Exam, 8 June 2023

TMA373 / MMG 801

① a) Discriminant $=0-4\cdot 1\cdot 1^2=-4<0 \Rightarrow$ elliptic 0.5 0.5

b) $\dim(P^{(3)}(0,1))=4$. Application: space of shape fct., etc. 0.5 0.5

c) $\|\pi_h f - f\|_{L^2} \le C\cdot h^2 \|f''\|_{L^2}$ or $C\cdot h\cdot \|f'\|_{L^2}$ 1

d) $\int_1^3 x^2\,dx \approx \frac{3-1}{2}(3^2+1^2)=10$ 1

e) $\|u-u_h\|_{L^2} \le C\cdot h$ 1

f) A posteriori 1

g) A matrix listing all nodes of a mesh 1

② $H^1(I) \to$ Neumann BC $\Rightarrow$ BVP reads $-u''(x)+c(x)u(x)=f(x)$ in I, $u'(0)=\alpha$, $u'(1)=\beta$ 1

by part on $\int u'v'$

③ a) $V_h \subset V$ is also Hilbert and a, l verify LM $\Rightarrow \exists!\, u_h \in V_h$ by LM 1

b) For $\varphi_h \in V_h \subset V$ one has $a(u,\varphi_h)=l(\varphi_h)=a(u_h,\varphi_h) \Rightarrow a(u-u_h,\varphi_h)=0$

Then, $\kappa\|u-u_h\|_V^2 \le a(u-u_h,u-u_h) = a(u-u_h,u) - \underbrace{a(u-u_h,u_h) - a(u-u_h,\varphi_h)}_{=0}$ 1

$\le \alpha\|u-u_h\|_V \cdot \|u-\varphi_h\|_V$ 1

$\Rightarrow \|u-u_h\|_V \le \frac{\alpha}{\kappa}\|u-\varphi_h\|_V$ 1

④ Line 6 → implicit/backward Euler scheme for ODE $\begin{cases} y'(t)=y(t)^2 \\ y(0)=1 \end{cases}$ 1 1

⑤ a) Hom. Dirichlet BC → test and trial spaces are H_0^1 ⇒ test w/ $v\in H_0^1$ + by part 1

(VF) Find $u\in H_0^1$ s.t. $(au',v')_{L^2}+(xu,v)_{L^2}=(f,v)_{L^2}$ $\forall v\in H_0^1$ 1

b) $V_h^0=\{v_h\in H_0^1 : v_h$ is cont. on $[0,1]$ and linear on $[x_n,x_{n+1}]$ $\forall n$ and $v_h(0)=v_h(1)=0\}$ 1

(FE) Find $u_h\in V_h^0$ s.t. $(au_h',v_h')_{L^2}+(xu_h,v_h)_{L^2}=(f,v_h)_{L^2}$ $\forall v_h\in V_h^0$ 1

c) $V_h=\mathrm{span}(\{\varphi_j\}_{j=1}^N)$ ~~Shape~~ hat fct. Set $u_h(x)=\sum_{j=1}^N \xi_j\varphi_j(x)$ and take $v_h=\varphi_i$ for $j=1,\dots,N$ into (FE) to get lin. syst. $S\xi+M\xi=F$, where 1

$S=((\varphi_j',\varphi_i')_{L^2})_{i,j=1}^N$, $M=((\varphi_j,\varphi_i))_{i,j=1}^N$ and $F=((f,\varphi_i)_{L^2})_{i=1}^N$ with 1

$(f,\varphi_i)_{L^2}=\int_0^1 f(x)\varphi_i(x)\,dx=\int_0^1 \varphi_i(x)\,dx=1\cdot h$ 1

⑥ a) For $v\in H_0^2(\Omega)$, one has $v'\in H_0^1$. Apply Poincaré to $v'\in H_0^1$ and get 1

$\|v'\|_{L^2} \le C\cdot\|v''\|_{L^2}$ 1

b) Since $H_0^2 \subset H_0^1$, one can apply Poincaré to $v \in H_0^2$ and get

$\|v\|_{L^2} \leq C\|v'\|_{L^2} \leq C\|v''\|_{L^2}$ 1

By def of H^2 one then has $\|v\|_{H^2} \leq C\|v''\|_{L^2}$ 1

c) $\|u-u_h\|_{H^2} \underset{\text{Hint 1}}{\leq} C\|u-\Pi_h u\|_{H^2} \underset{\Pi_h u \in H_0^2}{\leq} C\|(u-\Pi_h u)''\|_{L^2} \underset{\text{Hint 2}}{\leq} Ch^2\|u''''\|_{L^2} \underset{u'''' \in L^2 \text{ since } f \in H^1, H_0^1}{\leq} Ch^2\|f''\|_{L^2}$ 1

7 a) Hom. Dirichlet BC $\Rightarrow V = H_0^1$ and VF reads: $\forall t \in [0,T]$ 1

(VF) Find $u(\cdot,t) \in H_0^1$ s.t. $(u_{tt}(\cdot,t),v)_{L^2} + (\nabla u(\cdot,t),\nabla v)_{L^2} + (u(\cdot,t),v)_{L^2} = (f(\cdot,t),v)_{L^2}$ $\forall v \in H_0^1$

$u(x,0) = u_0(x)$

$u_t(x,0) = v_0(x)$ 1

b) $V_h^0 = \text{span}(\{\varphi_j\}_{j=1}^{n_i}) = \{v: \Omega \to \mathbb{R},\ v$ cont. pw l on triangulation $\mathcal{T}_h$ and $v|_{\partial\Omega} = 0\}$, where n_i = # of interior nodes 1

(FE) $\forall t$, find $u_h(\cdot,t) \in V_h^0$ s.t. $(u_{h,tt}(\cdot,t),v)_{L^2} + (\nabla u_h(\cdot,t),\nabla v)_{L^2} + (u_h(\cdot,t),v)_{L^2} = (f(\cdot,t),v_h)_{L^2}$ $\forall v_h \in V_h^0$

$u_h(x,0) = \Pi_h u_0(x)$, $u_{h,t}(x,0) = \Pi_h v_0(x)$ 1

c) See lecture $\to$ $M = ((\varphi_j,\varphi_i)_{L^2})_{i,j=1}^{n_i}$, $S = ((\nabla\varphi_j,\nabla\varphi_i)_{L^2})_{i,j=1}^{n_i}$, $F = ((f,\varphi_j)_{L^2})_{j=1}^{n_i}$,

$\xi(0) = (u_0(x_j))_{j=1}^{n_i}$, $\dot\xi(0) = (v_0(x_j))_{j=1}^{n_i}$ 1

8 Write $u = v + iw$ into Schrödinger,

$iv_t - w_t = v_{xx} + iw_{xx} \Leftrightarrow \begin{cases} v_t = w_{xx} & |\cdot v \text{ and } \int_0^1 \cdot\, dx \\ w_t = -v_{xx} & |\cdot w \text{ and } \int_0^1 \cdot\, dx \end{cases}$ 1

$\Leftrightarrow \begin{cases} \int_0^1 v_t v\,dx = \int_0^1 w_{xx} v\,dx \overset{\text{by parts}}{=} -\int_0^1 w_x v_x\,dx \\ \underbrace{\int_0^1 w_t w\,dx}_{\int_0^1 \frac{1}{2}\frac{d}{dt}(w^2)\,dx} = -\int_0^1 v_{xx} w\,dx = \int_0^1 v_x w_x\,dx \end{cases}$ 1

Add both lines: $\frac{1}{2}\frac{d}{dt}(\|u(\cdot,t)\|_{L^2}^2) = 0$ hence $\|u(\cdot,t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 = \text{Const}$ $\forall t > 0$ 1

9 a) Set $V = \{v: \Omega \to \mathbb{R};\ \|v\|_{H^1(\Omega)} < \infty,\ v(0,y) = 0 \text{ for } y \in [0,1]\}$.

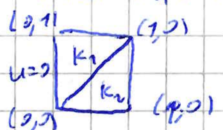
For $v \in V$ ~~[illegible]~~, one has $\int_\Omega f(x,y)\,v(x,y)\,dx\,dy = -\int_\Omega \Delta u(x,y)\,v(x,y)\,dx\,dy =$ 1

$= \int_\Omega \nabla u(x,y)\cdot\nabla v(x,y)\,dx\,dy - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} v\,ds}_{=0} = \int_\Omega \nabla u(x,y)\cdot\nabla v(x,y)\,dx\,dy$

VF: Find $u \in V$ with Neumann BC s.t. $\int_\Omega \nabla u\cdot\nabla v = \int_\Omega f v$ $\forall v \in V$ 1

b) Triangulation $\Omega = \bigcup_{j=1}^{n} K_j$ and $V_h = \{v \in V:\ v|_{K_j}$ linear and cont. on $\Omega\}$ 1

FE: Find $u_h \in V_h$ s.t. $\int_\Omega \nabla u_h \nabla v_h = \int_\Omega f v_h$ $\forall v_h \in V_h$ 1

c) 

Shape fct $\varphi_1(x,y) = \begin{cases} x, & y \geq x \\ y, & y < x \end{cases}$ and $\varphi_2(x,y) = \begin{cases} x-y, & x > y \\ 0, & x \leq y \end{cases}$ 1 $\quad \nabla\varphi_1 = \begin{cases} (1,0), & y > x \\ (0,1), & y < x \end{cases}$ $\quad \nabla\varphi_2 = \begin{cases} (1,-1), & x > y \\ 0, & x \leq y \end{cases}$

FE reads $A\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} = F$, where $u_h(x,y) = \xi_1\varphi_1(x,y) + \xi_2\varphi_2(x,y)$, $A_{ij} = (\nabla\varphi_j,\nabla\varphi_i)_{L^2}$ 2

$F = ((f,\varphi_j)_{L^2})_{j=1}^{2}$ or $\begin{pmatrix} 1 & -1/2 \\ \ldots & \ldots \end{pmatrix}\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} = \begin{pmatrix} 1/3 \\ \ldots \end{pmatrix}$ or $(\xi_1,\xi_2) = (5/9, 4/9)$ 3