Proba2 IIND-2027

Complementaria V

Iván Andrés Trujilllo Abella

ai-page.readthedocs.ios

ivantrujillo1229@gmail.com

Genetering moment function

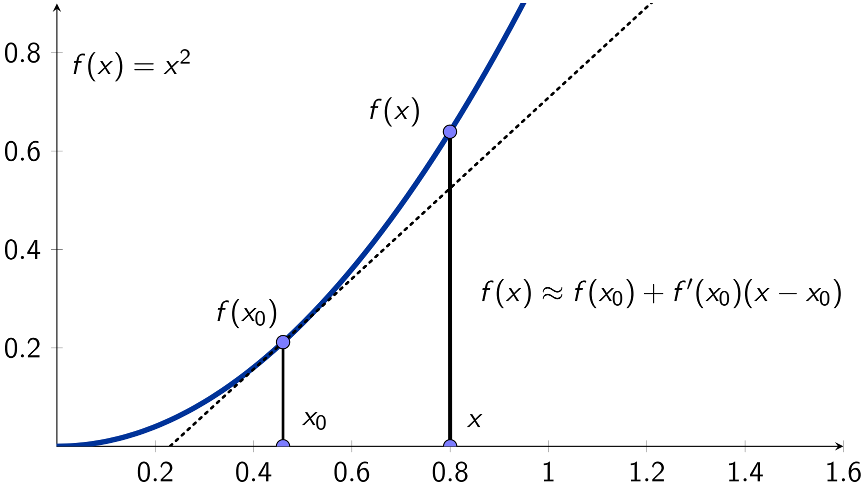
...

$$M_x(t) = E(e^{tx}) \tag{1}$$

$$E(X^n) = \frac{d^n}{dt^n} M_x(t)|_{t=0} \tag{2}$$

where n is n-th moment $E(X) = \mu$ and $E(X^2) = \sigma^2$.

Taylor expansion



...

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + Re \tag{3}$$

$$f(a) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \tag{4}$$

where $f^n = \frac{\partial^n}{\partial x^n} f(x)$
the taylor expansion of e^x around 0 is.

$$e^x = \sum_{i=0}^{n} \frac{x^n}{n!} \tag{5}$$

$$e^{tx} = \sum_{i=0}^{n} \frac{t^n x^n}{n!} \tag{6}$$

$$E(e^{tx}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(x^n) \tag{7}$$

assuming a as large as possible.

$$\frac{d^k}{dt^n} E(e^{tx})|_{t=0} = \left(\sum_{n=0}^{p} \frac{n!}{(n-k)!n!} t^{n-k} E(x^n) \right) |_{t=0} = E(x^n) \tag{8}$$

...

for the n-th term in the summation:

- if $n < k$ then $\frac{n!}{(n-k)!} = 0$ then the term in summation is zero
- if $n > k$ then $t^{\alpha > 0}$ but evaluated in $t = 0$ all term is zero.
- if $n = k$ then $\frac{n!}{(n-k)!} = n!$ and $t = 1$ therefore the term is $E(x^n)$

Theorem 1

X is a random variable with a pdf $f(x)$ then μ of $g(x)$ is

$$\mu_{g(x)} = E(g(x)) = \sum g(x)f(x) \tag{9}$$

Example of income and the probability of sell a product.

Example

$$E(X^2) = \sum x_i^2 P(X = x_i) \tag{10}$$

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generalized

$$E(X^n) = \sum_i x_i^n P(X = x_i) \tag{11}$$

for this case

$$g(x) = e^{tx} \tag{12}$$

Discrete case

$$E(e^{tx}) = \sum_i e^{tx_i} P(X = x_i) \tag{13}$$

for a discrete case.... (think a moment)

$$
\begin{aligned}
E(e^{tx}) &= \sum_i e^{tx_i} P(X = x_i) \\
&= \sum_i \left(\sum_{n=0}^{p\to\infty} \frac{t^n x^n}{b!} \right) P(X = x_i) \\
&= \sum_i \frac{t^0 x^0}{0!} P(X = x_i) + \ldots + \sum_i \frac{t^p x^p}{p!} P(X = x_i) \\
&= \frac{t^0}{0!} E(x^0) + \ldots + \frac{t^p x^p}{p!} E(X^p)
\end{aligned}
\tag{14}
$$

The last result has been developed in [8].

Important things

...

- Alternative to probability distribution
- not all variables have m.g.f

Bernoulli

m.g.f.

a random variable $X \sim Bernoulli(P)$ then remember the pmf of beronulli

$$pmf(x) = P^{x_i}(1-P)^{1-x_i} \tag{15}$$

...

$$\begin{aligned} M_x(t) &= \sum_i e^{tx_i} P(X = x_i) \\ &= e^{t0}(1-P) + e^t P \\ &= (1-P) + e^t P \end{aligned} \tag{16}$$

We can uses this function to get first and second centrized moment, mean and variance.

Theorem

X is a random variable with pdf $f(x)$ then then variance of $g(x)$ will be:

$$\sigma^2_{g(x)} = E\left((g(x) - \mu_{g(x)})^2)\right) \tag{17}$$

this is equation is derived of the definition of variance of a random variable, remember that $g(x)$ is a random variable with mean $\mu_{g(x)}$.

...

$$\begin{aligned} E\left((X - E(x))^2\right) &= E(X^2 - 2E(X)X + E(X)^2) \\ &= E(X^2 - 2E(X)^2 + E(X)^2) \\ &= E(X^2) - E(X)^2 \end{aligned} \tag{18}$$

Moments bernoulli

First Moment (Mean)

$$E(X) = \frac{\partial\left((1-P)+e^{t}P\right)}{\partial t}\Big|_{t=0} = P \tag{19}$$

Centrized second moment variance

$$\begin{aligned} E(X^2) - E(X)^2 &= \frac{\partial^2 M_x}{\partial t^2}\Big|_{t=0} - P^2 \\ &= \frac{\partial e^{t}P}{\partial t}\Big|_{t=0} - P^2 \\ &= P - P^2 \end{aligned} \tag{20}$$

Properties of m.g.f

Some important properties are:

- if X and Y two random variables have the same m.g.f $M_x(t) = M_y(t), \forall t$ then have the same distribution (see laplace transformation for proof).
- $M_{x+y}(t) = M_x(t)M_y(y)$

for a collection of $X_1, ...X_n$ of random variables:

...

$$
\begin{aligned}
M_{\sum X_i}(t) &= E(e^{t(\sum X_i)}) \\
&= E(e^{tX_1} e^{tX_2} ... e^{tX_n}) \ \textbf{independent} \\
&= E(e^{tX_1}) E(e^{tX_2}) ... E(e^{tX_n}) \\
&= M_{X_1}(t) M_{X_2}(t) ... M_{X_n}(t)
\end{aligned} \tag{21}
$$

sum of bernoullis are binomial

from a collection of $X_1, ..., X_n \overset{iid}{\sim} Bernoulli(P)$ we want describe $\sum X_i$.

...

$$M_{\sum X_i}(t) = \prod_i M_{X_i}(t) = \left((1 - P) + e^t P\right)^n \tag{22}$$

1-th moment

$$\frac{\partial \left(1 - P + e^t P\right)^n}{\partial t}\Big|_{t=0} = \left(n(1 - P + e^t P)^{n-1} e^t P\right)\Big|_{t=0} = nP \text{ (binomial expected value)} \tag{23}$$

Compute variance

Students!

Compute variance

Students!

2-th moment

$$\frac{\partial^2 (1 - P + e^t P)^n}{\partial t^2} = n(n-1)((1-P) + e^t P)^{n-2}(e^t P)^2 + n((1-P) + e^t P)^{n-1} e^t P \quad (24)$$

now with $t = 0$

$$E(X^2) = P^2 n^2 - nP^2 + np \quad (25)$$

Variance

$$var(X) = P^2 n^2 - nP^2 + nP - n^2 P^2 = nP - nP^2 \quad (26)$$

is binomial variance!

$\sum_i Z_i^2 \sim \chi_n^2$

proof

Now we need proof that $\sum Z_i^2 \sim \chi_n^2$ with:

$$M_{\sum Z_i^2}(t) = M_{\chi^2}(t) \tag{27}$$

Things to remember

$$\int_{-\infty}^{\infty} e^{xt} f(x) dx \tag{28}$$

$$pdf(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \tag{29}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) = 1 \tag{30}$$

...

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx \tag{31}$$

changing variable define

...

$$z = \frac{x-\mu}{\sigma} \tag{32}$$

$$dx = \sigma dz \tag{33}$$

$$x = \sigma z + \mu \tag{34}$$

$$M_x(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(t\sigma z + t\mu\right) \exp\left(-\frac{1}{2}z^2\right) \sigma dz \tag{35}$$

...

$$\frac{\exp(t\mu)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(z^2 - 2tz\sigma\right)\right) \tag{36}$$

Now complete the square $(a-b)^2 = a^2 - 2ab + b^2$;

$$-2zb = -2tz\sigma \tag{37}$$

therefore $b = t\sigma$. now to complete the square we have $(z-tz)^2 - (tz)^2$.

...

$$\begin{aligned} M_x(t) &= \frac{\exp(t\mu)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left((z-tz)^2 - (t\sigma)^2\right)\right) \\ &= exp\left(t\mu + \frac{\tau^2\sigma^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z-zt)^2\right) \\ &= exp\left(t\mu + \frac{\tau^2\sigma^2}{2}\right) \end{aligned} \tag{38}$$

note that

...

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z - zt)^2\right) = 1,\ \textbf{given that}\ x \sim N(zt, 1) \tag{39}$$

The area over all pdf is 1.

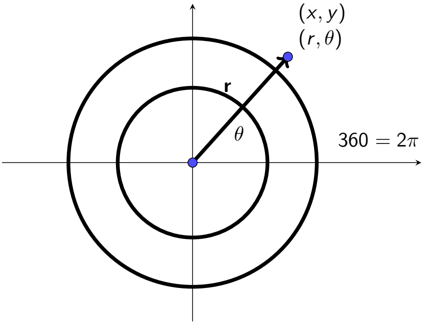
now to the sum

...

$Z_1, ..., Z_n \overset{iid}{\sim} N(0,1)$

$$M_{\sum Z_i^2} \tag{40}$$

Polar coordinates



...

- $\theta = \tan^{-1}$, $r^2 = x^2 + y^2$, $y = r\sin(\theta)$ and $x = r\cos(\theta)$.

...

important to note here in cartesian to cover all area we use $\int_{-\infty}^{\infty}\int_{\infty}^{\infty} f(x,y)dxdy$, however the transformation from (x,y) to (r,θ) (see slides about Jacobian to see the area transformation).

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r \tag{41}$$

and therefore:

$$dxdy = rdrd\theta. \tag{42}$$

Change domain

now the distance r goes from 0 to ∞ and the angle goes from 0 to 2π.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{0}^{2\pi}\int_{0}^{\infty} f(r,\theta)rdrd\theta \tag{43}$$

Gauss integral

...

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx \tag{44}$$

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-ax^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-ax^2} dx\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ax^2} e^{-ay^2} dxdy \ \textbf{(change the x-right by y)} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dxdy \ \textbf{(change to polar coordinates)} \\ &= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-ar^2} r dr d\theta \end{aligned} \tag{45}$$

...

$$\int_0^{2\pi}\int_0^{\infty} e^{-ar^2} r dr d\theta \tag{46}$$

$$\begin{aligned} u &= -ar^2 \\ du &= -2ardr \end{aligned} \tag{47}$$

if $r \to 0$ then $u \to 0$ and if $r \to \infty$ $u \to -\infty$.

$$\begin{aligned} \int_0^{2\pi}\int_0^{-\infty} -\frac{1}{2a} e^u du d\theta &= \int_0^{2\pi} \left(-\frac{1}{2a}\right) |_{u=0}^{u=\infty} d\theta \\ I^2 &= \int_0^{2\pi} \frac{1}{2a} d\theta = \frac{2\pi}{2a} \\ I &= \sqrt{\frac{\pi}{a}} \end{aligned} \tag{48}$$

mgf of z^2

Assume that $Z \sim N(0, \sigma^2)$ using [9]

...

$$
\begin{aligned}
M_{z^2}(t) = E(e^{tz}) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right) \\
= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{1}{2\sigma^2} - t\right) z^2\right) \\
= \frac{1}{\sqrt{2\pi\sigma}} \sqrt{\frac{\pi}{\frac{1}{2\sigma^2} - t}} \\
= \frac{1}{\sqrt{1 - t2\sigma^2}}, \quad t < \frac{1}{2\sigma^2}
\end{aligned} \tag{49}
$$

g.m.f $\sum Z_i^2$

...

$$
\begin{aligned}
M_{\sum Z_i}(t) &= E(e^{t\sum Z_i}) \\
&= E(\prod_i^n e^{tZ_i}) \\
&= \prod_i^n E(e^{tZ_i}) = \prod_i^n M^2_{Z_i}(t) \\
&= \prod \frac{1}{\sqrt{1-t2\sigma^2}} = \frac{1}{(1-t2\sigma^2)^{n/2}}
\end{aligned}
\tag{50}
$$

This will be, the g.m.f.

Gamma distribution

exponential, erlang and χ^2 are specific cases of gamma.

gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{51}$$

Gamma distribution

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}, \quad x > 0. \tag{52}$$

g.m.f gamma

...

$$E(e^{tx}) = \int_0^\infty \exp(tx)\exp\left(\frac{-x}{\beta}\right) x^{\alpha-1}dx \tag{53}$$

$$\frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty \exp\left(-x\left(\frac{1}{\beta}-t\right)\right) x^{\alpha-1}dx \tag{54}$$

note that is $gamma(\alpha, \frac{1}{\frac{1}{\beta}-t})$ and $\frac{1}{\frac{1}{\beta}-t} = \frac{\beta}{1-t\beta}$

$$\int_0^\infty \frac{1}{\Gamma(\alpha)\left(\frac{\beta}{1-t\beta}\right)^\alpha}\exp\left(-x\left(\frac{\beta}{1-t\beta}\right)\right) x^{\alpha-1}dx = 1 \tag{55}$$

...

$$M_x(t) = \frac{1}{(1-t\beta)^\alpha} \tag{56}$$