Logistic and Linear regression

Using python.

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Preamble

For this lesson you need remember some concepts as **Probability Distribution Function(PDF)**, **joint distribution function**, **expectation** and **variance** of random varaibles please check the following material:

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This note we are constructed with several references that are listed in references.

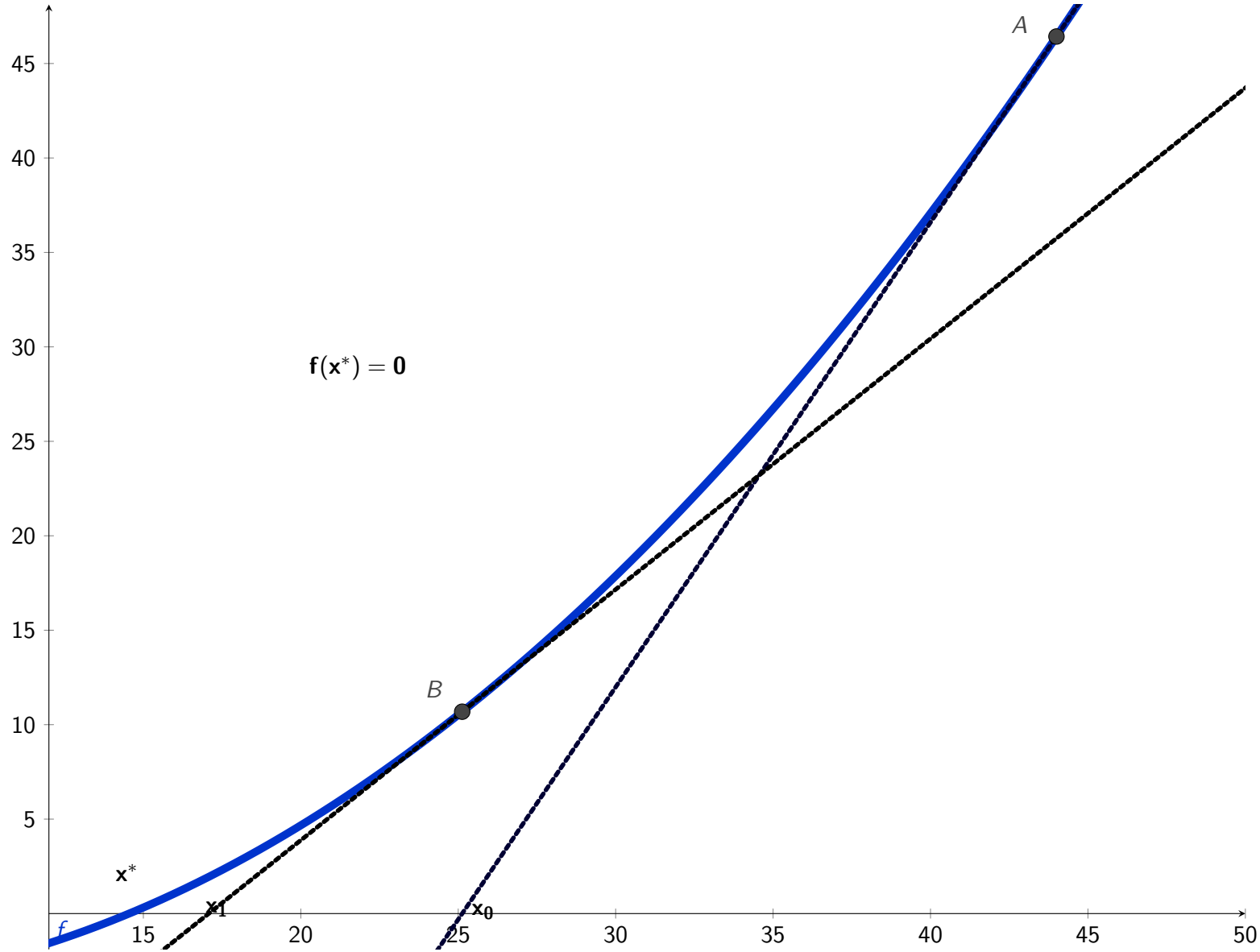
Introduction

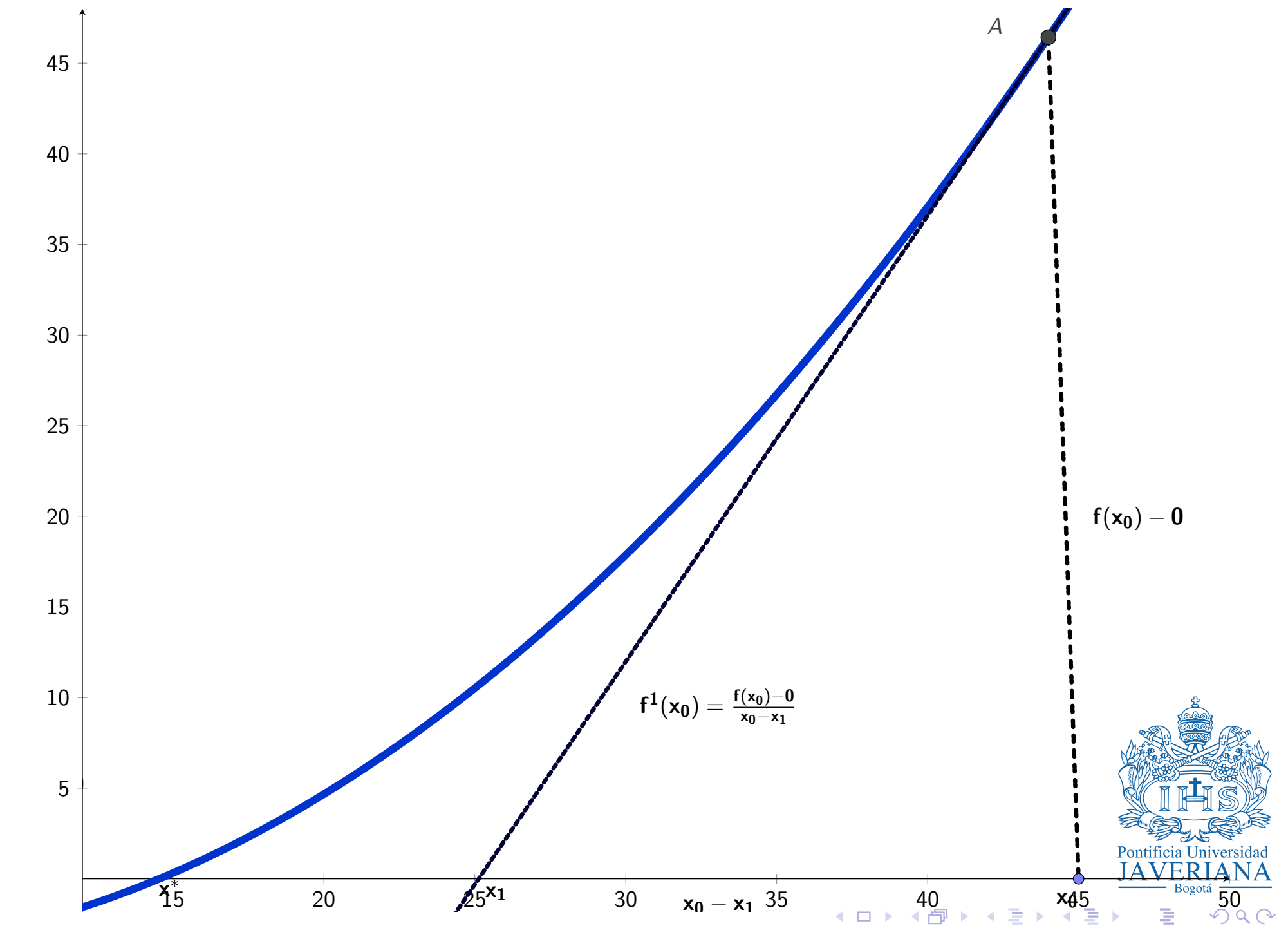
Logistic regression is used broadly in empirical works, it used in economics, engineering, epidemiology and clinical research. In a simplified way the logistic regression it is used to binary problems.

Insights

If we take a point near to the change in concavity the method could produce divergence.

illustration




A
45
40
35
30
25
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15
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15
20
25
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35
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45
50
x^*
x_1
$x_0 - x_1$
x_0
$f(x_0) - 0$
$f^1(x_0) = \frac{f(x_0)-0}{x_0-x_1}$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} \tag{1}$$

Thus in i iteration we have:

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} \tag{2}$$

We select the point A the tangent line cross the x-axi in x_0 and again in $f(x_0)$ and again the tangent line of this point (B) cross the x-axis in x_1 each step is also known as a iteration. The algorithm converge to the the value x^* think that this could be used to *optimization problems*.

Newton-raphson program

```
def quadratic(a,b,c, x):
      return a*x**2 + b*x + c
def dfQuadratic(a,b,x):
  return a*x**2 + b*x
a,b,c ,x0 = 1,-3,-4,8
def raphsonQuadratic(a,b,c,x0, error_max=0.0000015,
    iteration_max=100):
  xi = x0
  iter, error = 0, 100
  data = []
  while (iter < iteration_max) and (error > error_max):
    xj = xi - quadratic(a,b,c,xi) / dfQuadratic(a,b,xi)
    error = abs(xj - xi)
    iter += 1
    data.append((xj,xi,error,iter))
    xi = xj
  return data
raphsonQuadratic(a,b,c,4)[-1]
```

Gradient descendent

$P(\Theta)$ prior distribution, posterior distribution $P(\Theta \mid \mathbf{X})$. likelihood $P(\mathbf{X} \mid \Theta)$. Prior the belief before seen the data.

Max a posteriori

A set of features $\mathbf{X} = \{x_1, ...x_n\}$ assuming a distribution $P(\mathbf{X}, \Theta)$ where *Theta* is a parameter (a random variable).

$$\Theta_{ma} = \max P(\Theta \mid \mathbf{X}) \tag{3}$$

the last equation must be compared regarding the maximun likelihood estimation. that establish the $\max P(\mathbf{X} \mid \Theta)$.

Problem

Problem

We need find the probability that we draw exactly 3 damaged computers of 5 if in the batch there are 12 computers and 5 of them fail.

Generalization of the problem

The problem: *Find the probability of get* (i) *in* (j) *where* (m) *fill a property in a total of* (N).

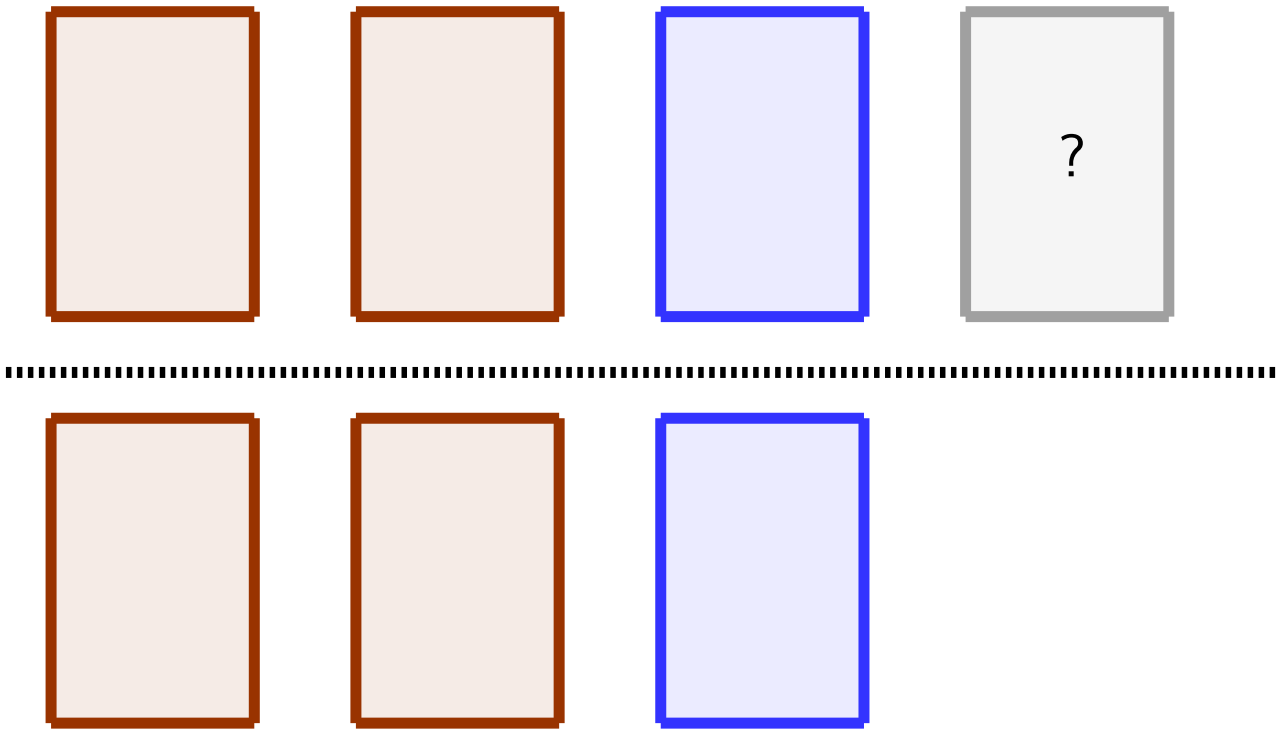
deriving

there are $\binom{m}{i}$ possible ways of get i among m and by each one of the possible combination there are $\binom{N-m}{j-i}$ in a total of $\binom{N}{j}$ combinations, therefore the probability will be:

$$\frac{\binom{m}{i}\binom{N-m}{j-i}}{\binom{N}{j}} \tag{4}$$

(4) is also known as **Hypergeometric distribution**.

Freund ilustration



Estimate Θ by maximun likelihood

The **Method of maximun likelihood** consist in estimate the parameter Θ the number of *red cards* that maximize the probability of see the **data** (three red cards and a blue car), in this case Θ could be 2 or 3.

$$\frac{\binom{3}{2}\binom{1}{1}}{\binom{4}{3}} > \frac{\binom{2}{2}\binom{2}{1}}{\binom{4}{3}} \tag{5}$$

Scipy

in Scipy the notation is:

$$p(k, M, n, N) = \frac{\binom{n}{k}\binom{M-n}{N-k}}{\binom{M}{N}} \tag{6}$$

```
from scipy.stats import hypergeom
k,M,n,N = 2,4,3,3
print(hypergeom.pmf(k,M,n,N))
k, M,n,N = 2,4,2,3
print(hypergeom.pmf(k,M,n,N))
```

Maximun Likelihood Estimation (MLE)

Suppose that you data it is generated by a theoretical distribution, the inverse problem is determine the most probable parameter that generate the data.

Insights about MLE

we are going to say in a general term that $f(x_i)$ is PDF or PMF of a random variable.

Problem

you have n balls in a bag where there are j reds and k black thus $n = j + k$. However you do not know the really proportion of each one color. if you draw balls and both are different what is the proportion of black balls Θ.

MLE insight

Then we choose a Θ among all posible values that maximize the probability of seen the data.

Problem

In a formal way we can assume that $x \sim B(n, \Theta)$ and we have seen the data results $\{x_1, x_2, x_3, ..., x_n\}$

$$P(X = x_1 \cap X = x_2, ..., \cap X = x_n) \tag{7}$$

The variables are i.i.d and therefore the joint probability is the result of multiply the marginal probabilities.

$$P(X = x_1 \cap X = x_2, ..., \cap X = x_n) = \prod_{i=1}^{n} P(x_i) \tag{8}$$

Problem

$$
\begin{aligned}
\prod_{i=1}^{n} P(x_i) &= \prod_{i=1}^{n} \binom{m}{x_i} \Theta^{x_i} (1-\Theta)^{m-x_i} \\
&= \prod \binom{m}{x_i} \prod \Theta^{x_i} \prod (1-\Theta)^{m-x_i} \\
&= \prod \binom{m}{x_i} \Theta^{\sum x_i} (1-\Theta)^{\sum (m-x_i)} \\
ln(\prod P(x_i)) &= ln(\binom{m}{x_i} + \sum x_i ln(\Theta) + (nm - \sum x_i) ln(1-\Theta)
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
L(\Theta) =& ln(\prod P(x_i) \\
\frac{dL(\Theta)}{d\Theta} =& \frac{\sum x_i}{\Theta} - \frac{nm - \sum x_i}{1-\Theta} = 0 \\
\frac{1}{\Theta} =& \frac{nm - \sum x_i}{\sum x_i} + 1 \\
\Theta^* =& \frac{\sum x_i}{nm}
\end{aligned}
\tag{10}
$$

Now to proof that $L(\Theta^*)$ is the maximun probability we must show that $\frac{d^2 L(\Theta)}{d\Theta^2} \mid_{\Theta=\Theta^*} < 0$.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About estimators

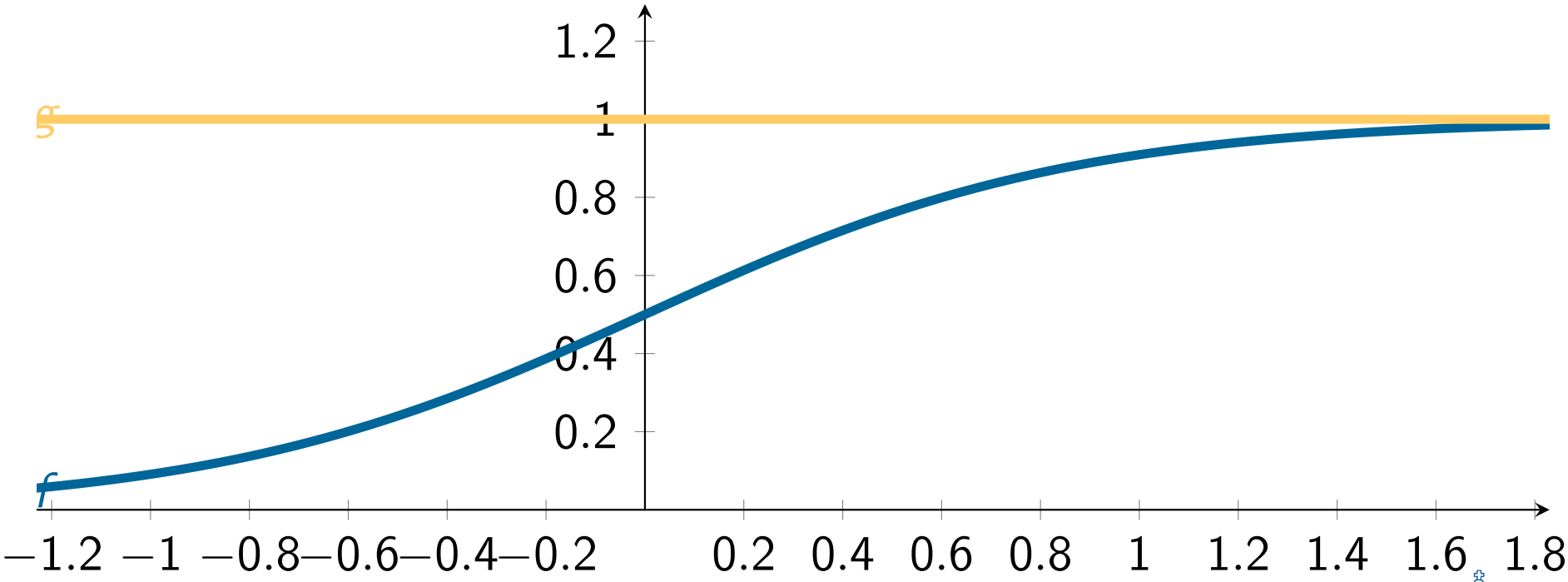
The quality of estimators:

- **Unbiased estimator :** The mean of the estimator is equal to the mean of parameter.
- **Variance:** the dispersion of the estimations regarding the mean value of the same.
- **quadratic value mean:** offer information about another two.

Suppose that do you have a function $f(x, y) = x^2 + y^2$, then we can define the gradient as follow:

$$\nabla f(x, y) = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix} \tag{11}$$

Logistic equation



logistic equation

$$f(x) = \frac{\kappa}{1 + e^{-\alpha(x - x_0)}} \tag{12}$$

Where κ it is the maximun value.

logistic equation

Population growth

$$\frac{dy}{dt} = ry(1 - \frac{y}{k}) \tag{13}$$

python implementation

Statsmodels

```
formula = 'died~studytime + C(drug) ' # logit died studytime
    i.drug
model = smf.logit(formula= formula, data=df)
results = model.fit()
print(results.summary())
coefs = pd.DataFrame({
    'coef': results.params.values,
    'odds ratio': np.exp(results.params.values),
    'name': results.params.index
})
coefs
```

Solve

with newton method solve the following:

$$e^x = 4 - x^2 \tag{14}$$

las dos soluciones son: x1 = 1.05 x2 = -1.96

Contingency table

		Diagnose	
		Disease	No-Disease
Risk Factor	Smoke	a	b
	Not Smoke	c	d

From this table we can calculate the odds, increase the probability of a diagnose smoke, or not?

$$odds_{smoke} = \frac{a}{b} = \frac{P(D \mid smoke)}{P(ND \mid smoke)} \quad (15)$$

e

Odds ratio

We can calculate the odds ratio as:

$$OR = \frac{odds_{smoke}}{odds_{NoSmoke}} \tag{16}$$

according to the information of the table, we can calculate the odds ratio as $OR = \frac{\frac{a}{b}}{\frac{c}{d}}$.
How we can interpret this OR?.

From odds to probability

$$\frac{P(A)}{P(A^c)} + 1 = odds + 1 \tag{17}$$

$$\frac{P(A) + P(A^c)}{P(A^c)} = odds + 1 \tag{18}$$

$$\frac{1}{P(A^c)} = odds + 1 \tag{19}$$

$$\frac{odds}{odds + 1} = P(A) \tag{20}$$

Odds Ratio

```
testing = pd.DataFrame({
'smoke':[1,1,0,1,1,0,1,1,0,1,0,0,1,1,1,0,0,1,1,1],
'diagnose':[1,0,1,1,0,0,1,1,0,0,0,1,1,0,0,0,0,0,1,0]})
pd.crosstab(testing['smoke'], testing['diagnose'])
```

```
# Odds ratio
oddsnum = 6/7
oddsdem = 2/5
print(oddsnum, oddsdem, oddsnum/oddsdem)
```

Logistic regression

Using statsmodels

```
import statsmodels.formula.api as smf
model_logit = smf.logit(formula="diagnose ~ smoke", data=testing)
res = model_logit.fit()
# How we can convert to dummy variable??
res.summary()
np.exp(res.params) # This give us the odds ratio getted
    previously.
```

ORs lessert than 1

$$(1 - OR) * 100 \tag{21}$$

Linear regression

When it is useful the linear regression?
for instance, the relation among the income and compsution, body mas index and levels of microalbum, etc, the wage and years of experience, unlike logistic regression the indepent variable not is dicotomy.

A linear model

Where are going consider that our prediction $\hat{y}_i$ differ from the real value y_i in u_i:

$$y_i - \hat{y}_i = u_i \tag{22}$$

and we are going to model $\hat{y}_i = \beta_0 + \beta_1 x_i$, therefore:

$$y_i = \beta_0 + \beta_1 x_i + u_i \tag{23}$$

Where y_i is our independent term, β_0 is a constant.

MLE

We have a collection of $\vec{y} = y_1, y_2, ..., y_n$, that our objective is build a model that allow us to find the value of β_0, β_1 and maximize the probability of seen the data($\vec{y}$).

Assume that

$$y_i = \beta_0 + \beta_1 x_i + u_i \tag{24}$$

and

$$u_i \sim N(0, \sigma^2) \tag{25}$$

Therefore the $E(y_i) = \beta_0 + \beta_1 x_i$ and $var(y_i) = \sigma^2$ (why?).

$$y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2) \tag{26}$$

If we make distributional assumptions we could make inference and hypotesis testing.

PMF normal distributon

The PMF of a normal variable is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \tag{27}$$

where μ is mean. Therefore for our problem.

$$f(y_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y_i - \beta_0 - \beta_1 x_i}{\sigma}\right)^2\right) \tag{28}$$

Likelihood function

$$L(\beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n} f(y_i) \tag{29}$$

$$\prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y_i - \beta_0 - \beta_1 x_i}{\sigma}\right)^2\right) \tag{30}$$

Now rememeber that $e^x.e^y = e^{x+y}$ therefore the *likelihood function* is rewritten as:

$$\frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2\right) \tag{31}$$

Log likelihood function

Apply natural logarithm to the likelihood function

$$\ln(L(\beta_0, \beta_1, \sigma^2)) = -n\ln(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 \qquad (32)$$

now we need establish the partial derivatives for β_0 and β_1. Notice that to maximize the expression we need minimize $\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$.

$$\frac{\partial ln(L)}{\partial \beta_0} = \sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) = 0 \tag{33}$$

$$\frac{\partial ln(L)}{\partial \beta_1} = \sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)x_i = 0 \tag{34}$$

We are going to reach the same equations minimizing the error among the prediction and the real value of y_i.

Solving the problem

We have an analytical solution!.

Ordinary least squares

$$u = \sum (y_i - \hat{y}_i)^2 \tag{35}$$

note that $\hat{y}_i = \beta_0 + \beta_1 x_i$.

$$u^2 = \sum (y_i - \beta_0 - \beta_1 x_i)^2 \tag{36}$$

the first order condition require $\frac{\partial u^2}{\partial \beta_0} = 0, \frac{\partial u^2}{\partial \beta_1} = 0$.

Chain rule

to get

$$\frac{\partial u^2}{\partial \beta_0} = \frac{\partial u^2}{\partial u}\frac{\partial u}{\partial \beta_0} \tag{37}$$

$$\frac{\partial u^2}{\partial \beta_1} = \frac{\partial u^2}{\partial u}\frac{\partial u}{\partial \beta_1} \tag{38}$$

Remember that $\frac{d\sum g_i(x)}{dx} = \frac{\sum dg_i(x)}{dx}$. Therefore $\frac{\partial u^2}{\partial u} = 2u$ and $\frac{\partial u}{\partial \beta_0} = -1$, $\frac{\partial u}{\partial \beta_1} = -x_i$.

Partial derivatives

$$\frac{\partial u^2}{\partial \beta_0} = -2 \sum (y_i - \beta_0 - \beta_1 x_i) \tag{39}$$

$$\frac{\partial u^2}{\partial \beta_1} = -2 x_i \sum (y_i - \beta_0 - \beta_1 x_i) \tag{40}$$

note that this will be zero if we know the exactly parameters.

Solving

Analytically

$$\hat{\beta}_1 = \frac{\sum(y_i - \bar{y})(x_i - \bar{x})}{\sum(x_i - \bar{x})^2} = \frac{cov(x, y)}{var(x)}. \tag{41}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \tag{42}$$

To see the full derivation of 41 and 42 **click here**

Causality

Correlation not imply cuasality!!

What is the correct interpretation?

Multiple Linear regression

$$y_i \sim N(\beta^\intercal x_i, \sigma^2) \tag{43}$$

The likelihood function $\beta, \boldsymbol{x} \in \mathbf{R}^d$.

$$\prod_{i=1}^{n} \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta^{\mathsf{T}}x_i)^2\right) \tag{44}$$

$$\frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta^{\mathsf{T}}x_i)^2\right) \tag{45}$$

You can check the appendix in matricial operations

$$\frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \exp\left(\frac{-1}{2\sigma^2}(\boldsymbol{y} - \beta^{\mathsf{T}}\boldsymbol{x})^{\mathsf{T}}(\boldsymbol{y} - \beta^{\mathsf{T}}\boldsymbol{x})\right) \tag{46}$$