Examination, 11 March 2024
TMA373 and MMG801

Read this before you start!
I'll try to come at ca. 09:15 and 11:15. You can ask for calling me (0317723021) in case of questions.
Aid: Chalmers approved calculators.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
*Use a proper pen and order your answers if possible.**Thank you!***
Feel free to write more than one solution on one piece of paper (save paper).
The test has 4 pages and a total of 50 points.
Preliminary grading limits: 3:20-29p, 4:30-39p, 5:40-50p (Chalmers) and G:20-34p, VG:35-50p (GU). Valid bonus points will be added to the total score if needed.
You will be informed via Canvas when the exams are corrected.
Good luck!

Some exercises were taken from, or inspired by, materials from *M. Asadzadeh*, *P. Knabner*, *Y. Wang*, *S. Zahedi*.

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1. Provide concise answers to the following short questions:

   (a) Give an example of a hyperbolic PDE. Justify your answer. (2p)

   (b) Give a nodal basis to the space of polynomials $\mathcal{P}^{(1)}(0,1)$ (a name is enough if you don't remember the formulas)? What is the dimension of this space? (2p)

   (c) State Poincaré's inequality on $\Omega \subset \mathbb{R}^2$ a bounded domain with smooth boundary (the value of the constant is not important). (1p)

   (d) Use (one step of) the midpoint rule to approximate the area under the function $f(x) = x^2$ between $x = 1$ and $x = 3$. (1p)

   (e) Give the general formula for the implicit Euler scheme, with time step k, when applied to the IVP $\dot{y}(t) = f(y(t)), y(0) = y_0$. (1p)

   (f) What is an estimate for the a priori error estimate, measured in the energy norm, of the (continuous and piecewise linear) FEM with mesh size h for the model problem (Poisson's equation in $1d$ with homogeneous Dirichlet BC) seen in the lecture? (1p)

   (g) What is the point matrix in a FEM implementation? (1p)

   (h) Give an example of a variational crime. (1p)

(i) Draw the stability region of the explicit Euler scheme. Don't forget the axes. (1p)

(j) Give a finite difference approximation of $u_{xx}(x, y)$. (1p)

2. Let $V = \{v \in H^1(0,1) : v(0) = 0\}$, $f \in L^2(0,1)$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Define $a(u,v) = \int_0^1 u'(x)v'(x)\,\mathrm{d}x + \alpha \int_0^1 u(x)v(x)\,\mathrm{d}x$ and $L(v) = \int_0^1 f(x)v(x)\,\mathrm{d}x + \beta v(1)$ for $u, v \in V$. Show that the problem:

$$\text{Find } u \in V \text{ such that } a(u,v) = L(v) \text{ for all } v \in V$$

has a unique solution in V. (6p)

<u>Hint</u>: *Don't forget that* $H^1(0,1) \subset C^{(0)}(0,1)$ *and the injection is continuous.*

3. Let us apply cG(1) to the BVP

$$\begin{cases} -u'' = f & \text{in } \Omega = (0,10) \\ u(0) = 0, u'(10) = 25, \end{cases}$$

where $f(x) = 10$ for $0 < x < 5$ and $f(x) = 0$ for $5 \leq x < 10$.

(a) Derive the variational formulation of this BVP (define all quantities). (2p)

(b) Derive the corresponding finite element problem (define all quantities). (2p)

(c) Consider an equidistant partition with two elements. Give the linear system of equations for the computation of the finite element solution. Compute the first entry, $s_{1,1}$, of the stiffness matrix and the last entry, b_2, of the load vector. (3p)

(d) Consider now an equidistant partition with four elements. Give the element stiffness matrices (no need to compute the entries of these matrices). (2p)

4. Let $(V, \|\cdot\|_V)$ be a normed linear space, a be a bounded (that is continuous) and coercive bilinear form on $V \times V$ and b a continuous linear form on V. Let u be the solution to $a(u,v) = b(v)$ for all $v \in V$. Let $V_h \subset V$ and u_h the solution to $a(u_h, v_h) = b(v_h)$ for all $v_h \in V_h$.

(a) First show that $a(u - u_h, w_h) = 0$ for all $w_h \in V_h \subset V$. In connection to the lecture, how is this relation called? (2p)

(b) Using the above and properties of a, prove that

$$\|u - u_h\|_V \leq C\|u - v_h\|_V$$

for all $v_h \in V_h \subset V$. (2p)

5. Let $\Omega \subset \mathbb{R}^2$ be a nice domain, $f\colon \Omega \to \mathbb{R}$ be a nice function, and $g\colon \Gamma \to \mathbb{R}$ be a nice function defined on the boundary $\Gamma = \partial\Omega$ of the domain. Consider the PDE

$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega, \\ n \cdot \nabla u = g, & \text{on } \Gamma, \end{cases}$$

where n in the outward unit normal to Γ. Show the following stability estimate: There exists a constant $C > 0$ such that

$$||\nabla u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)} \leq C\left(||f||^2_{L^2(\Omega)} + ||g||^2_{L^2(\Gamma)}\right). \qquad \text{(3p)}$$

Hint: You may use the trace theorem $||u||_{L^2(\Gamma)} \leq C||u||_{H^1(\Omega)}$ for $u \in H^1(\Omega)$. The inequality $ab \leq a^2 + b^2/4$ may be of use.

6. Let $\Omega \subset \mathbb{R}^2$ be a nice domain, $f, u_0, v_0 \colon \Omega \to \mathbb{R}$ be nice, $T > 0$. Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u + u = f \quad \text{in } \Omega \times (0,T] \\ u = 0 \quad \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) \quad \text{in } \Omega \\ u_t(x,0) = v_0(x) \quad \text{in } \Omega. \end{cases}$$

   (a) Write the variational formulation of the above problem in a suitable space V (don't forget to define all quantities). (2p)

   (b) Formulate the finite element problem based on a cG(1) approximation on the space V_h^0 (don't forget to define all quantities). (2p)

   (c) From the above, derive the linear system of ordinary differential equations

$$M\ddot{\zeta}(t) + S\zeta(t) + M\zeta(t) = F(t), \text{with initial value } \zeta(0), \dot{\zeta}(0).$$

   Don't forget to provide the entries of the matrices M, S and of the vectors $F(t), \zeta(0), \dot{\zeta}(0)$. (2p)

   (d) Let now $F(t) = 0$. Transform the above system of ODEs into a system of first order and apply the Crank–Nicolson scheme. (3p)

7. Let $\Omega \subset \mathbb{R}^2$, with its triangulation, be the domain from Figure 1 (information on the boundary is not needed here).

   (a) Provide the global mass matrix (do not compute the integrals). (1p)

   (b) What are the element mass matrices and element stiffness matrices for this triangulation (you do not need to compute the integrals)? (2p)

8. We want to describe linear rectangle elements with four nodes by a finite element (K, P, Σ). For this, consider the polygon K to be the reference rectangle (a square in fact) centered at $(0,0)$ with nodes at $N_1 = (-1,-1)$, $N_2 = (1,-1)$, $N_3 = (1,1)$, $N_4 = (-1,1)$. Determine P and the set of nodal values $\Sigma \colon P \to \mathbb{R}$. Give the shape function $\varphi_1(x,y)$ corresponding to the node N_1. (4p)

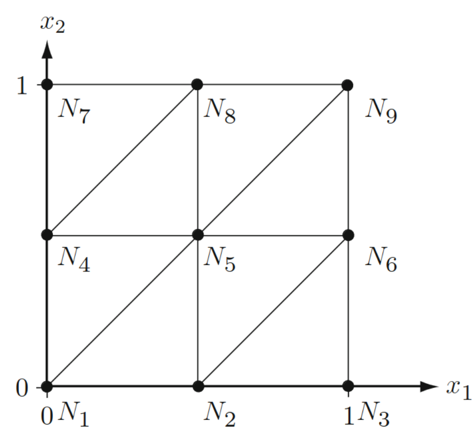

Figure 1: Courtesy from M.G. Larson.