

Chapter 11: FEM for wave equations in higher dimensions (summary)

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Goal: Study the exact solution to the wave equation (conservation of energy), derive a FE discretisation for this PDE and provide a semi-discrete error estimate for the FEM.

- Let $\Omega \subset \mathbb{R}^d$ be a nice domain with smooth or polygonal boundary. Consider the **homogeneous wave equation**

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \times \mathbb{R}_+ \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ u(\cdot, 0) = u_0 & \text{in } \Omega \\ u_t(\cdot, 0) = v_0 & \text{in } \Omega. \end{cases}$$

One has **conservation of energy** for solutions to the above PDE:

$$\frac{1}{2} \left(\|u_t(\cdot, t)\|_{L^2(\Omega)}^2 + \|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 \right) = \frac{1}{2} \left(\|v_0\|_{L^2(\Omega)}^2 + \|\nabla u_0\|_{L^2(\Omega)}^2 \right) = \text{Const} \quad \forall t > 0.$$

- The **variational formulation of the inhomogeneous wave equation**

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega \times [0, T] \\ u = 0 & \text{on } \partial\Omega \times [0, T] \\ u(\cdot, 0) = u_0 & \text{in } \Omega \\ u_t(\cdot, 0) = v_0 & \text{in } \Omega. \end{cases}$$

reads: For each $0 < t \leq T$ of interest,

Find $u(\cdot, t) \in H_0^1(\Omega)$ such that $(u_{tt}(\cdot, t), v)_{L^2(\Omega)} + (\nabla u(\cdot, t), \nabla v)_{L^2(\Omega)} = (f(\cdot, t), v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$ and $u(\cdot, 0) = u_0, u_t(\cdot, 0) = v_0$ in Ω for the initial values.

- Let now T_h denotes a mesh of Ω and V_h the space of continuous piecewise linear functions of T_h . Consider the space $V_h^0 = V_h^0(\Omega) = \{v: \Omega \rightarrow \mathbb{R} : v \text{ continuous pw linear on } T_h \text{ and } v = 0 \text{ on } \partial\Omega\}$ and observe that $V_h^0 = \text{span}(\{\varphi_j\}_{j=1}^{n_i})$, where n_i denotes the number of interior nodes. The **finite element problem** for the above wave equation reads: For each $0 < t \leq T$ of interest,

Find $u_h(\cdot, t) \in V_h^0(\Omega)$ such that $(u_{h,tt}(\cdot, t), v_h)_{L^2(\Omega)} + (\nabla u_h(\cdot, t), \nabla v_h)_{L^2(\Omega)} = (f(\cdot, t), v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h^0(\Omega)$ and $u_h(x, 0) = \pi_h u_0(x), u_{h,t}(x, 0) = \pi_h v_0(x)$ in Ω for the initial values.

As always, writing $u_h(x, t) = \sum_{j=1}^{n_i} \zeta_j(t) \varphi_j(x)$ and taking $v_h = \varphi_i$ in the FE problem gives the system of linear ODEs

$$\begin{cases} M\ddot{\zeta}(t) + S\zeta(t) = F(t) \\ \zeta(0) = \zeta_0, \dot{\zeta}(0) = \eta_0. \end{cases}$$

One has the following **a priori error estimate for the FE approximation of the inhomogeneous wave equation**

$$\|u_h(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq C \left(\|u_h(\cdot, 0) - R_h u_0\|_1 + \|u_{h,t}(\cdot, 0) - v_0\|_{L^2(\Omega)} \right) + Ch^2 \left(\|u(t)\|_{H^2(\Omega)} + \int_0^t \|u_{tt}(\cdot, s)\|_{H^2(\Omega)} ds \right),$$

where we recall that R_h denotes **Ritz projection**.

Applications: The wave equation can be used as a simple model for vibrating membranes and plates.