David Cohen (david.cohen@chalmers.se) Chalmers & GU

Chapter 12: The finite element (summary)

March 3, 2025

Goal: Study the concept of finite element.

- A finite element consists of the triplet (K, P, Σ), where
  - $K \subset \mathbb{R}^d$ is a polygon
  - P is a polynomial function space on K (of finite dimension)
  - Σ is a (P-unisolvent) set of linear functionals on P (that is maps from P to $\mathbb{R}$): $\Sigma = \{L_1, L_2, \ldots, L_n\}$, where $n = \dim(P)$. *Technical notes: Being unisolvent means, more or less, that one can find n linearly independent polynomials with $L_j(p_i) = \delta_{ij}$.*

  K is the element domain: line in $1d$, triangle or quadrilateral in $2d$, brick in $3d$, etc.

  P is the space of shape functions: $P^{(1)}(K)$ the set of polynomials of degree at most 1 on K, $P^{(2)}(K)$ the set of polynomials of degree at most 2 on K, etc.

  Σ is the set of nodal variables: This set uniquely specifies the basis functions/shape functions on each polygon K as well as the behaviour of these functions between adjacent polygons.

- Examples of finite elements are:
  - $1d$ Lagrange $P^{(k)}$ elements: Let $a < b$ and distinct points $x_0 = a < x_1 < \ldots < x_k = b$. The polygon K is the interval $[a, b]$, $P = P^{(k)}(a, b)$ is the set of polynomials of degree less or equal to k on $[a, b]$, and $\Sigma = \{L_0, L_1, \ldots, L_k\}$ with L_j defined by $L_j : P \to \mathbb{R}$ and $L_j(f) = f(x_j)$ for $j = 0, 1, \ldots, k$.
  - $2d$ linear Lagrange element: Here, K is the reference triangle, $P = P^{(1)}(K)$ the set of linear polynomials on K, and $\Sigma = \{L_1, L_2, L_3\}$ defined by $L_1(f) = f(0, 0)$, $L_2(f) = f(1, 0)$, and $L_3(f) = f(0, 1)$ for any $f \in P$. One then determines the shape functions $\{\varphi_j\}_{j=1}^3$ by the conditions $\varphi_j(x, y) = a_j + b_j x + c_j y$ and $L_i(\varphi_j) = \delta_{ij}$. This provides the hat functions seen in earlier chapters: $\varphi_1(x, y) = 1 - x - y$, $\varphi_1(x, y) = x$, $\varphi_1(x, y) = y$.

- Examples of more exotic finite elements are MWX elements.

- Using higher oder FE for Poisson's equation

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

  where $\Omega \subset \mathbb{R}^2$ is a convex polygon gives higher rates of convergence: Assume that $u \in H^{p+1}(\Omega)$ and consider the FE approximation u_h based on a mesh $T_h = \{K\}$ and $V_h^0 = \{v \in C^{(0)}(\bar{\Omega}) : v|_K \in P^{(p)}(K)\ \forall K \in T_h,\ v|_{\partial\Omega} = 0\}$. Then, the error of the FE reads

$$\|u - u_h\|_{H^1(\Omega)} \leq C h^p |u|_{H^{p+1}(\Omega)}.$$

- Variational crimes consist of errors done in a VF:

  Consider the variational problem:

$$\text{Find} \quad u \in U \quad \text{such that} \quad a(u, v) = \ell(v) \quad \forall v \in V.$$

In reality, one works with the following finite element problem

$$\text{Find} \quad u_h \in U_h \quad \text{such that} \quad a_h(u_h, \chi) = \ell_h(\chi) \quad \forall \chi \in V_h,$$

where the index h denotes possible errors coming from numerical integrations, triangulations, etc. Error estimates have to be extended in this situation, doable but not easy at all.

Further resources:

- FEM at wikipedia.org
- Finite Element Analysis at simscale.com
- Intro to FE at math.tamu.edu
- Shape functions at ethz.ch
- Variational crimes at youtube.com

Applications: See the rest of the lecture :-)