Examination, 29 August 2024
TMA373 and MMG801

Read this before you start!
I'll try to come at ca. 09:15 and 11:15. You can ask for calling me (0317723021) in case of questions.
Aid: Chalmers approved calculators.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Use a proper pen and order your answers if possible. ***Thank you!***
No need to use one piece of paper for only one exercise.
The test has 4 pages and a total of 50 points.
Preliminary grading limits: 3:20-29p, 4:30-39p, 5:40-50p (Chalmers) and G:20-34p, VG:35-50p (GU). You will be informed via Canvas when the exams are corrected.
Good luck!

Some exercises were taken from, or inspired by, materials from *M. Bader*, *F. Boyer*, and *M.G. Larson*.

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1. Provide concise answers to these short questions:

   (a) Give a concrete example of a parabolic PDE. Justify. (2p)

   (b) What is span$\{1, x\}$ (that is, which space do you get?)? What is the dimension of this space? Give a basis for this space. (3p)

   (c) Give an application of the Lagrange polynomials. (1p)

   (d) Give the definition of the midpoint rule to approximate $\int_a^b f(x)\,\mathrm{d}x$. (1p)

   (e) Choose your favorite IVP and apply one step of the explicit/forward Euler method to it. Define all quantities. (2p)

   (f) Give a possible application of a posteriori error estimates. (1p)

   (g) Why is the Crank–Nicolson scheme of interest when applied to the linear homogeneous wave equation? (1p)

   (h) Let K be a triangle. Define the space of linear functions on K denoted by $P_1(K)$. Give an application of this space. (2p)

   (i) Describe the main steps for the assembling procedure of the (mass or stiffness) matrix for the FEM algorithm for $2d$ problems seen in the lecture (you do not have to carry out any computations, just illustrate the main idea). (2p)

(j) Provide Dahlquist's test equation and explain why it is of interest (in the context of this course). (2p)

2. Define the strong form and the minimization problem of Poisson's equation in $1d$. Are these equivalent to the corresponding variational formulation? (3p)

3. Let $f \in L^2(0,1)$ and $k\colon (0,1) \to \mathbb{R}$ continuous, bounded, positive with $\alpha = \inf_{(0,1)} k > 0$. Consider the BVP
$$\begin{cases} -(k(x)u(x)')' = f(x) \quad \text{for } x \in (0,1) \\ u(0) = 0, u(1) = 0. \end{cases}$$

(a) Give the variational formulation of this BVP. Don't forget to define the test and trial spaces as well as the forms $a(u,v)$ and $\ell(v)$. (3p)

(b) Verify that all the assumptions of the Lax–Milgram theorem are satisfied for this problem. What can you conclude from this? (4p)

4. Let H be a Hilbert space and consider the variational problem
$$\text{Find } u \in H \text{ such that } a(u,v) = \ell(v) \text{ for all } v \in H,$$
where $a(\cdot,\cdot)$ is a symmetric bilinear form on H and $\ell(\cdot)$ a linear form. Assume that the conditions of Lax–Milgram are satisfied and define the a-inner product $\langle u,v\rangle_a = a(u,v)$ for all $u,v \in H$. Take now a finite dimensional subspace $V \subset H$ and consider the Galerkin variational problem
$$\text{Find } \hat{u} \in V \text{ such that } a(\hat{u},\hat{v}) = \ell(\hat{v}) \text{ for all } \hat{v} \in V.$$

(a) Show that the error $u - \hat{u}$ is a-orthogonal to the subspace V. (2p)

5. Consider the boundary value problem
$$\begin{cases} -u''(x) = 0 \quad \text{for} \quad x \in (0,2) \\ u'(0) = 1, \quad u(2) = 0. \end{cases}$$

(a) Integrate the problem twice and give its exact solution. (2p)

(b) State the variational formulation to the above problem. Define all quantities. (2p)

(c) Give the Galerkin (linear) FE problem in the case of one element of length $h = 2$. Define all quantities. (2p)

(d) Finally solve the obtained linear system of equation and provide the FE solution
$$u_h(x) = \zeta_0\varphi_0(x),$$
where one recalls that $\varphi_0(x) = 1 - x/2$ and ζ_0 is the unknown. (2p)

6. Let u_0 and f be given functions. Assume that there are nice solutions, denoted $u_1(x,t)$ and $u_2(x,t)$, to the heat equation

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0 < x < 1, 0 < t \leq T \\ u(0,t) = 0, u(1,t) = 0 & 0 < t \leq T \\ u(x,0) = u_0(x) & 0 < x < 1. \end{cases}$$

Define $w = u_1 - u_2$.

(a) Provide a linear partial differential equation (with initial and boundary conditions) for which w is a solution to. (1p)

(b) Use the stability estimates (seen in the lecture)

$$||w(\cdot,t)||_{L^2} \leq ||\text{initial value of above PDE}||_{L^2} + \int_0^t \left\| \text{right-hand side of above PDE} \right\|_{L^2} \,\mathrm{d}s$$

to show that $||w(\cdot,t)||_{L^2} = 0$ for all t. (1p)

7. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and consider the problem: Find $u = u(x,y)$ such that

$$\begin{cases} -\nabla \cdot (a\nabla u) + bu = f & \text{in } \Omega \\ -n \cdot (a\nabla u) = \gamma(u - g) & \text{on } \partial\Omega, \end{cases}$$

where $a = a(x,y) \geq \alpha > 0$, $b = b(x,y) \geq 0$, $f = f(x,y)$, $g = g(x,y)$, and $\gamma \geq 0$ are given. Here, $n = n(x,y)$ denotes the outward unit normal to Ω.

(a) Derive the variational formulation to the above problem. (1p)
Hint: Remember Green's formula from the lecture

$$\int_\Omega \Delta u v \, dxdy = \int_{\partial\Omega} (n \cdot \nabla u) v \, ds - \int_\Omega \nabla u \cdot \nabla v \, dxdy.$$

(b) Provide the FE problem that comes from a $cG(1)$ discretisation of this variational formulation. (1p)

(c) Derive the linear system of equations

$$(A + B + M)\,\zeta = \Gamma + R$$

resulting from the FE problem. Here, A, B, M are $(N_p + 1) \times (N_p + 1)$ matrices and Γ, R are $(N_p + 1) \times 1$ vectors, where one uses the basis functions $\{\varphi_j\}_{j=0}^{N_p}$ for the FE space. (2p)

(d) Consider now the above problem with $a = 1, b = 0, \gamma = 0$ and $\Omega = [0,6] \times [0,2]$ with the triangulation from Figure 1. Compute the entry $A_{1,2}$ of the (global) stiffness matrix. (3p)
Hint: The basis functions on a triangle are of the form $a_0 + a_1 x + a_2 y$ for some real numbers a_0, a_1, a_2. Observe that you can do this part of the exercise even if you have not done the previous parts.

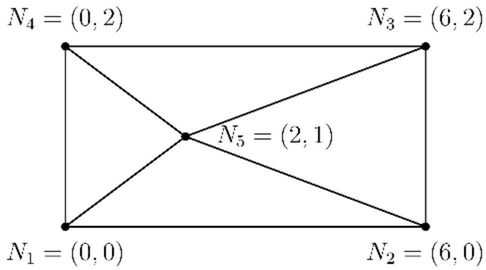

Figure 1: Courtesy from M.G. Larson.

8. Provide the nodal basis function $\Phi_1(x, y), \Phi_2(x, y), \Phi_3(x, y)$ for the finite element $\left(K, P^{(1)}(K), \Sigma\right)$, where K is the reference triangle with vertices $\{N_1, N_2, N_3\} = \{(0, 0), (1, 0), (0, 1)\}$, $P^{(1)}(K)$ denotes the set of polynomials of degree less or equal to one and $\Sigma = (L_1, L_2, L_3)$ is given by

$$L_1(f) = f(N_1), \quad L_2(f) = f(N_2), \quad L_3(f) = f(N_3). \qquad \text{(4p)}$$