Examination, 24 August 2023
TMA373 and MMG801

Read this before you start!
I'll try to come at ca. 09:30. You can ask for calling me (0317723021) in case of questions.
Aid: Chalmers approved calculators.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Use a proper pen and order your answers if possible. ***Thank you!***
No need to use one piece of paper for only one exercise.
The test has 4 pages and a total of 50 points.
Preliminary grading limits: 3:20-29p, 4:30-39p, 5:40-50p (Chalmers) and G:20-34p, VG:35-50p (GU). You will be informed via Canvas when the exams are corrected.
Good luck!

Some exercises were taken from, or inspired by, materials from *C. Cotter* and *J. Hoffman*.

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1. Provide concise answers to these short questions:

   (a) Give a simple example (it does not need to be realistic) of an initial value problem. For a given time step h, apply one step of the explicit/forward Euler method to your IVP. Define all quantities. (2 p)

   (b) Is the PDE $\partial_x \partial_y u - \partial_x u = 0$, with $u = u(x, y)$, elliptic, parabolic or hyperbolic? (1 p)

   (c) Is the minimization problem of Poisson's equation in $1d$ equivalent to its variational formulation? (1 p)

   (d) Which function lives in the trial space in the variational form of a PDE? (1 p)

   (e) Use (one step of) the trapezoidal rule to approximate the area under the function $f(x) = x^2$ between $x = 1$ and $x = 3$. (1 p)

   (f) What is an estimate for the a priori error estimate, measured in the energy norm, of the (continuous and piecewise linear) FEM with mesh size h for the model problem (Poisson's equation in $1d$ with homogeneous Dirichlet BC) seen in the lecture? (2 p)

   (g) Describe the main steps for the assembling procedure of the (mass or stiffness) matrix for the FEM algorithm for $2d$ problems seen in the lecture (you do not have to carry out any computations, just illustrate the main idea). (2 p)

(h) Formulate the Lax–Milgram theorem for the problem: Find $u \in H$ such that $a(u,v) = \ell(v)$ for all $v \in H$, where H is some Hilbert space. (3 p)

2. Let $I = (0,1)$ and a function $c \in L^\infty(I)$ such that

$$c(x) \geq c_0 \quad \forall x \in I,$$

for some $c_0 \in \mathbb{R}$. On the Sobolev space $H_0^1(I)$, we define the bilinear form

$$a(u,v) = \int_I u'(x)v'(x)\,\mathrm{d}x + \int_I c(x)u(x)v(x)\,\mathrm{d}x.$$

(a) Show that a is continuous on $H_0^1(I)$, that is $|a(u,v)| \leq c\|u\|_{H^1(I)}\|v\|_{H^1(I)}$ for all $u,v \in H_0^1(I)$. (2 p)

(b) Use the fact that, for $v \in H_0^1(I)$, $\|v\|_{L^2(I)} \leq \frac{1}{2}\|v'\|_{L^2(I)}$ to show that a is coercive on $H_0^1(I)$ if $c_0 > -4$. (4 p)

3. Let $a < b$ and $p,q \colon (a,b) \to \mathbb{R}$ two piecewise continuous functions with $0 < p_* \leq p(x) \leq p^* < \infty$ and $0 < q_* \leq q(x) \leq q^* < \infty$ for all $x \in (a,b)$. Let $\beta \geq 0$ and $f \in L^2(a,b)$. Consider the BVP

$$\begin{cases} -(p(x)u'(x))' + q(x)u(x) = f(x) \\ p(b)u'(b) + \beta u(b) = 0 \\ -p(a)u'(a) + \beta u(a) = 0. \end{cases}$$

(a) Write down the variational formulation of the above BVP. (1 p)
Hint: Consider trial and test functions in $H^1(a,b)$.

(b) We now consider a cG(1) FE approximation of this problem. Let N be a positive integer and define the mesh $h = \frac{b-a}{N}$ as well as the uniform grid $a = x_0 < x_1 < \ldots < b = x_N$ with $x_j = a + jh$ for $j = 0,1,\ldots,N$. Define the corresponding FE space V_h and provide a basis for this space. (2 p)

(c) Write the approximation (cG(1)) solution $u_h(x)$ in terms of elements of the above basis (define all quantities). (1 p)

(d) From the above FE formulation, derive a linear system of equations and give the entries of the matrix corresponding to the term $q(x)u(x)$ in the above BVP. (4 p)

(e) Assume now that $u \in H^2(a,b)$ and denote by u_h the cG(1) approximation. Denote by $\pi_h v$ the piecewise linear interpolant of some $v \in H^1(a,b)$. Use Galerkin's orthogonality to show that

$$A(\pi_h u - u, v) = A(\pi_h u - u_h, v) \quad \text{for all} \quad v \in V_h. \qquad (1\text{ p})$$

Here, the continuous (bounded) and coercive bilinear form $A(\cdot,\cdot)$ is defined as

$$A(u,v) = \int_a^b p(x)u'(x)v'(x)\,\mathrm{d}x + \int_a^b q(x)u(x)v(x)\,\mathrm{d}x + \beta u(b)v(b) + \beta u(a)v(a).$$

Hint: First write down Galerkin's orthogonality for the bilinear form $A(\cdot,\cdot)$.

(f) Take $v = \pi_h u - u_h \in V_h$ in the above equality and use properties (seen above) of the bilinear form $A(\cdot,\cdot)$ to prove that

$$\|\pi_h u - u_h\|_{H^1} \leq C_1 \|\pi_h u - u\|_{H^1} .$$

Using the above, deduce then that

$$\|u - u_h\|_{H^1} \leq C_2 \|\pi_h u - u\|_{H^1} . \qquad (2 \text{ p})$$

(g) Finally, recall that the interpolation error verifies $\|\pi_h u - u\|_{H^1} \leq C_3 h$, deduce the error estimate for the cG(1) FEM

$$\|u - u_h\|_{H^1} \leq Ch. \qquad (1 \text{ p})$$

4. Let $\Omega \subset \mathbb{R}^2$ be a nice domain, $f, u_0 \colon \Omega \to \mathbb{R}$ be nice, $T > 0$. Consider the inhomogeneous heat equation

$$\begin{cases} u_t - \Delta u + u = f \quad \text{in } \Omega \times (0,T] \\ u = 0 \quad \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) \quad \text{in } \Omega. \end{cases}$$

(a) Write down the variational formulation of the above PDE (define all quantities). (2 p)

(b) Formulate the finite element problem based on a cG(1) approximation on the space V_h^0 (don't forget to define this space). (2 p)

(c) From the above, derive the linear system of ordinary differential equations

$$M\dot{\zeta}(t) + S\zeta(t) + M\zeta(t) = F(t), \text{with initial value } \zeta(0).$$

Don't forget to provide the entries of the matrices M, S and of the vectors $F(t), \zeta(0)$ (you don't need to compute the integrals). (3 p)

5. Let $\Omega \subset \mathbb{R}^2$ be the square from Figure 1. Consider the problem

$$\begin{cases} -\Delta u(x) = 1 \quad \text{for } x \in \Omega \\ u(x) = 0 \quad \text{for } x \in \partial\Omega, \end{cases}$$

where $x = (x_1, x_2)$.

(a) Give the variational formulation of this problem. (2 p)

(b) Give the finite element approximation of this problem using a cG(1) FEM with the mesh from the figure. (2 p)

(c) Compute the 1×1 stiffness matrix, denoted by S, from the above FE problem. (3 p)

Hint: The mesh consists of reference triangles.

(d) Compute the corresponding load vector, denoted by b, from the above FE problem. (1 p)

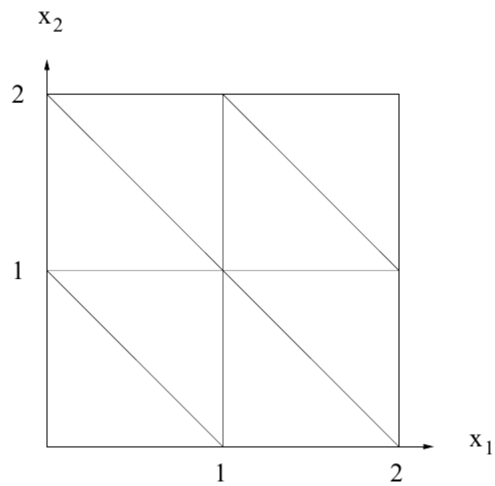

Figure 1: Courtesy from J. Hoffman.

(e) Solve the "linear system" $Ax = b$ and provide the cG(1) FE approximation of the solution to the above problem. (1 p)

6. Provide the nodal basis function $\Phi_1(x)$ for the finite element $\big(K = [0,1], P^{(2)}(K), \Sigma\big)$, where $P^{(2)}(K)$ denotes the set of polynomials of degree less or equal to two and $\Sigma = (L_1, L_2, L_3)$ is given by

$$L_1(f) = f'(0), \quad L_2(f) = f'(1), \quad L_3(f) = \int_0^1 f(x)\,\mathrm{d}x. \qquad (3\text{ p})$$

Hint: Determine the constants a, b, c in $\Phi_1(x) = ax^2 + bx + c$.