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Chapter 5: FEM for two-point BVP (summary)

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Goal: Present and analyse FEM for several classical BVPs.

- Consider the BVP

$$\begin{cases} -(a(x)u'(x))' = f(x) \quad \text{for} \quad x \in (0,1) \\ u(0) = 0, u(1) = 0, \end{cases}$$

where the given functions f, a are nice (for instance f is continuous or in $L^2(0,1)$, $a(x) \geq \alpha_0 > 0$ continuous or piecewise continuous on $(0,1)$ and bounded on $[0,1]$). The above boundary conditions are of the type homogeneous Dirichlet BC. This BVP serves as a simple model for the elastic deformation of a bar, where the solution $u(x)$ denotes the displacement of the bar.

In a nutshell, a Galerkin finite element method (FEM) for the above BVP is given by following the steps:

1. Multiply the DE by a test function $v \in H_0^1 = \{v\colon [0,1] \to \mathbb{R}\colon\ v, v' \in L^2(0,1) \text{ and } v(0) = v(1) = 0\}$. Integrate the above over the domain $[0,1]$ and get the variational formulation of the problem.
2. Specify the finite dimensional space $V_h^0 \subset H_0^1$, here defined as $V_h^0 = \text{span}(\varphi_1, \ldots, \varphi_m)$, where we recall that φ_j are the hat functions. This allows to consider the FE problem.
3. Insert the ansatz

$$u_h(x) = \sum_{j=1}^{m} \zeta_j \varphi_j(x)$$

for the FE solution into the FE problem and take test functions $v_h = \varphi_i$, for $i = 1, \ldots, m$, to get a linear system of equations, of the form $S\zeta = b$, for the unknown $\zeta = (\zeta_1, \ldots, \zeta_m)$.

For the above BVP we obtain the variational formulation (VF)

$$\text{Find} \quad u \in H_0^1 \quad \text{such that} \quad \int_0^1 a(x)u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \quad \text{for all} \quad v \in H_0^1.$$

The corresponding FE problem (FE) reads

$$\text{Find} \quad u_h \in V_h^0 \quad \text{such that} \int_0^1 a(x)u_h'(x)v_h'(x)\,\mathrm{d}x = \int_0^1 f(x)v_h(x)\,\mathrm{d}x \quad \forall\, v_h \in V_h^0.$$

This type of FEM is called a cG(1) FEM, for continuous Galerkin (using pw linear approximation).

Using the definition of the hat functions, the FE problem can be written as the linear system of equations

$$S\zeta = b.$$

Here, $S = (s_{i,j})_{i,j=1}^m$ is termed the stiffness matrix. This tridiagonal matrix has entries

$$s_{ij} = \int_0^1 a(x)\varphi_i'(x)\varphi_j(x)'\,\mathrm{d}x.$$

The vector $b = (b_i)_{i=1}^m$ is termed the load vector (with entries $b_i = (f, \varphi_i)_{L^2(0,1)}$). If no explicit formulas for these integrals can be found, one may use quadrature formulas seen in a previous chapter.

- Observing that $V_h^0 \subset H_0^1$, one gets Galerkin orthogonality condition (GO)

$$\int_0^1 a(x)\left(u'(x)-u_h'(x)\right)v_h'(x)\,\mathrm{d}x=0 \quad \forall v_h \in V_h^0$$

which says that the error of the FE approximation of the above BVP is orthogonal to V_h^0 in the energy inner product that we now define.

- For $f,g \in H^1$ and a as above, one defines

the weighted L_a^2 inner product

$$(f,g)_a=\int_0^1 f(x)g(x)a(x)\,\mathrm{d}x$$

the energy inner product

$$(f,g)_E=(f',g')_a$$

and the corresponding norms

$$\|f\|_a=\sqrt{(f,f)_a} \quad \text{and} \quad \|f\|_E=\sqrt{(f,f)_E}.$$

Observe that the definition of the energy norm $\|\cdot\|_E$ is problem dependent.

- The above cG(1) approximation is the best approximation of u in the space V_h^0 in the energy norm.

- A priori error estimate for cG(1): Let u, u_h be the solutions to (VF), resp. (FE). Assume $u'' \in L_a^2(0,1)$. Then, there exists a constant $C>0$ such that

$$\|u-u_h\|_E \le C\left\|hu''\right\|_a,$$

where we recall that $h=h(x)$ is the mesh function of the FE approximation. This result indicates that the error of the cG(1) FEM goes to zero as the mesh goes to zero.

- A posteriori error estimate for cG(1): For the above BVP, under technical assumptions on u and u_h, one has the following error estimate

$$\|u-u_h\|_E \le C\left(\int_0^1 \frac{1}{a(x)}h^2(x)R^2(u_h(x))\,\mathrm{d}x\right)^{1/2},$$

where R denotes the residual $R(u_h)=f(x)+(a(x)u_h'(x))'$ of the FE approximation to the BVP.

- The concept of adaptivity uses the above a posteriori error estimates to locally refine or modify the mesh in order to obtain a better numerical approximation u_h.

- Let us derive a FE approximation for the BVP

$$\begin{cases}-u''(x)=f(x) & \text{for} \quad x\in(0,1)\\ u(0)=\alpha \quad \text{and} & u'(1)=\beta,\end{cases}$$

where $\alpha,\beta \ne 0$ are given real numbers.

The variational formulation reads

Find $\quad u \in \{H^1(0,1), u(0)=\alpha\} \quad$ such that $\displaystyle\int_0^1 u'(x)v'(x)\,\mathrm{d}x=\int_0^1 f(x)v(x)\,\mathrm{d}x+v(1)\beta \quad \forall v \in \{H^1(0,1), v(0)=0\}.$

We use the ansatz $u_h(x) = \sum_{j=0}^{m+1} \zeta_j \varphi_j(x)$ for the FE approximation and observe that $\zeta_0 = \alpha$ due to the Dirichlet boundary condition. The FE problem then reads

$$\text{Find} \quad u_h(x) = \alpha\varphi_0(x) + \sum_{j=1}^{m+1} \zeta_j \varphi_j(x) \quad \text{such that} \quad (u_h', v_h')_{L^2} = (f, v_h)_{L^2} + v_h(1)\beta \quad \forall v_h \in \text{span}(\varphi_1, \dots, \varphi_{m+1}).$$

Taking $v_h = \varphi_i$, for $i = 1, 2, \dots, m+1$ above then gives the linear system of equations

$$S\zeta = F + G,$$

where $S = (s_{ij})_{i,j=1}^{m+1}$ is the stiffness matrix, the vectors $F = ((f, \varphi_i)_{L^2})_{i=1}^{m+1}$ and $G = (-\alpha(\varphi_0', \varphi_1')_{L^2}, 0, \dots, 0, \beta)^T$ (using properties of the hat functions).

- Let us now derive a FE approximation for the BVP

$$\begin{cases} -u''(x) + 4u(x) = 0 & \text{for} \quad x \in (0,1) \\ u(0) = \alpha \quad \text{and} \quad u(1) = \beta, \end{cases}$$

where $\alpha, \beta \neq 0$ are given real numbers. Such boundary conditions are called non-homogeneous Dirichlet boundary conditions.

The derivation of a numerical approximation for solutions to the above problem is given by

1. Define the trial space $V = \{v \colon [0,1] \to \mathbb{R} \colon \ v \in H^1(0,1), v(0) = \alpha, v(1) = \beta\}$ and the test space $V^0 = \{v \colon [0,1] \to \mathbb{R} \colon \ v \in H^1(0,1), v(0) = v(1) = 0\}$. Multiply the DE with a test function $v \in V^0$, integrate over the domain $[0,1]$ and get the VF

$$\text{Find} \quad u \in V \quad \text{such that} \quad \int_0^1 u'(x) v'(x) \, \mathrm{d}x + 4\int_0^1 u(x) v(x) \, \mathrm{d}x = 0 \quad \forall v \in V^0.$$

2. Next, define the finite dimensional spaces
$V_h = \{v \colon [0,1] \to \mathbb{R} \colon \ v \text{ is cont. pw. linear on } T_h \text{ and} v(0) = \alpha, v(1) = \beta\}$ and
$V_h^0 = \{v \colon [0,1] \to \mathbb{R} \colon \ v \text{ is cont. pw. linear on } T_h, v(0) = v(1) = 0\}$, where as before T_h is a uniform partition with mesh $h = \frac{1}{m+1}$. Observe that $V_h = \text{span}(\varphi_0, \varphi_1, \dots, \varphi_m, \varphi_{m+1}) \subset V$ and $V_h^0 = \text{span}(\varphi_1, \dots, \varphi_m) \subset V^0$ with the hat functions φ_j.
The FE problem then reads

$$\text{Find} \quad u_h \in V_h \quad \text{such that} \quad \int_0^1 u_h'(x) v_h'(x) \, \mathrm{d}x + 4\int_0^1 u_h(x) v_h(x) \, \mathrm{d}x \quad \forall v_h \in V_h^0.$$

3. Choosing $v_h = \varphi_i$, writing $u_h(x) = \sum_{j=0}^{m+1} \zeta_j \varphi_j(x)$ with $\zeta_0 = \alpha$ and $\zeta_{m+1} = \beta$ (due to the non-homogeneous Dirichlet BC), and inserting everything into the FE problem gives the following linear system of equations

$$(S + 4M)\zeta = b,$$

where the $m \times m$ stiffness matrix S has entries $s_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x) \, \mathrm{d}x$, the $m \times m$ mass matrix M has entries $m_{ij} = \int_0^1 \varphi_i(x)\varphi_j(x) \, \mathrm{d}x$, and the $m \times 1$ vector b has entries $b_i = -\alpha(\varphi_0', \varphi_i')_{L^2} - \beta(\varphi_{m+1}', \varphi_i')_{L^2} - 4\alpha(\varphi_0, \varphi_i)_{L^2} - 4\beta(\varphi_{m+1}, \varphi_i')_{L^2}$. Detailed formulas for these entries can be found in the book. Solving this system gives the vector ζ and in turn the numerical approximation u_h.

- Let us finally consider the problem of finding a numerical approximation of solutions to the BVP

$$\begin{cases} -au''(x) + bu'(x) = r & \text{for} \quad x \in (0,1) \\ u(0) = 0 \quad \text{and} & u'(1) = \beta, \end{cases}$$

where $\beta \neq 0$, $a, b > 0$, and r are given real numbers. One has a homogeneous Dirichlet boundary conditions for $x = 0$ and non-homogeneous Neumann boundary conditions for $x = 1$.

For ease of presentation we take $a = b = r = 1$ and derive a FE approximation as follows

1. Define the space $V = \{v\colon [0,1] \to \mathbb{R} : \ v \in H^1(0,1), v(0) = 0\}$. Multiply the DE with a test function $v \in V$, integrate over the domain $[0,1]$ and get the VF

$$\text{Find} \quad u \in V \quad \text{such that} \quad (u', v')_{L^2} + (u', v)_{L^2} = \int_0^1 v(x)\,\mathrm{d}x + \beta v(1) \quad \forall v \in V.$$

2. Next, define the finite dimensional space $V_h = \{v\colon [0,1] \to \mathbb{R} : \ v \text{ is cont. pw. linear on } T_h, v(0) = 0\}$, where as before T_h is a uniform partition with mesh $h = \frac{1}{m+1}$.
Observe that $V_h = \text{span}(\varphi_1, \ldots, \varphi_m, \varphi_{m+1}) \subset V$, with the hat functions φ_j.
The FE problem then reads

$$\text{Find} \quad u_h \in V_h \quad \text{such that} \quad (u_h', v_h')_{L^2} + (u_h', v_h)_{L^2} = \int_0^1 v_h(x)\,\mathrm{d}x + \beta v_h(1) \quad \forall v_h \in V_h.$$

3. Choosing $v_h = \varphi_i$, writing $u_h(x) = \sum_{j=1}^{m+1} \zeta_j \varphi_j(x)$, observing that φ_{m+1} is a half hat function, and inserting everything into the FE problem gives the following linear system of equations

$$(S + C)\zeta = b,$$

where the $(m+1) \times (m+1)$ stiffness matrix S has entries $s_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x$, the $(m+1) \times (m+1)$ convection matrix C has entries $c_{ij} = \int_0^1 \varphi_j'(x)\varphi_i(x)\,\mathrm{d}x$, and the $(m+1) \times 1$ vector b has entries $b_i = \int_0^1 \varphi_i(x)\,\mathrm{d}x + \beta\varphi_i(1)$. Detailed formulas for these entries can be found in the book. Solving this system gives the vector ζ and in turn the numerical approximation u_h.

- For indication, and for a uniform partition of $[0,1]$ denoted by T_h: $x_0 = 0 < x_1 < x_2 < \ldots < x_m < x_{m+1} = 1$ with element length/mesh denoted by h, we summarise the possible choices for the FE spaces:

1. Dirichlet BC $u(0) = 0, u(1) = 0$: test and trial spaces given by $span(\varphi_1, \ldots, \varphi_m)$.
2. Dirichlet BC $u(0) = \alpha \neq 0, u(1) = 0$: trial given by $span(\varphi_0, \varphi_1, \ldots, \varphi_m)$ and test by $span(\varphi_1, \ldots, \varphi_m)$.
3. Dirichlet BC $u(0) = 0, u(1) = \beta \neq 0$: trial given by $span(\varphi_1, \ldots, \varphi_m, \varphi_{m+1})$ and test by $span(\varphi_1, \ldots, \varphi_m)$.
4. Dirichlet BC $u(0) = \alpha \neq 0, u(1) = \beta \neq 0$: trial given by $span(\varphi_0, \varphi_1, \ldots, \varphi_{m+1})$ and test by $span(\varphi_1, \ldots, \varphi_m)$.
5. Dirichlet/Neumann BC $u(0) = 0, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_1, \ldots, \varphi_{m+1})$ and test by $span(\varphi_1, \ldots, \varphi_{m+1})$.
6. Neumann/Dirichlet BC $u'(0) = \alpha$ (zero or not), $u(1) = 0$: trial given by $span(\varphi_0, \ldots, \varphi_m)$ and test by $span(\varphi_0, \ldots, \varphi_m)$.

7. Dirichlet/Neumann BC $u(0) = \alpha \neq 0, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_0, \dots, \varphi_{m+1})$ and test by $span(\varphi_1, \dots, \varphi_{m+1})$.
8. Neumann/Dirichlet BC $u'(0) = \alpha$ (zero or not), $u(1) = \beta \neq 0$: trial given by $span(\varphi_0, \dots, \varphi_{m+1})$ and test by $span(\varphi_0, \dots, \varphi_m)$.
9. Neumann BC $u'(0) = \alpha, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_0, \dots, \varphi_{m+1})$ and test by $span(\varphi_0, \dots, \varphi_{m+1})$.

Further resources:

- FE at wikiversity.org
- FE at github.io
- FEM course notes at web.stanford.edu
- FEM for BVP at amath.unc.edu
- FEM by Gilbert Strang on youtube (good!)
- Galerkin method at wikipedia.org
- Error estimation at csc.kth-se
- Adaptivity at csc.kth-se

Applications: The FEM is use to find approximate solutions to complex problems in engineering. Areas of applications are multiple, for instance: Mechanical engineering design, CAD, fatigue and fracture mechanics, etc. See link.