Exam 13.03.23
TMA 373 + MMG 801

(1) a) discriminant $= 2^2 - 4\cdot 3\cdot 5 < 0 \Rightarrow$ elliptic

b) $\dim(\mathcal{P}^{(l)}(0,1)) = l+1$, application: space of shape fct. in $(K, \mathcal{P}, \Sigma)$, etc.

c) $\|u\|_{L^2(\Omega)} \leq C \cdot \|u'\|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega)$

d) $\|\pi_h f - f\|_{L^p(0,1)} \leq C \cdot h^2 \|f''\|_{L^p(0,1)}$ or $C\, h \|f'\|_{L^p}$ for $p = 2, 3$

e) $\int_1^3 \underbrace{x^2}_{f(x)} dx \approx (3-1) f\left(\frac{1+3}{2}\right) = 8$

f) $y_{n+1} = y_n + \frac{k}{2}\left(f(y_{n+1}) + f(y_n)\right)$

g) $\|u - u_h\|_{E} \leq C \cdot h$

h) Can provide error estimates of a desired tolerance at a possibly reduced cpu cost.

i) Yes, see lecture

j) Contains nodes of triangles and info on real/fake boundary

(2) 2nd term $\to$ integration by part and $H_0^1(\Omega) \Rightarrow$ homog. Dirichlet BC (1)

$74\, u'(x) - u''(x) + u(x) = f(x)$, $u(0) = 0 = u(1)$ (1)

(3) Dirichlet BC $\to$ $H = H_0^1(\Omega)$, set $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega \frac{\partial u}{\partial x_1} v\, dx$

and $L(v) = \int_\Omega f v\, dx$ (1)

Lax-Milgram: H Hilbert ✓, a bilinear ✓, L linear ✓ (1)

L bounded: $|L(v)| \overset{C\text{-}S}{\leq} \|f\|_{L^2} \|v\|_{L^2} \leq \underbrace{\|f\|_{L^2}}_{\leq \text{Const}} \|v\|_{H^1}$ ✓ (1)

a bounded: $|a(u,v)| \overset{CS}{\leq} \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|\frac{\partial u}{\partial x_1}\|_{L^2} \|v\|_{L^2} \leq 2 \cdot \|u\|_{H^1} \|v\|_{H^1}$ ✓ (1)

a coercive: $|a(u,u)| \underset{\text{Hint (detail } b)}{=} \int_\Omega |\nabla u|^2 dx \underset{\text{Poincaré}}{\geq} C \cdot \|u\|_{H^1}^2$ ✓ (1)

$\int_\Omega \frac{\partial u}{\partial x_1} u\, dx = \frac{1}{2} \int_\Omega \frac{\partial}{\partial x_1}(u^2) \cdot 1\, dx \overset{\text{Green}}{=} \frac{1}{2} \int_{\partial\Omega} u^2 n_1\, dx = 0$

$\Leftrightarrow$ $\exists!$ sol. $u \in H$.

(4) ODE: $\dot y(t) = y(t)^2$, $y(0) = 1$, Euler: $y_{n+1} = y_n + h y_n^2$ (1) + (1)

(5) a) Find $u \in H^1(\Omega)$ (1) with $u(0) = \alpha$, $u(1) = \beta$ s.t. $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in H_0^1$ (1)

b) Let $h = \frac{1}{m+1}$, T_h: $0 = x_0 < x_1 < \ldots < x_{m+1} = 1$, $V_h = \{v_h \in \mathcal{C}^0([0,1]) : v_h$ cont. p.w. lin. $\}$

$= \text{span}(\varphi_j)_{j=0}^{m+1}$, and $V_h^0 = \text{span}(\varphi_j)_{j=1}^{m}$ (1)

FE prob: Find $u_h \in V_h$ s.t. $(u_h', v_h')_{L^2} = (f, v_h)_{L^2} \quad \forall v_h \in V_h^0$ (1)

c) Write $u_h(x) = \sum_{j=0}^{m+1} \xi_j \varphi_j(x)$ where $\xi_0 = u(0) = \alpha$ and $\xi_{m+1} = u(1) = \beta \Rightarrow m$ unknown ξ_j (1)

Take $v_h = \varphi_i$ for $i = 1, \dots, m$ and get $S\xi = b$ where $S = (s_{ij})_{i,j=1}^{m}$, $b = (b_i)_{i=1}^{m}$ (1)

and $s_{ij} = (\varphi_j', \varphi_i')_{L^2}$ and $b_i = (f, \varphi_i)_{L^2}$ for $i = 2, \dots, m-1$, $b_1 = (f, \varphi_1)_{L^2} - \alpha(\varphi_0', \varphi_1')_{L^2}$ (2)

$b_m = (f, \varphi_m)_{L^2} - \beta(\varphi_{m+1}', \varphi_m')_{L^2}$

6) a) $\|u - u_h\|_{L^2}^2 = \int_\Omega |u - u_h|^2 dx \overset{(Aux)}{=} \int_\Omega \xi'(u' - u_h')dx \underset{\Pi_h \xi \in V_h^0}{\overset{(GO)}{=}} \int_\Omega (\xi - \Pi_h \xi)'(u' - u_h')dx$ (2)

b) $\|u - u_h\|_{L^2}^2 \overset{C-S}{\leq} \|(\xi - \Pi_h \xi)'\|_{L^2} \|(u - u_h)'\|_{L^2} \underset{\text{Interpolation}}{\leq} C \cdot h \|\xi''\|_{L^2} \|(u - u_h)'\|_{L^2} \leq$

$\underset{\text{Def } \xi''}{\leq} C \cdot h \|u - u_h\|_{L^2} \|(u - u_h)'\|_{L^2}$ (2)

7) a) Hom. Dirichlet BC $\Rightarrow H_0^1(\Omega)$ (1)

VF: Find $u(\cdot, t) \in H_0^1(\Omega)$ s.t. $(u_t(\cdot, t), v)_{L^2} + (\nabla u(\cdot, t), \nabla v)_{L^2} + (u(\cdot, t), v)_{L^2} = (f(\cdot, t), v)_{L^2} \quad \forall v \in H_0^1(\Omega)$ (1)

$u(x, 0) = u_0(x)$

b) $V_h^0 = \text{span}(\{\varphi_j\}_{j=1}^{n_i}) = \{v : \Omega \to \mathbb{R} : v \text{ cont. pw linear on triangulation } T_h \text{ and } v_{|\partial\Omega} = 0\}$ (1)

$n_i = \#$ Interior nodes

FE: Find $u_h(\cdot, t) \in V_h^0$ s.t. $(u_{h,t}(\cdot, t), v_h)_{L^2} + (\nabla u_h(\cdot, t), \nabla v_h)_{L^2} + (u_h(\cdot, t), v_h)_{L^2} = (f(\cdot, t), v_h)_{L^2} \quad \forall v_h \in V_h^0$ (1)

c) Since $V_h^0 \subset H_0^1 \rightarrow$ can take v_h as test fct in VF. (1)

VF−FE: $((u - u_h)_t, v_h)_{L^2} + (\nabla(u - u_h), \nabla v_h)_{L^2} + ((u - u_h), v_h)_{L^2} = 0$ ok. (1)

d) See lecture, $M = ((\varphi_j, \varphi_i)_{L^2})_{i,j=1}^{n_i}$, $S = ((\nabla\varphi_j, \nabla\varphi_i)_{L^2})_{i,j=1}^{n_i}$, $F = ((f, \varphi_i))_{i=1}^{n_i}$, $\xi(0) = (u_0(x_j))_{j=1}^{n_i}$

e) Take $v = u$ in VF: $\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 = (f, u)_{L^2} \overset{C-S}{\leq} \|f\|_{L^2}\|u\|_{L^2} \leq \frac{1}{2}\|f\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2$

Integrate $\int_0^T ds$: $\frac{1}{2}\|u(\cdot, T)\|_{L^2}^2 + \int_0^T \|\nabla u(\cdot, s)\|_{L^2}^2 ds + \frac{1}{2}\int_0^T \|u(\cdot, s)\|_{L^2}^2 ds \leq \frac{1}{2}\|u_0\|_{L^2}^2 + \frac{1}{2}\int_0^T \|f(\cdot, s)\|_{L^2}^2 ds$

8) a) $M = ((\varphi_j, \varphi_i)_{L^2(\Omega)})_{i,j=1}^{5}$ (1) b) $M_{K_1} = ((\varphi_j, \varphi_i))_{i,j=1,4,5}$, $M_{K_2} = ((\varphi_j, \varphi_i))_{i,j=1,2,4}$, $M_{K_3} = ((\varphi_j, \varphi_i))_{i,j=2,3,4}$ (2)

c) $\int_{K_3} \varphi_3 \varphi_3 dx$ (1) $\underset{\text{Ref triangle!}}{\overset{\Phi}{=}} \int_0^1 \int_0^{1-x} (1-x-y)(1-x-y)dy\,dx = \dots = \frac{1}{12}$ (0.5) (0.5)

9) Want $L_1(\Phi_1) = 1$, $L_2(\Phi_1) = 0$, $L_3(\Phi_1) = 0 \Rightarrow c = 1, a = 3, b = -4$ (1)

$\Rightarrow \Phi_1(x) = 3x^2 - 4x + 1$ (1)