Fast and Robust Earth Mover's Distances

Ofir Pele and Michael Werman
School of Computer Science and Engineering
The Hebrew University of Jerusalem
{ofirpele,werman}@cs.huji.ac.il

Abstract

We present a new family of Earth Mover's Distances. We show that members of this family have many desirable properties. First, they correspond to the way humans perceive distances. Second, they are robust to outlier noise and quantization effects. Third, we prove that they are metrics. We also propose a fast algorithm and a linear time lower bound computation which allows fast filtering of large distances. The algorithm runs in an order of magnitude faster than the original algorithm, which makes it possible to compute the EMD on large histograms and databases. Finally, we show results for image retrieval.

1. Introduction

Histograms are ubiquitous tools in numerous computer vision tasks. It is common practice to use distances such as L_2 or χ^2 for comparing histograms. This practice assumes that the histogram domains are aligned. However this assumption is violated through quantization, shape deformation, light changes, etc.

The Earth Mover's Distance (EMD) [8] is a cross-bin distance that addresses this alignment problem. EMD is defined as the minimal cost that must be paid to transform one histogram into the other, where there is a "ground distance" between the basic features that are aggregated into the histogram. The EMD as defined by Rubner is a metric only for normalized histograms. However, recently Pele and Werman [6] suggested $\widehat{EMD}$ and showed that it is a metric for all histograms.

A major issue that arises when using EMD is which ground distance to use for the basic features. This, of course, depends on the histograms, the task and practical considerations. In many cases we would like the distance to correspond to the way humans perceive distances (image retrieval, for example). In other cases we would like the distance to fit the distribution of the noise (keypoint matching, for example). Practical considerations include speed of computation, existence of easy to compute lower bounds (this allows fast filtering of large distances) and the metric property that enables fast algorithms for nearest neighbor searches.

We propose to use thresholded ground distances. *i.e*. distances that saturate to a constant value. These distances have many desirable properties. First, saturated distances correspond to the way humans perceive distances [11]. Second, many natural noise distributions have a heavy tail; *i.e*. outlier noise. Thresholded distances give different outliers the same large distance.

The usage of saturated distances with EMD is not new. However, usually the exponent function is used as it is a metric [8, 9]. We show that thresholded metrics are also metrics. In addition, we present a fast algorithm that computes an EMD with a thresholded ground distance.

Fast to compute lower bounds are also of importance as they allow fast filtering of large distances. Rubner *et al*. [8] have proposed the distance between the centers of mass of two normalized histograms as lower bound for EMD with a norm ground distance. We show a linear time lower bound for an EMD with a thresholded distance for any histograms.

The Earth Mover's Distance was used successfully in many applications such as image retrieval [8], edge and corner detection [9], SIFT matching [4, 6], classification of texture and object categories [17], contour matching [2] and as a melodic similarity measure [13]. The major contribution of this paper is the fast algorithm for the computation of the EMD with thresholded ground distance. We argue that thresholded distances has all the benefits of the exponent function that is usually used as a saturated distance and also a big advantage of its much shorter computation time.

This paper is organized as follows. Section 2 is an overview of previous work. Section 3 describes the Earth Mover's Distance. Section 4 discusses thresholded distances and proves that they are metrics. Section 5 describes the fast algorithm and the linear time lower bound computation. Section 6 presents the results. Finally, conclusions are drawn in Section 7.

2. Previous Work

Early work using cross-bin distances for histogram comparison can be found in [10, 16, 15, 7]. Shen and Wong [10] proposed to unfold two integer histograms, sort them and then compute the L_1 distance between the unfolded histograms. To compute the modulo matching distance between cyclic histograms they proposed taking the minimum from all cyclic permutations. This distance is equivalent to EMD between two normalized histograms. Werman *et al*. [16] showed that this distance is equal to the L_1 distance between the cumulative histograms. They also proved that matching two cyclic histograms by examining only cyclic permutations is in effect optimal. Werman *et al*. [15] proposed an $O(M \log M)$ algorithm for finding a minimal matching between two sets of M points on a circle. The algorithm was adapted by Pele and Werman [6] to compute the EMD between two N-bin, normalized histograms with time complexity $O(N)$.

Peleg *et al*. [7] suggested using the EMD for grayscale images and using linear programming to compute it. Rubner *et al*. [8] suggested using EMD for color and texture images and gave a definition of an EMD for non-normalized histograms. They computed the EMD using a specific linear programming algorithm - the transportation simplex. The algorithm worst case time complexity is exponential. Practical run time was shown to be super-cubic. Interior-point algorithms or Orlin's algorithm [5] are both with time complexity of $O(N^3 \log N)$ and can also be used. All of these algorithms have high computational cost.

Indyk and Thaper [3] proposed approximating certain EMD by embedding it into an Euclidean space. Embedding time complexity is $O(Nd \log \Delta)$, where N is the feature set size, d is the feature space dimension and Δ is the diameter of the union of the two feature sets. Shirdhonkar and Jacobs [12] proposed a linear time algorithm for approximating certain EMD for low dimensional histograms using the sum of absolute values of the weighted wavelet coefficients of the difference histogram.

Ling and Okada proposed EMD-L_1[4]; *i.e*. EMD with L_1 as the ground distance. To execute the EMD-L_1 computation, they propose a tree-based algorithm, Tree-EMD. Tree-EMD exploits the fact that a basic feasible solution of the simplex algorithm-based solver forms a spanning tree when the EMD-L_1 is modeled as a network flow optimization problem. The worst case time complexity is exponential. Empirically, they show that this new algorithm has an average time complexity $O(N^2)$.

Pele and Werman [6] proposed $\widehat{EMD}$ - a new definition of the EMD for non-normalized histograms. They showed that unlike Rubner's definition $\widehat{EMD}$ is a metric also for non-normalized histograms. In addition they proposed a linear time algorithm for an EMD that gives a transportation cost of 0 to corresponding bins, 1 for two nearby bins and 2 for farther bins and for the extra mass. They showed that this distance can be used to improve SIFT matching.

3. The Earth Mover's Distance

The Earth Mover's Distance (EMD) [8] is defined as the minimal cost that must be paid to transform one histogram into the other, where there is a "ground distance" between the basic features that are aggregated into the histogram.

Given two histograms P, Q the EMD as defined by Rubner *et al*. [8] is:

$$EMD(P,Q) = \min_{\{f_{ij}\}} \frac{\sum_{i,j} f_{ij} d_{ij}}{\sum_{i,j} f_{ij}} \quad s.t \tag{1}$$

$$\sum_j f_{ij} \le P_i \ , \ \sum_i f_{ij} \le Q_j \ , \tag{2}$$

$$\sum_{i,j} f_{ij} = \min(\sum_i P_i, \sum_j Q_j) \ , \ f_{ij} \ge 0 \tag{3}$$

where $\{f_{ij}\}$ denotes the flows. Each f_{ij} represents the amount transported from the ith supply to the jth demand. We call d_{ij} the *ground distance* between bin i and bin j in the histograms.

Pele and Werman [6] suggested $\widehat{EMD}$:

$$\widehat{EMD}_\alpha(P,Q) = (\min_{\{f_{ij}\}} \sum_{i,j} f_{ij} d_{ij}) \tag{4}$$

$$+|\sum_i P_i - \sum_j Q_j| \times \alpha \max_{i,j}\{d_{ij}\} \ \ s.t \text{ Eqs. } 2,3 \tag{5}$$

Pele and Werman proved that $\widehat{EMD}$ is a metric for any two histograms if the ground distance is a metric and $\alpha \ge 0.5$ [6].

4. Thresholded Distances

Thresholded distances are distances that saturate to a threshold; *i.e*. let $d(a,b)$ be a distance measure between two features - a, b. The thresholded distance with a threshold of t is defined as:

$$d_t(a,b) = \min(d(a,b), t)$$

We now prove that if d is a metric then d_t is also a metric. Non-negativity and symmetry hold trivially, so we only need to prove that the triangle inequality holds.

$d_t(a,b) + d_t(b,c) \ge d_t(a,c)$ if d is a metric.

We consider three cases:

1.

$$\Big(d_t(a,b) < t\Big) \wedge \Big(d_t(b,c) < t\Big) \wedge \Big(d_t(a,c) < t\Big) \Rightarrow$$
$$\Big(d_t(a,b) = d(a,b)\Big) \wedge \Big(d_t(b,c) = d(b,c)\Big) \wedge$$
$$\Big(d_t(a,c) = d(a,c)\Big) \Rightarrow$$
$$d_t(a,b) + d_t(b,c) \geq d_t(a,c)$$

2.

$$\Big(d_t(a,b) = t\Big) \vee \Big(d_t(b,c) = t\Big) \Rightarrow$$
$$d_t(a,b) + d_t(b,c) \geq t \geq d_t(a,c)$$

3.

$$d_t(a,c) = t \Rightarrow$$

Assume negatively that:

$$d_t(a,b) + d_t(b,c) < d_t(a,c) = t \Rightarrow$$
$$\Big(d_t(a,b) < t\Big) \wedge \Big(d_t(b,c) < t\Big) \Rightarrow$$
$$\Big(d_t(a,b) = d(a,b)\Big) \wedge \Big(d_t(b,c) = d(b,c)\Big) \Rightarrow$$
$$d(a,b) + d(b,c) < t = d_t(a,c) \leq d(a,c) \Rightarrow$$
$$d(a,b) + d(b,c) < d(a,c)$$

The last statement contradicts the triangle inequality of the metric d.

As every case as covered by at least one of the above cases, we proved that d_t is a metric if d is a metric.

5. Fast Computation of Thresholded Earth Mover's Distances

This section describes an algorithm that computes $\widehat{EMD}$ with a thresholded ground distance in a fast way and a linear time lower bound.

$\widehat{EMD}$ can be solved by a min-cost-flow algorithm. Our algorithm does a simple transformation to the flow network that reduces the number of edges. If N is the number of bins in the histogram, the flow network of $\widehat{EMD}$ has exactly $N^2 + N$ edges (see (a) in Fig. 1). N^2 edges connects all sources to all sinks. The extra N edges connects all sources to the sink that gets the difference between the total mass of the two histograms (we assume without the loss of generality that the source histogram total mass is greater or equal than the sink histogram total mass).

The transformation (see Fig. 1) first removes all edges with cost t. Second, it adds a new transhipment vertex. Finally we connect all sources to this vertex with edges of cost t and connect the vertex to all sinks with edges of cost 0. Let K be the number of edges going out of each bin that has a cost different than the threshold t. We removed $N(N-K)$ edges and added $2N$ edges. Thus the total number of edges is $N^2 - (K-2)N$. If $K = O(N)$ the number of edges is $O(N)$ as opposed to the original $O(N^2)$. Note that the new flow network is no longer a transportation problem, but a transhipment problem [1].

Let $K = O(N)$ the optimization problem can be solved [1, 5] with worst case time complexity of

$$O(\min((N^2 \log\log U \log(NC)), \quad (N^2 \log U \sqrt{\log C}, \\ (N^2 \log U \log N), \quad (N^2 \log^2 N)))$$

Algorithms with C term assumes integral cost coefficients that are bounded by C. Algorithms with U assumes integral supply and demands that are bounded by U.

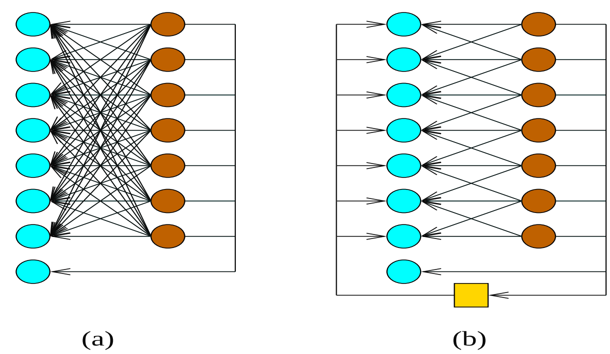


Figure 1. An example of the transformation on a flow network of an EMD with the ground distance: $d(a,b) = min(2, |a-b|)$. (a) is the original flow network with N^2 edges. Note that $N(N-3)$ of these edges has the cost of 2. The bottom cyan vertex on the left is the sink that gets the difference between the total mass of the two histograms. (b) is the transformed flow network. The yellow square is the new transhipment vertex.

We now describe a linear time lower bound computation. If the ground distance is a metric, we start by saturating all zero-cost edges. Second, we greedily flow from each source using the edges which has the minimum cost to all its neighbors. It is easy to see that the constraints $\sum_i f_{ij} \leq Q_j$ might be violated. Thus, the resulting cost of this flow is a lower bound of the $\widehat{EMD}$. As each source is connected to $O(1)$ sinks, the computation time is linear.

6. Results

In this section we present results for image retrieval. However, please note that we do not claim that this is the right way to do image retrieval. Image retrieval is used here as an example of an application where the EMD was already used to show that thresholded distances give good results. The major contribution is the faster algorithm. We

Image size	10×10	20×20	30×30	40×40	50×50
Our's algorithm	0.005	0.09	0.43	1.6	4
Rubner's algorithm	0.023	2.23	30.4	239.7	1084

Table 1. Run time results in seconds of a computation of EMD on greyscale images of various sizes.

show that using our algorithm the running time decrease in an order of magnitude.

We use a database that contains 800 landscape images from the COREL database, that were used also in Wang *et al*. [14]. All images are of size 256×384. We resized them to size 24×16 and converted them to L*a*b* space. In L*a*b* space small Euclidean distances correspond to perceptual distances. Then we selected 16 images as query (numbers $1, 51, 101, \ldots, 751$ from the landscapes images) and searched for the 5 most similar images. We used four different distances:

1. EMD-t is EMD where the ground distance for pixels $(x_1, y_1), (x_2, y_2)$ with colors $(L*_1, a*_1, b*_1), (L*_2, a*_2, b*_2)$ respectively is: $d = \min(||(L*_1, a*_1, b*_1) - (L*_2, a*_2, b*_2)||_2 + ||(x_1, y_1) - (x_1, y_1)||_2, 10)$.
2. EMD-e is EMD where the ground distance for pixels $(x_1, y_1), (x_2, y_2)$ with colors $(L*_1, a*_1, b*_1), (L*_2, a*_2, b*_2)$ respectively is: $d = ||(L*_1, a*_1, b*_1) - (L*_2, a*_2, b*_2)||_2 + ||(x_1, y_1) - (x_1, y_1)||_2$.
3. $\sum L_2$-t is $\sum_{(x_1,y_1)=(x_2,y_2)} \min(||(L*_1, a*_1, b*_1) - (L*_2, a*_2, b*_2)||_2, 10)$
4. $\sum L_2$ is $\sum_{(x_1,y_1)=(x_2,y_2)} ||(L*_1, a*_1, b*_1) - (L*_2, a*_2, b*_2)||_2$

The results are given in figures 2,3,4,5,6,7,8,9 and should be viewed in color. Each figure shows two queries and the 4 most similar images in the database (excluding the query itself) using each of the 4 distances described above. Above each image is the distance of the returned image to the query.

The results show that EMD-t; *i.e*. with a thresholded distances returns the greatest number of perceptually similar images. Second to it is EMD-e followed by $\sum L_2$ and $\sum L_2$-t.

In Fig. 2 top and bottom, both EMD-t and EMD-e performs good, where $\sum L_2$ and $\sum L_2$-t performance is poor. In Fig. 3 top all results are poor. In Fig. 3 bottom EMD-t performs best. In Fig. 4 top again EMD-t performs best while in the bottom EMD-e performs best. However, EMD-t performance is also very good. In Fig. 5 top EMD-t performs best. In Fig. 5 bottom and Fig. 6 all performs equally good. In Fig. 6 bottom $\sum L_2$ performs best. In Fig. 7 8 bottom and top all results are good, although $\sum L_2$-t and EMD-e each does one mistake. In Fig. 9 top all performance is poor. In the bottom EMD-e performs best, EMD-t also has a good performance with only one mistake, while both $\sum L_2$ and $\sum L_2$-t performance is very poor.

It should also be noted that as EMD-e ground distance is not thresholded, its computation is in order of magnitude slower. That is, the run time of computing 800 EMD-t computations is a minute, while for EMD-e it is an hour.

The algorithm we use for the computation of the EMD is successive shortest path [1]. This algorithm has a worst time complexity $O(N^2 U \log N)$. A comparison of the practical running time on a Pentium 3GHz of our algorithm and Rubner's implementation of the simplex algorithm is given in table 1. Our algorithm is in order of magnitude faster.

7. Conclusions

We presented a new family of Earth Mover's Distances. Members of this family correspond to the way humans perceive distances, robust to outlier noise and quantization effects. We proved that they are metrics. We also propose a fast algorithm and a linear time lower bound computation which allows fast filtering of large distances. The algorithm runs in an order of magnitude faster than the original algorithm, which makes it possible to compute the EMD on large histograms and databases.

We are working on a faster implementation of the algorithm and testing the proposed lower bound. In addition, we want to test the new Earth Mover's Distance in applications where it was not possible to use it before because of the long running time.

References

[1] R. Ahuja, T. Magnanti, and J. Orlin. *Network flows: theory, algorithms, and applications*. Prentice-Hall, Inc. Upper Saddle River, NJ, USA, 1993. 3, 4

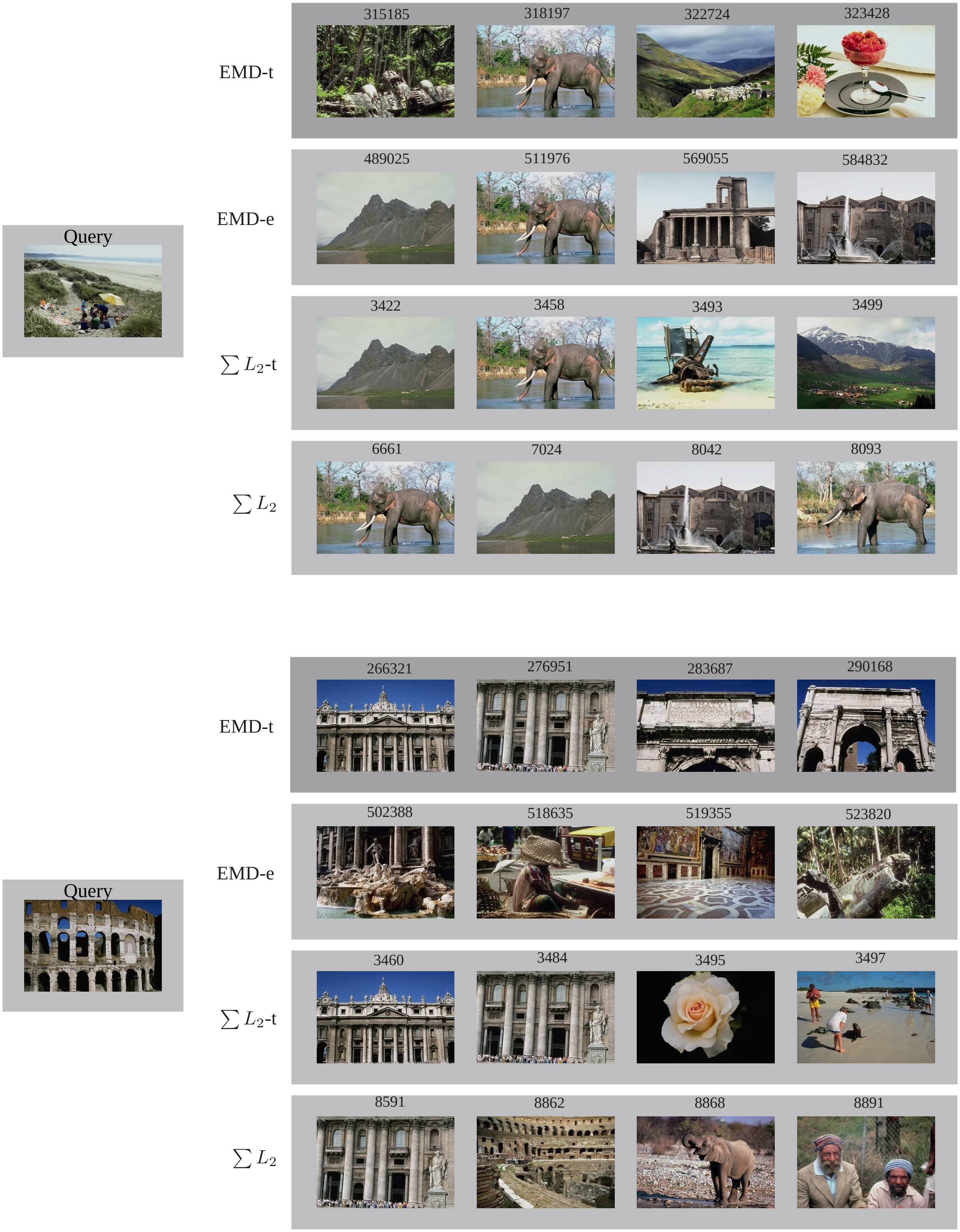
[2] K. Grauman and T. Darrell. Fast contour matching using approximate earth mover's distance. In *CVPR*, 2004. 1

[3] P. Indyk and N. Thaper. Fast image retrieval via embeddings. In *3rd International Workshop on Statistical and Computational Theories of Vision*, Oct 2003. 2

[4] H. Ling and K. Okada. An Efficient Earth Mover's Distance Algorithm for Robust Histogram Comparison. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 29(5):840–853, 2007. 1, 2


Query
EMD-t
337084 338008 338818 339750
EMD-e
546066 573677 575717 578270
$\sum L_2$-t
3676 3707 3713 3714
$\sum L_2$
10306 10389 10554 10601
Query
EMD-t
305949 307598 308029 310535
EMD-e
433033 446563 456403 475494
$\sum L_2$-t
3380 3407 3429 3432
$\sum L_2$
9029 9608 9728 9855

Figure 2.



Query
EMD-t
315185
318197
322724
323428
EMD-e
489025
511976
569055
584832
$\sum L_2$-t
3422
3458
3493
3499
$\sum L_2$
6661
7024
8042
8093
Query
EMD-t
266321
276951
283687
290168
EMD-e
502388
518635
519355
523820
$\sum L_2$-t
3460
3484
3495
3497
$\sum L_2$
8591
8862
8868
8891

Figure 3.

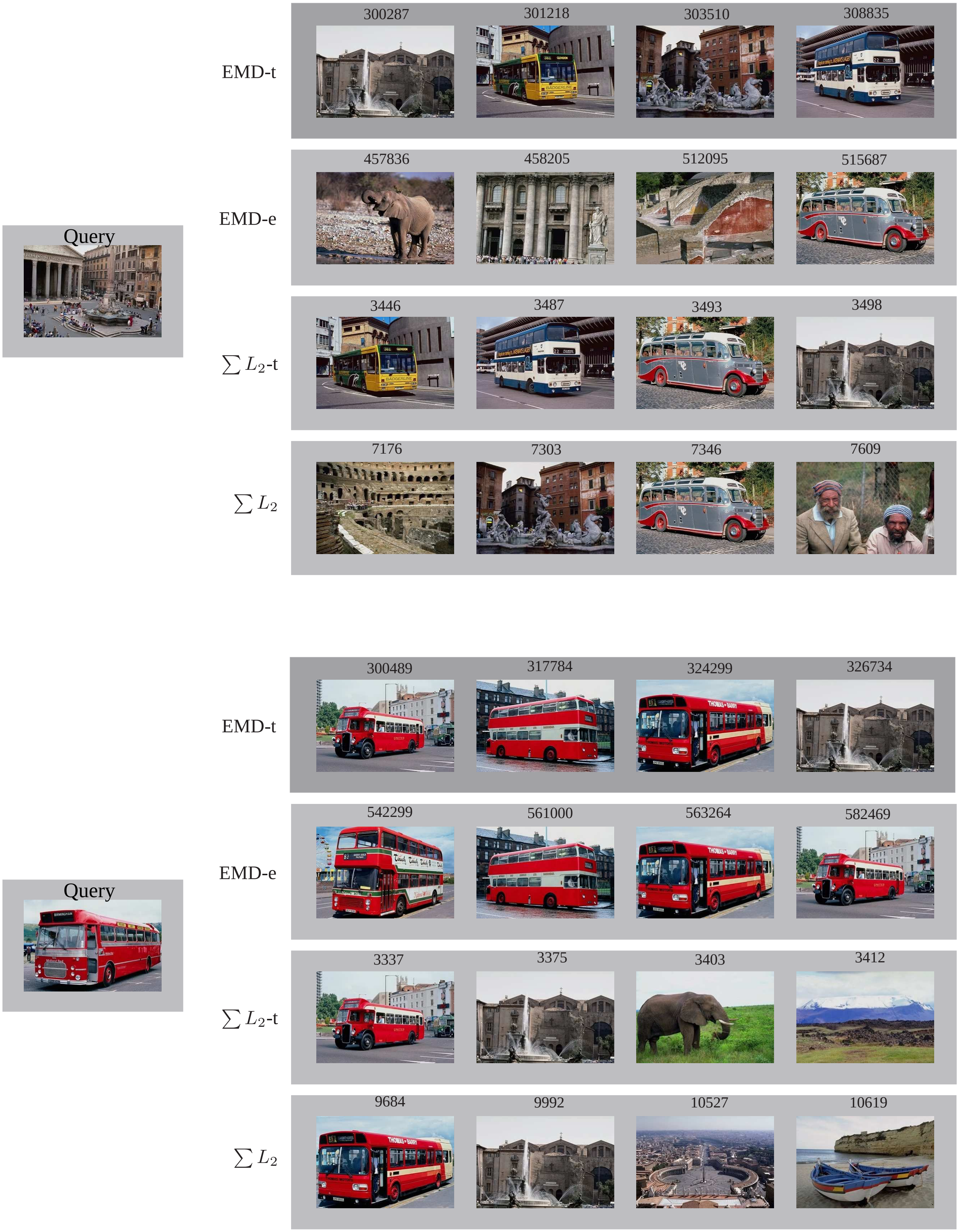

EMD-t
300287
301218
303510
308835
EMD-e
457836
458205
512095
515687
Query
$\sum L_2$-t
3446
3487
3493
3498
$\sum L_2$
7176
7303
7346
7609
EMD-t
300489
317784
324299
326734
EMD-e
542299
561000
563264
582469
Query
$\sum L_2$-t
3337
3375
3403
3412
$\sum L_2$
9684
9992
10527
10619

Figure 4.

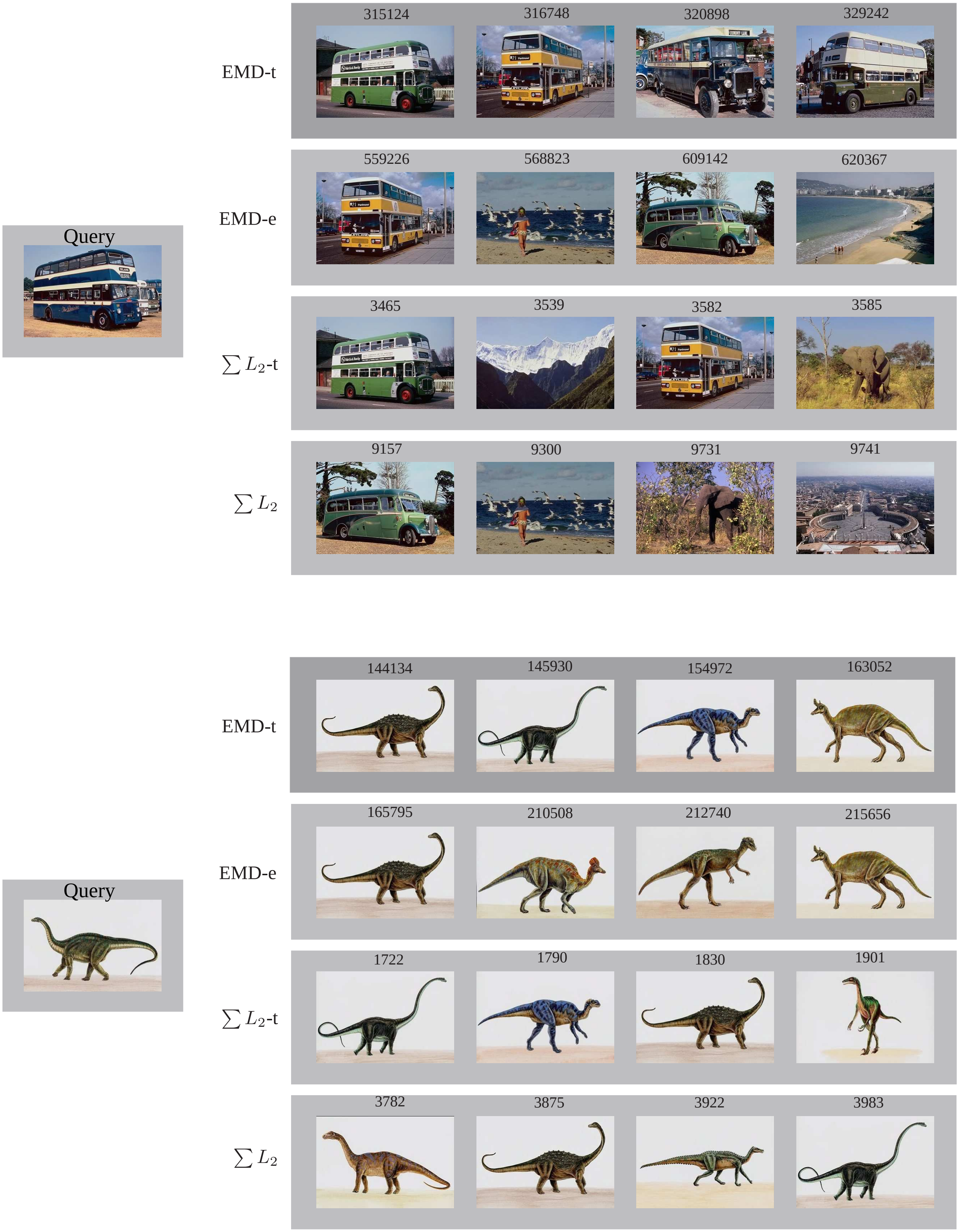

Query
EMD-t
315124
316748
320898
329242
EMD-e
559226
568823
609142
620367
$\sum L_2$-t
3465
3539
3582
3585
$\sum L_2$
9157
9300
9731
9741
Query
EMD-t
144134
145930
154972
163052
EMD-e
165795
210508
212740
215656
$\sum L_2$-t
1722
1790
1830
1901
$\sum L_2$
3782
3875
3922
3983

Figure 5.

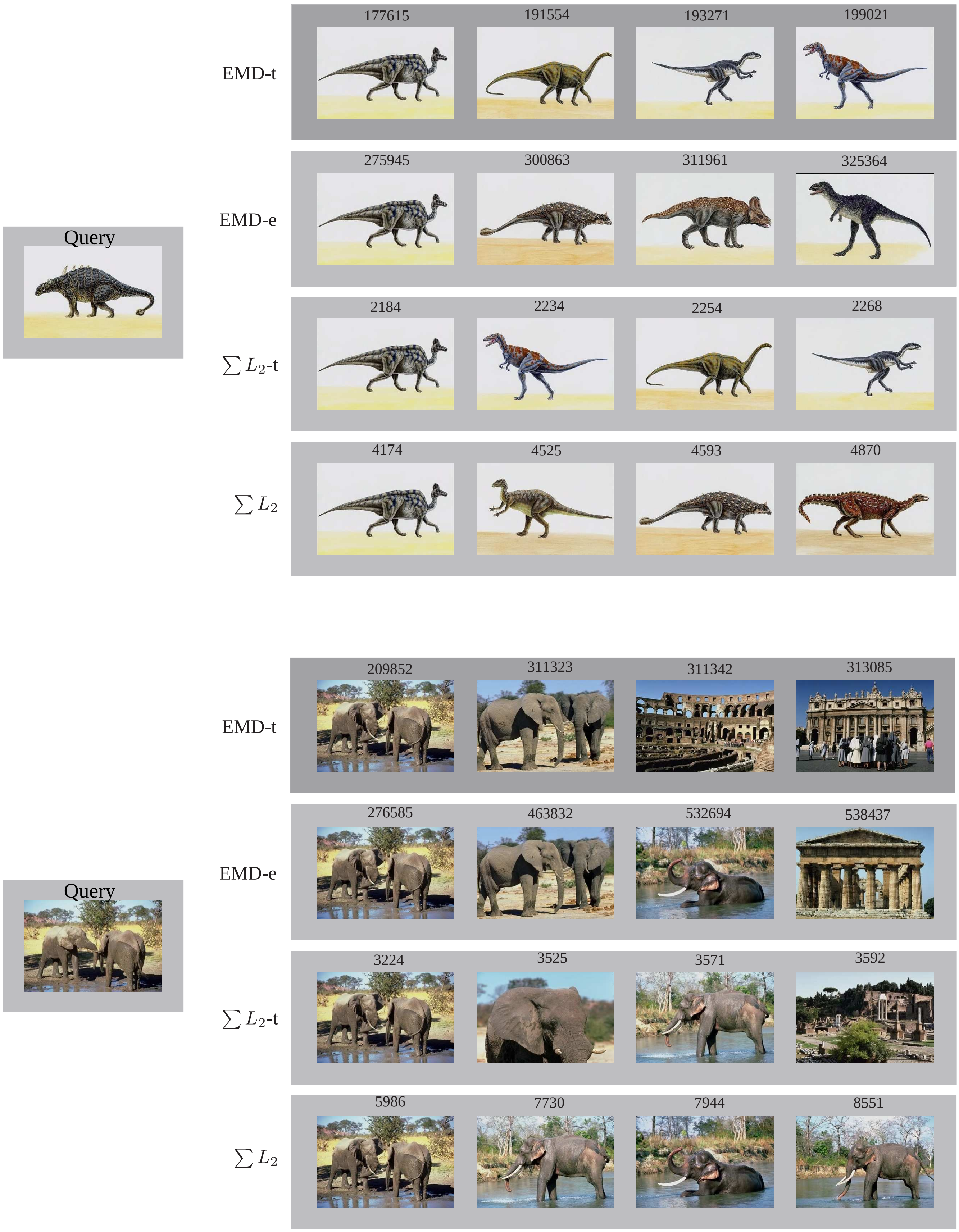

Query
EMD-t
177615
191554
193271
199021
EMD-e
275945
300863
311961
325364
$\sum L_2$-t
2184
2234
2254
2268
$\sum L_2$
4174
4525
4593
4870
Query
EMD-t
209852
311323
311342
313085
EMD-e
276585
463832
532694
538437
$\sum L_2$-t
3224
3525
3571
3592
$\sum L_2$
5986
7730
7944
8551

Figure 6.

Query
EMD-t
338997
339385
339826
344426
EMD-e
711751
773585
816637
830815
$\sum L_2$-t
3648
3689
3699
3705
$\sum L_2$
10914
10982
11091
12355
Query
EMD-t
253665
285690
286497
289033
EMD-e
360311
542958
567472
582197
$\sum L_2$-t
3100
3117
3142
3145
$\sum L_2$
6145
7255
7612
8033

Figure 7.


Query
EMD-t
313458
321511
330658
333711
EMD-e
509880
614632
649406
659818
$\sum L_2$-t
3436
3527
3556
3595
$\sum L_2$
8960
9310
9563
9736
Query
EMD-t
277239
279627
280028
286294
EMD-e
358028
379943
383648
409917
$\sum L_2$-t
3284
3311
3315
3328
$\sum L_2$
4522
4576
5645
5856

Figure 8.

Query
EMD-t
336175
340211
341600
345253
EMD-e
570052
696407
710617
710634
$\sum L_2$-t
3616
3623
3656
3657
$\sum L_2$
8882
10087
10235
10405
Query
EMD-t
348652
349593
351313
352492
EMD-e
613627
670169
686446
701732
$\sum L_2$-t
3722
3738
3741
3748
$\sum L_2$
10700
11433
11770
11827

Figure 9.

[5] J. Orlin. A faster strongly polynomial minimum cost flow algorithm. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, pages 377–387, 1988. 2, 3

[6] O. Pele and M. Werman. A linear time histogram metric for improved sift matching. In *ECCV*, 2008. 1, 2

[7] S. Peleg, M. Werman, and H. Rom. A unified approach to the change of resolution: Space and gray-level. *IEEE Trans. Pattern Analysis and Machine Intelligence.*, 11(7):739–742, 1989. 2

[8] Y. Rubner, C. Tomasi, and L. J. Guibas. The earth mover's distance as a metric for image retrieval. *International Journal of Computer Vision*, 40(2):99–121, 2000. 1, 2

[9] M. Ruzon and C. Tomasi. Edge, Junction, and Corner Detection Using Color Distributions. *IEEE Trans. Pattern Analysis and Machine Intelligence.*, pages 1281–1295, 2001. 1

[10] H. Shen and A. Wong. Generalized texture representation and metric. *Computer vision, graphics, and image processing*, 23(2):187–206, 1983. 2

[11] R. Shepard. Toward a universal law of generalization for psychological science. *Science*, 237(4820):1317–1323, 1987. 1

[12] S. Shirdhonkar and D. Jacobs. Approximate earth movers distance in linear time. In *CVPR*, pages 1–8, 2008. 2

[13] R. Typke, P. Giannopoulos, R. Veltkamp, F. Wiering, and R. van Oostrum. Using transportation distances for measuring melodic similarity. In *ISMIR*, pages 107–114, 2003. 1

[14] J. Wang, J. Li, and G. Wiederhold. SIMPLIcity: Semantics-Sensitive Integrated Matching for Picture LIbraries. *IEEE Trans. Pattern Analysis and Machine Intelligence.*, pages 947–963, 2001. 4

[15] M. Werman, S. Peleg, R. Melter, and T. Kong. Bipartite graph matching for points on a line or a circle. *Journal of Algorithms*, 7(2):277–284, 1986. 2

[16] M. Werman, S. Peleg, and A. Rosenfeld. A distance metric for multidimensional histograms. *Computer Vision, Graphics, and Image Processing*, 32(3), 1985. 2

[17] J. Zhang, M. Marszalek, S. Lazebnik, and C. Schmid. Local features and kernels for classification of texture and object categories: A comprehensive study. *Int. J. Comput. Vision*, 73(2):213–238, 2007. 1