

Problem 1. (Probabilistic PCA)

Principal Components Analysis (PCA) is a famous model applied to dimensionality reduction or visualization where we map data samples to a lower dimensional space¹. We will now leverage the linear-Gaussian framework to derive a probabilistic form of PCA. *Note:* for this problem, we will be denoting random variables with lower case letters, and bolded lowercase letters to represent vectors, and bolded uppercase letters to represent matrices.

For the probabilistic PCA model, we have D -dimensional data points \mathbf{x}_i for $i = 1, 2, \dots, N$ and we aim to find some reduced structure for the data. For each data point, we associate a M -dimensional latent variable (where often $M < D$) \mathbf{z}_i that has prior distribution,

$$p(\mathbf{z}_i) = \mathcal{N}(\mathbf{0}, \mathbf{I}).$$

We define each observed variable \mathbf{x} as,

$$\mathbf{x}_i = \mathbf{W}\mathbf{z}_i + \boldsymbol{\mu} + \boldsymbol{\epsilon}_i$$

where $\boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$. We can imagine that each data point is obtained by first sampling from the prior $p(\mathbf{z}_i)$ followed by an affine transformation and additive Gaussian noise.

Problem 1.a. Draw the DGM corresponding to the model above. *Hint:* use plate notation for the different data points.

Solution:

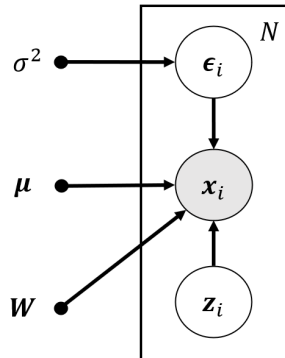


Figure 1: DGM

Problem 1.b. Show that the conditional distribution for each observed variable \mathbf{x}_i is given by:

$$p(\mathbf{x}_i | \mathbf{z}_i) = \mathcal{N}(\mathbf{x}_i | \boldsymbol{\mu} + \mathbf{W}\mathbf{z}_i, \sigma^2 \mathbf{I})$$

¹For more information about PCA, you can see various sources including <https://royalsocietypublishing.org/doi/10.1098/rsta.2015.0202>.

Solution:

We can apply the affine property of Gaussian distribution,

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma) \Leftrightarrow (\mathbf{A}\mathbf{x} + \mathbf{b}) \sim \mathcal{N}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\Sigma\mathbf{A}^T) \quad (1)$$

Since we are computing $p(\mathbf{x}_i|\mathbf{z}_i)$, we consider \mathbf{z}_i is observed (“deterministic”, i.e., zero uncertainty)
Given $\mathbf{x}_i = \mathbf{W}\mathbf{z}_i + \boldsymbol{\mu} + \boldsymbol{\epsilon}_i$ and $\boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$,

$$p(\mathbf{x}_i|\mathbf{z}_i) = \mathcal{N}(\mathbf{x}_i|\mathbf{W}\mathbf{z}_i + \boldsymbol{\mu} + \mathbf{0}, \sigma^2\mathbf{I}) \quad (2)$$

$$= \mathcal{N}(\mathbf{x}_i|\boldsymbol{\mu} + \mathbf{W}\mathbf{z}_i, \sigma^2\mathbf{I}) \quad (3)$$

Problem 1.c. To find the MLE values for the model parameters \mathbf{W} , $\boldsymbol{\mu}$, and σ^2 , we would need the marginal distribution $p(\mathbf{X}) = \prod_i^N p(\mathbf{x}_i)$ (assuming i.i.d. data). Due to the latent variables, we will use the EM algorithm². This requires us to marginalize out the latent \mathbf{z} ’s. To help us along,

1. First, show that the marginal distribution of each data point is again a Gaussian given by

$$p(\mathbf{x}_i) = \mathcal{N}(\boldsymbol{\mu}, \mathbf{C})$$

where $\mathbf{C} = \mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}$.

2. Then, show that the posterior distribution is also normally distributed,

$$p(\mathbf{z}_i|\mathbf{x}_i) = \mathcal{N}(\mathbf{M}^{-1}\mathbf{W}^\top(\mathbf{x}_i - \boldsymbol{\mu}), \sigma^2\mathbf{M}^{-1})$$

where $\mathbf{M} = \mathbf{W}^\top\mathbf{W} + \sigma^2\mathbf{I}$.

Hint: Given random variables \mathbf{x} and variable \mathbf{y} where:

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \Sigma_x) \quad (4)$$

$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{A}\mathbf{x} + \mathbf{b}, \Sigma_{y|x}) \quad (5)$$

The marginal distribution of \mathbf{y} and the conditional distribution of \mathbf{x} given \mathbf{y} are given by

$$p(\mathbf{y}) = \mathcal{N}(\mathbf{y}|\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \Sigma_{y|x} + \mathbf{A}\Sigma_x\mathbf{A}^T) \quad (6)$$

$$p(\mathbf{x}|\mathbf{y}) = \mathcal{N}\left(\mathbf{x}|\Sigma_{x|y}\left(\mathbf{A}^T\Sigma_{y|x}^{-1}(\mathbf{y} - \mathbf{b}) + \Sigma_x^{-1}\boldsymbol{\mu}\right), \Sigma_{x|y}\right) \quad (7)$$

where

$$\Sigma_{x|y} = \left(\Sigma_x^{-1} + \mathbf{A}^T\Sigma_{y|x}^{-1}\mathbf{A}\right)^{-1}$$

Solution:

1. Recall the rule of sum of normally distributed random variables:

If

$$x \sim \mathcal{N}(\mu_X, \sigma_X^2) \quad (8)$$

$$y \sim \mathcal{N}(\mu_Y, \sigma_Y^2) \quad (9)$$

$$z = \mathbf{X} + \mathbf{Y} \quad (10)$$

²We can actually find the solution for fully-observed \mathbf{X} via eigenvector decomposition. However, the EM algorithm can have computational advantages and can be extended to missing data scenarios. For more information, see Chapter 12 of Bishop’s Pattern Recognition and Machine Learning.

Then,

$$z \sim \mathcal{N}(\boldsymbol{\mu}_X + \boldsymbol{\mu}_Y, \sigma_X^2 + \sigma_Y^2) \quad (11)$$

By using the rule of sum above and also affine property of Gaussian distribution (equation 1), we can get:

$$p(\mathbf{x}_i) = \mathcal{N}(\mathbf{0} + \boldsymbol{\mu} + \mathbf{0}, \mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}) \quad (12)$$

$$= \mathcal{N}(\boldsymbol{\mu}, \mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}) \quad (13)$$

2. From above, we already have:

$$p(\mathbf{z}_i) = \mathcal{N}(\mathbf{0}, \mathbf{I}) \quad (14)$$

$$p(\mathbf{x}_i|\mathbf{z}_i) = \mathcal{N}(\mathbf{x}_i|\boldsymbol{\mu} + \mathbf{W}\mathbf{z}_i, \sigma^2\mathbf{I}) \quad (15)$$

$$(16)$$

Now, we compute:

$$p(\mathbf{z}_i|\mathbf{x}_i) \quad (17)$$

From equation 11,

$$p(\mathbf{z}_i|\mathbf{x}_i) = \mathcal{N}\left(\mathbf{z}_i|\boldsymbol{\Sigma}_{\mathbf{z}_i|\mathbf{x}_i}\left(\mathbf{W}^T\boldsymbol{\Sigma}_{\mathbf{x}_i|\mathbf{z}_i}^{-1}(\mathbf{x}_i - \boldsymbol{\mu}) + \boldsymbol{\Sigma}_{\mathbf{z}_i}^{-1}\mathbf{0}\right), \boldsymbol{\Sigma}_{\mathbf{z}_i|\mathbf{x}_i}\right) \quad (18)$$

where

$$\boldsymbol{\Sigma}_{\mathbf{x}_i|\mathbf{z}_i} = \sigma^2\mathbf{I} \quad (19)$$

$$\boldsymbol{\Sigma}_{\mathbf{z}_i|\mathbf{x}_i} = \left(\boldsymbol{\Sigma}_{\mathbf{x}_i}^{-1} + \mathbf{W}^T\boldsymbol{\Sigma}_{\mathbf{z}_i|\mathbf{x}_i}^{-1}\mathbf{W}\right)^{-1} \quad (20)$$

$$= (\mathbf{I}^{-1} + \mathbf{W}^T(\sigma^2\mathbf{I})^{-1}\mathbf{W})^{-1} \quad (21)$$

$$= \sigma^2(\mathbf{W}^T\mathbf{W} + \sigma^2\mathbf{I})^{-1} = \sigma^2\mathbf{M}^{-1} \quad (22)$$

Then, we have

$$p(\mathbf{z}_i|\mathbf{x}_i) = \mathcal{N}(\mathbf{z}_i|\sigma^2\mathbf{M}^{-1}(\mathbf{W}^T(\sigma^2\mathbf{I})^{-1})(\mathbf{x}_i - \boldsymbol{\mu}), \sigma^2\mathbf{M}^{-1}) \quad (23)$$

$$= \mathcal{N}(\mathbf{M}^{-1}\mathbf{W}^\top(\mathbf{x}_i - \boldsymbol{\mu}), \sigma^2\mathbf{M}^{-1}) \quad (24)$$

Problem 1.d. Finally, derive the E-step and the M-step for the EM algorithm applied to probabilistic PCA. *Hint:* If you are really stuck, refer to Chapter 12.2.2. of Bishop's Pattern Recognition and Machine Learning. This portion is not especially difficult but is notationally heavy and requires algebraic manipulation.

Solution:

In E-step, we need to evaluate $p(\mathbf{z}|\mathbf{x}, \mathbf{W}, \boldsymbol{\mu}, \sigma^2)$. We already know that

$$p(\mathbf{z}|\mathbf{x}, \mathbf{W}, \boldsymbol{\mu}, \sigma^2) = \mathcal{N}(\mathbf{M}^{-1}\mathbf{W}^T(\mathbf{x} - \boldsymbol{\mu}), \sigma^2\mathbf{M}^{-1})$$

where $\mathbf{M} = \mathbf{W}^T\mathbf{W} + \sigma^2\mathbf{I}$. Then, We assign $q(\mathbf{z}_n) = p(\mathbf{z}_n|\mathbf{x}, \mathbf{W}_{old}, \boldsymbol{\mu}_{old}, \sigma_{old}^2)$

In M-step, denote $\theta = \{\mathbf{W}, \boldsymbol{\mu}, \sigma^2\}$ and maximize:

$$\max_{\theta} \sum_{n=1}^N \mathbb{E}_{q(\mathbf{z}_n)} \left[\ln p(\mathbf{x}_n|\mathbf{z}_n, \theta) \right] + \mathbb{E}_{q(\mathbf{z}_n)} \left[p(\mathbf{z}_n|\theta) \right] \quad (25)$$

where

$$\ln p(\mathbf{x}_n | \mathbf{z}_n, \theta) = -\frac{D}{2} \ln(2\pi\sigma^2) - \frac{1}{2}(\mathbf{x}_n - \boldsymbol{\mu} - \mathbf{W}\mathbf{z}_n)^T (\sigma^2 \mathbf{I})^{-1} (\mathbf{x}_n - \boldsymbol{\mu} - \mathbf{W}\mathbf{z}_n) \quad (26)$$

$$= -\frac{D}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(\mathbf{x}_n - \boldsymbol{\mu})^T (\mathbf{x}_n - \boldsymbol{\mu}) - \frac{1}{2\sigma^2}(\mathbf{W}\mathbf{z}_n)^T (\mathbf{W}\mathbf{z}_n) + \frac{1}{\sigma^2} \mathbf{z}_n^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}) \quad (27)$$

$$= -\frac{D}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \|\mathbf{x}_n - \boldsymbol{\mu}\|^2 - \frac{1}{2\sigma^2} \text{trace}(\mathbf{z}_n \mathbf{z}_n^T \mathbf{W}^T \mathbf{W}) + \frac{1}{\sigma^2} \mathbf{z}_n^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}) \quad (28)$$

and

$$\ln p(\mathbf{z}_n | \theta) = -\frac{1}{2}(\mathbf{z}_n - 0)^T \mathbf{I}^{-1} (\mathbf{z}_n - 0) + \text{const} \quad (29)$$

$$= -\frac{1}{2} \mathbf{z}_n^T \mathbf{z}_n \quad (30)$$

$$= -\frac{1}{2} \text{trace}(\mathbf{z}_n^T \mathbf{z}_n) \quad (31)$$

$$= -\frac{1}{2} \text{trace}(\mathbf{z}_n \mathbf{z}_n^T) \quad (32)$$

Substitute them back,

$$Q(q, \theta) = \sum_{n=1}^N -\frac{D}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \|\mathbf{x}_n - \boldsymbol{\mu}\|^2 - \frac{1}{2\sigma^2} \mathbb{E}_{q(\mathbf{z}_n)} [\text{trace}(\mathbf{z}_n \mathbf{z}_n^T \mathbf{W}^T \mathbf{W})] + \quad (33)$$

$$\frac{1}{\sigma^2} \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu})] - \frac{1}{2} \mathbb{E}_{q(\mathbf{z}_n)} [\text{trace}(\mathbf{z}_n \mathbf{z}_n^T)] \quad (34)$$

$$= -\sum_{n=1}^N \left[\frac{D}{2} \ln(2\pi\sigma^2) + \frac{1}{2\sigma^2} \|\mathbf{x}_n - \boldsymbol{\mu}\|^2 \right] + \frac{1}{2\sigma^2} \text{trace} [\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \mathbf{W}^T \mathbf{W}] + \quad (35)$$

$$\frac{1}{\sigma^2} \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu})] + \frac{1}{2} [\text{trace}(\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T])] \quad (36)$$

Since $p(\mathbf{x}) \sim \mathcal{N}(\mathbf{x} | \boldsymbol{\mu}, \mathbf{C})$, where $\mathbf{C} = \mathbf{W}\mathbf{W}^T + \sigma^2 \mathbf{I}$ is positive definite and symmetric,

$$\max_{\boldsymbol{\mu}} \sum_{n=1}^N \ln p(\mathbf{x}_n) = -\frac{ND}{2} \ln(2\pi) - \frac{N}{2} \ln |\mathbf{C}| - \frac{1}{2} \sum_{n=1}^N (\mathbf{x}_n - \boldsymbol{\mu})^T \mathbf{C}^{-1} (\mathbf{x}_n - \boldsymbol{\mu}) \quad (37)$$

Since \mathbf{C}^{-1} is positive definite and it's a quadratic form, there exists a unique global optimal $\boldsymbol{\mu}_M$.

Let

$$\frac{\partial}{\partial \boldsymbol{\mu}} \left(\sum_{n=1}^N \ln p(\mathbf{x}_n) \right) = \sum_{n=1}^N 2(\mathbf{x}_n - \boldsymbol{\mu})^T \mathbf{C}^{-1} = 0 \quad (38)$$

and we have

$$\boldsymbol{\mu}_M = \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n = \bar{\mathbf{x}} \quad (39)$$

that maximizes Q (if other parameters are fixed).

$$\frac{\partial Q}{\partial \sigma^2} = - \sum_{n=1}^N \left[\frac{\partial}{\partial \sigma^2} \left(\frac{D}{2} \ln(2\pi\sigma^2) \right) + \frac{1}{2} \|\mathbf{x}_n - \boldsymbol{\mu}_M\|^2 \frac{\partial}{\partial \sigma^2} \left(\frac{1}{\sigma^2} \right) + \frac{1}{2} \text{trace} \left[\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \mathbf{W}^T \mathbf{W} \right] \frac{\partial}{\partial \sigma^2} \left(\frac{1}{\sigma^2} \right) - \right. \quad (40)$$

$$\left. \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n]^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}_M) \frac{\partial}{\partial \sigma^2} \left(\frac{1}{\sigma^2} \right) \right] \quad (41)$$

$$= - \sum_{n=1}^N \left[\frac{D}{2} \cdot \frac{1}{\sigma^2} - \left[\frac{1}{2} \|\mathbf{x}_n - \boldsymbol{\mu}_M\|^2 + \frac{1}{2} \text{trace} [\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \mathbf{W}^T \mathbf{W}] - \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n]^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}_M) \right] \left(\frac{1}{\sigma^2} \right)^2 \right] \quad (42)$$

$$= - \sum_{n=1}^N \frac{1}{\sigma^2} \left[\frac{D}{2} \cdot - \left[\frac{1}{2} \|\mathbf{x}_n - \boldsymbol{\mu}_M\|^2 + \frac{1}{2} \text{trace} [\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \mathbf{W}^T \mathbf{W}] - \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n]^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}_M) \right] \left(\frac{1}{\sigma^2} \right) \right] \quad (43)$$

$$= 0 \quad (44)$$

As a result,

$$\sigma_M^2 = \frac{1}{ND} \sum_{n=1}^N \left[\|\mathbf{x}_n - \boldsymbol{\mu}_M\|^2 + \text{trace} [\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \mathbf{W}^T \mathbf{W}] - 2 \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n]^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}_M) \right] \quad (45)$$

$$\frac{\partial Q(q, \theta)}{\partial \mathbf{W}} = - \sum_{n=1}^N \frac{1}{2\sigma^2} \frac{\partial}{\partial \mathbf{W}} \left[\text{trace} [\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \mathbf{W}^T \mathbf{W}] \right] - \frac{1}{\sigma^2} \frac{\partial}{\partial \mathbf{W}} \left[\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n]^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}_M) \right] \quad (46)$$

$$= - \frac{1}{2\sigma^2} \sum_{n=1}^N \frac{\partial}{\partial \mathbf{W}} \left[\text{trace} [\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \mathbf{W}^T \mathbf{W}] \right] - 2 \frac{\partial}{\partial \mathbf{W}} \left[\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n]^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}_M) \right] \quad (47)$$

Since

$$\frac{\partial}{\partial \mathbf{X}} (\text{trace}(\mathbf{A} \mathbf{X} \mathbf{B} \mathbf{X}^T \mathbf{C})) = \mathbf{B} \mathbf{X}^T \mathbf{C} \mathbf{A} + \mathbf{B} \mathbf{X}^T \mathbf{C} \mathbf{A}$$

Set $\mathbf{A} = \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T]$, $\mathbf{B} = \mathbf{I}$, $\mathbf{C} = \mathbf{I}$, $\mathbf{X} = \mathbf{W}^T$.

Then,

$$\frac{\partial}{\partial \mathbf{W}} \left(\text{trace} [\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \mathbf{W}^T \mathbf{W}] \right) = \mathbf{W} \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] + \mathbf{W} \left(\mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \right)^T \quad (48)$$

$$= 2 \mathbf{W} \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n \mathbf{z}_n^T] \quad (49)$$

Since

$$\frac{\partial}{\partial \mathbf{x}} (\mathbf{a}^T \mathbf{x}^T \mathbf{b}) = \mathbf{a} \mathbf{b}^T$$

Set $\mathbf{a} = \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n]$, $\mathbf{b} = (\mathbf{x}_n - \boldsymbol{\mu}_M)$, $\mathbf{x} = \mathbf{W}$.

Then,

$$\frac{\partial}{\partial \mathbf{W}} \left[\mathbb{E}_{q(\mathbf{z}_n)}^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}_M) \right] = (\mathbf{x}_n - \boldsymbol{\mu}_M) \mathbb{E}_{q(\mathbf{z}_n)} [\mathbf{z}_n]^T \quad (50)$$

$$(51)$$

Substitute this back into $\frac{\partial}{\partial \mathbf{W}}(Q(q, \theta))$ and set $\frac{\partial}{\partial \mathbf{W}}(Q(q, \theta)) = 0$. We obtain

$$\frac{\partial}{\partial \mathbf{W}}(Q(q, \theta)) = -\frac{1}{2\sigma^2} \sum_{n=1}^N \left[2\mathbf{W} \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n \mathbf{z}_n^T] - (\mathbf{x}_n - \boldsymbol{\mu}_M) \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n]^T \right] = 0 \quad (52)$$

$$\mathbf{W}_M = \left[\sum_{n=1}^N (\mathbf{x}_n - \boldsymbol{\mu}_M) \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n]^T \right] \left[\sum_{n=1}^N \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n \mathbf{z}_n^T] \right]^{-1} \quad (53)$$

Now let's simplify the process and put everything together.

EM for PPCA

E-step: we evaluate

$$\mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n] = \mathbf{M}^{-1} \mathbf{W}_{old}^T (\mathbf{x}_n - \boldsymbol{\mu}_M) \quad (54)$$

where $\boldsymbol{\mu}_M = \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n = \bar{\mathbf{x}}$ and $\mathbf{M} = \mathbf{W}_{old}^T \mathbf{W}_{old} + \sigma_{old}^2 \mathbf{I}$.

$$\mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n \mathbf{z}_n^T] = \text{Cov}(\mathbf{z}_n, \mathbf{z}_n) + \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n] \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n]^T \quad (55)$$

$$= \sigma_{old}^2 \mathbf{M}^{-1} + \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n] \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n]^T \quad (56)$$

M-step: we update model parameters by:

$$\boldsymbol{\mu}_M = \bar{\mathbf{x}} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n \quad (57)$$

$$\mathbf{W}_{new} = \left[\sum_{n=1}^N (\mathbf{x}_n - \boldsymbol{\mu}_M) \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n]^T \right] \left[\sum_{n=1}^N \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n \mathbf{z}_n^T] \right]^{-1} \quad (58)$$

$$\sigma_{new}^2 = \frac{1}{ND} \sum_{n=1}^N \left[\|\mathbf{x}_n - \boldsymbol{\mu}_M\|^2 + \text{trace}[\mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n \mathbf{z}_n^T] \mathbf{W}^T \mathbf{W}] - 2 \mathbb{E}_{q(\mathbf{z}_n)}[\mathbf{z}_n]^T \mathbf{W}^T (\mathbf{x}_n - \boldsymbol{\mu}_M) \right] \quad (59)$$