Introduction to probability, counting and Bayes theorem

Iván Andrés Trujilllo Abella

ivantrujillo1229@gmail.com

Probability

The sample space (Ω) represent the total possibles outcomes in a experiment. An **event** (A_i) is a subset of Ω $(A \subseteq \Omega)$ and the number of favorable cases to the event is (a_i).

Rolling dice - Probability of get an even number

$$\begin{aligned} \Omega &= \{1, 2, 3, 4, 5, 6\} \\ A_i &= \{2, 4, 6\} \end{aligned} \tag{1}$$

Probability

The sample space (Ω) represent the total possibles outcomes in a experiment. An **event** (A_i) is a subset of Ω $(A \subseteq \Omega)$ and the number of favorable cases to the event is (a_i).

Rolling dice - Probability of get an even number

$$\begin{aligned} \Omega &= \{1, 2, 3, 4, 5, 6\} \\ A_i &= \{2, 4, 6\} \end{aligned} \tag{1}$$

...

$$P(A_i) = \frac{n(A_i)}{n(\Omega)} = \frac{3}{6} \tag{2}$$

Probability

Naive Definition

The probability that occur the event A_i is equal to $P(A_i) = \dfrac{a_i}{\Omega}$ we can see that $\Omega = n(\Omega)$ is the number of favorable and unfavorable cases.

Note that we refer to the number of elements, specifically to the cardinally of the sets, then $a_i = n(A_i)$.

Probability

To compute

- What is the probability of get 12 in two dice rolling
- What is the probability of get two tails flipping two times a coin

Those above listed problems are counting problems!

Multipliation rule

...

If one action can be make it of p ways and later other can be make it of q ways then the total numbers of ways that the actions can make it jointly is $p.q$

There are "q**" ways to carry out the second action, by to "**p**" ways of make the first action, therefore there are** $p.q$ **total ways of carry out both actions.**

Paiting doors

if you have 4 doors and 3 different colors of paint, how many ways you can painted the doors?

Paiting doors

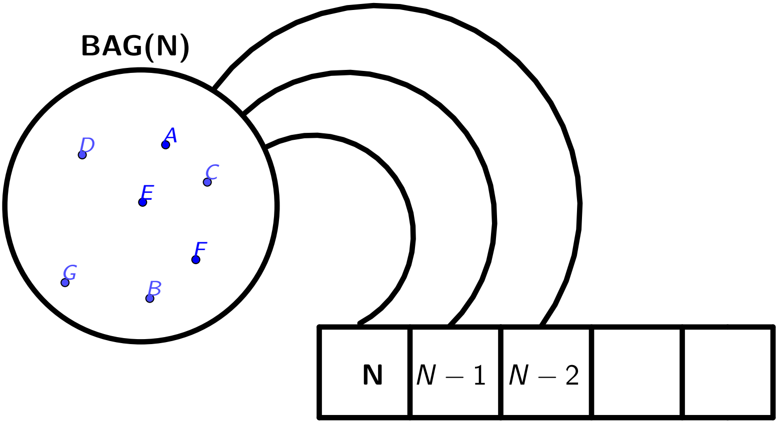
if you have 4 doors and 3 different colors of paint, how many ways you can painted the doors?

Bag model

for n marbles and drawing k of them one by one, how many arragments we can get?... the first time we can get n marbles, after $(n-1)$, the third time $(n-2)$ thus the $k-time$ we have $(n-k+1)$ ways of drawing one.

$$p(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)...(n-k+1) \tag{3}$$

Bag model ilustration



...

How many possibles arrangements are result of drawing two marbles in a bag with letters (a,b,c).

first time we can get three ways of drawing a marble, there are two ways of draw the second marble (don't insert the marble again - without replacement) multiplication rule total of possibles results are six:

...

How many possibles arrangements are result of drawing two marbles in a bag with letters (a,b,c).

first time we can get three ways of drawing a marble, there are two ways of draw the second marble (don't insert the marble again - without replacement) multiplication rule total of possibles results are six:

Arragements	(a,b)	(a,c)	(b,a)	(b,c)	(c,a)	(c,b)

Permutations

...

$$\begin{aligned} n! &= n(n-1)! \\ \frac{n!}{n-1!} &= n \\ \frac{n}{n-2!} &= n(n-1) \\ \frac{n}{n-3!} &= n(n-1)(n-2) \end{aligned} \tag{4}$$

The generalization:

Formula

$$p(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)...(n-k+1) \tag{5}$$

Combinations

in a permutation there is not replacement and order matter, However for each arrangement of k longitude have $k!$ ways of ordered therefore the number of possible ways of different total elements it is a combination.

Formula

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{(n-k!)k!} \tag{6}$$

Binomial coefficient

Note that combinations are also noted as follow:

$$C(n,k) = \binom{n}{k} \tag{7}$$

Summary

We need consider the following in counting problems:

...

- Repetition
- order

	order	not-order
Replacement	n^k	$\binom{n+k-1}{n-1}$
Without replacement	$\dfrac{n!}{(n-k)!}$	$\binom{n}{k}$

Combinatorial identity

Using the following binomial theorem:

$$(a+b)^n = \sum_{i=0}^{n} a^{n-i} b^i C(n,i) \tag{8}$$

$$(1+1)^n = \sum_{r=0}^{n} C(n,r) = 2^n - C(n,0) = \sum_{r=1}^{n} C(n,r) \tag{9}$$

...

$$2^n - 1 = \sum_{r=1}^{n} C(n,r) \tag{10}$$

What is the practical importance of this identity?

Naive is a limited definition

...

The problem with **naive definition;** is that require that all events are equally likely...

No naive defintion

probability space, a new concept, that include a function f such that:

$$x \subseteq \Omega; \quad f(x) \in [0, 1] \tag{11}$$

Assumptions

According to the literature $f() = P()$.

...

- $P(A_i) \in [0,1] \quad \forall A_i$.
- $P(\emptyset) = 0$ the probability that something that never will happened is equal to zero.
- $P(\Omega) = 1$ the probability of something that always occur is equal to one.
- $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$ if and only if $\bigcap_{i=1}^{n} A_i = \emptyset$.

Kolgomorov approach

The certainty, it is one (1), for instance; the following statement *A person is live or dead it is certainty*

$$P(A \cup A^c) = 1 \tag{12}$$

Could be a natural property that two two two events mutually exclusive could be sum up their probabilities.

$$P(A \cup B) = P(A) + P(B) \tag{13}$$

You can extend this properties to n events.

Independece

Independence not is equal not occurrence, the independence is related with the change of the probability that occur a event given that another occur.

$$P(A \cap B) = P(A)P(B) \tag{14}$$

Tossing coins it is a bernoulli trial, and the occurrence of a event not affect the another event.

Birthday paradox

Programming and mathematics are practical. Statistics it is a field in which we can interact with real data, the **birthday paradox (See simulation)** it is a practical example of this.
If we have k persons then the probability that two of them born in the same day is; if the year have 360 days:

$$1 - \frac{360.359.358...(360 - k + 1)}{360^k}$$

In the case of :

$$1 - \left(\prod_{i=311}^{360} i(360^{-50}) \right) \approx 0,97 \tag{15}$$

is 97% is a high probability that at least two person born the same day.

The following Figure 1 show the relation among the probability of two coincidences among two person in the sample.

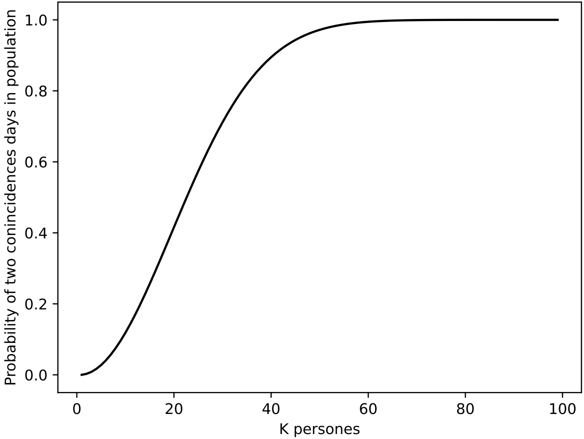


Figure: Birthday paradox with 360 days

Bayes Theorem

Uses

- Inference!
- Filter spam!
- Medical diagnosis!
- ...

What we need?

- Conditional probability
- Law of total probability

Conditional Probability

Probability of A occur given B occurs or B already happened.

Conditional Probability

Probability of A occur given B occurs or B already happened.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{16}$$

Misused

$$P(A \mid B) = P(B \mid A) \tag{17}$$

Contingency table

		Diagnose	
		Disease	No-Disease
Risk Factor	Smoke	a	b
	Not Smoke	c	d

Contingency table

		Diagnose	
		Disease	No-Disease
Risk Factor	Smoke	a	b
	Not Smoke	c	d

...

- $P(Smoke) = \frac{a+b}{(a+b+c+d)}$
- $P(Disease \cap Smoke) = \frac{a}{(a+b+c+d)}$
- $P(Disase \mid Smoke) = \frac{a}{a+b}$

Bayes theorem

Bayes theorem

Note that $P(A \cap B)$ it is equal to $P(B \cap A)$.

$$P(A \mid B)P(B) = P(B \mid A)P(A) \tag{18}$$

...

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \tag{19}$$

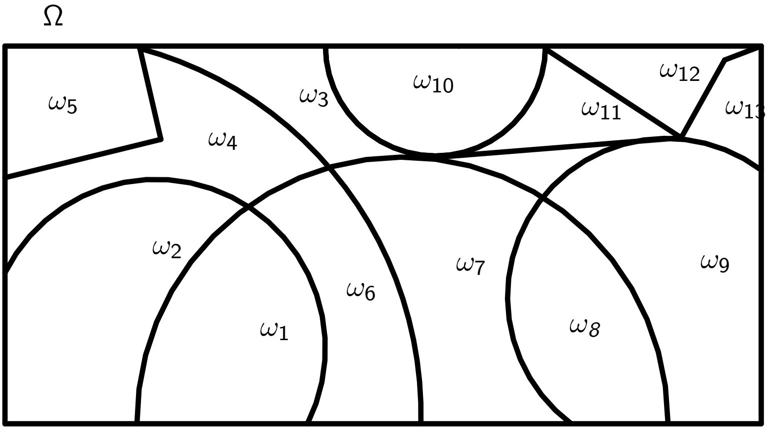
Using bayes theorem compute:

$$P(Smoke \mid Disease) \tag{20}$$

Law of total probability

The sample space defined as Ω if we split omega in ω k subsets in order that each *subset* no overlap with others. $\bigcup_{i=1}^{k} \omega_i = \Omega$ y $\bigcap_{i=1}^{k} \omega_i = \emptyset$ For instance the sample space defined as $\Omega = \{a, b, c, d, e, f\}$ $\omega_1 = \{a, f\}$ $\omega_2 = \{b, c, d\}$ $\omega_3 = \{e\}$.

Split Ω



Ω
ω_{5}
ω_{3}
ω_{10}
ω_{12}
ω_{11}
ω_{13}
ω_{4}
ω_{2}
ω_{7}
ω_{9}
ω_{6}
ω_{1}
ω_{8}

$P(A)$

Total law probability

$P(A)$
Total law probability

We need remember by set theory that a event A could be rewrite as $A = (A \cap B) \cup (A \cap C)$. if $(B \cup C) = \Omega$. We can rewrite for k:

$$\begin{aligned} A &= (A \cap \omega_1) \cup (A \cap \omega_2) ... (A \cap \omega_k) \\ P(A) &= P(A \cap \omega_1) + ... + P(A \cap \omega_k) \\ P(A) &= P(A \mid \omega_1) P(\omega_1) + P(A \mid \omega_2) P(\omega_2) + ... + P(A \mid \omega_k) P(\omega_k) \end{aligned} \tag{21}$$

note that by total law $P(A)$

Practical problem:

Medical diagnosis

Clinical and epidemiological research

Provides information about patterns of diseases, when analyzing medical history records, you can compute:

$$P(Symptom \mid Disease) \tag{22}$$

Consider that an a symptom could be associated with several diseases, there in a practical context you need compute:

$$P(Disease \mid Symptom) \tag{23}$$

Medical diagnosis

An a test have a sensitivity $P(\hat{+} \mid +)$ but in a practical sense the really question is:

...

$$P(+ \mid \hat{+}) \tag{24}$$

In simple words: **The probability that the person really present a condition given that the test is positive.**

A daily practical context!

Bayes gives us a way of compute it!.

...

$$P(+ \mid \hat{+}) = \frac{P(\hat{+} \mid +)P(+)}{P(\hat{+})} \tag{25}$$

Now split our population in Disease(D) and Not disease(ND) persons and using **Total probability law** we have:

...

$$P(+ \mid \hat{+}) = \frac{P(\hat{+} \mid +)P(+)}{P(\hat{+} \mid D)P(D) + P(\hat{+} \mid ND)P(ND)} \tag{26}$$

Diagnose

Activity

Given a $P(\hat{+} \mid Disease) = 0.98$, and the prevalence about 3% , y is known that 80% of time the test is positive independent if have or not the disease, find:

$$P(Disease \mid \hat{+}) \tag{27}$$