16th International Science Youth Forum

Colloborative Science Activty

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Pentagon Drawing Solutions

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Construction

General notation: Let the 5 vertices on the pentagon be ABCDE

Steps:

Draw a line segment of arbitrary length and label the 2 ends as A and B.

Construct the perpendicular bisector of AB. Let the midpoint of A and B be M.

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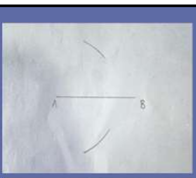
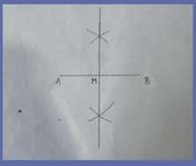
Construction

To do this, extend the compass and set it to a length which is more than half of the length of AB. Draw arcs from both points A and B with that specific length above and below the line.

After which, 2 points of intersection will be formed between the arcs drawn. Join the 2 points of intersection and the perpendicular bisector of AB is constructed.





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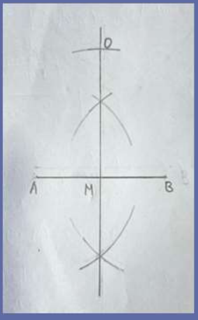
Construction

Construct point O, the point on the perpendicular bisector such that MO = AB.



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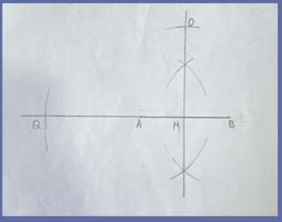
Construction

Extend AB through A.

Let Q be the point on AB extended such that AO = AQ.



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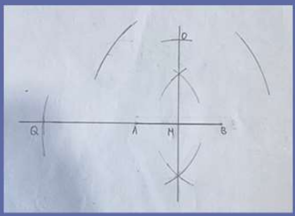
Construction

Measure the length MQ with the compass. Then draw an arc from B such that all points on the arc have a distance MQ from B.

Repeat the above with point A.



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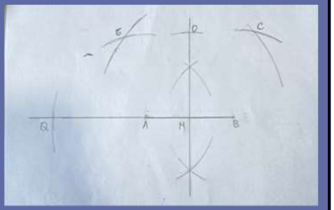
Construction

Measure the length AB with the compass. From both points A and B, draw arcs such that all points on the arc have a distance AB from A and B respectively.

The 2 points of intersection between the arcs drawn in steps 6, 7 and 8 will be points C and E.



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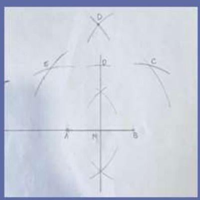
Construction

To find point D, repeat step 8 with points C and E respectively. The intersection point between the 2 arcs will be point D.



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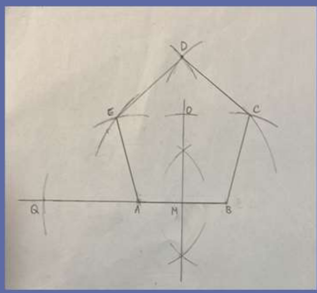
Construction

Final construction



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Proof of Correctness

In a pentagon, the ratio of its diagonal length to its side length is exactly equal to $\left(1 + \sqrt{5}\right) : 2$. Hence, we will be using the side lengths of $1,\ 2\ and\ \sqrt{5}$, which are the 3 sides to a right angle triangle, to construct a regular pentagon (seen in Triangle AMO).

Let AB have length $2x$. Then

$$AM = MB = x,$$
$$MO = 2x, \text{ and}$$
$$AO = AQ = \sqrt{5}x. \text{ Hence,}$$
$$MQ = AM + AQ = \left(1 + \sqrt{5}\right)x.$$

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Proof of Correctness

Utilising the property of a pentagon that

$$\frac{Diagonal\ Length}{Side\ Length} = \frac{1+\sqrt{5}}{2},$$

we can deduce that

$$BE = AC = \frac{1+\sqrt{5}}{2} \times 2x = (1 + \sqrt{5})x.$$

Furthermore, since all side lengths of a regular pentagon are equal, we know that

$$BC = AE = 2x.$$

Assuming both points C and E lie above the line AB, we can uniquely find points C and E which fulfil both equations above respectively by drawing arcs to find the intersection points.

Lastly, to find point D, we need a point that is above all other points on the pentagon and is equidistant with distance AB from both point E and point C. We can hence uniquely determine point D and the regular pentagon is constructed.

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Bonus

Why cant a heptagon be constructed?

Note that 7 = 2 x 3 + 1. Then we would need to be trisecting an angle to construct the heptagon, which cannot be done with a compass.

A more comprehensive proof can be found here: https://proofwiki.org/wiki/Compass_and_Straightedge_Construction_for_Regular_Heptagon_does_not_exist

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2 Falling Ruler Solutions

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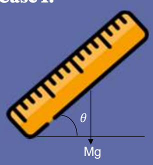
Solution 1: Torque/Force Analysis

Case 1:



$$\frac{d^2\theta}{dt^2} = \frac{dw}{dt} = \frac{dw}{d\theta}\left(\frac{d\theta}{dt}\right) = w\frac{dw}{d\theta}$$

For a ruler with uniform mass and length L, rotating about the end,

$$I = \int_0^L r^2 dm = \int_0^L r^2 \frac{M}{L} dr = \frac{1}{3}ML^2$$

Torque Analysis:

$$\mathrm{M}g\frac{L}{2}cos\theta = I\alpha$$

$$Mg\frac{L}{2}cos\theta = Iw\frac{dw}{d\theta}$$

$$Mg\frac{L}{2}\int cos\theta\, d\theta = I\int w\, dw$$

$$Mg\frac{L}{2}sin\theta = \frac{1}{2}Iw^2 + c$$

When t=0, w=0, $\theta = \alpha$,

$$c = Mg\frac{L}{2}sin\alpha$$

$$\mathrm{M}g\frac{L}{2}(sin\theta - sin\alpha) = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\left(\frac{d\theta}{dt}\right)^2$$

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Solution 1

$$g(sin\theta - sin\alpha) = \frac{1}{3}L\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{d\theta}{dt} = \sqrt{\frac{3g(sin\theta - sin\alpha)}{L}}$$

$$\frac{dt}{d\theta} = \sqrt{\frac{L}{3g(sin\theta - sin\alpha)}}$$

$$\int dt = \int \sqrt{\frac{L}{3g}}\frac{1}{\sqrt{(sin\theta - sin\alpha)}}d\theta$$

$$t = \sqrt{\frac{L}{3g}}\int \frac{1}{\sqrt{(sin\theta - sin\alpha)}}d\theta$$

Notice that the time taken for the ruler to fall is independent of the mass of the ruler!
The longer the ruler, the more time it will take to topple!

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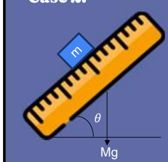
Solution 1: Torque/Force Analysis

Case 2:



$$\frac{d^2\theta}{dt^2} = \frac{dw}{dt} = \frac{dw}{d\theta}\left(\frac{d\theta}{dt}\right) = w\frac{dw}{d\theta}$$

Torque Analysis:

Extra Mass!

$$\mathrm{M}g\frac{L}{2}cos\theta + \boxed{mg\frac{L}{2}cos\theta} = I\alpha$$

$$(M+m)g\frac{L}{2}cos\theta = Iw\frac{dw}{d\theta}$$

$$(\mathrm{M}+\mathrm{m})g\frac{L}{2}\int cos\theta\, d\theta = I\int w\, dw$$

$$(\mathrm{M}+\mathrm{m})\, g\frac{L}{2}sin\theta = \frac{1}{2}Iw^2 + c$$

Due to the additional mass,

$$I = \frac{1}{3}ML^2 + m\frac{L^2}{4}$$

Similarly,

$$(\mathrm{M}+\mathrm{m})g\frac{L}{2}(sin\theta - sin\alpha) = \frac{1}{2}\left(\frac{1}{3}ML^2 + m\frac{L^2}{4}\right)\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{d\theta}{dt} = \sqrt{\frac{gL(M+m)(sin\theta - sin\alpha)}{\left(\frac{ML^2}{3} + \frac{mL^2}{4}\right)}}$$

$$= \sqrt{\frac{3g(M+m)((sin\vartheta - sin\alpha)}{L(M+\frac{3m}{4})}}$$

$$t = \sqrt{\frac{L}{3g}}\sqrt{\frac{(M+\frac{3m}{4})}{(M+m)}}\int\frac{1}{\sqrt{(sin\theta - sin\alpha)}}d\theta$$

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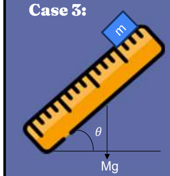
Solution 1: Torque/Force Analysis

Case 3:



$$\frac{d^2\theta}{dt^2} = \frac{dw}{dt} = \frac{dw}{d\theta}\left(\frac{d\theta}{dt}\right) = w\frac{dw}{d\theta}$$

Torque Analysis:

Different Length!

$$\mathrm{M}g\frac{L}{2}cos\theta + \boxed{mgLcos\theta} = I\alpha$$

$$(M+2m)g\frac{L}{2}cos\theta = Iw\frac{dw}{d\theta}$$

$$(\mathrm{M}+2\mathrm{m})g\frac{L}{2}\int cos\theta\, d\theta = I\int w\, dw$$

$$(\mathrm{M}+2\mathrm{m})\, g\frac{L}{2}sin\theta = \frac{1}{2}Iw^2 + c$$

Due to the additional mass,

$$I = \frac{1}{3}ML^2 + mL^2$$

Similarly,

$$(\mathrm{M}+2\mathrm{m})g\frac{L}{2}(sin\theta - sin\alpha) = \frac{1}{2}\left(\frac{1}{3}ML^2 + mL^2\right)\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{d\theta}{dt} = \sqrt{\frac{gL(M+2m)(sin\theta - sin\alpha)}{\left(\frac{ML^2}{3} + mL^2\right)}}$$

$$= \sqrt{\frac{3g(M+2m)((sin\vartheta - sin\alpha)}{L(M+3m)}}$$

$$t = \sqrt{\frac{L}{3g}}\sqrt{\frac{(M+3m)}{(M+2m)}}\int\frac{1}{\sqrt{(sin\theta - sin\alpha)}}d\theta$$

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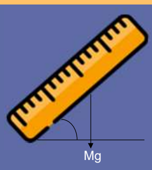
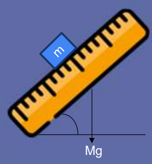
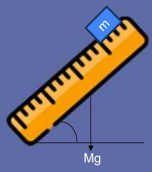
Solution Analysis







$$t = \sqrt{\frac{L}{3g}} \int \frac{1}{\sqrt{(sin\theta - sin\alpha)}} d\theta$$

$$t = \sqrt{\frac{L}{3g}} \sqrt{\frac{(M + \frac{3m}{4})}{(M + m)}} \int \frac{1}{\sqrt{(sin\theta - sin\alpha)}} d\theta$$

$$t = \sqrt{\frac{L}{3g}} \sqrt{\frac{(M + 3m)}{(M + 2m)}} \int \frac{1}{\sqrt{(sin\theta - sin\alpha)}} d\theta$$

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Solution Analysis

$$\sqrt{\frac{(M + \frac{3m}{4})}{(M + m)}} < 1 < \sqrt{\frac{(M + 3m)}{(M + 2m)}}$$

Hence, when the mass is placed in the center of the ruler, the ruler falls down the fastest.
However, if the mass is placed at the edge at the top, it will take the most time to topple!

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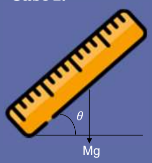
Solution 2: Energy Conservation

Case 1:



By the conservation of energy,

$$\mathrm{M}g\frac{L}{2}sin\alpha = \frac{1}{2}Iw^2 + \mathrm{M}g\frac{L}{2}sin\theta$$
$$Mg\frac{L}{2}(sin\alpha - sin\theta) = \frac{1}{2}Iw^2$$
$$Mg\frac{L}{2}(sin\alpha - sin\theta) = \frac{1}{6}ML^2w^2$$
$$g\ (sin\alpha - sin\theta) = \frac{1}{3}Lw^2$$
$$\frac{d\theta}{dt} = \sqrt{\frac{3g(sin\alpha - sin\theta)}{L}}$$

$$\Rightarrow \quad t = \sqrt{\frac{L}{3g}} \int \frac{1}{\sqrt{(sin\alpha - sin\theta)}} d\theta$$

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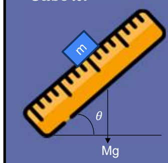
Solution 2: Energy Conservation

Case 2:



By the conservation of energy,

$$(\mathrm{M} + \mathrm{m})g\frac{L}{2}sin\alpha = \frac{1}{2}Iw^2 + (\mathrm{M} + \mathrm{m})g\frac{L}{2}sin\theta$$
$$(\mathrm{M} + \mathrm{m})\ g\frac{L}{2}(sin\alpha - sin\theta) = \frac{1}{2}\left(\frac{1}{3}ML^2 + m\frac{L^2}{4}\right)(\frac{d\theta}{dt})^2$$
$$\frac{d\theta}{dt} = \sqrt{\frac{3g(M + m)(sin\alpha - sin\theta)}{L(M + \frac{3m}{4})}}$$

$$\Downarrow$$

$$t = \sqrt{\frac{L}{3g}} \sqrt{\frac{(M + \frac{3m}{4})}{(M + m)}} \int \frac{1}{\sqrt{(sin\alpha - sin\theta)}} d\theta$$

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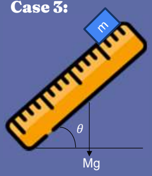
Solution 2: Energy Conservation

Case 3:



By the conservation of energy,

$$Mg\frac{L}{2}sin\alpha + mgLsin\alpha = \frac{1}{2}Iw^2 + Mg\frac{L}{2}sin\theta + mgLsin\theta$$
$$(\mathrm{M} + 2\mathrm{m})\ g\frac{L}{2}(sin\alpha - sin\theta) = \frac{1}{2}\left(\frac{1}{3}ML^2 + mL^2\right)(\frac{d\theta}{dt})^2$$
$$\frac{d\theta}{dt} == \sqrt{\frac{3g(M + 2m)(sin\alpha - sin\theta)}{L(M + 3m)}}$$

$$\Downarrow$$

$$t = \sqrt{\frac{L}{3g}} \sqrt{\frac{(M + 3m)}{(M + 2m)}} \int \frac{1}{\sqrt{(sin\alpha - sin\theta)}} d\theta$$

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Thank You!

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