Examination, 13 March 2023
TMA373 and MMG801

Read this before you start!
I'll try to come at ca. 09:15.
Aid: Chalmers approved calculators.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Use a proper pen and order your answers if possible. ***Thank you!***
No need to use one piece of paper for only one exercise.
The test has 4 pages and a total of 50 points.
Preliminary grading limits: 3:20-29p, 4:30-39p, 5:40-50p (Chalmers) and G:20-34p, VG:35-50p (GU). Valid bonus points will be added to the total score if needed.
You will be informed via Canvas when the exams are corrected.
Good luck!

Some exercises were taken from, or inspired by, materials from *C. Cotter*, *P. Frey*, *A. Larcher*, *M.G. Larson*, *M. Nazarov*.

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1. Provide concise answers to the following short questions:

   (a) Is the following PDE $3\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y} = 0$ elliptic or parabolic? (1p)

   (b) Let an integer $\ell > 0$. What is the dimension of the polynomial space $\mathcal{P}^{(\ell)}(0,1)$? Give an application of such spaces. (1p)

   (c) State Poincaré's inequality on $\Omega = (0,1)$ (the value of the constant is not important). (1p)

   (d) Consider a uniform partition of $[0,1]$ with mesh h. What is the size (in term of h) of the error of the continuous piecewise linear interpolant? (1p)

   (e) Use (one step of) the midpoint rule to approximate the area under the function $f(x) = x^2$ between $x = 1$ and $x = 3$. (1p)

   (f) Give the general formula for the Crank–Nicolson scheme, with time step k, when applied to the IVP $\dot{y}(t) = f(y(t)), y(0) = y_0$. (1p)

   (g) What is an estimate for the a priori error estimate, measured in the energy norm, of the (continuous and piecewise linear) FEM with mesh size h for the model problem (Poisson's equation in $1d$ with homogeneous Dirichlet BC) seen in the lecture? (1p)

   (h) Why is adaptivity useful? (1p)

(i) Is the energy of the homogeneous linear wave equation with homogeneous Dirichlet BC a conserved quantity? (1p)

(j) What is the connectivity matrix in a FEM implementation? (1p)

2. Let $\Omega = (0,1)$, $\kappa \in \mathbb{R}$, and $f \in L^2(\Omega)$. Consider the problem: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \kappa \frac{\partial u}{\partial x} v \,\mathrm{d}x + \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x \quad \text{for all } v \in H_0^1(\Omega).$$

Formulate the strong problem corresponding to the above weak formulation/variational formulation. (2p)

3. Let $\Omega \subset \mathbb{R}^2$ a bounded open smooth set and denote $\partial\Omega$ its boundary. Let $f \in L^2(\Omega)$. Vectors $x \in \mathbb{R}^2$ are denoted by $x = (x_1, x_2)$.

Let $w \in H^2(\Omega)$ be a solution to the problem

$$\begin{cases} -\Delta w + \frac{\partial w}{\partial x_1} = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Define a bilinear form $a(\cdot,\cdot)$, a linear form L, and a suitable functional space H such that the hypothesis of the Lax–Milgram theorem are satisfied (and you have to show them) and such that one has, for all $v \in H$, that

$$a(w,v) = L(v). \qquad \text{(6p)}$$

<u>Hint</u>: Green's formula $\int_\Omega \frac{\partial u}{\partial x_1} v \,dx = -\int_\Omega u \frac{\partial v}{\partial x_1}\,dx + \int_{\partial\Omega} uvn_1 \,ds$, *where* $n = (n_1, n_2)$, *may be of use to show that* $\int_\Omega \frac{\partial u}{\partial x_1} u \,dx = 0$.
Poincaré's inequality may be of use.

4. Consider the following pseudo-code

```
function [t,y] = solveit(t0, y0, T, n)
  t = zeros(n+1,1); y = zeros(n+1,1);
  t(1) = t0; y(1) = y0; h = (T-t0)/n;
  for i = 1:n
    t(i+1) = t(i)+h;
    y(i+1) = y(i)+h*(y(i))^2;
  end
```

Suppose that the input values are $t0 = 0, y0 = 1, T = 1$, and $n = 10$.

What is the initial-value problem being approximated numerically? What is the numerical method being used? (2p)

5. Let $\alpha, \beta \neq 0$ and let us apply cG(1) to the BVP

$$\begin{cases} -u'' = f & \text{in } \Omega = (0,1) \\ u(0) = \alpha, u(1) = \beta. \end{cases}$$

(a) Derive the variational formulation of this BVP (define all quantities). (2p)

(b) Derive the corresponding finite element problem (define all quantities). (2p)

(c) Give the linear system of equations for the computation of the finite element solution. You don't need to compute the integals in the stiffness matrix but must give all coordinates of the load vector. (4p)

6. Let $\Omega = (0,1)$ and $f \in L^2(\Omega)$. Consider the problem

Find $u \in H_0^1(\Omega)$ such that $\int_\Omega u'(x)v'(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x$ for all $v \in H_0^1(\Omega)$.

Let $u_h \in V_h^0$ be the corresponding cG(1) approximation to u on a uniform partition with mesh size h. Consider then the auxiliary problem

Find $\zeta \in H_0^1(\Omega)$ such that $\int_\Omega \zeta'(x)v'(x)\,\mathrm{d}x = \int_\Omega (u(x) - u_h(x))\,v(x)\,\mathrm{d}x$ for all $v \in H_0^1(\Omega)$.

(a) Using the above auxiliary problem and Galerkin's orthogonality, first show that

$$\|u - u_h\|_{L^2(\Omega)}^2 = \int_\Omega (u(x) - u_h(x))'(\zeta(x) - \pi_h\zeta(x))'\,\mathrm{d}x,$$

where we recall that $\pi_h\zeta \in V_h^0$ denotes the continuous piecewise linear interpolant of ζ. (2p)

(b) Next, using the above and an interpolation error estimate (observe that $\zeta \in H^2(\Omega)$ since $-\zeta'' = u - u_h$), show the following error estimate for the cG(1) approximation:

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch\,\|(u - u_h)'\|_{L^2(\Omega)}\,. \quad (2p)$$

7. Let $\Omega \subset \mathbb{R}^2$ be a nice domain, $f, u_0 \colon \Omega \to \mathbb{R}$ be nice, $T > 0$. Consider the inhomogeneous heat equation

$$\begin{cases} u_t - \Delta u + u = f & \text{in } \Omega \times (0,T] \\ u = 0 & \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

(a) Write the variational formulation of the above problem in a suitable space V. (2p)

(b) Formulate the finite element problem based on a cG(1) approximation on the space V_h^0 (don't forget to define this space). (2p)

(c) Derive Galerkin's orthogonality

$$\int_\Omega (e_t v_h + \nabla e \cdot \nabla v_h + e v_h)\,\mathrm{d}x = 0 \quad \text{for all } v_h \in V_h^0,$$

where $e = e(x,t) = u(x,t) - u_h(x,t)$ and u_h denotes the finite element approximation of u. (2p)

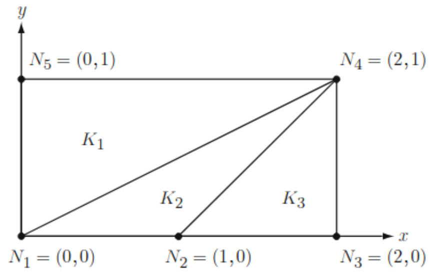

Figure 1: Courtesy from M.G. Larson.

(d) From the above, derive the linear system of ordinary differential equations

$$M\dot{\zeta}(t) + S\zeta(t) + M\zeta(t) = F(t), \text{with initial value } \zeta(0).$$

Don't forget to provide the entries of the matrices M, S and of the vectors $F(t), \zeta(0)$. (2p)

(e) Prove the following stability estimate

$$\|u(\cdot,T)\|^2_{L^2(\Omega)} + 2\int_0^T \|\nabla u(\cdot,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \int_0^T \|u(\cdot,s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \leq \|u_0\|^2_{L^2(\Omega)} + \int_0^T \left\|f(\cdot,s)\right\|^2_{L^2(\Omega)}\,\mathrm{d}s.$$

(3p)

Hint: Use an appropriate test function in the variational formulation of the problem. You may also use Young's inequality ($ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for nonnegative real numbers a, b) for the right-hand side.

8. Let $\Omega \subset \mathbb{R}^2$, with its triangulation, be the domain from Figure 1 (information on the boundary is not needed here).

   (a) Provide the global mass matrix (do not compute the integrals). (1p)

   (b) Give all element mass matrices (do not compute the integrals). (2p)

   (c) Compute the entry of the element mass matrix corresponding to the node N_3 for the triangle K_3. (2p)

9. Provide the nodal shape function $\Phi_1(x)$ for the finite element $\left(K = [0,1], \mathcal{P}^{(2)}(K), \Sigma\right)$, where $\mathcal{P}^{(2)}(K)$ denotes the set of polynomials of degree less or equal to two and $\Sigma = (L_1, L_2, L_3)$ is given by

$$L_1(f) = f(0), \quad L_2(f) = f(1), \quad L_3(f) = \int_0^1 f(x)\,\mathrm{d}x. \qquad (2p)$$

Hint: Determine the constants a, b, c in $\Phi_1(x) = ax^2 + bx + c$.