Introduction to economic growth

Theory and empirical insights

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References

The following lectures are only material support and are build over the following works:

- Acemoglu, D. (2008). Introduction to modern economic growth. Princeton university press.
- Barro, R., Sala-i-Martin, X. (2004). Economic growth second edition.

Growth rate and time

Problem 1

If an economy grows at rate of 3.7% per year, how many years are needed to duplicates its current GDP?

...

$$n = \frac{ln(\lambda)}{ln(1+r)} \tag{1}$$

λ is any number and r is the growth rate, for our problem $\lambda = 2$ and $r = 0.037$. **¿Why?**.

Problem 2

$$\frac{\partial n(\lambda, r)}{\partial r}? \tag{2}$$

Euler, interest rate, and exponential growth?

Final value

$$Fv = Pv(1+r)^n \tag{3}$$

Exponential growth is an importan model!.

Problem 3

Solve

$$\frac{dy}{dt} = \alpha y \tag{4}$$

Production function and technology

The technology have a multiplicative effect over the production factors...

technology also is....

- Algorithms
- Theorems (Where is the Pythagorean theorem applied?)
- Equations....
- Ideas, Insights...

Production Function

Our production function need satisfy some intuitive properties

...

- Produce λ times more if we increase input factor λ times.
- Marginal diminish productivity of inputs
- Inada conditions

Cobb-Douglas function

Cobb-Douglas production function is defined as:

$$Z = AK^{\alpha}L^{\beta} \tag{5}$$

A is technology, K capital and L labor. Whats is the economic interpretation of α, β?

Marginal product

what means?

$$PM_{x_i} = \frac{\partial f(x_1, ..., x_n)}{\partial x_i} \tag{6}$$

if $f(x_1, ..., x_n)$ is a production function with n input factors then PM_{x_i} is *The changes in production given a unitary increase of the input x_i.*

cobb-douglas (Capital marginal productivity)

$$PM_K = \alpha K^{\alpha-1} L^{\beta} \text{ (for labor?)} \tag{7}$$

Marginal Rate of Technical Substitution

Insight

How we can substitue one factor by another holding the production fixed?. for instance we can produce $z_0 = f(K, L)$

We need change the factors holding the total changes in production equal to zero.

$$df(K, L) = \frac{\partial f(K, L)}{\partial K} dK + \frac{\partial f(K, L)}{\partial L} dL = 0 \tag{8}$$

...

$$MRTS = \frac{dL}{dK} = -\frac{\frac{\partial f(K,L)}{\partial K}}{\frac{\partial f(K,l)}{\partial L}} = -\frac{PM_K}{PM_L} \tag{9}$$

Is useful this information?

We need duplicate the production!

We need duplicate the actual level of production, then is enough duplicates the input factors?

We need susbtitute one factor!

Given the scarcity of one factor we need replace one of them holding the production in the current level, we can substitute one unit of one by one unit of anoter?

Answer the quesionts!

To solve the questions we need only a basic introduction to linear regression, we could find more information here and the database here.

Homogeneity of a function

Insights

What happens to production level if we increase both factors by λ time?.

$$
\begin{aligned}
f(\lambda K, \lambda L) &= (\lambda K)^{\alpha}(\lambda L)^{\beta} \\
&= \lambda^{\alpha+\beta} K^{\alpha} L^{\beta} \\
&= \lambda^{\alpha+\beta} z
\end{aligned} \tag{10}
$$

Notice that if $\alpha + \beta = 1$ means that a increase of λ in input factors increase the production at the same level. This property is called *constant return to scale*.

Homogenous degree

The following function

Excersice

What is the homogeneous degree of the following function?

$$h(K, L) = K^{\tau} + L^{\tau} \tag{11}$$

Excersice

Check that

$$h(K, L) = \frac{1}{\lambda^{\tau}} f(\lambda K, \lambda L) \tag{12}$$

Homogeneous degree propierties

assume that $f(x_1,..x_i..,x_n)$ is homogeneous of ϕ degree therfore:

$$f(\lambda x_1,..\lambda x_i..,\lambda x_n) = \lambda^{\phi} f(x_1,..x_i..,x_n) \tag{13}$$

assuming that $\lambda = \frac{1}{x_i}$.

$$f\left(\frac{x_1}{x_i},..\frac{x_i}{x_i}..,\frac{x_n}{x_i}\right) = \left(\frac{1}{x_i}\right)^{\phi} f(x_1,..x_i..,x_n) \tag{14}$$

Intensive form

$$x_i^{\phi}\left(\frac{x_1}{x_i},..1..,\frac{x_n}{x_i}\right) = f(x_1,..x_i..,x_n) \tag{15}$$

Intensive form

Homogeneity in cobb-douglas

what happend if replace $\lambda = \frac{1}{L}$?

Intensive form

Homogeneity in cobb-douglas

what happend if replace $\lambda = \frac{1}{L}$?

$$\left(\frac{1}{L}\right)^{\alpha+\beta} K^{\alpha} L^{\beta} = \left(\frac{K}{L}\right)^{\alpha} = k^{\alpha} \tag{16}$$

Now we define k as capital per worker and K as agregate capital and k^{α} is the cob douglas in terms of capital per worker.

Intensive form

Cobb-douglas

$$f(K,L) = f\left(\frac{K}{L},1\right)L^{\alpha+\beta} \text{ why?}$$

Intensive form

Cobb-douglas

$$f(K, L) = f\left(\frac{K}{L}, 1\right) L^{\alpha+\beta} \text{ why?} \tag{17}$$

if $\alpha + \beta = 1$

$$f(K, L) = k^{\alpha} L \tag{18}$$

This last equation will be important to model economic growth.

Inada conditions

...

$$\lim_{K\to\infty} PM_K = 0 \qquad \lim_{L\to\infty} PM_L = 0$$

$$\lim_{K\to 0} PM_K = \infty \qquad \lim_{L\to 0} PM_L = \infty$$

Excersice

Check inada conditions in cobb-douglas production function!

Inada conditions

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Excersice

Check inada conditions in cobb-douglas production function!

$$\begin{aligned} PM_K &= \alpha A \left(\frac{L}{K}\right)^{1-\alpha} \\ PM_L &= (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \end{aligned} \tag{19}$$

Solow model

The evolution of capital per worker in time.

$$\frac{dk}{dt} = \frac{\frac{dK}{dt}.L - \frac{dL}{dt}K}{L^2} = \frac{\frac{dK}{dt}}{L} - \frac{\frac{dL}{dt}}{L}$$

remember that k is equal to $\frac{K}{L}$, assuming that population growth is constant (n).

Main equation [1]

$$\frac{dk}{dt} = \frac{\frac{dK}{dt}}{L} - nk \tag{20}$$

the product Z is divided in two important componets

Investment + Consumption

$$Z = I + C \tag{21}$$

I, C investment and consumption respectively.
Now investment also have two components, depreciation and net investment.

...

$$I = K^* + \delta K \tag{22}$$

Assume that depreciation rate δ is constant. Also, consumption is the unsaved product; $(1 - \psi)Z$.

The product will be expressed as:

$$Z = (1-\psi)Z + K^* + \delta K$$

$$Z = Z - \psi Z + K^* + \delta K$$

Main equation [2]

$$\frac{dK}{dt} = K^* = \psi Z - \delta K \tag{23}$$

The net capital, is the part of savings that not is accounting in depreciation.

Combining Main equations [1] and [2]

$$\frac{dk}{dt} = \frac{K^*}{L} - nk \tag{24}$$

$$\frac{dk}{dt} = \frac{\psi Z - \delta K}{L} - nk \tag{25}$$

Fundamental growth model equation

$$\frac{dk}{dt} = \psi \frac{Z}{L} - (\delta + n)k \tag{26}$$

you could uses cobb-douglas production function for Z, thus;

Fundamental growth model equation with cobb-douglas

$$\frac{dk}{dt} = Ak^{\alpha} - (\delta + n)k \tag{27}$$

Note that the evolution of the capital in time is the difference among the save by worker and the depretiation and population growth.

Rate of growth

Growth capital rate is inverse to the level of capital

$$\Omega k = \psi \frac{A}{k^{1-\alpha}} - (\delta + n) \tag{28}$$

What this means ?

Rate of growth

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What this means ?

Insight

Major levels of capital lesser rate of growth!

Steady-state capital

Save is equal to depreciation or $\Omega k = 0$.

Steady state capital

$$k^* = \left(\frac{\psi A}{\delta + n}\right)^{\frac{1}{1-\alpha}} \tag{29}$$

Product

Homogeneity

what is the degree of homogeneity of $\prod_{i=1}^{n} x_i^{\alpha_i}$?

Product

Homogeneity

what is the degree of homogeneity of $\prod_{i=1}^{n} x_i^{\alpha_i}$?

...

$$\lambda^{\sum_i^n \alpha_i} \tag{30}$$

Neoclasical asumptions

- Diminishing marginal returns
- Homogeneity degree equal to one

Production function with externalities

...

$$Y = AK^{\alpha}L^{1-\alpha}E^{\tau} \tag{31}$$

$$Y = A\left(\frac{K}{L}\right)^{\alpha}\left(\frac{E}{L}\right)^{\tau}L^{\tau+\alpha+\ 1-\alpha} = Ak^{\alpha}\left(\frac{E}{L}\right)^{\tau}L^{\tau+1} \tag{32}$$

The homogenity degree is $\tau + 1$, and is suggested $E = k$.

$$Y = Ak^{\alpha}\left(\frac{k^{\tau}}{L^{\tau}}\right)L^{\tau+1} = Ak^{\alpha+\tau}L$$

Thefore the production by worker is $y = Ak^{\alpha+\tau}$.

Fundamental equation with externalities

$$\frac{dk}{dt} = \psi A k^{\alpha+\tau} - (\delta + n)k$$

Fundamental equation with externalities

$$\frac{dk}{dt} = \psi A k^{\alpha+\tau} - (\delta + n)k \tag{33}$$

Analize the rate of growt for this equation.

growth rate with externalities

$$\Omega k = \psi A k^{\alpha+\tau-1} - (\delta + n) \tag{34}$$

Fundamental equation with externalities

$$\frac{dk}{dt} = \psi A k^{\alpha+\tau} - (\delta + n)k \tag{33}$$

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Analize the following cases:

- $\tau + \alpha = 1$
- $\tau + \alpha < 1$
- $\tau + \alpha > 1$

AK model

Production Function

$$Y = AK^{\alpha} \tag{35}$$

Excersice

Using [26] analize the growth rate of capital.

Solow model

Python simulation

Computing solow model in colab.

Solow model: Lab in Colab (click aquí)

Golden rule

Golden rule is the rate of saving that maximize the consumption:

$$C = Y - sY \tag{36}$$

In percapita terms $sy = y - c$ and considering the steady- state $\frac{dk}{dt} = 0$.

$$\frac{dk}{dt} = sy - (n + \delta)k$$
$$0 = y^* - c^* - (n + \delta)k^*$$

...

$$c^* = f(k^*) - (n + \delta)k^* \tag{38}$$

Golden rule

$$c^* = f(k^*) - (n+\delta)k^* \tag{39}$$

The first order condition to maximize the function is

$$\frac{\partial c}{\partial k^*} = f'(k^*) - (n+\delta) = 0 \tag{40}$$

Therefore the capital of steady stade that maximize the compsution percapita is

...

$$f'(k_{gold}) = n + \delta \tag{41}$$

k_{gold}

Excersice

Using cobb-douglas production function and the condition of k_{gold}:

$$f'(k_{gold}) = \delta + n \tag{42}$$

k_{gold}

Excersice

Using cobb-douglas production function and the condition of k_{gold}:

$$f'(k_{gold}) = \delta + n \tag{42}$$

$$A\alpha k^{\alpha-1} = n + \delta \tag{43}$$

cobb- douglas k_{gold}

$$k_{gold} = \left(\frac{\alpha A}{n+\delta}\right)^{\frac{1}{1-\alpha}} \tag{44}$$

Saving rate ψ

...

$$\psi f(k_{gold}) = (n + d)k_{gold} \tag{45}$$

Excersice

According to the last equation, cobb-douglas k_{gold} check that

$$\psi_{gold} = \alpha \tag{46}$$

Euler's theorem

Assume a homogenous function with degree ϕ

$$f(\lambda x_1, ..., \lambda x_n) = \lambda^{\phi} f(x_1, ..., x_n) \tag{47}$$

if we derivate with respect λ in both sides:

$$\sum_{i=1}^{n} \frac{\partial f(\lambda x_1, ..., \lambda x_n)}{\partial \lambda x_i} \frac{d\lambda x_i}{d\lambda} = \phi \lambda^{\phi - 1} f(x_1, ..., x_n) \tag{48}$$

...

$$\sum_{i=1}^{n} \frac{\partial f(\lambda x_1, ..., \lambda x_n)}{\partial \lambda x_i} x_i = \phi \lambda^{\phi - 1} f(x_1, ..., x_n) \tag{49}$$

Euler's theorem

...

$$\sum_{i=1}^{n} \frac{\partial f(\lambda x_1, ..., \lambda x_n)}{\partial \lambda x_i} x_i = \phi \lambda^{\phi-1} f(x_1, ..., x_n) \quad (50)$$

If we assume that $\lambda = 1$.

...

$$\sum_{i=1}^{n} \frac{\partial f(x_1, ..., x_n)}{\partial x_i} x_i = \phi f(x_1, ..., x_n) \quad (51)$$

if $f(x_1, ...x_2)$ is neoclasical production function then $\phi = 1$ then:

$$\sum_{i=1}^{n} PMx_i \mathbf{x_i} = f(x_1, ..., x_n) = Y \quad (52)$$

Cobb-douglas excersice

Show that the euler's theorem in the cobb-douglas funciton.

$$Y = Ak^{\alpha}L^{\beta}$$

Cobb-douglas excersice

Show that the euler's theorem in the cobb-douglas funciton.

$$
\begin{aligned}
Y &= Ak^{\alpha}L^{\beta} \\
&= PM_K K + PM_L L
\end{aligned}
\tag{53}
$$

Topics to research

- Apply numerical solution to the first differential equation

Appendix

$$\frac{dy}{dt} = \alpha y \tag{54}$$

A variable that change in time with respects its value arrive to a exponetial law!.