# Examination, 8 June 2023
# TMA373 and MMG801

**Read this before you start!**
*I'll try to come at ca. 15:15.*
*Aid: Chalmers approved calculators.*
*Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.*
*I tried to use the same notation as in the lecture.*
*Answers may be given in English, French, German or Swedish.*
*Write down all the details of your computations clearly so that each steps are easy to follow.*
*Do not randomly display equations and hope for me to find the correct one. Justify your answers.*
*Write clearly what your solutions are and in the nicest possible form.*
*Don't forget that you can verify your solution in some cases.*
*Use a proper pen and order your answers if possible.**Thank you!***
*No need to use one piece of paper for only one exercise.*
*The test has 5 pages and a total of 50 points.*
*Preliminary grading limits: 3:20-29p, 4:30-39p, 5:40-50p (Chalmers) and G:20-34p, VG:35-50p (GU). Valid bonus points will be added to the total score if needed.*
*You will be informed via Canvas when the exams are corrected.*
*Good luck!*

Some exercises were taken from, or inspired by, materials from *A. Szepessy*.

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1. Provide concise answers to the following short questions:

   (a) Is the following PDE $u_{xx}(x,y)+u_{yy}(x,y) = x^2+y^3$ hyperbolic, elliptic or parabolic? (1p)

   (b) What is the dimension of the polynomial space $\mathcal{P}^{(3)}(0,1)$? Give an application of the polynomial space $\mathcal{P}^{(l)}(0,1)$, for a positive integer $l$. (1p)

   (c) Consider a uniform partition of $[0,1]$ with mesh $h$. What is the size (in term of $h$) of the error of the continuous piecewise linear interpolant $\left\|\pi_h f - f\right\|_{L^2(0,1)}$ for a nice function $f$? (1p)

   (d) Use (one step of) the trapezoidal rule to approximate the area under the function $f(x) = x^2$ between $x = 1$ and $x = 3$. (1p)

   (e) What is an estimate for the a priori error estimate, measured in the energy norm, of the (continuous and piecewise linear) FEM with mesh size $h$ for the model problem (Poisson's equation in 1$d$ with homogeneous Dirichlet BC) seen in the lecture? (1p)

   (f) Which type of error estimates can be used for adaptivity? (1p)

   (g) What is the point matrix in a FEM implementation? (1p)

2. Let $\Omega = (0,1)$, $\alpha, \beta \in \mathbb{R}$, and $c, f \colon \Omega \to \mathbb{R}$ be nice functions. Consider the problem: Find $u \in H^1(\Omega)$ such that
$$\int_\Omega u'(x)v'(x)\,\mathrm{d}x + \int_\Omega c(x)u(x)v(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x + \beta v(1) - \alpha v(0) \quad \text{for all } v \in H^1(\Omega).$$
Formulate the strong problem corresponding to the above variational formulation. (2p)

3. Let $V$ be an Hilbert space with inner product and norm denoted by $(\cdot,\cdot)_V$ and $\|\cdot\|_V$. On this space, let a bilinear form $a(\cdot,\cdot)$ and a functional $\ell(\cdot)$ verifying the assumptions of Lax–Milgram that we recall: There exist $\alpha > 0, \beta \geq 0, \kappa > 0$ such that
$$\begin{aligned} |a(u,v)| &\leq \alpha \|u\|_V \|v\|_V \quad && \forall u,v \in V \\ a(u,u) &\geq \kappa \|u\|_V^2 \quad && \forall u \in V \\ |\ell(v)| &\leq \beta \|v\|_V \quad && \forall v \in V. \end{aligned}$$
Consider the variational problem: Find $u \in V$ such that
$$a(u,\varphi) = \ell(\varphi) \quad \forall \varphi \in V.$$
Let now $V_h \subset V$ be a finite dimensional subspace of $V$ and $u_h \in V_h$ the solution to Galerkin's equation
$$a(u_h,\varphi_h) = \ell(\varphi_h) \quad \forall \varphi_h \in V_h.$$
   (a) Show that the discrete solution $u_h$ exists and is unique in $V_h$. (3p)
   (b) Show the following bound
   $$\|u - u_h\|_V \leq \frac{\alpha}{\kappa} \left\|u - \varphi_h\right\|_V$$
   for all $\varphi_h \in V_h$. (3p)

4. Consider the following pseudo-code

```
function [t,y] = solveit(t0, y0, T, n)
  t = zeros(n+1,1); y = zeros(n+1,1);
  t(1) = t0; y(1) = y0; h = (T-t0)/n;
  for i = 1:n
    t(i+1) = t(i)+h;
    y(i+1) = y(i)+h*(y(i+1))^2;
  end
```

Suppose that the input values are $t0 = 0, y0 = 1, T = 1$, and $n = 10$.

What is the initial-value problem being approximated numerically? What is the numerical method being used? (2p)

5. Let $a, r > 0$ and $f \in L^2(0,1)$. Consider the BVP

$$\begin{cases} -au''(x) + ru(x) = f & \text{in } \Omega = (0,1) \\ u(0) = 0, u(1) = 0. \end{cases}$$

   (a) Derive the variational formulation of this BVP (define all quantities). (2p)

   (b) Derive the corresponding cG(1) finite element problem (define all quantities). (2p)

   (c) Let now $a = r = 1$ and $f(x) = 1$ for $x \in (0,1)$. Give the linear system of equations for the computation of the finite element solution. You don't need to compute the integals in the matrices but must give all coordinates of the right-hand side vector ("load vector"). (4p)

6. Let $\Omega = (0,1)$, $f \in L^2(\Omega)$, and the space $H_0^2(\Omega) = \{v \in H^2(\Omega) : v(0) = v(1) = 0, v'(0) = v'(1) = 0\}$. Consider the variational problem

$$\text{Find } u \in H_0^2(\Omega) \text{ such that } \int_\Omega u''(x)v''(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^2(\Omega).$$

   (a) First, show that

$$\|v'\|_{L^2(\Omega)} \leq C\,\|v''\|_{L^2(\Omega)}$$

   for all $v \in H_0^2(\Omega)$. (2p)

   *Hint: One may use Poincaré inequality.*

   (b) Observing that $H_0^2(\Omega) \subset H_0^1(\Omega)$ and the above inequality, show that

$$\|v\|_{H^2(\Omega)} \leq C\,\|v''\|_{L^2(\Omega)}$$

   for all $v \in H_0^2(\Omega)$. (2p)

   *Hint: One may use Poincaré inequality.*

   (c) Let a uniform partition of $\Omega$ with mesh size $h$. Consider now the finite element space $W_{0h} = \{v_h \in C^1(\bar{\Omega}) : v_h|_{[x_j, x_{j+1}]} \in P^3 \quad \text{and} \quad v_h(0) = v_h(1) = 0, v_h'(0) = v_h'(1) = 0\}$, where $P^3$ is the set of polynomials of degree $\leq 3$ on an element. Denote by $u_h$ the corresponding finite element approximation to the above given variational problem.

   Using the fact that

$$\|u - u_h\|_{H^2(\Omega)} \leq C\,\|u - \pi_h u\|_{H^2(\Omega)} \quad \text{and} \quad \|(v - \pi_h v)''\|_{L^2(\Omega)} \leq Ch^2\,\|v''''\|_{L^2(\Omega)}$$

   for $v \in H^4(\Omega) \cap H_0^2(\Omega)$ and the interpolation operator $\pi_h$ and the fact that $u'''' = f$ (in $L^2(\Omega)$), show the final error estimate

$$\|u - u_h\|_{H^2(\Omega)} \leq Ch^2\,\|f\|_{L^2(\Omega)}. \quad \text{(2p)}$$

7. Let $\Omega \subset \mathbb{R}^2$ be a nice domain, $f, u_0, v_0 \colon \Omega \to \mathbb{R}$ be nice, $T > 0$. Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u + u = f \quad \text{in } \Omega \times (0,T] \\ u = 0 \quad \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) \quad \text{in } \Omega \\ u_t(x,0) = v_0(x) \quad \text{in } \Omega. \end{cases}$$

(a) Write the variational formulation of the above problem in a suitable space $V$ (don't forget to define this space). (2p)

(b) Formulate the finite element problem based on a cG(1) approximation on the space $V_h^0$ (don't forget to define this space). (2p)

(c) From the above, derive the linear system of ordinary differential equations

$$M\ddot{\zeta}(t) + S\zeta(t) + M\zeta(t) = F(t), \text{with initial value } \zeta(0), \dot{\zeta}(0).$$

Don't forget to provide the entries of the matrices $M, S$ and of the vectors $F(t), \zeta(0), \dot{\zeta}(0)$ (you don't need to compute the integrals). (2p)

8. Let $i = \sqrt{-1}$ and a given (complex) initial value $u_0$. Consider the linear Schrödinger equation (for $x \in [0,1]$ and $t \in [0,T]$, where $T > 0$)

$$\begin{aligned} iu_t(x,t) - u_{xx}(x,t) &= 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

with homogeneous Dirichlet boundary conditions.

Show that the (square of the) $L^2$-norm of the solution $\|u(\cdot,t)\|_{L^2(0,1)}^2$ is a conserved quantity for all time $t > 0$. This can be proven directly or you may start by writing the complex-valued function $u$ as $u = v + iw$ with two real-valued functions $v$ and $w$ and get a system of linear PDEs for $(v, w)$. (3p)

*<u>Hint</u>: For complex-valued functions $f, g$ the $L^2$-inner product reads*

$$(f,g) = \int f(x)\bar{g}(x)\,dx.$$

*Try to inspire yourself with what we did for the linear wave equation in the lecture.*

9. Let $u \colon [0,1] \times [0,1] \to \mathbb{R}$ be the solution to the problem

$$\begin{cases} -\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y) \quad \text{in } \Omega = [0,1] \times [0,1] \\ u(0,y) = 0 \quad \text{for } y \in [0,1] \\ \frac{\partial u}{\partial n}(x,y) = 0 \quad \text{for } (x,y) \in (\{1\} \times [0,1]) \cup ((0,1] \times \{0,1\}), \end{cases}$$

where $f \colon [0,1] \times [0,1] \to \mathbb{R}$ is a given function and $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of $u$.

(a) Give the variational formulation of the above problem (define all quantities). (2p)

(b) Give the corresponding finite element formulation for the approximate (cG(1)) solution $u_h(x,y)$ (define all quantities). (2p)

(c) Let now $f(x,y) = 1$ for $(x,y) \in \Omega$. Compute the (two by two) linear system of equations from the above FE formulation as well as the above FE approximation $u_h$ for the triangulation of $\Omega$ given by the two triangles:

$$K_1 = \{(x,y)\colon x < y < 1, x \in [0,1]\} \quad \text{and} \quad K_2 = \{(x,y)\colon 0 \le y \le x, x \in [0,1]\}.$$ (6p)