David Cohen (david.cohen@chalmers.se) Chalmers & GU

Chapter 2: Mathematical tools (summary)

February 2, 2025

Goal: Introduce some spaces of functions and several mathematical tools and results. This will help us to study and solve (numerically) differential equations in the next chapters.

- A set V is called a vector space or linear space (VS or LS) if, for all $u, v, w \in V$ and for all $\alpha, \beta \in \mathbb{R}$ one has

  1. $(u+v)+w = u+(v+w) = u+v+w$ (associativity)
  2. $u+v = v+u$ (commutativity)
  3. There exists an element $0 \in V$ such that $u+0 = 0+u = u$ for all $u \in V$ (zero element)
  4. For all $u \in V$, there exists an element $(-u) \in V$ such that $u+(-u) = 0$ (inverse element)
  5. $\alpha(\beta u) = (\alpha\beta)u$
  6. There exists $1 \in \mathbb{R}$ such that $1u = u$ for all $u \in V$.
  7. $\alpha(u+v) = \alpha u + \alpha v$
  8. $(\alpha+\beta)u = \alpha u + \beta u$.

  The elements in V are sometimes called vectors (but they can be something else, like "normal" vectors, matrices, functions, or sequences). The elements in $\mathbb{R}$ are called scalars. The above axioms (rules) tell us that we can do anything reasonable with vectors or functions and scalars.

  Example: The linear space of all polynomials, defined on $\mathbb{R}$, of degree $\leq n$ is denoted by

  $$\mathscr{P}^{(n)}(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : \ a_0, a_1, \ldots, a_n \in \mathbb{R}\}.$$

- A subset $U \subset V$ of a LS V is called a subspace of V if $\alpha u + \beta v \in U$ for all $u, v \in U$ and $\alpha, \beta \in \mathbb{R}$.

- Let V be a LS. The space of all linear combinations of the elements $v_1, v_2, \ldots, v_n \in V$ is denoted by

  $$\mathrm{span}(v_1, \ldots, v_n) = \{a_1 v_1 + a_2 v_2 + \ldots + a_n v_n : \ a_1, \ldots, a_n \in \mathbb{R}\}.$$

  Example: $\mathrm{span}(1, x, x^2) = \{a_0 1 + a_1 x + a_2 x^2 : \ a_0, a_1, a_2 \in \mathbb{R}\} = \mathscr{P}^{(2)}(\mathbb{R})$.

- A set $\{v_1, v_2, \ldots, v_n\}$ in a LS V is linearly independent if the equation

  $$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0 \in V$$

  has only the trivial solution $a_1 = a_2 = \ldots = a_n = 0 \in \mathbb{R}$. Else it is called linearly dependent.

  Example: The set $\{1, x, x^2\} \subset \mathscr{P}^{(2)}(\mathbb{R})$ is linearly independent.

- A set $\{v_1, v_2, \ldots, v_n\}$ in a LS V is called a basis of V if the set is linearly independent and $\mathrm{span}(v_1, \ldots, v_n) = V$. The dimension of V is then given by the number of elements of the basis, here $\dim(V) = n$.

  Example: The set $\{1, x, x^2\}$ is a basis of $\mathscr{P}^{(2)}(\mathbb{R})$ and thus $\dim(\mathscr{P}^{(2)}(\mathbb{R})) = 3$.

- A scalar product or inner product on a LS V is a map $(\cdot,\cdot)\colon V \times V \to \mathbb{R}$ such that, for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$,

  1. $(u, v) = (v, u)$ (symmetry)

2. $(u+\alpha v, w) = (u, w) + \alpha(v, w)$ (linearity)
3. $(u, u) \geq 0$ (positivity)
4. $(u, u) = 0 \in \mathbb{R}$ if and only if $u = 0 \in V$.

- A LS V with an inner product is called an inner product space, which is denoted by $(V, (\cdot,\cdot))$ or $(V, (\cdot,\cdot)_V)$ or $(V, \langle\cdot,\cdot\rangle_V)$.

  Such space has a norm defined by $\|v\| = \sqrt{(v, v)}$ for all $v \in V$.

- Let $(V, (\cdot,\cdot))$ be an inner product space and $u, v \in V$. u and v are orthogonal if $(u, v) = 0$. Notation: $u \perp v$.

- Let $(V, (\cdot,\cdot))$ be an inner product space and $u, v \in V$. Cauchy–Schwarz inequality (CS) reads

$$|(u, v)| \leq \|u\| \cdot \|v\|.$$

- Let $(V, (\cdot,\cdot))$ be an inner product space and $u, v \in V$. The triangle inequality ($\triangle$) reads

$$\|u+v\| \leq \|u\| + \|v\|.$$

- Example of a LS: The space of square integrable functions defined on the interval $[a, b]$ is denoted by

$$L^2([a,b]) = L^2(a,b) = L_2(a,b) = \left\{ f\colon [a,b] \to \mathbb{R} \text{ (measurable)} : \int_a^b |f(x)|^2 \,\mathrm{d}x < \infty \right\}.$$

  It is equipped with the inner product

$$(f, g)_{L^2} = \int_a^b f(x) g(x) \,\mathrm{d}x$$

  which induces the norm

$$\|f\|_{L^2} = \sqrt{(f, f)_{L^2}} = \sqrt{\int_a^b |f(x)|^2 \,\mathrm{d}x}.$$

  More generally, for an open (or a nice set) $\Omega \subset \mathbb{R}^n$ (or a measurable space), one defines the space of square integrable functions

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ (measurable)} : \|f\|_{L^2(\Omega)} < \infty \right\},$$

  where $\|f\|_{L^2(\Omega)} = \sqrt{(f, f)_{L^2(\Omega)}}$ and $(f, g)_{L^2(\Omega)} = \int_\Omega f(x) g(x) \,\mathrm{d}x$.

  Similarly, one can also define the spaces $L^p(\Omega)$, for a real number $1 \leq p < \infty$, as well as the space $L^\infty(\Omega)$. These two spaces are equipped with their corresponding norms.

- The space of continuous function defined on $[a, b]$ is given by

$$C([a,b]) = C^0([a,b]) = \mathscr{C}^0([a,b]) = \mathscr{C}^{(0)}(a,b) = \{f\colon [a,b] \to \mathbb{R} : f \text{ is continuous}\}$$

  and equipped with the norm

$$\|f\|_{C^0([a,b])} = \max_{a \leq x \leq b} |f(x)|.$$

Similarly, for $\Omega \subset \mathbb{R}^n$ an open set and k a positive integer, one defines the space of kth continuously differentiable functions

$$C^k(\Omega) = \mathscr{C}^k(\Omega) = \{f : \Omega \to \mathbb{R} : D^\alpha f \text{ are continuous for all } |\alpha| \leq k\}$$

and we equipe this space with the norm

$$\|f\|_{C^k(\Omega)} = \sum_{|\alpha|\leq k} \sup_{x\in\Omega} |D^\alpha f(x)|.$$

One can also use the following space

$$C^k(\overline{\Omega}) = \mathscr{C}^k(\overline{\Omega}) = \left\{f \in C^k(\Omega) : D^\alpha f \text{ can be extended continuously from } \Omega \text{ to its closure } \overline{\Omega}\right\}$$

and equipped with the norm

$$\|f\|_{C^k(\overline{\Omega})} = \sum_{|\alpha|\leq k} \sup_{x\in\overline{\Omega}} |D^\alpha f(x)|.$$

- For a positive integer k and $\Omega \subset \mathbb{R}^n$ open, one considers the Sobolev space

  $$H^k(\Omega) = \{f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \text{ for } |\alpha| \leq k\}$$

  with the inner product

  $$(f, g)_{H^k} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha f(x) D^\alpha g(x) \, \mathrm{d}x$$

  and norm

  $$\|f\|_{H^k} = \sqrt{(f, f)_{H^k}}.$$

  For $k = 1$ and $n = 1$ and in dimension one, the above norm reads

  $$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|f'\|_{L^2}^2.$$

- The triangle inequality as well as Cauchy–Schwarz can be extended to L^p spaces.

  Minkowski's inequality: Consider a nice set $\Omega \subset \mathbb{R}^n$ (or a measure space), $1 \leq p \leq \infty$ and $f, g \in L^p(\Omega)$. One then has

  $$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

  Hölder's inequality: Consider a nice set $\Omega \subset \mathbb{R}^n$ (or a measure space), $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, and $g \in L^q(\Omega)$. One then has

  $$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

  This is Cauchy–Schwarz for $p = q = 2$.

- Poincaré inequality ($1d$): Let $M > 0$ and consider the open interval $\Omega = (0, M)$. One then has

  $$\|u\|_{L^2(\Omega)} \leq \frac{M}{\sqrt{2}} \|u'\|_{L^2(\Omega)}$$

  for all $u \in H_0^1 = \{v \in H^1(\Omega) : v(0) = 0, v(M) = 0\}$.

- Trace theorem ($p = 2$): Let an integer $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ (bounded domain with Lipschitz boundary). One then has
$$\|u\|^2_{L^2(\partial\Omega)} \leq C \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}$$
for all $u \in H^1(\Omega)$.

- The strong form of Poisson's equation in dimension one reads
$$\begin{cases} -u''(x) = f(x) & \text{for } x \in \Omega = (0,1) \\ u(0) = 0, u(1) = 0, \end{cases}$$
where $f : \Omega \to \mathbb{R}$ is a given function (bounded and continuous for instance).

  The weak form or variational formulation (VF) reads
$$\text{Find } u \in H^1_0(\Omega) \text{ s.t. } (u', v')_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \text{ for all } v \in H^1_0(\Omega).$$
  The minimisation problem (MP) reads
$$\text{Find } u \in H^1_0(\Omega) \text{ s.t. } F(u) \text{ is minimal,}$$
  where the functional $F: H^1_0(\Omega) \to \mathbb{R}$ is defined by $F(v) = \frac{1}{2}(v', v')_{L^2(\Omega)} - (f, v)_{L^2(\Omega)}$ for $v \in H^1_0(\Omega)$.

  We have proved that
$$\text{Strong} \implies \text{VF} \iff \text{MP}$$
  and if in addition u is two times continuously differentiable, one has
$$\text{Strong} \impliedby \text{VF}.$$

- Lax–Milgram theorem: Consider a Hilbert space H, a bounded and coercive bilinear form $a : H \times H \to \mathbb{R}$, and a bounded linear functional $\ell : H \to \mathbb{R}$. Then, there exists a unique element $u \in H$ solution to the equation
$$a(u, v) = \ell(v) \quad \text{for all } v \in H.$$
  Lax–Milgram's theorem can be used, for instance, to find a unique solution in $H^1_0(\Omega)$ to the VF of Poisson's equation seen above.

Further resources:

- linear space in wikipedia.org
- inner product space in wikipedia.org
- Lp spaces in wikipedia.org
- CS in wikipedia.se
- VS and InnerProduct
- Sobolev space in youtube.com
- function spaces by Terry Tao (good!)
- Sobolev spaces and PDE (a little bit more advanced)

- application and proof of LM (more advanced)
- Trace theorem (more advanced)

Applications: Sobolev spaces are useful in the theoretical analysis of PDEs and variational problems (existence, stability and regularity of solutions or convergence of numerical methods) as well as in optimization and control problem, see for instance link.