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Chapter 1: Terminology (summary)

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Goal: Present/recall notions and problems we shall consider in the lecture.

- A differential equation (DE) is an equation that relates an unknown function (or more) and its derivative(s).
- An ordinary differential equation (ODE) is a DE, where the unknown function depends only on *one* variable (say $y(x)$ or $x(t)$ for instance).
- To determine a unique solution to an ODE, one needs to specify additional conditions:

  An initial value problem (IVP) consists of an ODE with an initial value or initial condition. The Malthusian growth model for bacteria reads (t_0, T, P_0 and λ are given, $t \in [t_0, T]$, and $P(t)$ is unknown)

  $$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} P(t) = \lambda P(t) \\ P(t_0) = P_0. \end{cases}$$

  Here, P_0 is the size of the initial population of bacteria and $P(t)$ describes the size of the population at time t.

  A boundary value problem (BVP) consists of an ODE with boundary conditions. For instance ($u(x)$ is unknown)

  $$\begin{cases} -u''(x) + 4u(x) = \cos(x) \quad \text{for} \quad x \in (0,1) \\ u(0) = 0 \quad \text{and} \quad u(1) = 5. \end{cases}$$

  Here, one specifies the values of the solution $u(x)$ at the boundaries 0 and 1.

- A partial differential equation (PDE) is a DE, where the unknown function depends on *2 or more* variables (say $u(x, y)$ or $u(t, x, y, z)$ for instance).
- Typical examples of PDEs are

  Laplace's equation

  $$\Delta u = 0,$$

  with the Laplace operator Δ defined by $\Delta u(x) = \sum_{k=1}^{n} u_{x_k,x_k}(x)$ for $x = (x_1, x_2, \ldots, x_n)$ and $u\colon \mathbb{R}^n \to \mathbb{R}$.

  In $2d$, the above reads $u_{x_1,x_1}(x_1, x_2) + u_{x_2,x_2}(x_1, x_2) = 0$ or (other notation) $u_{xx}(x, y) + u_{yy}(x, y) = 0$.

  The heat equation

  $$u_t - \Delta u = f.$$

  In $1d$, the above reads $u_t(x, t) - u_{xx}(x, t) = f(x)$, where $u(x, t)$ could describe the temperature at time t and position x of a thin wire (more on this later).

  The wave equation

  $$u_{tt} - \Delta u = g.$$

  In $1d$, the above reads $u_{tt}(x, t) - u_{xx}(x, t) = g(x)$, where $u(x, t)$ could describe the motion of a guitar string at time t and position x on the string (more on this later).

- To specify a unique solution to a PDE, one also needs additional conditions. For the example of the $1d$ heat equation on $[0,1]$, one gets

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x) \quad \text{for} \quad x \in (0,1), t \in (0,T] \\ u(0,t) = 0, u(1,t) = \sin(t) \quad \text{for} \quad t \in (0,T] \\ u(x,0) = 3x \quad \text{for} \quad x \in (0,1). \end{cases}$$

  The last condition is an initial value/condition (it specifies the initial temperature profile). The other conditions are boundary conditions (they specify the value of the temperature at the ends of the wire). Observe that one has one initial value (since the DE has one derivative in time) and two boundary conditions (since the DE has two derivatives in space).

- Finally, we provide a classification of linear second-order PDE with constant coefficients

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

  where $u = u(x,y)$ and $A, B, C, D, E, F, G \in \mathbb{R}$ are given. Looking at the discriminant $d = B^2 - 4AC$, one says that the above problem is

  1. elliptic if $d < 0$ (Laplace equation for instance)
  2. parabolic if $d = 0$ (heat equation for instance)
  3. hyperbolic if $d > 0$ (wave equation for instance).

  This classification can be adapted if the coefficients $A = A(x,y), B = B(x,y), \ldots$.

Further resources:

- DE at wikipedia.org/
- ODE at sv.wikipedia.org
- PDE at sv.wikipedia.org
- DE at tutorial.math.lamar.edu
- DE at tutorial.math.lamar.edu/DE2
- DE at khanacademy.org
- DE at analyzemath.com

Applications: ODE are used in thermodynamics, classical mechanics, medicine, population dynamics, electrical engineering, etc. see for instance link. PDE are used in fluid dynamics, quantum mechanics, electrodynamics, elasticity, etc. see for instance link.