Exam 29.08.24
TMA373 + MMG801

1

① a) Heat eq. $u_t - u_{xx} = 5x$, where $u = u(x,t)$ (+IC+BC) 1
Discriminant $= 0$ → parabolic 1

b) $\text{Span}\{1,x\} = \mathcal{P}^{(1)}(\mathbb{R})$ = polyn. of degree ≤ 1. 1 dim = 2 1. Basis $= \{1, x\}$ 1 :-)

c) Interpolation 1

d) $\int_a^b f(x)\,dx \approx (b-a)\, f\left(\frac{a+b}{2}\right)$ 1

e) $y' = y$, $y(0) = y_0$. 1 Explicit Euler reads $y_{n+1} = y_n + h\, y_n$, $n = 0,1,2,\dots$ where $y_n \approx y(t_n)$, $t_n = n \cdot h$ 1
stepsize

f) Adaptivity 1

g) Conserve energy 1

h) $P_1(K) = \{ v : K \to \mathbb{R} : v(x_1,x_2) = c_0 + c_1 x_1 + c_2 x_2,\ (x_1,x_2) \in K,\ c_0, c_1, c_2 \in \mathbb{R} \}$ 1

Applications: Definition of space V_h of cont. pw linear polynomials 1

i) Write global stiffness matrix as sum of element stiffness matrices
Compute element stiffness matrices 1
Add them at the appropriate location to build the global stiffness matrix 1

j) $\dot{y} = \lambda y$, $y(0) = y_0$, where $\lambda \in \mathbb{C}$. 1

Used to analyze stability region of time integrators 1

② Strong $\begin{cases} -u''(x) = f(x) & \text{on } (0,1) \\ u(0) = 1,\ u(1) = 0 \end{cases}$ 1

Minimization prob: Find $u \in H_0^1(0,1)$ st. $F(u)$ is minimal, where $F(v) = \frac{1}{2}(v', v')_{L^2} - (f, v)_{L^2}$ 1

We have Strong $\Rightarrow$ VF $\Leftrightarrow$ MP 1
$\Leftarrow$ + additional cdt on u.

③ a) Homog. Dirichlet BC ⇒ test = trial $= H_0^1(0,1)$

Test with $v \in H_0^1$, integrate by part to get 1

(VF) Find $u \in H_0^1$ s.t. $a(u,v) = \ell(v)$ $\forall v \in H_0^1$, where

$a(u,v) = \int_0^1 k(x)\, u'(x)\, v'(x)\, dx$ 1 and $\ell(v) = \int_0^1 f(x) v(x)\, dx$ 1

b) $a(\cdot,\cdot)$ is bilinear ok
$\ell(\cdot)$ is linear ok
H_0^1 is Hilbert ok
0.5

ℓ is bounded: $|\ell(v)| \overset{C-S}{\leq} \|f\|_{L^2} \cdot \|v\|_{L^2} \leq \underbrace{\|f\|_{L^2}}_{\leq \text{const} < \infty} \cdot \|v\|_{H^1}$ $\forall v \in H_0^1$ ⇒ ℓ bnd. 1

a is continuous / bounded:
$|a(u,v)| \underset{\substack{C-S \\ k \text{ bnd}}}{\leq} \|k\|_\infty \|u'\|_{L^2} \|v'\|_{L^2} \leq \underbrace{\|k\|_\infty}_{\leq \text{const} < \infty} \cdot \|u\|_{H^1} \cdot \|v\|_{H^1}$ $\forall u, v \in H_0^1$
ok 1

a is coercive : $a(u,u) = \int_0^1 k(x)|u'(x)|^2 dx \underset{\text{Hyp. } k}{\geq} \alpha \|u'\|_{L^2} \underset{\text{Poincaré}, u \in H_0^1}{\geq} C\|u\|_{L^2} \geq C \cdot \|u\|_{H^1}$ 1

H^1-norm equivalent to semi-norm $|u|_{H^1} = \|u'\|_{L^2}$ see lecture

$\underset{L\text{-}M}{\Rightarrow}$ $\exists!$ sol. of (VF) in $H_0^1(0,1)$. 0.5

(4) Let $w \in V_h$: $\langle u-\hat{u}, w\rangle_a \underset{\text{linearity}}{=} \langle u, w\rangle_a - \langle \hat{u}, w\rangle_a \underset{\text{Def VF / Galerkin}}{=} l(w) - l(w) = 0 \underset{\text{Def orthog.}}{\Rightarrow} u - \hat{u} \perp V$ 1 1

(5) a) Integrate $\Rightarrow u(x) = c_0 + c_1 x$ 1, Look at BC $\Rightarrow c_0 = -2, c_1 = 1 \Rightarrow$ solution reads $u(x) = x-2$. 1

b) Multiply with test fct $v \in V$, integrate by parts:

$$-\underset{\text{set}}{u'(2)}\,\underset{=0}{v(2)} + \underbrace{\overset{=1}{u'(0)}v(0)}_{\text{keep}} + \int_0^2 u'(x)v'(x)dx = 0 \quad \forall v \in V$$ 1

(VF) Find $u \in V$ s.t. $(u', v')_{L^2} = 0 - v(0)$ $\forall v \in V$, where $V = \{v \in H^1 \text{ s.t. } v(2) = 0\}$ 1

c) $0 = x_0 \underbrace{\qquad}_{h} x_1 = 2 \Rightarrow 1$ hat fct $\varphi_0(x)$ 1

(FE) Find $u_h = \xi_0 \varphi_0 \in \operatorname{span}\{\varphi_0\}$ s.t. $(u_h', \varphi_0')_{L^2} = -\varphi_0(0)$ 1

d) FE gives $\xi_0 (\varphi_0', \varphi_0')_{L^2} = -1$ 1 or $\xi_0 = -2$ hence $u_h(x) = x-2$ (exact sol. :-)) 1

(6) a) By linearity, w solves the linear PDE

$$\begin{cases} w_t - w_{xx} = 0 & \text{on } (0,1) \\ w(0,t) = 0 = w(1,t) \\ w(x,0) = 0 \end{cases}$$ 1

b) $\|w(\cdot,t)\|_{L^2} \leq 0 + \int_0^t 0\, ds = 0$ 1

(7) a) Test with $v \in H^1(\Omega)$

$$\int_\Omega f v \,dxdy = -\int_\Omega \nabla\cdot(a\nabla u) v \,dxdy + \int_\Omega buv\,dxdy \overset{\text{Green}}{=} \int_\Omega a\nabla u \cdot \nabla v \,dxdy - \int_{\partial\Omega} (n \cdot a\nabla u) v\,ds + \int_\Omega buv\,dxdy \overset{BC}{=}$$

$$= \int_\Omega a\nabla u \cdot \nabla v\,dxdy + \int_{\partial\Omega} \gamma(u-g)v\,ds + \int_\Omega buv\,dxdy$$

(VF) Find $u \in H^1$ s.t. $\int_\Omega a\nabla u \nabla v\,dxdy + \int_\Omega buv\,dxdy + \int_{\partial\Omega} \gamma uv\,ds = \int_\Omega fv\,dxdy + \int_{\partial\Omega} \gamma g v\,ds$ $\forall v \in H^1$ 1

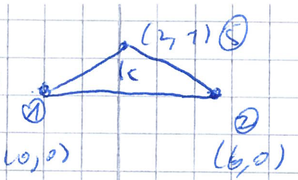
b) Find $u_h \in \bar{V}_h$ s.t. $\int_\Omega a\nabla u_h \nabla v_h\,dxdy + \int_\Omega b u_h v_h\,dxdy + \int_{\partial\Omega} \gamma u_h v_h\,ds = \int_\Omega f v_h\,dxdy + \int_{\partial\Omega} \gamma g v_h\,ds$ $\forall v \in \bar{V}_h$,

where $\bar{V}_h = \operatorname{span}(\{\varphi_j\}_{j=0}^{N_p}) \subset H^1$ and φ_j hat fct on given triangulation 1

c) Set $u_h = \sum_{j=0}^{N_h} \xi_j \varphi_j$ and take $v_h = \varphi_i$ in FE formulation, see lecture 1

$$A = \left(\int_\Omega a\nabla\varphi_j\nabla\varphi_i\right)_{ij=1}^{N_p}, \quad B = \left(\int_\Omega b\varphi_j\varphi_i\right)_{ij=1}^{N_p}, \quad M = \left(\int_{\partial\Omega} \gamma\varphi_j\varphi_i\right)_{ij=1}^{N_p}, \quad F = \left(\int_{\partial\Omega} \gamma g\varphi_i\right)_{i=1}^{N_p}$$ 1

$$R = \left(\int_\Omega f\varphi_i\right)_{i=1}^{N_p}$$

d) 

$A_{12} = \int_K \nabla\varphi_1 \cdot \nabla\varphi_2 \, dx\,dy \overset{P}{=} \dots = \frac{7}{36}\,\text{area}(K) = \frac{7}{36} \cdot \frac{6 \cdot 1}{2} = \frac{7}{12}$ ② ①

$\varphi_1(x,y) = 1 - \frac{1}{6}x - \frac{2}{3}y$

$\varphi_2(x,y) = \frac{1}{6}x - \frac{1}{3}y$

⑧ See lecture.

This is the definition of 2d linear Lagrange element.

Shape fct Φ_1, Φ_2, Φ_3 are given by $\Phi_j(x,y) = a_j + b_j x + c_j y$ (linear on K) ①

and

$L_i(\Phi_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ ①

Sol. are standard hat fct:

$\Phi_1(x,y) = 1 - x - y, \quad \Phi_2(x,y) = x, \quad \Phi_3(x,y) = y$ ②