Calculus applied to microeconomic

Main insights

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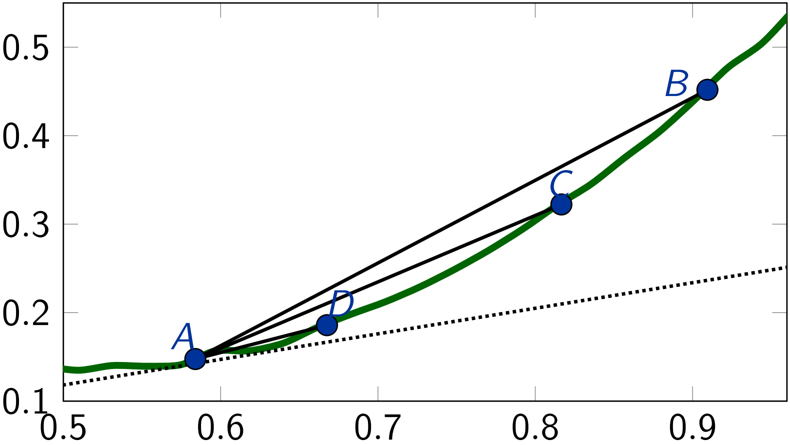
Insights

The main objective of these notes is give important concepts in calculus and optimization using economic theory, take decisions using data methodologies; Almost all are rely on in the bibliography presented here.

Geometric concept

Derivate



Definition

Some properties

All the properties are defined using the definition.

Production Function

Assume that the production process could be encapsulate in:

$$Y = AK^{\alpha}L^{\beta} \tag{1}$$

How we can represented the "black box" in product?

Elasticities

what is the definition?

What are the implications

you can find how change the production whit changes of capital and labor.

Properties

suppose two functions:

$$[f(x) + g(x)]' = f'(x) + g'(x) \tag{2}$$

Profit

$$\pi(q) = IT(q) - CT(q) \tag{3}$$

We could be interested in:

$$\frac{d\pi(q)}{dq} \tag{4}$$

Now applying the last property we have:

$$\frac{d\pi(q)}{dq} = IT'(q) - CT'(q) \tag{5}$$

Marginalism

$IT(q)$ and $CT(q)$ allow us known what is the increase of income and cost when the production increase in one unity, to the increase now are called marginal income $\frac{dIT(q)}{dq} = IM$ and marginal cost respectively.

$$\frac{dCT(q)}{q} = CT'(q) = CM. \tag{6}$$

$$\frac{dIT(q)}{q} = IT'(q) = IM. \tag{7}$$

Applying

$$PM_I = \frac{\partial Y}{\partial K} \tag{8}$$

Optimize

From economic perspective we have that the if the income percived by unit of one product is greater than the cost of production.

$$IM > CM \tag{9}$$

the firm could produce more, otherwise

$$CM > IM \tag{10}$$

Therefore the condition:

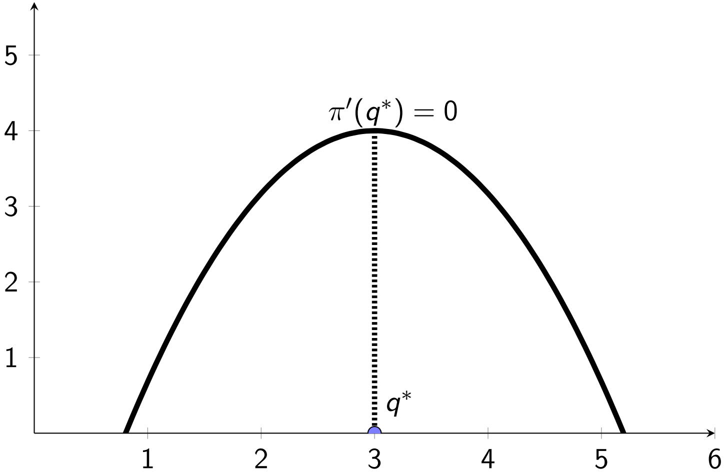
$$IM = CM \tag{11}$$

that is derived from:

$$\frac{d\pi(q)}{dq} = IM - CM = 0. \tag{12}$$

Geometric

First condition



First condition

For simplicity (de aquí en adelante) we are going to assume (asumiremos) that the first condition is enough to optimize the functions, therefore to optimize $f(x)$ (minimize or maximize) then:

$$f'(x^*) = 0 \tag{13}$$

for instance, we have the following function $f(x) = ax^2 + bx + c$.

$$\frac{df(x)}{dx} = 2ax + b = 0$$

$$= x^* = \frac{-b}{2a}$$

Nash equilibrium

$$\forall i, U(e^{*}, e_{-i}) > U(e, e_{-i}) \tag{14}$$

Note here that one condition is the **stability** not is a NE if at least one of the players have incentives to change of strategy

pseudo code

```
respond to the strategy of A:
select the better decision
keep the result
respond to the the set of strategy of B:
select the better decison
```

Nash equilibrium

Python implementation

```
import matplotlib.pyplot as plt
mat = [[2,1],[0,0],
       [0,0],[1,2]]
colB0 =[mat[0], mat[2]]
colB1 =[mat[1],mat[3]]
Sb = [colB0, colB1]
rowA0 =[mat[0],mat[1]]
rowA1 =[mat[2] , mat[3]]
Sa = [rowA0, rowA1]
```

Nash Equilibrium

python implementation

```python
def dnash(i,col):
  aux = []
  init = col[0][i]
  aux =[col[0]]
  for e in col:
    if e[i]>init:
      aux = []
      aux.append(e)
      init = e[i]
  return aux
sb  = [dnash(1,row) for row in Sa]
sa  =  [dnash(0,col) for col in Sb ]
eq = [x for x in sb if x in sa]
print(eq)
```

Collab

dominated strategies

Theorem

if exist a dominated equilibrium then this is unique and is also a Nash Equilibrium (NE)

Existence and uniqueness of NE

tragedy of the commons

Gibbons chapter - Nicholson

This allow us to know the efficiency of public resource or overuse, this a example join prisoner dilemma that individuals strategies not guaranteed a social optimum.

there a n individuals that have livestock each i individual could take $s_i = (0, G_{max}]$ we assume that this set of strategies are continuous. We assume that each individual could have a profit of each cattle that rely on in the number to total $Q = \sum_{i=1}^{n} q_i$ where q_i is the number of cattle's of the i individual.

the value per capita is given by:

$$v(q_i + \sum_{j \neq i}^{n} q_j) \tag{15}$$

$v()$ function have some assumptions, $v'() < 0$ and $v''() < 0$ and $\exists \hat{Q}$ that for $\forall Q > \hat{Q}$ then $V(Q) = 0$.
note for instance that the a function of the form $y = \sqrt{1 - x^2}$ could be useful to model $v()$.
then the profit of individual i could be written as:

$$\pi_i = q_i v(Q) - c q_i \tag{16}$$

if c is the fixed cost by cattle.

better strategies

Note that Q also is related with q_i

$$\frac{d\pi_i}{dq_i} = v(Q) + q_i v'(Q) - c = 0 \tag{17}$$

sum all the first conditions and divide by n we have:

$$v(Q) + \frac{1}{n} Q v'(Q) - c \tag{18}$$

we can said that is Q^{NE},but if we consider a public administrator the problem will be:

$$\pi_Q = Qv(Q) - cQ \tag{19}$$

and the first condition will be $V(Q) + QV'(Q) - c$.

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Which is better?

remember that $v'() < 0$ suppose that $Q^{NE} > Q$ then $v(Q^{NE}) < v(Q)$

$$\frac{dF}{Q} = 0 \tag{20}$$

where $Q = \sum_{i=1}^{n} q_i$

Insights Monopoly

in perfect competition $\frac{dq}{dp} = 0$ otherwise $\frac{dq}{dp} < 0$. The demand curve of monopoly is equal to the market.

Marginal income

How much we can increase the price of a product?
Income is equal to $I = p(q)q$

$$IM = \frac{dI}{dq} = \frac{dp}{dq}q + p \tag{21}$$

Note that in perfect competition $\frac{dp}{dq} = 0$ and therefore the marginal income is equal to the price.

Elasticity

increase of q due the reduction of p.

$$\xi = \frac{dq}{dp}\frac{p}{q} \tag{22}$$

how increase or reduce the demand quantity when the price increase in one percent.

Elasticity and Marginal income

$$IM = p(\frac{1}{\xi} + 1) \tag{23}$$

we hope that $\xi < 0$ due a increase in price in **normal** goods, notice that the marginal income in perfect competition was equal to p therefore $\xi \to \infty$. this give us idea about the importance.

Power of market

$$\pi(q) = I(q) - C(q) \tag{24}$$

the first order condition give us to maximize the profit $IM = CM$.

$$I_l = \frac{p - CM}{p} \tag{25}$$

there are a relationship among the elasticity of demand and power of market.

$$\begin{aligned} CM &= p(\frac{1}{\xi} + 1) \\ \frac{CM - p}{p} &= \frac{1}{\xi} \end{aligned} \tag{26}$$

what is the power of the monopoly to set up a price higher of its marginal cost.

How many produce monopoly

The monopoly produce less remember that given any market $IM = CM$ and marginal income is always lesser than price in a competitive market.

Discrimination prices

if we have $\pi(q_1 + q_2) = p_1 q_1 + p_2 q_2 - CQ$ where $Q = q_1 + q_2$.

Insights Cournot

Cournot(1838) each firm maximize the profit, the question is: **How many q_i unities of a product could produce i according to the q_j unities of j?** .

Each firm known that $\frac{\partial P}{\partial q_i} \neq 0$, but $\frac{\partial q_j}{\partial q_i} = 0$.

Cournot

n -firms

Assume a linear demand:

$$p = \Theta - \beta Q \tag{27}$$

take in mind that $Q = \sum q_i$ (by two firms $Q = q_1 + q_2$).
Now the profit of $i - th$ firm is:

$$\pi_i = pq_i - cq_i \tag{28}$$

note here that, c is a constant and there $MC = c$.

$$\pi_i = (\Theta - \beta Q)q_i - cq_i \tag{29}$$

first condition

$$\pi_i = \Theta q_i - \beta q_i (\sum_j q_j) - c q_i \tag{30}$$

$$\frac{\partial \pi_i}{d q_i} = 0 \tag{31}$$

$$\Theta - \beta(\sum_{j \neq i} q_j)) - 2\beta q_i - c \tag{32}$$

thus the strategy is

$$q_i^* = \frac{\Theta - c}{2\beta} - \frac{\sum_{j \neq i} q_j}{2} \tag{33}$$

solution

2-firms

$$\frac{\Theta - c}{2\beta} = \frac{\sum_{j \neq i} q_j}{2} + q_i \tag{34}$$

with $n = 2$ we can express the system in the following way:

$$\begin{aligned} \frac{\Theta - c}{2\beta} &= q_1 + \frac{q_2}{2} \\ \frac{\Theta - c}{2\beta} &= \frac{q_1}{2} + q_2 \end{aligned} \tag{35}$$

therefore

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{\Theta - c}{2\beta} \\ \frac{\Theta - c}{2\beta} \end{bmatrix}$$

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Python implementation

Cournot model

```python
import numpy as np
import matplotlib.pyplot as plt
A = np.array([[1,1/2],[1/2,1]])
def solution(A,theta, cost, beta):
  Ai = np.linalg.inv(A)
  cons = (theta - cost) / (2 * beta)
  mat = np.array([[cons],[cons]])
  solv = Ai @ mat
  return solv
solution(A, theta=100, cost=1, beta=1)
qi = [solution(A, 100, x, 3)[0][0] for x in range(100)]
plt.plot(list(range(100)),qi)
```

Nicholson example

Solve the example of the book and assume that $CM = 0$, and $Q = q_1 + q_2 = 120 - p$. See the solution (click here).

Homework

Find the solution of 6 firms that face a linear function of demand.