CS5478 Homework 1

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1 1.

1.1 a.

The rotation matrix for the first rotation (about the $\hat{Z}_C$ axis by θ_1 degrees) is:

$$R_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotation matrix for the second rotation (about the new $\hat{Y}_C$ axis by θ_2 degrees) is:

$$R_2 = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

So the total transformation matrix ${}^B_C R$ can be obtained by multiplying these two rotation matrices in the reverse order (since the latest rotation is applied first):

$${}^B_C R = R_2 \cdot R_1 = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1\cos\theta_2 & \sin\theta_1\cos\theta_2 & -\sin\theta_2 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ \cos\theta_1\sin\theta_2 & \sin\theta_1\sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

Thus the matrix ${}^B_C R$ can be used to transform a vector ${}^C\mathbf{P}$ from frame $\{C\}$ to frame $\{B\}$ as follows:

$${}^B\mathbf{P} = {}^B_C R \cdot {}^C\mathbf{P}$$

1.2 b.

Given $\theta_1 = 45°$ and $\theta_2 = 60°$, then

$${}^B_C R = \begin{bmatrix} \cos\theta_1\cos\theta_2 & \sin\theta_1\cos\theta_2 & -\sin\theta_2 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ \cos\theta_1\sin\theta_2 & \sin\theta_1\sin\theta_2 & \cos\theta_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & \frac{1}{2} \end{bmatrix}$$

1.3 c.

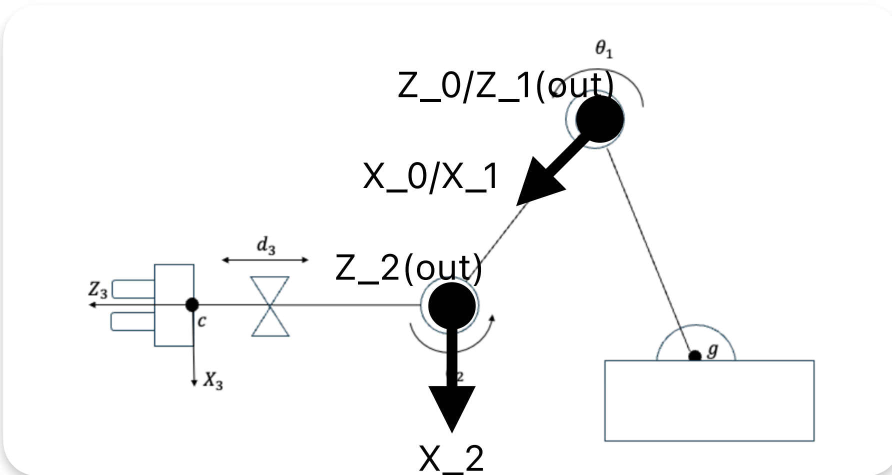
Given that frame A translates from frame B along the vector ${}^B\mathbf{q} = [q_1, q_2, q_3]^T$, we can get ${}^A_C T$ as follows:

$${}^A_C T = \begin{bmatrix} {}^B_C R & {}^B\mathbf{q} \\ 0 & 1 \end{bmatrix}$$

Where ${}^B_C R$ is the rotation matrix that transforms from frame C to frame B, ${}^B\mathbf{q}$ is the translation vector from frame B to frame A.

2 2.

2.1 a.



2.2 b.

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	l_2	0	0	θ_2
3	0	$\frac{\pi}{2}$	d_3	0

2.3 c.

$$
{}^0_3T = {}^0_1T \cdot {}^1_2T \cdot {}^2_3T = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & -\frac{1}{4} \\ 0 & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & \frac{2}{5} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

3 3.

3.1 a.

Let

$$
{}^0_ET = \begin{bmatrix} R & {}^0P_E \\ 0 & 1 \end{bmatrix}
$$

Then

$$^0P_E = \begin{bmatrix} 2c_{13} - d_2 s_1 \\ 2s_{13} + d_2 c_1 \\ 1 \end{bmatrix}$$

Then

$$\begin{aligned} {}^0J_v &= \frac{\partial^0 P_E(i)}{\partial q_j} \\ &= \begin{bmatrix} -2s_{13} - d_2 c_1 & -s_1 & -2s_{13} \\ 2c_{13} - d_2 s_1 & c_1 & 2c_{13} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Then

$$V_0 = {}^0J_v \cdot \dot{\theta} = \begin{bmatrix} -2s_{13} - d_2 c_1 & -s_1 & -2s_{13} \\ 2c_{13} - d_2 s_1 & c_1 & 2c_{13} \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Besides, according to ${}^0_1T$ and ${}^0_3T$ and the second joint is prismatic joint, we can get

$$^0J_\omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then

$$\omega = {}^0J_\omega \cdot \dot{\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

3.2 b.

These is no problem b.

3.3 c.

$$\tau = J_v^T \cdot {}^0F = \begin{bmatrix} -2s_{13} - d_2 c_1 & 2c_{13} - d_2 s_1 & 0 \\ -s_1 & c_1 & 0 \\ -2s_{13} & 2c_{13} & 0 \end{bmatrix} \cdot \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} (-2s_{13} - d_2 c_1)F_x + (2c_{13} - d_2 s_1)F_y \\ -s_1 F_x + c_1 F_y \\ -2s_{13} F_x + 2c_{13} F_y \end{bmatrix}$$

4 4.

4.1 a.

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	L_1	0	0	θ_2
3	L_2	$\pi/2$	0	θ_3
4	L_3	0	0	0

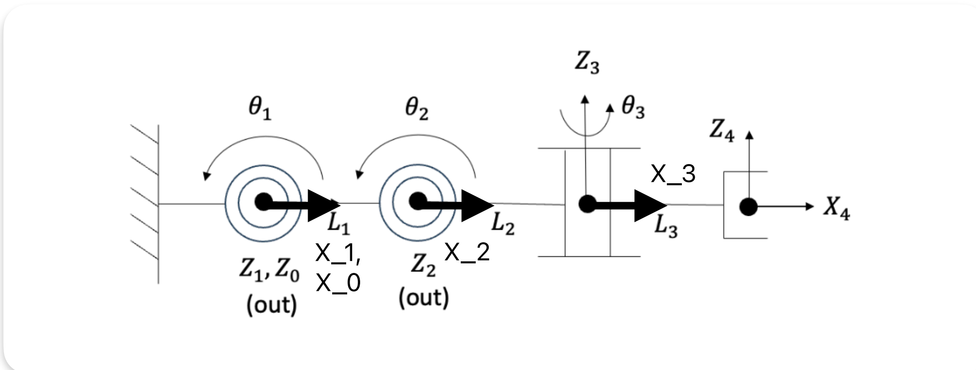
4.2 b.

Since

$$^0P_4 = \begin{bmatrix} L_1 c_1 + L_2 c_{12} + L_3 c_{12} c_3 \\ L_1 s_1 + L_2 s_{12} + L_3 s_{12} c_3 \\ -L_3 s_3 \end{bmatrix}$$

Then

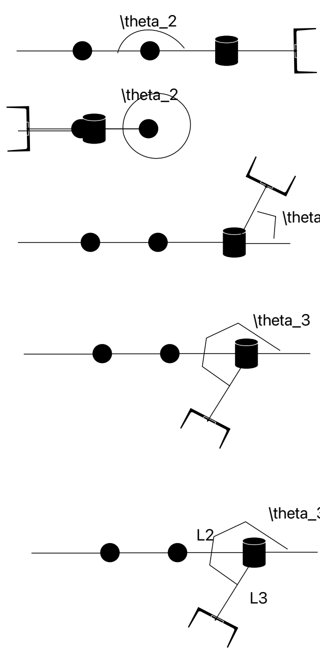
$$\begin{aligned} {}^0J_v &= \frac{\partial^0 P_4(i)}{\partial q_j} \\ &= \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{12} c_3 & -L_2 s_{12} - L_3 s_{12} c_3 & -L_3 c_{12} s_3 \\ L_1 c_1 + L_2 c_{12} + L_3 c_{12} c_3 & L_2 c_{12} + L_3 c_{12} c_3 & -L_3 s_{12} s_3 \\ 0 & 0 & -L_3 c_3 \end{bmatrix} \end{aligned}$$



4.3 c.

$$
\begin{aligned}
0 &= \det|{}^2J_v| \\
&= \det|\begin{bmatrix} -L_1 s_2 & 0 & -L_3 s_3 \\ L_1 c_2 + L_2 + L_3 c_3 & L_2 + L_3 c_3 & 0 \\ 0 & 0 & -L_3 c_3 \end{bmatrix}| \\
&= (-1)^{1+1}(-L_1 s_2)\det|| \begin{bmatrix} L_2 + L_3 c_3 & 0 \\ 0 & -L_3 c_3 \end{bmatrix} \\
&= L_1 s_2 (L_2 L_3 c_3 + L_3^2 c_3^2) \\
&= L_1 L_3 s_2 c_3 (L_2 + L_3 c_3)
\end{aligned}
$$

So there are 3 possible situations. First, $s_2 = 0, \theta_2 = k\pi$. Second, $c_3 = 0, \theta_3 = \frac{\pi + k\pi}{2}$. Third, $L_2 + L_3 c_3 = 0, \theta_3 = -\arccos \frac{L_2}{L_3}$. As shown in the figure, in the first two cases, the singularity occurs.



In this case, the movement of the manipulator is restricted in the direction of the frame axis due to $\sin \theta_2$ being zero.

The next two lines show when $c_3 = 0$ and $\theta_3 = \frac{\pi + k\pi}{2}$, the singularity occurs. Here, the manipulator's movement is constrained along the frame axes due to $\cos\theta_3$ becoming zero.

The last line shows when $\theta_3 = -\arccos\frac{L_2}{L_3}$, the singularity occurs. In this case, the manipulator experiences movement restrictions along the frame axes because of the relationship between the lengths L_2 and L_3 and the joint angle θ_3.

These three situations represent the singular configurations of the manipulator, where certain joint values or geometric conditions lead to limitations in the robot's movement along its frame axes.

5 5.

First, according to the figure, we can get the Denavit-Hartenberg parameters for this manipulator.

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	L_1	θ_1
2	0	$\frac{\pi}{2}$	d_2	$\frac{\pi}{2}$
3	0	$\frac{3\pi}{2}$	0	θ_3
4	0	$\frac{\pi}{2}$	d_4	0

Then, we can compute the forward kinematics.

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Then

$$ {}^0_2T = {}^0_1T \cdot {}^1_2T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -c_1 & s_1 & s_1 d_2 \\ 0 & -s_1 & -c_1 & -c_1 d_2 \\ 1 & 0 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_3T = {}^0_1T \cdot {}^1_2T \cdot {}^2_3T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 & -c_1 & s_1 & s_1 d_2 \\ 0 & -s_1 & -c_1 & -c_1 d_2 \\ 1 & 0 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -s_3 s_1 & -c_3 s_1 & -c_1 & s_1 d_2 \\ c_1 s_3 & c_1 c_3 & -s_1 & -c_1 d_2 \\ c_3 & -s_3 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_4T = {}^0_1T \cdot {}^1_2T \cdot {}^2_3T \cdot {}^3_4T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$= \begin{bmatrix} -s_3 s_1 & -c_1 & c_3 s_1 & s_1(c_3 d_4 + d_2) \\ c_1 s_3 & -s_1 & -c_1 c_3 & -c_1(c_3 d_4 + d_2) \\ c_3 & 0 & s_3 & s_3 d_4 + L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then

$$Z = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix} = \begin{bmatrix} 0 & s_1 & -c_1 & c_3 s_1 \\ 0 & -c_1 & -s_1 & -c_1 c_3 \\ 1 & 0 & 0 & s_3 \end{bmatrix}$$

Let P_{1E} denotes the position vector from the base frame to the origin of the end-effector frame 4, taking into account the motions of joints θ_1, d_2, and θ_3, as well as the fixed link lengths L_1 and d_4. So this vector can be extracted from the position part of ${}^0_4T$, which is the first three rows of the fourth column of the 4^0T matrix.

And Let P_{3E} denotes the position vector from the base frame to the origin of the end-effector frame 4 but only considers the motion of joint d_4. So this vector can be extracted from the same place without accounting for the influence of joint d_2..

So

$${}^0J_v = \begin{bmatrix} z_1 \times P_{1E} & z_2 & z_3 \times P_{3E} & z_4 \end{bmatrix} = \begin{bmatrix} c_1 c_3 d_4 + c_1 d_2 & s_1 & -s_1 s_3 d_4 & s_1 c_3 \\ s_1 c_3 d_4 + s_1 d_2 & -c_1 & c_1 s_3 d_4 & -c_1 c_3 \\ 0 & 0 & c_3 d_4 (s_1^2 + c_1^2) & s_3 \end{bmatrix}$$

Besides, according to the second joint and the fourth joint are prismatic joints, we can get

$${}^0J_\omega = \begin{bmatrix} 0 & 0 & -c_1 & 0 \\ 0 & 0 & -s_1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$