Exam 03.06.24
TMA373, MMG801

1 a) Poisson eq. in 2D: $\Delta u = f$, (1) $d = B^2 - 4AC = 0 - 4 \cdot 1 \cdot 1 = -4 < 0 \Rightarrow$ classification elliptic PDE (1)

b) Lagrange polyn. (1) $\mathrm{Dim}(\mathcal{P}^{(1)}) = 2$ (1)

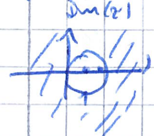
c) $\int_1^3 x^2\,dx \approx (3-1)\cdot \frac{1}{2}(3^2+1^2) = 10$ (1)

d) $y_{n+1} = y_n + k f(y_n)$ for $n = 0,1,2,\dots$ (1)

e) $\|u - u_h\|_{L^2} \leq C \cdot h$ (1)

f) Matrix that contain nodes of all triangles and 0 or 1 indicating a fake or real boundary (1)

g) Approximating integrals by quadrature formulas in VF/FE. (1)

h)  (1)

i) $u_{yy}(x,y) \approx \dfrac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$ (1)

2 a) Homogeneous Dirichlet BC $\Rightarrow$ test = trial = $H_0^1(0,1) = \{v : [0,1] \to \mathbb{R} : v \in H^1(0,1), v(0) = v(1) = 0\}$ (1)

See lecture: (VF) Find $u \in H_0^1$ s.t. $a(u,v) = l(v)$ $\forall v \in H_0^1$ (1)

where $a(u,v) = \int_0^1 u'(x)v'(x)\,dx$ and $l(v) = \int_0^1 f(x)v(x)\,dx$ (1)

b) Assumptions:

$a(\cdot,\cdot)$ bilinear and $l(\cdot)$ linear ok (clear) (0,5)

l bounded: $|l(v)| \underset{C-S}{\leq} \|f\|_{L^2} \cdot \|v\|_{L^2} \underset{\text{Def } H^1\text{-norm}}{\leq} \underbrace{\|f\|_{L^2}}_{\leq C < \infty} \|v\|_{H^1} \leq C \cdot \|v\|_{H^1}$ ok (1)

a bounded/continuous: $|a(u,v)| \underset{C-S}{\leq} \|u'\|_{L^2} \cdot \|v'\|_{L^2} \underset{\text{Def } H^1\text{-norm}}{\leq} \|u\|_{H^1} \|v\|_{H^1}$ ok (1)

a coercive/elliptic: For $u \in H_0^1$, $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 \underset{\text{Poincaré}}{\leq} \left(\frac{1}{2}+1\right)\|u'\|_{L^2}^2$ (1)

$\leq \frac{3}{2}(u',u')_{L^2} \leq \frac{3}{2} a(u,u) \Rightarrow a(u,u) \geq \frac{2}{3}\|u\|_{H^1}^2$ ok

$\underset{L-M}{\Longrightarrow}$ $\exists!$ VF in H_0^1 (0,5)

3 Test with v TBD: $\int_0^1 1 \cdot v(x)\,dx = \int_0^1 u'(x)v'(x)\,dx + \int_0^1 u(x)v(x)\,dx - \underbrace{u'(x)v(x)\Big|_0^1}_{u'(1)v(1) - u'(0)v(0)}$ (1)

(VF) Find $u \in H^1(0,1)$ with $u'(1) = 0$, $u'(0) = 3$ s.t. $\int_0^1 u'(x)v'(x)\,dx + \int_0^1 u(x)v(x)\,dx = \int_0^1 v(x)\,dx - 3v(0)$ (1)

$\forall v \in H^1(0,1)$

b) Test = trial space = $V_h = \mathrm{span}(\varphi_0, \varphi_1, \dots, \varphi_{m+1})$ with hat fct φ_j on $x_0 = 0 < x_1 < \dots < x_{m+1} = 1$ 1

and $x_j - x_{j+1} = h = \frac{1}{m+1}$

(FE) Find $u_h \in V_h$ with $u_h'(1) = 0$, $u_h'(0) = -3$ s.t. $\int_0^1 u_h'(x) v_h'(x)\,dx + \int_0^1 u_h(x) v_h(x)\,dx = \int_0^1 v_h(x)\,dx - 3 v_h(0)$ $\forall v_h \in H_0^1$ 1

c) Write $u_h(x) = \sum_{j=0}^{m+1} \xi_j \varphi_j(x)$ and take $v_h = \varphi_i$ for $i = 0, 1, \dots, m+1$ in (FE) $\Leftarrow$ 1

$$\sum_{j=0}^{m+1} \underbrace{\left(\int_0^1 \varphi_j'(x) \varphi_i'(x)\,dx + \int_0^1 \varphi_j(x) \varphi_i(x)\,dx \right)}_{a_{ij}} \xi_j = \underbrace{\int_0^1 \varphi_i(x)\,dx - 3\varphi_i(0)}_{b_i} \quad \text{for } i = 0, 1, \dots, m+1$$

1 1

$\Rightarrow$ $A = (a_{ij})_{i,j=0}^{m+1}$, $b = (b_i)_{i=0}^{m+1}$

with $b_0 = \int_0^1 \varphi_0(x)\,dx - 3\varphi_0(0) = \frac{h}{2} - 3$ 1 and $b_{m+1} = \int_0^1 \varphi_{m+1}(x)\,dx - 3\varphi_{m+1}(0) = \frac{h}{2}$ by def of φ_j 1

4 a) $V_h = \mathrm{span}(\{\varphi_j\}_{j=1}^{n_p})$, φ_j hat fct on node j. 1

b) $\Pi_1 f = \sum_{j=1}^{3} f(N_j)\varphi_j$ 1

c) $\|f - \Pi_h f\|_{L^2(\Omega)}^2 = \sum_{K \in \mathcal{T}_h} \|f - \Pi_1 f\|_{L^2(K)}^2 \underset{\text{Hint}}{\leq} \sum_{K \in \mathcal{T}_h} C_K^2 h_K^4 \|f\|_{H^2(K)}^2 \leq C \cdot h^4 \cdot \|f\|_{H^2(\Omega)}^2$ 1

$\Pi_h f = \Pi_1 f$ on triangle K 1

5 Test with u:

$$0 = -\int_\Omega \Delta u\, u\,dx \underset{\text{Green}}{=} \int_\Omega \nabla u \cdot \nabla u\,dx - \int_{\partial\Omega} \underbrace{(\nabla u \cdot n)}_{g-u \text{ on } \Gamma = \partial\Omega} u\,ds = \underbrace{\int_\Omega \nabla u \cdot \nabla u\,dx}_{\|\nabla u\|_{L^2(\Omega)}^2} - \int_\Gamma g u\,ds + \underbrace{\int_\Gamma u^2\,ds}_{\|u\|_{L^2(\Gamma)}^2}$$ 1

$\Rightarrow \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Gamma)}^2 \leq \int_\Gamma g u\,ds \underset{\text{C-S}}{\leq} \|g\|_{L^2(\Gamma)} \cdot \|u\|_{L^2(\Gamma)} \underset{\text{Hint}}{\leq} \frac{1}{2}\|g\|_{L^2(\Gamma)}^2 + \frac{1}{2}\|u\|_{L^2(\Gamma)}^2$ 1

$\Rightarrow \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u\|_{L^2(\Gamma)}^2 \leq \frac{1}{2}\|g\|_{L^2(\Omega)}^2$ ok 1

6 a) Hom. Dirichlet BC $\Rightarrow$ test = trial = $H_0^1(\Omega)$, see lecture... 1

(VF) For all $0 < t \leq T$, find $u(\cdot,t) \in H_0^1$ s.t. $(u_t(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v')_{L^2} + (u(\cdot,t), v)_{L^2} = (f, v)_{L^2}$ $\forall v \in H_0^1$ 1

$u(x,0) = u_0(x)$

b) $V_h^0 = \mathrm{span}(\{\varphi_j\}_{j=1}^{N})$, where φ_j hat fct on triangulation $\mathcal{T}_h = \{K_j\}_{j=1}^{\sim}$ of Ω. 1

(FE) For all $0 < t \leq T$, find $u_h(\cdot,t) \in V_h^0$ s.t. $(u_{h,t}(\cdot,t), v_h)_{L^2} + (u_{h,x}(\cdot,t), v_h')_{L^2} + (u_h(\cdot,t), v_h)_{L^2} = (f, v_h)_{L^2}$ $\forall v_h \in V_h^0$ 1

$u_h(x,0) = \Pi_h u_0(x) \in V_h^0$ pw cont. linear interpolant of u_0

c) Set $u_h(x,t) = \sum_{j=1}^{N} \xi_j(t)\varphi_j(x)$ and take $v_h(x) = \varphi_i(x)$ for $i = 1, \dots, N$ into (FE) $\Leftarrow$ 1

$$\sum_{j=1}^{N} \left\{ \dot{\xi}_j(t) \underbrace{(\varphi_j, \varphi_i)_{L^2}}_{M_{ij}} + \xi_j(t) \underbrace{(\varphi_j', \varphi_i')_{L^2}}_{S_{ij}} + \xi_j(t)(\varphi_j, \varphi_i)_{L^2} \right\} = \underbrace{(f, \varphi_i)_{L^2}}_{b_i} \quad \text{for } i = 1, \dots, N$$ 1

where $M = (m_{ij})_{i,j=1}^{N}$, $S = (s_{ij})_{i,j=1}^{N}$, $F = (b_i)_{i=1}^{N}$, $\xi_j = \Pi_h u_0(x_j) = u_0(x_j)$ for $j = 1, \dots, N$. 1 2

d) Step size $k \Rightarrow$ $\xi^{(n+1)} = \xi^{(n)} + k M^{-1}\left(F(t_{n+1}) - S\xi^{(n+1)} - M\xi^{(n+1)}\right)$ 1

where $\xi^{(n)} \approx \xi(t_n)$ with $t_n = n \cdot k$ for $n = 0, 1, 2, \dots$

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a) $V_h = \operatorname{span}(\varphi_j)$, φ_j hat fct on triangulation $\rightsquigarrow$ 9 nodes $\rightsquigarrow \dim(V_h) = 9$ 1

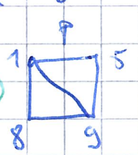
b) By def, stiffness matrix $A = (A_{ij})_{i,j=1}^{9}$ where $A_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx$ $\Rightarrow$ 81 elements 1 1

c) $A = \begin{pmatrix} A_{11} & \dots & A_{19} \\ \vdots & & \\ A_{91} & \dots & A_{99} \end{pmatrix}$ Symmetry $\Rightarrow$ only diagonal and above needed

disjoint support of hat fct $\Rightarrow A_{12} = A_{16} = A_{13} = 0$ f.ex 2

$\rightsquigarrow$ many 0 entries

d) $A_{11} = \int_\Omega \underbrace{\nabla\varphi_1 \cdot \nabla\varphi_1}_{\text{const, const, 1}} dx = 2 \cdot \frac{1}{2} = 1$ 1 , $A_{99} = \int_\Omega \nabla\varphi_9 \cdot \nabla\varphi_9 \, dx = 8 \cdot \frac{1}{2} = 4$ 1



1

2 triangles of area $\frac{1}{2}$

8 triangles

8 $L_1(\Phi_1) = 1$, $L_2(\Phi_1) = 0$, $L_3(\Phi_1) = 0$ 0.5

$L_1(\Phi_1) = 1 = \Phi_1'(0) = b \Rightarrow b = 1$

$0 = L_2(\Phi_1) = \Phi_1'(1) = 2a + b \Rightarrow a = -\frac{1}{2}$ 1 $\Rightarrow \Phi_1(x) = -\frac{1}{2}x^2 + 1 \cdot x - \frac{1}{3}$ 0.5

$0 = \int_0^1 \Phi_1(x)\,dx = -\frac{1}{6} + \frac{1}{2} + c \Rightarrow c = -\frac{1}{3}$