A Novel Fixed-Time Protocol for First-Order Consensus Tracking With Disturbance Rejection

Yong Xu, *Member, IEEE*, Zhaozhan Yao, Renquan Lu, *Senior Member, IEEE*, and Bijoy K. Ghosh, *Fellow, IEEE*

***Abstract*—This technical note studies fixed-time consensus tracking with disturbance rejection for first-order multi-agent systems. The communication topology among the leader and followers contains a directed spanning tree. The control input to the leader is time-varying and unknown to the followers, except that its upper bound is known a priori. A novel fixed-time protocol is devised based on discontinuous and nonlinear control. A fixed-time stability analysis for consensus tracking with disturbance rejection is completed as a one-step control process using non-smooth analysis. An upper bound estimate for the settling time, independent of the initial conditions, is aesthetically pleasing, in comparison to previous results derived based on two-step control design technique using sliding modes. Finally, a numeric example confirms the theoretical results.**

***Index Terms*—Fixed-time stability, consensus tracking, disturbance rejection, directed graphs, non-smooth analysis.**

I. Introduction

RESEARCH on consensus for multi-agent systems (MASs) has acquired considerable attention in control system society, owing to its widespread practical applications, such as in coordination of mobile robots [1]–[3], in formation control of UAV [4]–[6], in sensor network localization [7]–[9], and in many other engineering problems. A critical issue for consensus is to devise appropriate distributed protocols in order that all agents can reach a common state. In the published literature [10]–[12], a few nonlinear protocols have been described to realize finite-time consensus. The corresponding *settling time estimate* (i.e., an upper bound estimate for the settling time) is dependent on the initial conditions, thus finite-time consensus might not be useful in practical applications when the initial conditions are not available. Since settling time is an important performance measure, fixed-time consensus under different scenarios has been investigated in several papers [13]–[15], where the settling time estimate can be determined without knowledge of the initial conditions. Therefore, it is preferable to research fixed-time consensus for practical consideration. Related works are now discussed to bring out the motivation and contribution of this technical note.

A. Related Work

1) Leaderless Consensus and Consensus Tracking: Generally speaking, there are two categories of consensus, as stated in the subtitle. *Leaderless consensus* under different scenarios has been studied in the published literature, e.g., [16]–[18], whereas a remarkable disadvantage of leaderless consensus derived therein is that the final common state of all agents cannot be selected. As a result, the system trajectory might not be desirable. By contrast, a rather practical scenario is to consider the so-called *consensus tracking* by introducing one leader into MASs, where the leader is tracked by other followers. Therefore, it is feasible to derive any desired final common state and particular system trajectory by selecting a control input to the leader. In [19], the fixed-time consensus tracking problem for a group of perturbed first-order followers has been addressed, whereas the leader is assumed to be static, which facilitates the stability analysis and weakens the generality of the results. It is more general to study the case that the control input to the leader is time-varying and unknown but its *a priori* upper bound is available to the followers, and the corresponding stability analysis is challenging.

2) Disturbance Rejection: The second related research direction is on the study of *consensus with disturbance rejection*, since disturbance always exists in physical systems and might have a severe impact on system performance. Up to date, consensus for MASs perturbed by external disturbance has been considered widely, e.g., [19]–[21]. In [19], fixed-time consensus tracking with disturbance rejection (CTDR) for single integrator dynamics has been studied. To the best of our knowledge, most existing results concerning consensus with disturbance rejection are derived based on sliding mode control, where the control objectives of consensus and disturbance rejection are handled and realized separately in a *two-step control process*. Technically speaking, the first step is to assume that the agent dynamics are without any external disturbance and the objective is to devise a nominal protocol to realize consensus. In the second step, a sliding mode surface is designed and the protocol is composed of a nominal term (devised in the first step) and an additional discontinuous term, which drives the state of the agent with external disturbance onto the sliding mode surface. Nevertheless, to realize consensus with disturbance rejection in a *one-step control process* is much more charming. A detailed discussion is put in the motivation part.

3) Directed Spanning Tree: Another typical issue in the literature is to find necessary and/or sufficient conditions on communication topology of MASs in order that consensus can be realized. It has been reported in [11] that a minimum requirement on communication topology, for the stated purpose, is the existence of a directed spanning tree. This requirement is milder than connected graphs and strongly connected graphs, and therefore is more suitable in the analysis of physical systems. In [22], consensus tracking for a group of *heterogeneous* agents with first-order or second-order nonlinear dynamics has been realized within a fixed time, in which the *underlying* communication topology among the followers, however, is required to be strongly connected. Finite-time leaderless consensus for a group of perturbed first-order agents in a directed spanning tree has been investigated in [23], where a discontinuous protocol is described to realize leaderless consensus with disturbance rejection in a one-step control process. Owing to the advantages of both fixed-time and consensus tracking, it would be interesting to use such a

This work was supported in part by the National Natural Science Foundation of China under Grants (62121004,61876041), the Local Innovative and Research Teams Project of Guangdong Special Support Program (2019BT02X353), Key Area Research and Development Program of Guangdong Province (2021B0101410005), and the Natural Science Foundation of Guangdong Province (2021B1515420008).

Yong Xu, Zhaozhan Yao, and Renquan Lu are with the Provincial Key Laboratory of Intelligent Decision and Cooperative Control, School of Automation, Guangdong University of Technology, Guangzhou 510006, China (e-mail: xuyong809@163.com; iszzyao@163.com; rqlu@gdut.edu.cn).

Bijoy K. Ghosh is with the Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042, USA (e-mail: bijoy.ghosh@ttu.edu).

one-step control idea to study fixed-time CTDR.

B. Motivation of the Paper

The aforementioned one-step control process is developed in [23]. The highlight is that the control objective of leaderless consensus with disturbance rejection is not realized based on sliding mode control (as most previous papers do, e.g., [19]–[21]), but instead uses non-smooth analysis. The resulting advantages are that a finite-time stability analysis is completed as a one-step control process, and a settling time estimate is aesthetically pleasing in form. When it comes to consensus with disturbance rejection, it has been observed that the settling time estimate derived by using sliding mode control is tedious in form than other results derived by using non-smooth analysis. Since the use of discontinuous control is necessary to realize disturbance rejection no matter exploiting sliding mode control or non-smooth analysis, the immediate next thought is to address the CTDR problem raised in this paper by using non-smooth analysis, rather than sliding mode control. Thus, the stability analysis for CTDR can be summarized in one step, and redundant sliding mode surface design can be removed.

Motivated by the discussion on the aforementioned three issues, this paper focuses on investigating fixed-time CTDR for first-order MASs in a directed spanning tree. As previously mentioned, each issue has its corresponding feature and motivation. Consensus tracking admits the demand for producing any particular system trajectory in practical applications. The external disturbance is not avoidable in practice, therefore consensus with disturbance rejection also needs to be considered for improving system robustness. A directed spanning tree is a minimum requirement to guarantee the realization of consensus tracking. Through comprehensive consideration of these three features, this paper intends to research consensus for MASs based on a practical perspective.

C. Contribution of the Paper

This technical note aims to investigate the aforesaid three issues (i.e., consensus tracking, disturbance rejection and directed spanning tree) in a unified framework. The contribution is twofold. First, in contrast to [19], in which only a static leader is considered, this paper studies consensus tracking in the presence of a moving leader with time-varying and unknown control input, by using a novel protocol based on discontinuous and nonlinear control, together with a non-smooth Lyapunov function. Both the protocol and the Lyapunov function are quite similar to but different from the correspondences in [19]. Secondly, in contrast to most previous papers, e.g., [19]–[21], in which the control objectives of consensus and disturbance rejection are realized separately in a two-step control process, in this paper a fixed-time stability analysis for CTDR is completed as a one-step control process using non-smooth analysis. Furthermore, a settling time estimate can be determined without knowledge of the initial conditions. The estimate given herein is aesthetically pleasing, in comparison to previous results derived based on two-step control design technique using sliding modes.

The remainder of this paper is structured as follows. In Section II, the CTDR problem for first-order MASs is formulated and preliminaries required in this paper are given. In Section III, a fixed-time protocol based on discontinuous and nonlinear control is devised, and a fixed-time stability analysis for CTDR is developed as a one-step control process using non-smooth analysis. In Section IV, a numeric example is provided to validate the theoretical results. Section V draws conclusions.

Notation: $\mathbb{R}$ represents real number set. $\mathbb{R}_{\geq 0} \triangleq \{x \in \mathbb{R} : x \geq 0\}$. $\mathbb{R}_{+} \triangleq \{x \in \mathbb{R} : x > 0\}$. Given a positive scalar $p \in \mathbb{R}_{\geq 0}$, for $x \in \mathbb{R}$, $x^{[p]} \triangleq \text{sign}(x)|x|^p$, and for $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n$, $\boldsymbol{x}^{[p]} \triangleq [x_1^{[p]}, x_2^{[p]}, \ldots, x_n^{[p]}]^\top$, $\boldsymbol{x}^\top$ denotes its transpose. $\mathbf{1}$ represents the all-one vector with appropriate dimension. For $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, $\mathcal{A}^\top$, $\mathcal{A}^{-1}$, $\|\mathcal{A}\|$, $\|\mathcal{A}\|_\infty$ and $\lambda_{\min}(\mathcal{A})$ denote its associated transpose, inverse, 2-norm, ∞-norm, and minimum eigenvalue, respectively. $\text{diag}(\boldsymbol{x})$ represents the diagonal matrix generated from $\boldsymbol{x}$, and is said to be positive if all its diagonal entries are positive.

II. Problem Formulation and Preliminaries

A. Problem Formulation

Consider MASs with one leader and n followers. The dynamics of the leader is given by

$$\dot{x}_0(t) = u_0(t), \tag{1}$$

where $x_0(t) \in \mathbb{R}$ and $u_0(t) \in \mathbb{R}$ represent the state and control input, respectively. The dynamics of the i^{th} follower is described by

$$\dot{x}_i(t) = u_i(t) + d_i(t), \quad i \in \mathcal{V}, \tag{2}$$

where $x_i(t) \in \mathbb{R}$, $u_i(t) \in \mathbb{R}$, and $d_i(t) \in \mathbb{R}$ stand for the state, control input, and external disturbance, respectively. Denote the state vector, control input vector, and external disturbance vector of the followers by $\boldsymbol{x}(t) = [x_1, x_2, \ldots, x_n]^\top$, $\boldsymbol{u}(t) = [u_1, u_2, \ldots, u_n]^\top$ and $\boldsymbol{d}(t) = [d_1, d_2, \ldots, d_n]^\top$, respectively.

A directed graph $\mathcal{G}$ consisting of a pair $(\mathcal{V}, \mathcal{E})$ is used to depict the underlying communication topology among the followers, where $\mathcal{V} = \{1, 2, ..., n\}$ and $\mathcal{E} \subseteq (\mathcal{V} \times \mathcal{V})$ represent the set of nodes and edges, respectively. If the i^{th} follower can receive information from the j^{th} follower, then there exists an edge $(j, i) \in \mathcal{E}$, which can be represented as an arrow starting from node j and pointing towards node i. Assume that $\mathcal{G}$ has no self-loops, i.e., $(i, i) \notin \mathcal{E}, \forall i \in \mathcal{V}$. The adjacency matrix $\mathcal{A}_{dj} = [a_{ij}] \in \mathbb{R}^{n\times n}$ of $\mathcal{G}$ is defined such that $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. A directed path starting from the i_1^{th} follower and ending with the i_k^{th} follower is an ordered sequence of distinct directed edges $(i_1, i_2)(i_2, i_3) \ldots (i_{k-1}, i_k)$. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$ of $\mathcal{G}$ is defined such that $l_{ij} = -a_{ij}, j \neq i$ and $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$. Furthermore, use a diagonal matrix $\mathcal{A} = \text{diag}(a_{10}, a_{20}, \ldots, a_{n0})$ to describe the communication relationship between the leader and followers, where $a_{i0} > 0$ if the i^{th} follower can receive information from the leader, otherwise $a_{i0} = 0$.

The following assumptions are required throughout this paper to guarantee fixed-time CTDR for MASs (1)–(2).

Assumption 1: The control input $u_0(t)$ to the leader is time-varying and unknown. However, its upper bound, denoted by $u_0^{\max} \in \mathbb{R}_{+}$, is an a priori knowledge for the followers, i.e.,

$$|u_0(t)| \leq u_0^{\max}. \tag{3}$$

Assumption 2: The external disturbance $d_i(t)$ of each follower is uniformly bounded, i.e., there exists a positive scalar $\varepsilon \in \mathbb{R}_{+}$ such that

$$|d_i(t)| \leq \varepsilon, \quad i \in \mathcal{V}. \tag{4}$$

Assumption 3: There exists a directed path starting from the leader and ending with any follower, i.e., the communication topology among the leader and followers contains a directed spanning tree, with the leader being the root.

Lemma 1: [24] Define $\mathcal{L}_{\mathcal{A}} \triangleq \mathcal{L} + \mathcal{A}$. Under *Assumption 3*, there exists a positive diagonal matrix $\Gamma = \text{diag}(\boldsymbol{\gamma})$ generated from $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, \ldots, \gamma_n]^\top$, where γ_i equals to the reciprocal of the i^{th} component of $\mathcal{L}_{\mathcal{A}}^{-1}\mathbf{1}$, such that $\hat{\mathcal{L}}_{\mathcal{A}} = \Gamma\mathcal{L}_{\mathcal{A}} + \mathcal{L}_{\mathcal{A}}^\top\Gamma$ is symmetric and positive definite.

B. Definitions and Lemmas

Consider a nonlinear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{5}$$

where $\boldsymbol{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ represents the system state, and the vector filed $\boldsymbol{f} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable and locally essentially bounded. The case that $\boldsymbol{f}(t, \boldsymbol{x})$ is discontinuous is not excluded, therefore the solutions of (5) are interpreted in Filippov sense [25]. The origin $\mathbf{0}$ is assumed to be a unique equilibrium point of (5).

Definition 1: [25] Let $\mathfrak{B}(\mathbb{R}^n)$ be the collection of subsets of $\mathbb{R}^n$. For possibly discontinuous function $\boldsymbol{f}(t, \boldsymbol{x})$ in (5), define its Filippov set-valued map $\mathbb{K}[\boldsymbol{f}](t, \boldsymbol{x}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathfrak{B}(\mathbb{R}^n)$ by

$$\mathbb{K}[\boldsymbol{f}](t, \boldsymbol{x}) \triangleq \bigcap_{r>0} \bigcap_{\mu(S)=0} \overline{co} \left\{ \boldsymbol{f}(t, B(\boldsymbol{x}, r) - S) \right\}, \tag{6}$$

where $\overline{co}$ represents convex closure, μ denotes Lebesgue measure, and $B(\boldsymbol{x}, r)$ is the open ball with radius r centered at $\boldsymbol{x}$. In addition, $S \subset \mathbb{R}^n$ is a set with zero Lebesgue measure which can be arbitrarily chosen. An absolutely continuous function $\boldsymbol{x} : [t_0, t_1] \to \mathbb{R}^n$ is called a Filippov solution of (5) on $[t_0, t_1]$, if $\dot{\boldsymbol{x}} \in \mathbb{K}[\boldsymbol{f}](t, \boldsymbol{x})$ holds for almost all $t \in [t_0, t_1]$.

Definition 2: [26] If the origin of (5) is asymptotically stable, furthermore, any solution $\mathcal{X}(t, \boldsymbol{x}_0)$ of (5) converges to the origin after a settling time of finite length, i.e. $\mathcal{X}(t, \boldsymbol{x}_0) = \mathbf{0}$, $\forall t \geq T(\boldsymbol{x}_0)$, where $T : \mathbb{R}^n \to \mathbb{R}_+$ is the so-called settling time function, then the origin is finite-time stable.

Definition 3: [27] If the origin of (5) is finite-time stable, furthermore, there exists $T_{\max} \in \mathbb{R}_+$ such that $T(\boldsymbol{x}_0) \leq T_{\max}$, $\forall \boldsymbol{x}_0 \in \mathbb{R}^n$, then the origin is fixed-time stable.

Lemma 2: If there exists a continuous radially unbounded and positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that any Filippov solution $\boldsymbol{x} : [t_0, t_1] \to \mathbb{R}^n$ of (5) meets

$$\dot{V}(\boldsymbol{x}) \leq -k_1 V^p(\boldsymbol{x}) - k_2, \tag{7}$$

where $k_1 > 0$, $k_2 > 0$, and $p > 1$, then the origin of (5) is fixed-time stable and an estimate for $T_{\max}$ is given by

$$T_{\max} = \frac{1}{k_1(p-1)} + \frac{1}{k_2}, \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n. \tag{8}$$

Proof. Due to (7) one has $\dot{V}(\boldsymbol{x}) \leq -k_1 V^p(\boldsymbol{x})$ if $V(\boldsymbol{x}) \geq 1$ and $\dot{V}(\boldsymbol{x}) \leq -k_2$ if $V(\boldsymbol{x}) \leq 1$. Hence, for any solution such that $V(\boldsymbol{x}_0) \geq 1$, the former inequality guarantees $V(\boldsymbol{x}) \leq 1, \forall t \geq 1/(k_1(p-1))$, and for any solution such that $V(\boldsymbol{x}_0) \leq 1$, the latter inequality guarantees $V(\boldsymbol{x}) = 0, \forall t \geq 1/k_2$. Therefore, for any solution of (5), it holds that $V(\boldsymbol{x}) = 0, \forall t \geq 1/(k_1(p-1)) + 1/k_2$, and the proof is completed. ∎

Remark 1: Lemma 2 borrows its idea from the preliminary results in [27]. Note that the specific form of (7) given herein is different from its correspondence in [27] as $\dot{V}(\boldsymbol{x}) \leq -k_1 V^p(\boldsymbol{x}) - k_2 V^q(\boldsymbol{x})$ with $0 < q < 1$.

Definition 4: [28] For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, its generalized gradient is defined as

$$\partial V(\boldsymbol{x}) \triangleq \overline{co} \left\{ \lim_{i \to \infty} \nabla V(\boldsymbol{x}_i) | \boldsymbol{x}_i \to \boldsymbol{x}, \boldsymbol{x}_i \notin \Omega_V \cup S \right\}, \tag{9}$$

where $\overline{co}$ represents convex hull, ∇ stands for nabla operator, and Ω_V denotes the set where the definition of $\nabla V(\boldsymbol{x})$ is not given. In addition, $S \subset \mathbb{R}^n$ is a set with zero Lebesgue measure which can be arbitrarily chosen.

Lemma 3: [29] Let $\boldsymbol{x} : [t_0, t_1] \to \mathbb{R}^n$ be a Filippov solution of (5), and $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz continuous, and in addition, regular function, then the generalized time derivative of $V(\boldsymbol{x})$ exists almost everywhere and takes the form

$$\dot{V}(\boldsymbol{x}) = \boldsymbol{p}^\top \dot{\boldsymbol{x}}, \quad \boldsymbol{p} \in \partial V(\boldsymbol{x}). \tag{10}$$

Lemma 4: [30] Let $x_1, x_2, \ldots, x_n \geq 0$ and $p > 1$, then it holds that

$$\sum_{i=1}^{n} x_i^p \geq n^{1-p} \left(\sum_{i=1}^{n} x_i \right)^p. \tag{11}$$

After that necessary definitions and lemmas have been introduced, the control objective for MASs (1)–(2) is stated below.

Problem: This paper aims to realize fixed-time CTDR in a one-step control process, i.e., there exists $T_{\max} \in \mathbb{R}_+$, independent of the initial conditions of MASs (1)–(2), such that

$$\|\boldsymbol{x}(t) - x_0(t)\mathbf{1}\| = 0, \quad \forall t \geq T_{\max}. \tag{12}$$

III. Main Results

The novel fixed-time protocol devised for the i^{th} follower takes the form

$$\begin{aligned} u_i(t) = &-k \left(\sum_{j=0}^{n} a_{ij}(x_i(t) - x_j(t)) \right)^{[0]} \\ &- k \left(\sum_{j=0}^{n} a_{ij}(x_i(t) - x_j(t)) \right)^{[p]}, \end{aligned} \tag{13}$$

where $k > 0$ and $p > 1$. Note that the signum function here is written as $(\cdot)^{[0]}$ for the sake of convenience. $(\cdot)^{[0]}$ is a discontinuous term and $(\cdot)^{[p]}$ is a nonlinear term. Define the tracking error of the i^{th} follower by

$$e_i(t) \triangleq x_i(t) - x_0(t). \tag{14}$$

Note that the explicit dependence of the variables and vectors on time t will be omitted hereafter for the sake of convenience. By taking derivative of (14) with respect to time t, the dynamics of the tracking error e_i is derived as

$$\begin{aligned} \dot{e}_i = &-k \left(\sum_{j=1}^{n} a_{ij}(e_i - e_j) + a_{i0} e_i \right)^{[0]} \\ &- k \left(\sum_{j=1}^{n} a_{ij}(e_i - e_j) + a_{i0} e_i \right)^{[p]} + d_i - u_0. \end{aligned} \tag{15}$$

Let $\boldsymbol{e} = [e_1, e_2, \ldots, e_n]^\top$ be the tracking error vector, taking its time derivative gives that

$$\dot{\boldsymbol{e}} = -k(\mathcal{L}_{\mathcal{A}} \boldsymbol{e})^{[0]} - k(\mathcal{L}_{\mathcal{A}} \boldsymbol{e})^{[p]} + \boldsymbol{d} - u_0 \mathbf{1}. \tag{16}$$

The fixed-time stability analysis for CTDR is demonstrated below.

Theorem 1: Under *Assumptions 1–3*, the realization of fixed-time CTDR for MASs (1)–(2) can be guaranteed by the proposed protocol (13) with $k > 0$ satisfying

$$\vartheta = \frac{1}{2} k \lambda_{\min} \left(\hat{\mathcal{L}}_{\mathcal{A}} \right) - \eta > 0, \tag{17}$$

where $\eta = \|\Gamma \mathcal{L}_{\mathcal{A}}\|_\infty \varepsilon + \|\Gamma \mathcal{A}\|_\infty u_0^{\max}$, and a settling time estimate is given by

$$T_{\max} = \frac{p+1}{k_1(p-1)} + \frac{1}{k_2}, \tag{18}$$

where $k_1 = n\vartheta(2n\gamma_{\max})^{-2p/(p+1)}$ and $k_2 = \vartheta/2$, in which $\gamma_{\max}$ denotes the maximum component of $\boldsymbol{\gamma}$ introduced in *Lemma 1*.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2021.3131549, IEEE Transactions on Automatic Control

Proof. Define $\delta_i \triangleq \mathcal{L}_{\mathcal{A}}^{i}\boldsymbol{e}$, in which $\mathcal{L}_{\mathcal{A}}^{i}$ represents the i^{th} row of $\mathcal{L}_{\mathcal{A}}$. Let $\boldsymbol{\delta} = [\delta_1, \delta_2, \ldots, \delta_n]^{\top}$. By taking into account (16), a compact form of the dynamics of the variable $\boldsymbol{\delta}$ is derived as

$$\dot{\boldsymbol{\delta}} = \mathcal{L}_{\mathcal{A}}\left(-k\boldsymbol{\delta}^{[0]} - k\boldsymbol{\delta}^{[p]} + \boldsymbol{d} - u_0\mathbf{1}\right). \tag{19}$$

The candidate Lyapunov function of the variable $\boldsymbol{\delta}$ is designed as

$$V(\boldsymbol{\delta}) = \sum_{i=1}^{n}\gamma_i|\delta_i| + \sum_{i=1}^{n}\frac{\gamma_i|\delta_i|^{p+1}}{p+1}. \tag{20}$$

For $\delta_1, \delta_2, \ldots, \delta_n \neq 0$, the time derivative of $V(\boldsymbol{\delta})$ is well-defined and given by

$$\dot{V}(\boldsymbol{\delta}) = \sum_{i=1}^{n}\gamma_i \mathrm{sign}(\delta_i)\dot{\delta}_i + \sum_{i=1}^{n}\gamma_i \mathrm{sign}(\delta_i)|\delta_i|^p\dot{\delta}_i. \tag{21}$$

The presented analysis hereafter makes use of arguments from non-smooth analysis [25, Ch. 3, Th. 3, pp. 157–158]. If there exists at least one $\delta_i = 0$, the definition of $\dot{V}(\boldsymbol{\delta})$ becomes non-trivial since the corresponding righthand side becomes discontinuous. Let us define two sets $\Delta_0 \triangleq \{i \in \mathcal{V} : \delta_i = 0\}$ and $\Delta_{\neq} \triangleq \{i \in \mathcal{V} : \delta_i \neq 0\}$. We interpret the solutions of (1)–(2) and (13) in Filippov sense, by using one of these solutions, say $\boldsymbol{x}$, the quantites $\boldsymbol{e}$, $\boldsymbol{\delta}$, $V(\boldsymbol{\delta})$ are naturally determined. Consider the generalized gradient of $V(\boldsymbol{\delta})$ in accordance with *Definition 4*, which takes the form

$$\partial V(\boldsymbol{\delta}) = \Gamma\left(\boldsymbol{p} + \boldsymbol{\delta}^{[p]}\right), \tag{22}$$

in which $\boldsymbol{p} = [p_1, p_2, \ldots, p_n]^{\top}$ is a vector whose i^{th} component is $p_i \in \mathrm{SIGN}(\delta_i)$ if $\delta_i \in \Delta_0$ and $p_i = \mathrm{sign}(\delta_i)$ if $\delta_i \in \Delta_{\neq}$, where $\mathrm{SIGN}(\cdot)$ is the Filippov set-valued map of $\mathrm{sign}(\cdot)$

$$\mathrm{SIGN}(\delta_i) = \begin{cases} 1, & \text{if} \quad \delta_i > 0, \\ [-1, 1], & \text{if} \quad \delta_i = 0, \\ -1, & \text{if} \quad \delta_i < 0. \end{cases} \tag{23}$$

Notice that $V(\boldsymbol{\delta})$ is locally Lipschitz continuous and regular (a function that is convex is regular), we derive its generalized time derivative according to *Lemma 3* as

$$\begin{aligned}\dot{V}(\boldsymbol{\delta}) = &\sum_{i\in\Delta_0}\gamma_i \mathrm{SIGN}(\delta_i)\dot{\delta}_i + \sum_{i\in\Delta_{\neq}}\gamma_i \mathrm{sign}(\delta_i)\dot{\delta}_i \\ &+ \sum_{i=1}^{n}\gamma_i \mathrm{sign}(\delta_i)|\delta_i|^p\dot{\delta}_i.\end{aligned} \tag{24}$$

Since the solutions are interpreted in Filippov sense, the case in which $\delta_i = 0$ holds for isolated time instants with zero Lebesgue measure can be disregarded. On the other hand, as noticed in [25, Ch. 3, Th. 3, pp. 157–158], if any of the conditions $\delta_i = 0$ holds along a time interval with positive Lebesgue measure, then at these time instants $\dot{\delta}_i$ exists and takes zero value, namely $\dot{\delta}_i = 0, \forall i \in \Delta_0$. Therefore, the term $\mathrm{SIGN}(\delta_i)$ can be modified as $\mathrm{sign}(\delta_i)$ by using the fact that in our particular case $i \in \Delta_0$ indicates $\dot{\delta}_i = 0$. The vector $\boldsymbol{p}$ can be further modified as $\boldsymbol{\delta}^{[0]}$. Then one finds that in our particular case the set-valued function $\dot{V}(\boldsymbol{\delta})$ defined in (24) takes values consists in a singleton corresponding to

$$\begin{aligned}\dot{V}(\boldsymbol{\delta}) &= \left(\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right)^{\top}\Gamma\dot{\boldsymbol{\delta}} \\ &= \left(\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right)^{\top}\Gamma\mathcal{L}_{\mathcal{A}}\left(-k\boldsymbol{\delta}^{[0]} - k\boldsymbol{\delta}^{[p]} + \boldsymbol{d} - u_0\mathbf{1}\right) \\ &= -\frac{1}{2}k\left(\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right)^{\top}\left(\Gamma\mathcal{L}_{\mathcal{A}} + \mathcal{L}_{\mathcal{A}}^{\top}\Gamma\right)\left(\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right) \\ &\quad + \left(\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right)^{\top}\Gamma\mathcal{L}_{\mathcal{A}}\left(\boldsymbol{d} - u_0\mathbf{1}\right) \\ &\leq -\frac{1}{2}k\lambda_{\min}\left(\hat{\mathcal{L}}_{\mathcal{A}}\right)\left\|\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right\|^2 \\ &\quad + \left|\left(\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right)^{\top}\Gamma\mathcal{L}_{\mathcal{A}}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right|,\end{aligned} \tag{25}$$

where $\hat{\mathcal{L}}_{\mathcal{A}} = \Gamma\mathcal{L}_{\mathcal{A}} + \mathcal{L}_{\mathcal{A}}^{\top}\Gamma$ is a symmetric and positive definite matrix introduced in *Lemma 1*.

For the upper bound of the coupling term in the last inequality of (25), it is estimated as

$$\begin{aligned}&\left|\left(\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right)^{\top}\Gamma\mathcal{L}_{\mathcal{A}}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right| \\ &= \left|\sum_{i=1}^{n}\gamma_i\left(\delta_i^{[0]} + \delta_i^{[p]}\right)\mathcal{L}_{\mathcal{A}}^{i}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right| \\ &\leq \sum_{i=1}^{n}\left|\gamma_i\left(\delta_i^{[0]} + \delta_i^{[p]}\right)\mathcal{L}_{\mathcal{A}}^{i}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right|.\end{aligned} \tag{26}$$

Note that both $\delta_i^{[0]}$ and $\delta_i^{[p]}$ have the same sign, which implies that

$$\begin{cases} \left|\delta_i^{[0]} + \delta_i^{[p]}\right| = 0, & \text{if} \quad \delta_i = 0, \\ \left|\delta_i^{[0]} + \delta_i^{[p]}\right| \geq 1, & \text{if} \quad \delta_i \neq 0. \end{cases} \tag{27}$$

Therefore, it follows that

$$\left|\delta_i^{[0]} + \delta_i^{[p]}\right| \leq \left(\delta_i^{[0]} + \delta_i^{[p]}\right)^2. \tag{28}$$

Substituting (28) into (26) gives that

$$\begin{aligned}&\left|\left(\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right)^{\top}\Gamma\mathcal{L}_{\mathcal{A}}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right| \\ &\leq \sum_{i=1}^{n}\left|\delta_i^{[0]} + \delta_i^{[p]}\right|\left|\gamma_i\mathcal{L}_{\mathcal{A}}^{i}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right| \\ &\leq \max_{i\in\mathcal{V}}\left\{\left|\gamma_i\mathcal{L}_{\mathcal{A}}^{i}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right|\right\}\sum_{i=1}^{n}\left(\delta_i^{[0]} + \delta_i^{[p]}\right)^2 \\ &= \max_{i\in\mathcal{V}}\left\{\left|\gamma_i\mathcal{L}_{\mathcal{A}}^{i}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right|\right\}\left\|\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right\|^2.\end{aligned} \tag{29}$$

The upper bound estimate for the maximum term is derived as

$$\begin{aligned}&\max_{i\in\mathcal{V}}\left\{\left|\gamma_i\mathcal{L}_{\mathcal{A}}^{i}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right|\right\} \\ &= \max_{i\in\mathcal{V}}\left\{\left|\gamma_i\left(\sum_{j=1}^{n}a_{ij}(d_i - d_j) + a_{i0}(d_i - u_0)\right)\right|\right\} \\ &\leq \max_{i\in\mathcal{V}}\left\{\left|\gamma_i(2l_{ii} + a_{i0})\varepsilon\right| + \left|\gamma_i a_{i0}u_0^{\max}\right|\right\} \\ &\leq \left\|\Gamma\mathcal{L}_{\mathcal{A}}\right\|_{\infty}\varepsilon + \left\|\Gamma\mathcal{A}\right\|_{\infty}u_0^{\max}.\end{aligned} \tag{30}$$

By substituting (30) into (29), one has

$$\left|\left(\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right)^{\top}\Gamma\mathcal{L}_{\mathcal{A}}\left(\boldsymbol{d} - u_0\mathbf{1}\right)\right| \leq \eta\left\|\boldsymbol{\delta}^{[0]} + \boldsymbol{\delta}^{[p]}\right\|^2, \tag{31}$$

where $\left\|\Gamma\mathcal{L}_{\mathcal{A}}\right\|_{\infty}\varepsilon + \left\|\Gamma\mathcal{A}\right\|_{\infty}u_0^{\max}$ is denoted by η for the ease of representation. Further, combining (25) and (31) yields that

$$\begin{aligned}\dot{V}(\boldsymbol{\delta}) &\leq -\frac{1}{2}k\lambda_{\min}\left(\hat{\mathcal{L}}_{\mathcal{A}}\right)\left\|\boldsymbol{\delta}^{[0]}+\boldsymbol{\delta}^{[p]}\right\|^2+\eta\left\|\boldsymbol{\delta}^{[0]}+\boldsymbol{\delta}^{[p]}\right\|^2\\ &= -\vartheta\sum_{i=1}^{n}\left(\delta_i^{[0]}+\delta_i^{[p]}\right)^2\\ &= -\vartheta\sum_{i=1}^{n}\left(|\delta_i|^{2p}+2|\delta_i|^p+|\mathrm{sign}(\delta_i)|\right),\end{aligned} \tag{32}$$

where $\vartheta = k\lambda_{\min}\left(\hat{\mathcal{L}}_{\mathcal{A}}\right)/2-\eta>0$.

The remainder of this proof presents that $\dot{V}(\boldsymbol{\delta})$ follows (7) given in *Lemma 2*, which gives rise to the fixed-time stability of the variable $\boldsymbol{\delta}$.

Due to $p-2p/(p+1)>0$, one has $|\delta_i|^p \geq |\delta_i|^{2p/(p+1)}$ if $|\delta_i|\geq 1$. Furthermore, it holds that $|\mathrm{sign}(\delta_i)|\geq|\delta_i|^{2p/(p+1)}$ if $|\delta_i|\leq 1$. Therefore, it follows that

$$\sum_{i=1}^{n}\left(|\delta_i|^{2p}+2|\delta_i|^p+|\mathrm{sign}(\delta_i)|\right)\geq\sum_{i=1}^{n}\left(|\delta_i|^{\frac{2p}{p+1}}+|\delta_i|^{2p}\right). \tag{33}$$

Due to $2p/(p+1)>1$, by invoking *Lemma 4*, it holds that

$$\begin{aligned}&\sum_{i=1}^{n}\left(|\delta_i|^{\frac{2p}{p+1}}+|\delta_i|^{2p}\right)\\ &\geq(2n)^{1-\frac{2p}{p+1}}\left(\sum_{i=1}^{n}|\delta_i|+\sum_{i=1}^{n}|\delta_i|^{p+1}\right)^{\frac{2p}{p+1}}.\end{aligned} \tag{34}$$

Look back on the candidate Lyapunov function (20), one has

$$\gamma_{\max}\left(\sum_{i=1}^{n}|\delta_i|+\sum_{i=1}^{n}|\delta_i|^{p+1}\right)\geq V(\boldsymbol{\delta}). \tag{35}$$

By combining a series of inequalities (33)–(35), it yields that

$$\sum_{i=1}^{n}\left(|\delta_i|^{2p}+2|\delta_i|^p+|\mathrm{sign}(\delta_i)|\right)\geq 2n\left(\frac{V(\boldsymbol{\delta})}{2n\gamma_{\max}}\right)^{\frac{2p}{p+1}}. \tag{36}$$

On the other hand, if CTDR for MASs (1)–(2) has not been realized, then there must exist at least one $\delta_i\neq 0$ such that $|\mathrm{sign}(\delta_i)|=1$, which follows that

$$\sum_{i=1}^{n}\left(|\delta_i|^{2p}+2|\delta_i|^p+|\mathrm{sign}(\delta_i)|\right)\geq 1. \tag{37}$$

Finally, by substituting (36)–(37) into (32), the relationship between the candidate Lyapunov function $V(\boldsymbol{\delta})$ and its corresponding generalized time derivative $\dot{V}(\boldsymbol{\delta})$ is derived as

$$\begin{aligned}\dot{V}(\boldsymbol{\delta}) &\leq -\frac{\vartheta}{2}\times 2\sum_{i=1}^{n}\left(|\delta_i|^{2p}+2|\delta_i|^p+|\mathrm{sign}(\delta_i)|\right)\\ &\leq -k_1V^{\frac{2p}{p+1}}(\boldsymbol{\delta})-k_2,\end{aligned} \tag{38}$$

where $k_1 = n\vartheta(2n\gamma_{\max})^{-2p/(p+1)}$ and $k_2=\vartheta/2$. According to *Lemma 2*, it follows that the variable $\boldsymbol{\delta}$ converges to the origin within a fixed time $T_{\max}$ and

$$T_{\max}=\frac{p+1}{k_1(p-1)}+\frac{1}{k_2}. \tag{39}$$

Note that $\boldsymbol{e}=\mathcal{L}_{\mathcal{A}}^{-1}\boldsymbol{\delta}=\boldsymbol{x}-x_0\mathbf{1}$ and $\mathcal{L}_{\mathcal{A}}^{-1}$ exists, therefore the tracking error vector $\boldsymbol{e}$ converges to the origin $\forall t\geq T_{\max}$, namely fixed-time CTDR for MASs (1)–(2) can be realized in a one-step control process, which completes the proof. ∎

Remark 2: It has been demonstrated that a fixed-time stability analysis for CTDR can be completed as a one-step control process using non-smooth analysis, and discontinuous control is necessary to guarantee disturbance rejection. A settling time estimate has been derived, and is aesthetically pleasing, in comparison to previous results derived based on two-step control design technique using sliding modes, e.g., [19]–[21].

Remark 3: It is noteworthy that the settling time estimate given by (39) is independent of the initial conditions, and the estimate can be determined once ϑ and p have been selected, then k will be tuned in accordance with (17), namely the value of k is dependent on the global topology information $\mathcal{L}$ and $\mathcal{A}$, together with the a priori knowledge ε and $u_0^{\max}$.

IV. Simulations

In this section, a numeric example for the proposed protocol is provided to verify the theoretical results.

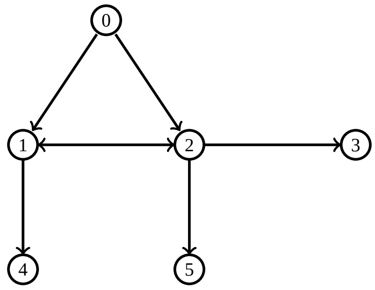

Fig. 1. Communication topology among the leader and followers.

A MAS with one leader and five followers labeled by 0 and 1–5, respectively, is considered in simulations. Fig. 1 shows the directed communication topology among the leader and followers. The corresponding Laplacian matrix of the underlying communication topology among the followers is

$$\mathcal{L}=\begin{bmatrix}1 & -1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ -1 & 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0 & 1\end{bmatrix}.$$

The communication relationship between the leader and followers is given by $\mathcal{A}=\mathrm{diag}(1,1,0,0,0)$. Suppose that the control input to the leader is $u_0=-\sin(0.5t)$ and the external disturbance of each follower is $d_i=0.1\sin(2t)$. The value of k and p are selected as $k=10$ and $p=1.2$, respectively. In accordance with (18), the theoretical estimate for $T_{\max}$ is less than 14s. The initial condition of the leader is set as $x_0=2$. For the sake of verifying that the settling time under the proposed protocol is independent of the initial conditions, two scenarios with different initial conditions are examined, i.e., (i) $\boldsymbol{x}_0=[-20,-10,-15,7,15]^{\top}$; (ii) $\boldsymbol{x}_0=[-100,-50,-75,35,75]^{\top}$.

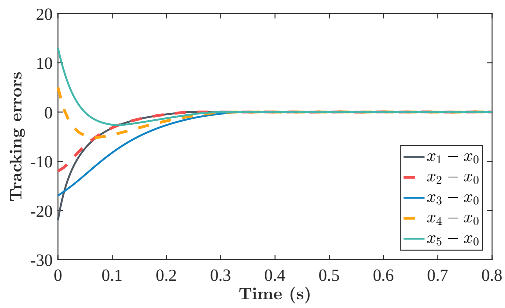

Fig. 2. The tracking errors in scenarios (i).

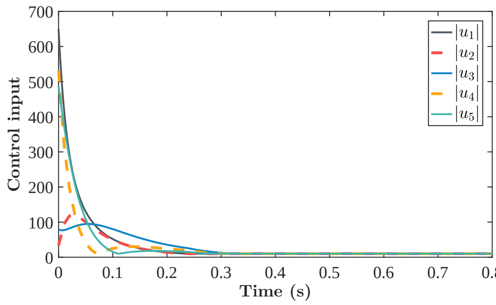

Fig. 3. The absolute value of control input in scenarios (i).

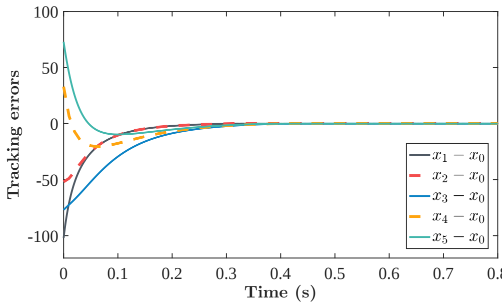

Fig. 4. The tracking errors in scenarios (ii).

As shown in Fig. 2, the tracking errors in scenarios (i) converge to the origin less than 0.8s, which implies that the proposed protocol guarantees fixed-time CTDR for the considered MAS. From Fig. 3, however, one may observe that even though the initial conditions are not large, the nonlinear term in the proposed protocol still results in a huge control input. Fig. 4 shows that the settling time in scenarios (ii) is quite identical to scenarios (i), which verifies that the settling time is independent of the initial conditions.

V. Conclusions

In this paper, fixed-time CTDR for first-order MASs was studied. A fixed-time protocol was devised based on discontinuous and nonlinear control. A fixed-time stability analysis for CTDR was completed as a one-step control process using non-smooth analysis. Global topology information $\mathcal{L}$ and $\mathcal{A}$, together with the a priori upper bounds ε and $u_0^{\max}$ were required to derive a settling time estimate, and the estimate is independent of the initial conditions. Finally, a numeric example of the proposed protocol was provided to validate the theoretical results. Further consideration includes an extension to high-order MASs.

References

[1] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, pp. 988–1001, Jun. 2003.

[2] W. Burgard, M. Moors, C. Stachniss, and F. E. Schneider, "Coordinated multi-robot exploration," *IEEE Trans. Robot.*, vol. 21, pp. 376–386, Jun. 2005.

[3] B. Ning, Q. Han, Z. Zuo, J. Jin, and J. Zheng, "Collective behaviors of mobile robots beyond the nearest neighbor rules with switching topology," *IEEE Trans. Cybern.*, vol. 48, pp. 1577–1590, May 2018.

[4] Z. Lin, L. Wang, Z. Han, and M. Fu, "A graph laplacian approach to coordinate-free formation stabilization for directed networks," *IEEE Trans. Autom. Control*, vol. 61, pp. 1269–1280, May 2016.

[5] T. Han, Z. Lin, R. Zheng, and M. Fu, "A barycentric coordinate-based approach to formation control under directed and switching sensing graphs," *IEEE Trans. Cybern.*, vol. 48, pp. 1202–1215, Apr. 2018.

[6] Z. Han, K. Guo, L. Xie, and Z. Lin, "Integrated relative localization and leaderfollower formation control," *IEEE Trans. Autom. Control*, vol. 64, pp. 20–34, Jan. 2019.

[7] U. A. Khan, S. Kar, and J. M. Moura, "Distributed sensor localization in random environments using minimal number of anchor nodes," *IEEE Trans. Signal Process.*, vol. 57, pp. 2000–2016, May 2009.

[8] Y. Diao, Z. Lin, and M. Fu, "A barycentric coordinate based distributed localization algorithm for sensor networks," *IEEE Trans. Signal Process.*, vol. 62, Sept.15. 2014.

[9] G. Chai, C. Lin, Z. Lin, and M. Fu, "Single landmark based collaborative multi-agent localization with time-varying range measurements and information sharing," *Syst. Control Lett.*, vol. 87, pp. 56–63, 2016.

[10] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, pp. 1520–1533, Sept. 2004.

[11] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, pp. 655–661, May 2005.

[12] W. Yu, G. Chen, M. Cao, and J. Kurths, "Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics," *IEEE Trans. Syst., Man, Cybern. B. Cybern.*, vol. 40, pp. 881–891, Jun. 2010.

[13] M. Defoort, A. Polyakov, G. Demesure, M. Djemai, and K. Veluvolu, "Leader-follower fixed-time consensus for multi-agent systems with unknown non-linear inherent dynamics," *IET Contr. Theory Appl.*, vol. 9, no. 14, pp. 2165–2170, 2015.

[14] M. Defoort, G. Demesure, Z. Zuo, A. Polyakov, and M. Djemai, "Fixed-time stabilisation and consensus of non-holonomic systems," *IET Contr. Theory Appl.*, vol. 10, no. 18, pp. 2497–2505, 2016.

[15] B. Ning and Q. Han, "Prescribed finite-time consensus tracking for multiagent systems with nonholonomic chained-form dynamics," *IEEE Trans. Autom. Control*, vol. 64, pp. 1686–1693, Apr. 2019.

[16] S. Su and Z. Lin, "Distributed consensus control of multi-agent systems with higher order agent dynamics and dynamically changing directed interaction topologies," *IEEE Trans. Autom. Control*, vol. 61, pp. 515–519, Feb. 2016.

[17] Y. Zhao, Y. Liu, G. Wen, W. Ren, and G. Chen, "Designing distributed specified-time consensus protocols for linear multiagent systems over directed graphs," *IEEE Trans. Autom. Control*, vol. 64, pp. 2945–2952, Jul. 2019.

[18] F. Chen and J. Chen, "Minimum-energy distributed consensus control of multiagent systems: A network approximation approach," *IEEE Trans. Autom. Control*, vol. 65, pp. 1144–1159, Mar. 2020.

[19] H. Wang, W. Yu, G. Wen, and G. Chen, "Fixed-time consensus tracking of multi-agent systems under a directed communication topology," in *IEEE Int. Conf. Control Automat.*, pp. 186–191, 2016.

[20] Y. Liu and S. Yu, "Finite-time consensus of multi-agent systems with disturbances under directed communication topology," in *Chin. Control Conf.*, pp. 752–756, 2016.

[21] Z. Zuo, B. Tian, M. Defoort, and Z. Ding, "Fixed-time consensus tracking for multiagent systems with high-order integrator dynamics," *IEEE Trans. Autom. Control*, vol. 63, pp. 563–570, Feb. 2018.

[22] H. Du, G. Wen, D. Wu, Y. Cheng, and J. Lü, "Distributed fixed-time consensus for nonlinear heterogeneous multi-agent systems," *Automatica*, vol. 113, p. 108797, 2020.

[23] M. Franceschelli, A. Pisano, A. Giua, and E. Usai, "Finite-time consensus with disturbance rejection by discontinuous local interactions in directed graphs," *IEEE Trans. Autom. Control*, vol. 60, pp. 1133–1138, Apr. 2015.

[24] Z. Qu, *Cooperative control of dynamical systems: Applications to autonomous vehicles*. London, U.K.: Springer-Verlag, 2009.

[25] A. Filippov, *Differential equations with discontinuous righthand sides*. Norwell, MA, USA: Kluwer, 1988.

[26] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[27] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Trans. Autom. Control*, vol. 57, pp. 2106–2110, Aug. 2012.

[28] F. H. Clarke, *Optimization and nonsmooth analysis*. New York, NY, USA: Wiley, 1983.

[29] B. Paden and S. Sastry, "A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Trans. Circuits Syst.*, vol. 34, pp. 73–82, Jan. 1987.

[30] G. H. Hardy, G. Polya, and J. E. Littlewood, *Inequalities*. Cambridge, U.K.: Cambridge Univ. Press, 1952.