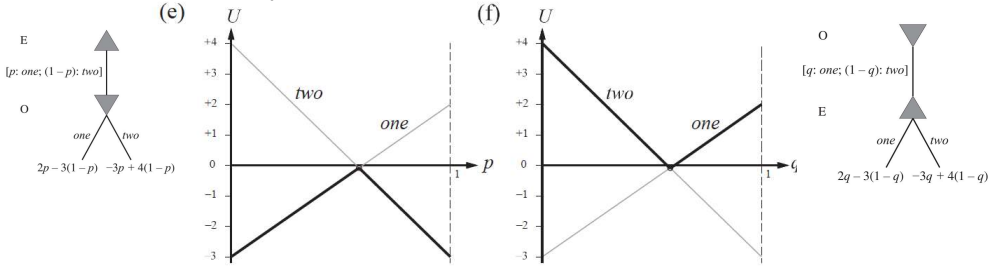




Source: RN Figure 18.12 (c) and (d)

Two Finger Morra: Utility Profile

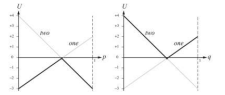
- Analyze utility functions to find optimal strategy
  - E always chooses probability parameter for mixed strategy at intersection point



Source: RN Figure 18.12 (e) and (f)

Two Finger Morra: Maximin Equilibrium

- E first: O selects action with smaller expected value, i.e., min of two linear functions.
  - The best that E can do is to select the value p that maximized the min of the two functions (maximin strategy)
  - $p = 7/12$
- O first: E finds max of the two actions
  - The best that O can do is to select the value q that minimizes the function (minimax strategy)
  - $q = 7/12$

E first  *O* first

- Note: Max of the first function is the same as min of the second! (A special case)
- True utility is attained by the mixed strategy, which should be played by both players.
  - $[7/12: one;\ 5/12: two]$
- This strategy is called the maximin equilibrium of the game, and is a Nash equilibrium.

Minimax Theorem (von Neumann 1928)

- In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value

- It follows that:
  - Each player's maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the value of the game;
  - For both players, the set of maxmin strategies coincides with the set of minmax strategies; and
  - Any maxmin strategy profile (or minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria.
    Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game)

Source: [SLB] 3.4.1

Complexity of Computing Nash Equilibrium

- Every finite game is known to have at least one Nash equilibrium
  - For 2-player zero sum game, we know how to find NE efficiently

- However, the problem of finding the NE in the general case is believed to be computationally intractable
  - It is PPAD-complete[1]
    - No known efficient algorithm in general
    - Computationally hard even for 2 player games

[1]https://en.wikipedia.org/wiki/PPAD_(complexity)