Exam 11.03.24, TMA373 + MMG801

1) a) Wave eq. $u_{tt} - u_{xx} = 0$ (1). Discriminant $d = B^2 - 4AC = 0 - 4(-1)(1) = 4 > 0$ (1)

b) Lagrange polynomials (1). $\dim(\mathbb{P}^{(1)}(0,1)) = 2$ (1)

c) $\|u\|_{L^2(\Omega)} \leq C \cdot \|\nabla u\|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega)$ (1)

d) $\int_1^3 f(x)\,dx \approx 3-1\, f\left(\frac{3+1}{2}\right) = 8$ (1)

e) $y_{n+1} = y_n + k f(y_{n+1})$ for $n = 0, 1, 2, \ldots$ (1)

f) $\|u - u_h\| \leq C \cdot h$ (1)

g) Point matrix lists coordinates of all nodes (1)

h) When approximating integrals by quadrature formulas in VF/FE (1)

i) Circle centered at $(-1, 0)$ (1)

j) $u_{xx}(x,y) \approx \dfrac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$ (1)

2) Use Lax-Milgram: V Hilbert space OK, (0.5) $a(\cdot,\cdot)$ bilinear OK, (0.5) $L(\cdot)$ linear OK (0.5) (0.5)

L bnd (1): $|L(v)| \underset{CS}{\leq} \underbrace{\|f\|_{L^2}}_{\leq C} \cdot \|v\|_{L^2} + \underbrace{|\beta|}_{\leq C} \underbrace{|v(1)|}_{\leq \|v\|_{C^0}} \leq C \cdot \|u\|_{H^1} + C \cdot \|v\|_{C^0} \leq C \cdot \|v\|_{H^1} \quad \forall v \in V$ (1)

Def H^1-norm, Hint

a bnd/cont: $|a(u,v)| \underset{C\text{-}S}{\leq} \|u'\|_{L^2} \cdot \|v'\|_{L^2} + \alpha \|u\|_{L^2} \cdot \|v\|_{L^2} \underset{\text{Def } H^1}{\leq} C \cdot \|u\|_{H^1} \|v\|_{H^1} \quad \forall u, v \in V$ (1)

a coercive: $a(u,u) = \int_0^1 (u'(x))^2\,dx + \alpha \int_0^1 (u(x))^2\,dx \geq \min(1,\alpha)\left(\|u'\|_{L^2}^2 + \|u\|_{L^2}^2\right) \geq C \cdot \|u\|_{H^1}^2 \quad \forall u \in V$ (1)

(L) L-M implies $\exists!$ sol.

3) a) Test with v: $\int_0^{10} f(x)v(x)\,dx = \int_0^{10} u'(x)v'(x)\,dx - u'(x)v(x)\Big|_{x=0}^{10}$

keep!

$\rightsquigarrow\ u'(10)v(10) - \overbrace{u'(0)v(0)}^{\text{sol}=0}$, with $u'(10) = 25$ (1)

(VF) Find $u \in H^1(I)$ with $u(0) = 0$ s.t. $\int_0^{10} u'(x)v'(x)\,dx = \int_0^{10} f(x)v(x)\,dx + 25\,v(10) \quad \forall v \in H^1(I)$ with $v(0) = 0$

b) Let m integer, $h = \frac{100}{m+1}$, uniform partition $x_0 = 0 < x_1 < \ldots < x_{m+1} = 100$ with $h = x_j - x_{j-1}$.

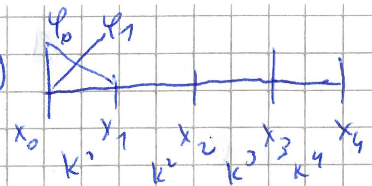
Set $V = \mathrm{span}\left(\{\varphi_j\}_{j=1}^{m+1}\right)$ with hat fct φ_j. (1)

(FE) Find $u_h \in V_h$ s.t. $\int_0^{10} u_h'(x)v_h'(x)\,dx = \int_0^{10} f(x)v_h(x)\,dx + 25\,v_h(10) \quad \forall v_h \in V_h$ (1)

c) Set $u_h(x) = \xi_1\varphi_1(x) + \xi_2\varphi_2(x)$ and take $v_h = \varphi_i$, $i = 1, 2$, in (FE):

$$\begin{pmatrix} (\varphi_1', \varphi_1')_{L^2} & (\varphi_1', \varphi_2')_{L^2} \\ (\varphi_2', \varphi_1')_{L^2} & (\varphi_2', \varphi_2')_{L^2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} (f, \varphi_1)_{L^2} \\ (f, \varphi_2)_{L^2} + 25\,\varphi_2(10) \end{pmatrix}$$ (1)

Def f and $\varphi_2 \Rightarrow (f, \varphi_2)_{L^2} + 25\,\varphi_2(10) = 0 + 25 \cdot 1 = 25$ (1) and $(\varphi_1', \varphi_1')_{L^2} = \int_0^5 \frac{1}{h^2}\,dx + \int_5^{10} \frac{1}{h^2}\,dx = \frac{2}{5}$ (1)

d) 

$$S_{K^1} = \begin{pmatrix} (\varphi_0', \varphi_0')_{L^2} & (\varphi_0', \varphi_1')_{L^2} \\ (\varphi_0', \varphi_1')_{L^2} & (\varphi_1', \varphi_1')_{L^2} \end{pmatrix} = S_{K^2} = S_{K^3} = S_{K^4}$$ (1) (1)

and adapt BC when assembling

(4) a) Difference of VF and FE with test fct $v = v_h \in V_h \subset V$ & $a(u-u_h, v_h) = b(v_h) - b(v_h) = 0$ (1)

This is Galerkin orthogonality (1)

b) $a(\cdot,\cdot)$ coercive $\Rightarrow a(u,u) \geq \alpha \|u\|_V^2 \Rightarrow \|u-u_h\|_V^2 \leq \frac{1}{\alpha} a(u-u_h, u-u_h) \overset{v_h \in V_h}{\leq}$ (1)

$\leq \frac{1}{\alpha} a(u-u_h, u-v_h+v_h-u_h) \underset{a \text{ bilinear}}{\leq} \frac{1}{\alpha}\big(a(u-u_h, u-v_h) + \underbrace{a(u-u_h, v_h-u_h)}_{\in V_h,\ =0 \text{ by a)}}\big) \underset{a \text{ bnd}}{\leq}$ (0.5) (0.5)

$\leq \frac{C}{\alpha} \|u-u_h\|_V \cdot \|u-v_h\|_V \Rightarrow \|u-u_h\|_V \leq \frac{C}{\alpha} \|u-v_h\|_V$

(5) Test ~~with~~ u and Green: $\int_\Omega f u\,dx = -\int_\Omega \Delta u\, u\,dx + \int_\Omega u u\,dx = \int_\Omega \nabla u \cdot \nabla u\,dx - \int_\Gamma \underbrace{n\nabla u}_{g}\, u\,ds + \int_\Omega u u\,dx$ (1)

Def L^2-norm $\Rightarrow \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \overset{CS}{\leq} \|f\|_{L^2}\|u\|_{L^2} + \|g\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)} \underset{\text{Trace}}{\overset{\text{Wint}}{\leq}} \|f\|_{L^2}^2 + \frac{1}{4}\|u\|_{L^2}^2 + C\|g\|_{L^2}^2 + \frac{1}{4}\|u\|_{H^1}^2$ (1)

Using def H^1-norm: $\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \leq C\cdot\left(\|f\|_{L^2}^2 + \|g\|_{L^2(\Gamma)}^2\right)$ (1)

(6) a) See lecture (1)

(VF) For $0 < t \leq T$, find $u(\cdot,t) \in H_0^1(\Omega)$ s.t. $(u_{tt}(\cdot,t), v)_{L^2} + (\nabla u(\cdot,t), \nabla v)_{L^2} + (u(\cdot,t), v)_{L^2} = (f(\cdot,t), v)_{L^2}\ \forall v \in H_0^1$ (1)

$u(x,0) = u_0(x),\ u_t(x,0) = v_0(x)$

b) See lecture (1)

(FE) For $0 < t \leq T$, find $u_h(\cdot,t) \in V_h^0$ s.t. $(u_{h,tt}(\cdot,t), v_h)_{L^2} + (\nabla u_h(\cdot,t), \nabla v_h)_{L^2} + (u_h(\cdot,t), v_h)_{L^2} = (f(\cdot,t), v_h)_{L^2}\ \forall v_h \in V_h^0$ (1)

$u_h(x,0) = \Pi_h u_0(x),\ u_{h,t}(x,0) = \Pi_h v_0(x)$

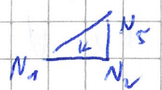
c) See lecture: $M_{ij} = (\varphi_j, \varphi_i)_{L^2}$, $S_{ij} = (\nabla\varphi_j, \nabla\varphi_i)_{L^2}$, $F_i(t) = (f(\cdot,t), \varphi_i)_{L^2}$, $\xi_j(0) = u_0(x_j)$, $\dot\xi_j(0) = v_0(x_j)$ (1) (1)

for $i,j = 1,2,\dots,n_i$

d) $\eta(t) = \dot\xi(t) \Rightarrow \begin{cases} M\dot\xi(t) = M\eta(t) \\ M\dot\eta(t) = (-S-M)\xi(t) \end{cases}$ (1) (1) $\overset{C-N}{\Longrightarrow}$ $M\dfrac{\xi^{(n+1)}-\xi^{(n)}}{k} = \dfrac{M}{2}\left(\eta^{(n+1)}+\eta^{(n)}\right)$ (1)

$M\dfrac{\eta^{(n+1)}-\eta^{(n)}}{k} = \dfrac{(-S-M)}{2}\left(\xi^{(n+1)}+\xi^{(n)}\right)$

(7) a) $M = \left((\varphi_i, \varphi_j)_{L^2(\Omega)}\right)_{i,j=1}^{9}$ with φ_i hat on node N_i (1)

b) For 

$M_K = \left((\varphi_i, \varphi_j)_{L^2(K)}\right)_{i,j=1,2,5}$ and $S_K = \left((\nabla\varphi_i, \nabla\varphi_j)_{L^2(K)}\right)_{i,j=1,2,5}$ (1) (1)

Same for all 8 elements

(8) Linear element $\Rightarrow P = \{(a+bx)\cdot(c+dy) : a,b,c,d \in \mathbb{R}, (x,y) \in K\}$, $\dim(P) = 4 \Rightarrow$ (1)

$\Sigma = \{L_1, L_2, L_3, L_4\}$ with $L_j(p) = p(N_j)$ (nodal basis) for $j = 1,2,\dots,4$ and $p \in P$ (1)

$\varphi_1(x,y) = (a+bx)\cdot(c+dy)$ must statisfy $L_j(\varphi_1) = \delta_{j1}$ for $j = 1$ and $= 0$ else (1)

$\Rightarrow \varphi_1(x,y) = \frac{1}{4}(1-x)(1-y)$ (1)