David Cohen (david.cohen@chalmers.se) Chalmers & GU

Chapter 8: On our way to FEM in 2d (summary)

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Goal: Extend integration by parts, piecewise linear function, linear interpolation in $2d$. Prepare for FE discretisation of PDEs in higher dimensions.

- Green's formula can be seen as a generalisation of integration by parts in $2d$ (or higher). Under some technical assumptions, one has

$$\int_\Omega \Delta u v \,\mathrm{d}x = \int_{\partial\Omega} (n\cdot\nabla u) v \,\mathrm{d}s - \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x,$$

  where $n = n(x) = n(x_1, x_2)$ is the outward unit normal vector of the boundary at a point $x = (x_1, x_2) \in \partial\Omega$, the first and last integrals are double integral on $\Omega \subset \mathbb{R}^2$, while the second integral is a line integral, the dot $\cdot$ stands for the dot product/scalar product between two vectors.

- An application of Green's formula can be used to derive the variational formulation to Poisson's equation on a nice domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

  where f is a given (nice enough) function.

  The variational formulation of the above PDE reads

$$\text{Find} \quad u \in H_0^1(\Omega) \quad \text{such that} \quad (\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

- Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\partial\Omega$ (or smooth boundary). A triangulation or mesh T_h of Ω is a set $\{K\}$ of triangles K such that $\Omega = \bigcup_{K\in T_h} K$ and the intersection of two triangles is either empty, a corner, or an edge.

  The corner of the triangles are called the nodes and will be denoted by N_j below. The local mesh size of a triangle K is denoted by h_K and is the length of the longest edge of the triangle K. The global mesh size is denoted by $h = \max_{K\in T_h} h_K$.

  Any polygon can be triangulated thanks to the fan triangulation for example. Else, one may need to use a mesh generator.

  All the triangle seen in the lecture will be regular (i. e. nice enough to do what we need to do).

- For a triangle K, one defines

$$P^{(1)}(K) = \{v\colon K \to \mathbb{R}\colon\ v(x_1, x_2) = c_0 + c_1 x_1 + c_2 x_2, \quad (x_1, x_2) \in K, \quad c_0, c_1, c_2 \in \mathbb{R}\}$$

  the space of linear functions on K. Observe that any function $v \in P^{(1)}(K)$ is uniquely determined by its nodal values.

  A nodal basis, for the above space, on the reference triangle with nodes/vertex $(0,0)$, $(1,0)$ and (0.1) consists of the following three functions

$$\lambda_1(x_1, x_2) = 1 - x_1 - x_2, \quad \lambda_2(x_1, x_2) = x_1, \quad \lambda_3(x_1, x_2) = x_2.$$

- Let $T_h = \{K\}$ be a triangulation of a bounded domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary. The space of continuous piecewise linear polynomials is defined by
$$V_h = \{v \in C^0(\Omega):\ v|_K \in P^{(1)}(K) \quad \forall K \in T_h\}.$$
Again, any function $v \in V_h$ can be written as
$$v = \sum_{j=1}^{n_p} \alpha_j \varphi_j,$$
where n_p denotes the number of nodes in the triangulation T_h, $\{\varphi_j\}_{j=1}^{n_p}$ are hat functions, and $\alpha_j = v(N_j)$, for $j = 1,\dots,n_p$, are the nodal values of v.

- Consider a continuous function f on a triangle K with nodes N_j, $j = 1,2,3$. The linear interpolant of f, denoted $\pi_1 f \in P^{(1)}(K)$, is defined by
$$\pi_1 f = \sum_{j=1}^{3} f(N_j)\varphi_j.$$
One has the following interpolation errors
$$\begin{aligned}\|\pi_1 f - f\|_{L^2(K)} &\le C_K h_K^2 \|f\|_{H^2(K)}\\ \|\nabla(\pi_1 f - f)\|_{L^2(K)} &\le C_K h_K \|f\|_{H^2(K)}\end{aligned}$$
for any $f \in H^2(K)$.

- For a continuous function $f\colon \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with a triangulation T_h, one defines the continuous piecewise linear interpolant of f by
$$\pi_h f = \sum_{j=1}^{n_p} f(N_j)\varphi_j.$$
Observe that $\pi_h f \in V_h$.

  For the errors of the pw linear interpolant, one has
$$\begin{aligned}\|\pi_h f - f\|^2_{L^2(\Omega)} &\le C \sum_{K\in T_h} h_K^4 \|f\|^2_{H^2(K)}\\ \|\nabla(\pi_h f - f)\|^2_{L^2(\Omega)} &\le C \sum_{K\in T_h} h_K^2 \|f\|^2_{H^2(K)}\end{aligned}$$
for any $f \in H^2(K)$.

  A further result is: Let an integer $r \ge 2$ and $f \in H^r(\Omega)$. The errors of a piecewise polynomial of degree $r-1$ reads
$$\|\pi_{r,h} f - f\|_{L^2(\Omega)} \le C h^r \|f\|_{H^r(\Omega)}.$$

- Let $\Omega \subset \mathbb{R}^d$ be a nice domain with a triangulation and $f \in L^2(\Omega)$. The L^2-projection $P_h f \in V_h$ of f is defined by
$$\int_\Omega (f - P_h f)\, v \,\mathrm{d}x = 0 \quad \text{for all} \quad v \in V_h.$$
The L^2-projection is the best approximation of f in V_h in the L^2-norm. Furthermore, the error of the L^2-projection is
$$\|P_h f - f\|_{L^2(\Omega)} \le C h^2 \|f\|_{H^2(\Omega)}$$
for all $f \in H^2(\Omega)$.

Further resources:

- Application of Poisson eq. at wikipedia
- Application of Poisson eq. at makmanx.github.io
- Finite element in $2d$ and $3d$ at github.io
- FEM part 1 at what-when-how.com

Applications: Poisson's equation can be used to calculate the electrostatic or gravitational (force) field. It is also used to perform surface reconstruction, see the links above for details.