Chapter 4: Numerical methods for IVP (summary)

February 7, 2025

Goal: Present basic numerical methods for the IVP

$$\begin{cases} \dot{y}(t) = f(y(t)) \quad \text{for} \quad t \in (0, T] \\ y(0) = y_0. \end{cases}$$

Here, $T > 0, y_0 \in \mathbb{R}, f\colon \mathbb{R} \to \mathbb{R}$ are given and $\dot{y}(t) = \frac{\mathrm{d}}{\mathrm{d}t} y(t)$. Observe that what is presented below can be adapted to the situation, where $f(t, y)$ and $y(t_0)$ or for vector-valued problems.

- The variation of constant (voc) formula for the IVP

$$\begin{cases} \dot{u}(t) + a(t)u(t) = f(t) \quad \text{for} \quad t \in (0, T] \\ u(0) = u_0, \end{cases}$$

  where f, a, u_0 are given (for instance f and a are continuous and $a(t) \geq 0$ is bounded), reads

$$u(t) = u_0 \mathrm{e}^{-A(t)} + \int_0^t \mathrm{e}^{-(A(t)-A(s))} f(s) \, \mathrm{d}s,$$

  where $A(t) = \int_0^t a(s) \, \mathrm{d}s$.

  The above formula can be extended to second order problems, systems of IVP, PDEs (see later in the lecture), or nonlinear problems.

  The variation of constant formula is used, for instance, to show stability results for the exact solutions of DEs (see later in the lecture).

- Consider the IVP

$$\begin{cases} \dot{y}(t) = f(y(t)) \quad \text{for} \quad t \in (0, T] \\ y(0) = y_0. \end{cases}$$

  Let $N \in \mathbb{N}$ and define the time step $k = \frac{T}{N}$ as well as the time grid $0 = t_0 < t_1 < \ldots < t_N = T$, where $t_n = nk$ for $n = 0, 1, \ldots, N$.

  We define the following time integrators for the above IVP (starting with $y_0 = y(0)$):

  The (forward/explicit) Euler scheme

$$y_{n+1} = y_n + kf(y_n).$$

  The backward/implicit Euler scheme

$$y_{n+1} = y_n + kf(y_{n+1}).$$

  The Crank–Nicolson scheme

$$y_{n+1} = y_n + \frac{k}{2}\big(f(y_n) + f(y_{n+1})\big).$$

  These provide numerical approximations $y_n \approx y(t_n)$ to the exact solution of the IVP on the time grid $(t_n)_{n=0}^N$.

Further resources:

- In the film Hidden Figures, Katherine Goble resorts to the Euler method in calculating the re-entry of astronaut John Glenn from Earth orbit link
- FD and Euler at ocw.mit.edu
- Euler at math.lamar.edu
- Euler at calcworkshop.com
- Euler at intmath.com

Applications: IVP are everywhere in sciences and engineering. Since many IVP cannot be solved exactly, numerical methods such as Euler's numerical schemes, have several applications in, for instance, physics, chemistry, biology, or economics.