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Chapter 10: FEM for heat equations in higher dimensions (summary)

February 26, 2025

Goal: Study the exact solution to the heat equation (stability), derive a FE discretisation for this PDE, and provide a semi-discrete error estimate for the FEM.

- Let $\Omega \subset \mathbb{R}^d$ be a nice domain with smooth or polygonal boundary. The solution to the inhomogeneous heat equation

$$\begin{cases} u_t - \Delta u = f \quad \text{in} \quad \Omega \times \mathbb{R}_+ \\ u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+ \\ u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega \end{cases}$$

  may be expressed thanks to Duhamel's formula/the variation of constants formula

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(s)\,\mathrm{d}s,$$

  where one writes $u(t)$ for $u(\cdot, t)$ and similarly for $f(s)$. Here, $E(t) = \mathrm{e}^{\Delta t}$ denotes the solution operator to the linear part of the above heat equation: $u_t - \Delta u = 0$.

- Since $\|E(t)v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}$ for any $v \in L^2(\Omega)$ and all $t > 0$, from the above one directly gets the stability estimates

$$\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} + \int_0^t \|f(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

  In addition, when $f \equiv 0$ (see page 263 in the book if interested in details), one has the following estimate

$$\int_0^t \|\nabla u(\cdot, s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \leq \frac{1}{2}\|u_0\|_{L^2(\Omega)}.$$

- The variational formulation of the above heat equation reads: For each $t > 0$ of interest,

  Find $\quad u(\cdot, t) \in H_0^1(\Omega), \quad$ such that $\quad (u_t(\cdot, t), v)_{L^2(\Omega)} + (\nabla u(\cdot, t), \nabla v)_{L^2(\Omega)} = (f(\cdot, t), v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$

  and $u(\cdot, 0) = u_0$ in Ω for the initial value. We also denote $a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)}$ for the energy inner product.

- Let now T_h denote a mesh of Ω and V_h the space of continuous piecewise linear functions of T_h. Consider the space $V_h^0 = \{v\colon \Omega \to \mathbb{R}:\ v \quad \text{continuous pw linear on } T_h \text{ and } v = 0 \text{ on } \partial\Omega\}$ and observe that $V_h^0 = \operatorname{span}(\{\varphi_j\}_{j=1}^{n_i})$, where n_i denotes the number of interior nodes. The finite element problem for the above heat equation reads: For each $t > 0$ of interest,

  Find $\quad u_h(\cdot, t) \in V_h^0(\Omega) \quad$ such that $\quad (u_{h,t}(\cdot, t), v_h)_{L^2(\Omega)} + (\nabla u_h(\cdot, t), \nabla v_h)_{L^2(\Omega)} = (f(\cdot, t), v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h^0(\Omega)$

  and $u_h(x, 0) = \pi_h u_0(x)$ in Ω for the initial value.

  As usual, writing $u_h(x, t) = \sum_{j=1}^{n_i} \zeta_j(t)\varphi_j(x)$ and taking $v_h = \varphi_i$ in the FE gives the system of linear ODEs

$$\begin{cases} M\dot{\zeta}(t) + S\zeta(t) = F(t) \\ \zeta(0) = \zeta_0. \end{cases}$$

  Finally, one can use the backward Euler scheme or the Crank–Nicolson scheme to find a numerical approximation of ζ at discrete times, see the $1d$ case.

- Under appropriate assumptions, one has the following a priori error estimate for the FE approximation of the heat equation

$$\|u_h(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)} \leq \|\pi_h u_0 - u_0\|_{L^2(\Omega)} + Ch^2\left(\|u_0\|_{H^2(\Omega)} + \int_0^t \|u_t(\cdot,s)\|_{H^2(\Omega)}\,\mathrm{d}s\right),$$

  where we recall that $\pi_h u_0$ denotes the continuous pw linear interpolant of u_0. Remembering a result on the interpolation error, the above says, more or less, that the error for the FEM for the inhomogeneous heat equation is of the size h^2.

- A useful tool in the proof of the above result, and in general, is the Ritz projection $R_h \colon H_0^1(\Omega) \to V_h^0(\Omega)$. This is defined as the orthogonal projection with respect to the energy inner product: For $v \in H_0^1(\Omega)$ one has

$$a(R_h v - v, \chi) = 0 \quad \forall \chi \in V_h^0.$$

  Under some assumptions on the domain $\Omega \subset \mathbb{R}^2$ and v, one has the estimate

$$\|R_h v - v\|_{L^2(\Omega)} \leq Ch^2 \|v\|_{H^2(\Omega)}.$$

Further resources:

- Duhamel at wikipedia.org
- Duhamel at pims.math.ca (link seems down?)
- FEM/VT for heat eq. at wikiversity.org
- FEM/exact sol. for heat eq. at wikiversity.org
- FEM/Euler-type scheme for heat eq. at wikiversity.org
- FEM for heat eq. with dolfinx
- Error estimates for FEM at math.uci.edu

Applications: Natural applications of the heat equations are: models for temperature changes of plates, simulation of temperature of rectangular or circular cakes link, or the study of chemical diffusion.