Chapter 6: The heat equation in $1d$ (summary)

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Goal: Briefly study the exact solution to some heat equations and present a numerical discretisation.

- Let us start with some stability estimates for the following heat equation

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0 < x < 1, 0 < t \leq T \\ u(0,t) = 0, u_x(1,t) = 0 & 0 < t \leq T \\ u(x,0) = u_0(x) & 0 < x < 1, \end{cases}$$

  where u_0 and f are given functions.

  The solution to the above problem statisfy the following estimates

$$\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)} \,\mathrm{d}s.$$
$$\|u_x(\cdot,t)\|^2_{L^2(0,1)} \leq \|u_0'\|^2_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|^2_{L^2(0,1)} \,\mathrm{d}s.$$

  When $f = 0$, one gets

$$\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} \,\mathrm{e}^{-2t}.$$

  When $f = 0$ and for some fixed $\varepsilon > 0$, one gets for all $t \in [\varepsilon, T]$

$$\int_\varepsilon^t \|u_t(\cdot,s)\|_{L^2(0,1)} \,\mathrm{d}s \leq \frac{1}{2}\sqrt{\ln\left(\frac{t}{\varepsilon}\right)} \|u_0\|_{L^2(0,1)}.$$

  The last inequality can be used to show a posteriori error estimates of FEM for the heat equation.

- Next, we discretise the inhomogeneous heat equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0 < x < 1, 0 < t \leq T \\ u(0,t) = u(1,t) = 0 & 0 < t \leq T \\ u(x,0) = u_0(x) & 0 < x < 1, \end{cases}$$

  where u_0 and f are given functions.

  Since it is seldom possible to find the exact solution $u(x,t)$ to the above problem, we need to find a numerical approximation of it. We proceed as follows

  1. To get a VF of the heat equation, consider the test/trial space $H_0^1 = \{v \colon [0,1] \to \mathbb{R} \colon\ v, v' \in L^2(0,1), v(0) = v(1) = 0\}$. Then, multiply the DE by a test function $v \in H_0^1$, integrate over $[0,1]$, and use integration by parts to get the VF: For each $0 < t \leq T$

$$\text{Find} \quad u(\cdot,t) \in H_0^1 \quad \text{s.t.} \quad (u_t(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = \big(f(\cdot,t), v\big)_{L^2} \quad \forall v \in H_0^1 \tag{VF}$$

     with the initial condition $u(x,0) = u_0(x)$.

2. To get a FE problem, we consider the following subspace of the above space H^1_0
$V^0_h = \left\{ v\colon [0,1] \to \mathbb{R}:\ v \text{ cont. pw. linear on unif. partition } T_h, v(0) = v(1) = 0 \right\} = \text{span}(\varphi_1, \dots, \varphi_m)$,
where $h = \frac{1}{m+1}$ is the mesh and φ_j are the hat functions.
The FE problem then reads: For each $0 < t \leq T$
$$\text{Find} \quad u_h(\cdot,t) \in V^0_h \quad \text{s.t.} \quad \left(u_{h,t}(\cdot,t), \chi\right)_{L^2} + \left(u_{h,x}(\cdot,t), v_{h,x}\right)_{L^2} = \left(f(\cdot,t), v_h\right)_{L^2} \quad \forall v_h \in V^0_h \quad \text{(FE)}$$
with the initial condition $u_h(x,0) = \pi_h u_0(x)$ the cont. pw. linear interpolant of u_0.
3. From the above FE problem, we obtain a system of linear ODE by choosing the test functions $v_h = \varphi_i$ for $i = 1, \dots, m$ and writing $u_h(x,t) = \sum_{j=1}^m \zeta_j(t)\varphi_j(x)$ with unknown coordinates $\zeta_j(t)$.
Inserting everything in (FE), one gets the ODE
$$\begin{aligned} M\dot{\zeta}(t) + S\zeta(t) &= F(t) \\ &\zeta(0), \end{aligned} \quad \text{(ODE)}$$
where M is the (already seen) $m \times m$ mass matrix, S is the (already seen) $m \times m$ stiffness matrix, $F(t)$ is an $m \times 1$ vector with entries $F_i(t) = (f(\cdot,t), \varphi_i)_{L^2}$ for $i = 1, \dots, m$, the initial condition is given by
$$\zeta(0) = \begin{pmatrix} u_0(x_1) \\ \vdots \\ u_0(x_m) \end{pmatrix},$$
and the unknown vector reads
$$\zeta(t) = \begin{pmatrix} \zeta_1(t) \\ \vdots \\ \zeta_m(t) \end{pmatrix}.$$
4. To find a numerical approximation of $\zeta(t)$ at some discrete time grid $t_0 = 0 < t_1 < \ldots < t_N = T$, with $t_j - t_{j-1} = k = \frac{T}{N}$, one can for instance use backward Euler scheme which reads
$$\begin{aligned} \zeta^{(0)} &= \zeta(0) \\ (M + kS)\zeta^{(n+1)} &= M\zeta^{(n)} + kF(t_{n+1}) \quad \text{for} \quad n = 0,1,2,\dots,N-1. \end{aligned}$$
Solving these linear systems at each time step provides numerical approximations $\zeta^{(n)} \approx \zeta(t_n)$ that can be inserted in the FE solution to get approximations to the exact solution to the heat equation $u^k_h(x,t_n) = u^{(n)}_h(x) = \sum_{j=1}^m \zeta^{(n)}_j \varphi_j(x) \approx u(x,t_n)$.
Instead of backward Euler scheme, one can also use the Crank-Nicolson scheme, but perhaps not the explicit Euler scheme (for stability reason, see later in the lecture).
5. The order of the error of the above numerical discretisation is:
$$\|u(\cdot,t) - u_h(\cdot,t)\|_{L^2} \leq Ch^2 \quad \text{and} \quad \left\| u(\cdot,t_n) - u^{(n)}_h \right\|_{L^2} \leq C_1 h^2 + C_2 k.$$
Details can be found in Theorem 5.1 and Theorem 5.2 in the book by Larson and Bengzon for instance.

Further resources:

- heat eq. at wikipedia.org
- heat eq. at math.lamar.edu

Applications: The heat equation (in $1d$ and higher dimension) is used to model heat transfer in, for instance: engineering, fluid dynamics/mechanics, atmospheric science, climate physics, weather forecasting, option pricing, geophysics, solar physics, or computer graphics, see for instance link.