# Examination, 5 June 2024
# TMA373 and MMG801

**Read this before you start!**
*I'll try to come at ca. 15:00 and 16:20. You can ask for calling me (0317723021) in case of questions.*
*Aid: Chalmers approved calculators.*
*Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.*
*I tried to use the same notation as in the lecture.*
*Answers may be given in English, French, German or Swedish.*
*Write down all the details of your computations clearly so that each steps are easy to follow.*
*Do not randomly display equations and hope for me to find the correct one. Justify your answers.*
*Write clearly what your solutions are and in the nicest possible form.*
*Don't forget that you can verify your solution in some cases.*
*Use a proper pen and order your answers if possible.**Thank you!***
*Feel free to write more than one solution on one piece of paper (save paper).*
*The test has 4 pages and a total of 50 points.*
*Preliminary grading limits: 3:20-29p, 4:30-39p, 5:40-50p (Chalmers) and G:20-34p, VG:35-50p (GU). Valid bonus points will be added to the total score if needed.*
*You will be informed via Canvas when the exams are corrected.*
*Good luck!*

Some exercises were taken from, or inspired by, materials from *M. Asadzadeh*, *C. Cotter*, *P. Frey*.

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1. Provide concise answers to the following short questions:

   (a) Give an example of an elliptic PDE. Justify your answer. (2p)

   (b) Give a nodal basis to the space of polynomials $\mathcal{P}^{(1)}(0,1)$ (a name is enough if you don't remember the formulas)? What is the dimension of this space? (2p)

   (c) Use (one step of) the trapezoidal rule to approximate the area under the function $f(x) = x^2$ between $x = 1$ and $x = 3$. (1p)

   (d) Give the general formula for the explicit Euler scheme, with time step $k$, when applied to the IVP $\dot{y}(t) = f(y(t)), y(0) = y_0$. (1p)

   (e) What is an estimate for the a priori error estimate, measured in the energy norm, of the (continuous and piecewise linear) FEM with mesh size $h$ for the model problem (Poisson's equation in 1$d$ with homogeneous Dirichlet BC) seen in the lecture? (1p)

   (f) What is the connectivity matrix in a FEM implementation? (1p)

   (g) Give an example of a variational crime. (1p)

   (h) Draw the stability region of the implicit Euler scheme. Don't forget the axes. (1p)

   (i) Give a finite difference approximation of $u_{yy}(x, y)$. (1p)

2. Let $f \in L^2(0,1)$. Consider the BVP

$$\begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = 0, u(1) = 0. \end{cases}$$

   (a) Give the variational formulation of this BVP. Don't forget to define the test and trial spaces as well as the forms $a(u,v)$ and $\ell(v)$. (3p)

   (b) Verify that the assumptions of the Lax–Milgram theorem are satisfied for this problem. What can you conclude from this? (4p)

3. Consider the boundary value problem

$$\begin{cases} -u''(x) + u(x) = 1 & \text{for} \quad x \in (0,1) \\ u'(0) = 3, \quad u'(1) = 0. \end{cases}$$

   (a) State the variational formulation to the above problem. Don't forget to define all quantities. (2 p)

   (b) Give the corresponding Galerkin (linear) FE problem. Don't forget to define all quantities. (2 p)

   (c) Derive the resulting linear system of equations $A\zeta = b$, where the components of the vector $\zeta$ are the coordinates of the FE solution $u_h(x)$ in the basis of the FE space. Compute the first and last components of the vector $b$. (You don't need to compute the other components of the vector $b$ and of the matrix $A$.) (5 p)

4. Let $T_h = \{K\}$ be a nice triangulation of a nice domain $\Omega \subset \mathbb{R}^2$. Define the space $V_h = \{v_h \in C^0(\Omega) : v_h|_K \in \mathcal{P}^{(1)}(K), K \in T_h\}$. Let $n_p$ denote the number of nodes in the triangulation $T_h$.

   (a) Give a basis for the space $V_h$. (1p)

   (b) Consider a continuous function $f$ on a triangle $K$ with nodes $N_i$ for $i = 1,2,3$. Denote by $\{\varphi_j\}_j$ the hat functions on this triangle. How do you define the linear interpolant $\pi_1 f \in \mathcal{P}^{(1)}(K)$ of $f$? (1p)

   (c) Let now $f \in H^2(\Omega)$ and denote by $\pi_h f \in V_h$ the continuous piecewise linear interpolant of $f$. Show that

$$\left\| f - \pi_h f \right\|_{L^2(\Omega)}^2 \leq C h^4 \left\| f \right\|_{H^2(\Omega)}^2,$$

   where $C_K \leq C$ for all triangles $K \in T_h$ and $h = \max_K h_K$. (2p)
   *Hint: It is known that $\left\|\pi_1 f - f\right\|_{L^2(K)} \leq C_K h_K^2 \left\|f\right\|_{H^2(K)}$, where $h_K$ is the largest edge of the triangle K.*

5. Let $d = 2,3$. Consider $\Omega \subset \mathbb{R}^d$ a nice domain, and $g\colon \Gamma \to \mathbb{R}$ a nice function defined on the boundary $\Gamma = \partial\Omega$ of the domain. Consider the PDE

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ n \cdot \nabla u + u = g, & \text{on } \Gamma, \end{cases}$$

where $n$ in the outward unit normal to $\Gamma$. Show the following stability estimate:

$$\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Gamma)} \leq \frac{1}{2}\|g\|^2_{L^2(\Gamma)}. \qquad \text{(3p)}$$

*Hint: You may test the PDE with an appropriate test function and use Green's formula. The inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, for $a, b \in \mathbb{R}$, may be of interest.*

6. Let $\Omega \subset \mathbb{R}^2$ be a nice domain, $f, u_0 \colon \Omega \to \mathbb{R}$ be nice, $T > 0$. Consider the inhomogeneous heat equation

$$\begin{cases} u_t - \Delta u + u = f \quad \text{in } \Omega \times (0,T] \\ u = 0 \quad \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) \quad \text{in } \Omega. \end{cases}$$

   (a) Write down the variational formulation of the above PDE (define all quantities). (2p)

   (b) Formulate the finite element problem based on a cG(1) approximation on the space $V_h^0$ (don't forget to define this space). (2p)

   (c) From the above, derive the linear system of ordinary differential equations

$$M\dot{\zeta}(t) + S\zeta(t) + M\zeta(t) = F(t), \text{with initial value } \zeta(0).$$

   Don't forget to provide the entries of the matrices $M, S$ and of the vectors $F(t), \zeta(0)$ (you don't need to compute the integrals). (3p)

   (d) Apply the implicit Euler scheme to the above system of ordinary differential equations. Don't forget to define all quantities. (1p)

7. Let $\Omega = (-1,1) \times (-1,1)$ be the square from Figure 1. Consider the problem

$$\begin{cases} -\Delta u(x) = 0 \quad \text{for } x \in \Omega \\ n \cdot \nabla u(x) = 0 \quad \text{for } x \in \partial\Omega, \end{cases}$$

where $x = (x_1, x_2)$ and $n$ in the outward unit normal to $\partial\Omega$. The goal of this exercise is to compute the stiffness matrix of the linear (cG(1)) Galerkin finite element method.

   (a) Denote by $V_h$ the space of cG(1) FE on this triangulation of $\Omega$. What is the dimension of $V_h$? (1p)

   (b) Using the numbering of Figure 1, define the global stiffness matrix $A$ (no need to compute the entries of the matrix yet). Don't forget to give the dimensions of this matrix. (2p)

   (c) Using properties of the hat functions (and symmetries in the matrix $A$), one observes that only some coefficients $A_{ij} = \int_\Omega \nabla\varphi_i(x) \cdot \nabla\varphi_j(x)\,\mathrm{d}x$ are non-zero. Explain why this is the case. (2p)

   (d) Compute the coefficients $A_{1,1}$ and $A_{9,9}$ of the stiffness matrix. (3p)

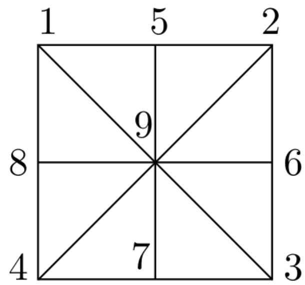


Figure 1: Courtesy from P. Frey.

8. Provide the nodal basis function $\Phi_1(x)$ for the finite element $\left(K=[0,1], P^{(2)}(K), \Sigma\right)$, where $P^{(2)}(K)$ denotes the set of polynomials of degree less or equal to two and $\Sigma=(L_1, L_2, L_3)$ is given by

$$L_1(f)=f'(0), \quad L_2(f)=f'(1), \quad L_3(f)=\int_0^1 f(x)\,\mathrm{d}x. \qquad \text{(2p)}$$

<u>*Hint*</u>*: Determine the constants $a, b, c$ in $\Phi_1(x)=ax^2+bx+c$.*