Calculus applied to microeconomics

Main insights

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Insights

The main objective of these notes is give important concepts in calculus and optimization using economic theory, to improve input data and have a better understanding about some data analysis techniques in some fields as finance, health and education.

Rate of change

Assume that we have a function $y = f(x)$, a value of x_0 that change to x_1 then $\Delta x = x_1 - x_0$, therefore the change of x could be expressed as $x_0 + \Delta x$, namely, the value of the function change from $f(x_0)$ to $f(x_0 + \Delta x)$.

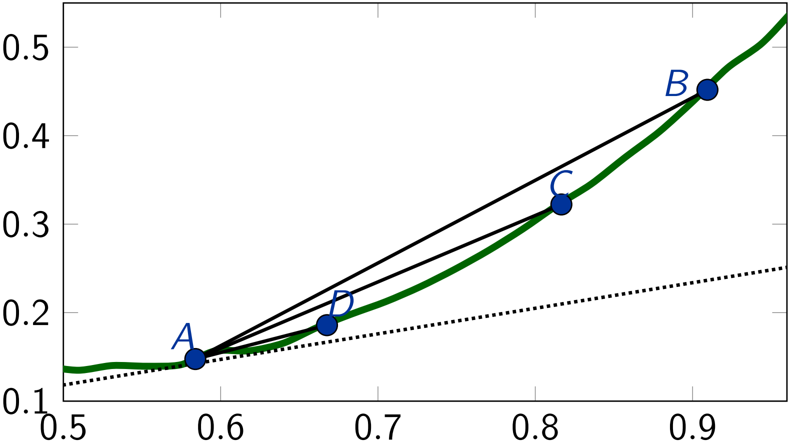
The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{1}$$

The mean rate of change; this mean that for instance if $\frac{\Delta y}{\Delta x} = \eta$.
On average y change in η unities by unitary change in x.

Geometric concept

Derivative



Definition

Derivate

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{df(x)}{dx} = f'(x). \tag{2}$$

The lagrange notation $f'(x)$, or leibniz notation $\frac{dy}{dx}$ can be used indiscriminately.

Intuition

if x increase by Δx then, $\Delta y \approx \frac{dy}{dx}\Delta x$.

Derivate of a constant

By definition $f(x) = \lambda, \forall x$.

$$\lim_{\Delta x \to \infty} \frac{f(x + \Delta x) - f(x))}{\Delta x} = \frac{\lambda - \lambda}{\Delta x} = 0 \tag{3}$$

Derivative of (λx)

$$f(x) = \lambda x \tag{4}$$

using definition,

$$\lim_{\Delta x \to \infty} \frac{\lambda(x + \Delta x) - \lambda x}{\Delta x} = \lambda. \tag{5}$$

Rewriting difference quotient

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{6}$$

Now replace Δx by $(x - x_0)$ and $\Delta x \to 0$ by $x \to x_0$.

Derivative of (x^{λ})

$$f(x) = x^{\lambda} \tag{7}$$

try first with $\frac{x^4 - x_0^4}{x - x_0}$ to find the following pattern:

$$\frac{x^{\lambda} - x_0^{\lambda}}{x - x_0} = x^{\lambda-1} + x_0 x^{\lambda-2} + x_0^2 x^{\lambda-3} + x_0^3 x^{\lambda-4} + ... + x_0^{\lambda-1} \tag{8}$$

to apply $\lim_{x \to x_0}(x^{\lambda-1} + x_0 x^{\lambda-2} + x_0^2 x^{\lambda-3} + x_0^3 x^{\lambda-4} + ... + x_0^{\lambda-1})$.
take in mind that $\lim_{x \to x_0} x = x_0$.

Derivative of x^{λ}

Remember that

$$x_0^k x_0^{\lambda-(k+1)} = x_0^{\lambda-1} \tag{9}$$

therefore:

$$\lim_{x \to x_0} (x^{\lambda-1} + x_0 x^{\lambda-2} + x_0^2 x^{\lambda-3} + x_0^3 x^{\lambda-4} + ... + x_0^{\lambda-1}) \tag{10}$$

$$x_0^{\lambda-1} + x_0^{\lambda-1} + ... + x_0^{\lambda-1} = \lambda x_0^{\lambda-1} \tag{11}$$

therefore;

$$\frac{dx^{\lambda}}{dx} = \lambda x^{\lambda-1} \tag{12}$$

Partial derivative

In this case the variable Y is related with x, y variables:

$$Z = f(x, y) \tag{13}$$

Now the the symbol is ∂ but the logic is the same,buy now we only change one variable while holding the others constants.

$$\begin{aligned} \frac{\partial Z}{\partial x} &= f_x \\ \frac{\partial Z}{\partial y} &= f_y \end{aligned} \tag{14}$$

Production Function

The production process could be encapsulated in:

$$Y = AK^{\alpha}L^{\beta} \tag{15}$$

Where A represent technology, K capital and L labor.

Applying

$$PM_k = \frac{\partial Y}{\partial K} \tag{16}$$

The changes in production by changes in capital is also denominated as marginal productivity of capital.

$$PML_K = \alpha k^{\alpha-1} L^{\beta} \tag{17}$$

The same derivation is applied to get the marginal productivity of labor.

Properties

suppose two functions:

$$[f(x) + g(x)]' = f'(x) + g'(x) \tag{18}$$

Derivative of a sum is the sum of derivatives.

Profit

$$\pi(q) = IT(q) - CT(q) \tag{19}$$

We could be interested in:

$$\frac{d\pi(q)}{dq} \tag{20}$$

Now applying the last property we have:

$$\frac{d\pi(q)}{dq} = IT'(q) - CT'(q) \tag{21}$$

Marginalism

$IT(q)$ and $CT(q)$ allow us known what is the increase of income and cost when the production increase in one unity, henceforth called marginal income and marginal cost respectively.

$$\frac{dCT(q)}{dq} = CT'(q) = CM. \tag{22}$$

$$\frac{dIT(q)}{dq} = IT'(q) = IM. \tag{23}$$

Optimize

From a economic perspective we have that the if the income perceived by unit of one product is greater than the cost of production.

$$IM > CM \tag{24}$$

the firm could produce more, otherwise;

$$CM > IM \tag{25}$$

Therefore the condition:

$$IM = CM \tag{26}$$

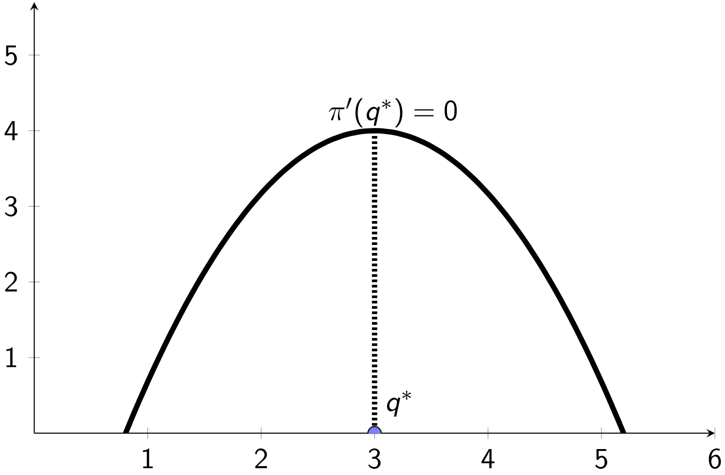
that is derived from:

$$\frac{d\pi(q)}{dq} = IM - CM = 0. \tag{27}$$

We can illustrate finding the maximum distance between the curves of total income and total costs.

Geometric

First condition



First condition

For simplicity, henceforth, we are going to assume that the first condition is enough to optimize the functions, therefore to optimize $f(x)$ (minimize or maximize) then:

$$f'(x^*) = 0 \tag{28}$$

for instance, we have the following function $f(x) = ax^2 + bx + c$.

$$\frac{df(x)}{dx} = 2ax + b = 0$$
$$x^* = \frac{-b}{2a}$$

Inverse function

In the previous slides we can see that y is determined by values of x now, we are interested in get the values of x given y. for $y = \Theta x + \beta$ its inverse function will be:

$$x = \frac{y - \beta}{\Theta}. \tag{29}$$

generally, for $f()$ its inverse will be denoted by $f^{-1}()$.
Namely, $y = f(x)$ and $x = f^{-1}(y)$.

Linear demand

$$q = \frac{\theta}{\beta} - \frac{p}{\beta} \tag{30}$$

get the price from:

$$p = \theta - \beta q \tag{31}$$

Now total income is:

$$IT = pq = \theta q - \beta q^2 \tag{32}$$

The marginal income is:

$$IM = \theta - 2\beta q \tag{33}$$

$IM < P$

$$\begin{aligned} \theta - 2\beta q &< \theta - \beta q \\ -2\beta q &< \beta q \\ -2 &< 1. \end{aligned} \tag{34}$$

This have a important relation.

Optimal production

Linear demand

Previously we defined that the $IM = CM$ in an optimal condition, therefore for linear demand, assuming that CM is fixed by a particular firm then:

$$q^* = \frac{\theta - CM}{2\beta}. \tag{35}$$

Insights Monopoly

In perfect competition $\frac{dp}{dq} = 0$ otherwise $\frac{dp}{dq} < 0$. The demand curve of monopoly is equal to the market.

Rule of product

Suppose that we have the following relation $T(x) = g(x)f(x)$ then;

$$\frac{dT(x)}{dx} = g'(x)f(x) + g(x)f'(x) \tag{36}$$

for instance, $g(x) = 2x$ and $f(x) = x^2$ see that $g'(x) = 2$ and $f'(x) = 2x$ therefore $[g(x)f(x)]' = 2x^2 + 4x^2$.

Marginal income

How much we can increase the price of a product?.
Income is equal to $IT = p(q)q$

$$IM = \frac{dIT}{dq} = \frac{dp}{dq}q + p \tag{37}$$

Note that in perfect competition $\frac{dp}{dq} = 0$ and therefore the marginal income is equal to the price, therefore price is exogenous.

Percentage change

if x change to Δx therefore we have that the percentage change is defined as $\frac{\Delta x}{x}$, in a numerical example, $x = 10$ and $\Delta x = 5$ therefore $\%\Delta = 0.5$.

Derivative of logarithm

This definition is obtained using the difference quotient and chain rule(explained later).

$$\frac{d ln(g(x))}{dx} = \frac{g'(x)}{g(x)} \tag{38}$$

Logarithm (proportional change)

Assume that $y = ln(x)$:

$$\frac{dln(x)}{dx} = \frac{1}{x} \tag{39}$$

remember that $\Delta y = \frac{dy}{dx}\Delta x$ then:

$$\Delta ln(x) \approx \frac{1}{x}\Delta x \tag{40}$$

This if x increase η percent then the logarithm have a absolute change of η in other words if x increase η percent then y increase in η.

Elasticity

Dollars, euros, and pesos are units of measure.

$$\epsilon = \frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}} = \frac{dln(q)}{dln(p)}$$

(41)

It is important in data analysis due not have units and meaning a increase of one percent in p increase q in ϵ percent.

Elasticity of the product

$Y = K^{\alpha}L^{\beta}$ applying a logarithm transformation we have:

$$ln(Y) = \alpha ln(K) + \beta ln(L) \tag{42}$$

$$\frac{dln(Y)}{dln(K)} = \alpha \tag{43}$$

This means on increase of one percent in the capital will be in crease the production in α percent.

Chain rule

It is also a important concept, suppose that $T(x) = f(g(x))$ and now we are interested in study $T'(x)$.

$$\frac{dT(x)}{dx} = f'(g(x))g'(x) \tag{44}$$

for instance; $z = y^\lambda$ and $y = x^2 + \beta$ we are interested in $\frac{dz}{dx}$:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} \tag{45}$$

$$\frac{dz}{dx} = \lambda y^{\lambda-1} 2x = (x^2 + \beta)2x. \tag{46}$$

Derivative of inverse function

Remember that $f^{-1}(f(x)) = x$.

$$\frac{df^{-1}(f(x))}{dx} = \frac{dx}{dx} \tag{47}$$

$$f^{-1\prime}(f(x))f'(x) = 1 \tag{48}$$

$$f^{-1\prime}(f(x)) = \frac{1}{f'(x)} \tag{49}$$

This is a practical implication, due that the derivative of an inverse function.

Marginal income

We defined previously that

$$IM = \frac{dIT}{dq} = \frac{dp}{dq}q + p \tag{50}$$

thus:

$$IM = p(\frac{dp}{dq}\frac{q}{p} + 1) \tag{51}$$

remember that $\frac{dp}{dq}\frac{q}{p}$ is the elasticity of the inverse demand function.

Elasticity and Marginal income

$$IM = p(\frac{1}{\epsilon} + 1) \tag{52}$$

we hope that $\epsilon < 0$ due a increase in price in **normal** goods, notice that the marginal income in perfect competition was equal to p therefore $\epsilon \to \infty$. this give us idea about the importance.

Marginal income

In theory, we can think that $\frac{dq}{dp} < 0$ we can rewrite marginal income as:

$$IM = p(1 - \frac{1}{|\epsilon|}) \tag{53}$$

Note that when $\epsilon = 1$ then $IM = 0$.
Analyze $|\epsilon|$ greater and lesser than 1.

Power of market

$$\pi(q) = I(q) - C(q) \tag{54}$$

the first order condition give us to maximize the profit $IM = CM$. there are a relationship among the elasticity of demand and power of market.

$$\begin{aligned} CM &= p(\frac{1}{\epsilon} + 1) \\ \frac{CM - p}{p} &= \frac{1}{\epsilon} \end{aligned} \tag{55}$$

the power of the monopoly to set up a price higher of its marginal cost.

Insights Cournot

Cournot(1838) each firm maximize the profit, the question is: **How many q_i unities of a product could produce i according to the q_j unities of j?** .
Each firm known that $\frac{\partial P}{\partial q_i} \neq 0$, but $\frac{\partial q_j}{\partial q_i} = 0$.

Cournot

n -firms

Assume a linear demand:

$$p = \Theta - \beta Q \tag{56}$$

take in mind that $Q = \sum q_i$ (by two firms $Q = q_1 + q_2$).
Now the profit of $i - th$ firm is:

$$\pi_i = pq_i - cq_i \tag{57}$$

note here that, c is a constant and there $MC = c$.

$$\pi_i = (\Theta - \beta Q)q_i - cq_i \tag{58}$$

First condition

$$\pi_i = \Theta q_i - \beta q_i (\sum_j q_j) - c q_i \tag{59}$$

$$\frac{\partial \pi_i}{d q_i} = 0 \tag{60}$$

$$\Theta - \beta(\sum_{j \neq i} q_j)) - 2\beta q_i - c \tag{61}$$

thus the strategy is

$$q_i^* = \frac{\Theta - c}{2\beta} - \frac{\sum_{j \neq i} q_j}{2} \tag{62}$$

solution

2-firms

$$\frac{\Theta - c}{2\beta} = \frac{\sum_{j \neq i} q_j}{2} + q_i \tag{63}$$

with $n = 2$ we can express the system in the following way:

$$\begin{aligned} \frac{\Theta - c}{2\beta} &= q_1 + \frac{q_2}{2} \\ \frac{\Theta - c}{2\beta} &= \frac{q_1}{2} + q_2 \end{aligned} \tag{64}$$

therefore

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{\Theta - c}{2\beta} \\ \frac{\Theta - c}{2\beta} \end{bmatrix}$$

Python implementation

Cournot model

```
import numpy as np
import matplotlib.pyplot as plt
A = np.array([[1,1/2],[1/2,1]])
def solution(A,theta, cost, beta):
  Ai = np.linalg.inv(A)
  cons = (theta - cost) / (2 * beta)
  mat = np.array([[cons],[cons]])
  solv = Ai @ mat
  return solv
solution(A, theta=100, cost=1, beta=1)
qi = [solution(A, 100, x, 3)[0][0] for x in range(100)]
plt.plot(list(range(100)),qi)
```

Nicholson example

Solve the example of the book and assume that $CM = 0$, and $Q = q_1 + q_2 = 120 - p$. See the solution (click here).

Homework

Find the solution of 6 firms that face a linear function of demand.

References

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