David Cohen (david.cohen@chalmers.se) Chalmers & GU

Chapter 13: Further topics (summary)

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Goal: First stability analysis of time integrators. Present finite difference (FD) for Poisson's equation in $2d$.

- We have motivated the use of Dahlquist's test equation

  $$\dot{y}(t) = \lambda y(t), \quad y(0) = y_0$$

  where $\lambda \in \mathbb{C}$. Observe that the solution to the above IVP remains bounded if $\mathrm{Re}(\lambda) \leq 0$.

  An application of a time integrator (explicit/implicit Euler scheme, Crank–Nicolson scheme, θ-scheme, Runge–Kutta scheme) to the test equation gives the recursion

  $$y_{n+1} = R(\lambda \Delta t) y_n,$$

  where Δt denotes the time step size.

  One then defines the stability region of a numerical method to be the set $S = \{z \in \mathbb{C} \text{such that} |R(z)| \leq 1\}$.

  When using the explicit Euler method one must thus choose the time step size in relation to the parameter λ for the numerical solution to be stable.

  As a consequence, when discretizing the heat equation by a FEM in space (the largest eigenvalue of the problem satisfy $\lambda \sim \frac{1}{h^2}$), for instance, one has a time step size restriction. This condition is called a CFL condition and it is of the form $\Delta t \sim h^2$, where h is the mesh of the FEM.

- We recall forward and backward finite difference:

  $$y'(x) \approx \frac{y(x+h) - y(x)}{h} \quad \text{and} \quad y'(x) \approx \frac{y(x) - y(x-h)}{h},$$

  where $y \colon \mathbb{R} \to \mathbb{R}$ is a differentiable function and $h > 0$ is a given (small) real number.

- Let $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square, $f, g \colon \Omega \to \mathbb{R}$ nice functions. Consider Poisson's equation

  $$\begin{cases} \Delta u = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

  where $u = u(x,y)$ is the unknown function and $\Delta u = u_{xx} + u_{yy}$ is the Laplacian.

  To find a numerical approximation to the solution u of Poisson's equation, consider a mesh size $h = \frac{1}{n+1}$ for some (large) integer n and the grid $x_i = ih$ and $y_j = jh$ for $i, j = 0, \ldots, n$.

  We first approximate the derivatives in the Laplacian using forward and backward finite difference:

  $$u_{xx} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$$

  resulting in the discrete Laplace operator Δ_h

  $$\Delta u(x,y) \approx \Delta_h u(x,y) = \frac{u(x+h,y) + u(x,y+h) - 4u(x,y) + u(x-h,y) + u(x,y-h)}{h^2}.$$

Denote then $u_{ij} \approx u(x,y)$ at the grid point (x_i, y_j) and $f_{ij} = f(x_i, y_j)$, resp. $g_{ij} = g(x_i, y_j)$, one gets the linear system of equations

$$\Delta_h u_{ij} = f_{ij} \quad \text{for} \quad i,j = 1,\ldots,n.$$

Using the boundary condition, one then ends up with the linear system of equations

$$A\mathbf{u} = F,$$

with a block tridiagonal matrix A of size $n^2 \times n^2$, a vector F of size $n^2 \times 1$ (containing f_{ij} and g_{ij}) and the unknown vector

$$\mathbf{u} = (u_{11}, u_{12} \ldots, u_{1n}, \ldots, u_{n1}, \ldots, u_{nn})^T.$$

- Under some assumptions, one can prove convergence of the above finite difference approximation to the solution to Poisson's equation on the unit square:

$$\left\| u(x_i, y_j) - u_{ij} \right\|_\infty \leq Ch^2.$$

- The above can be generalised in various directions, for instance: high order FD, Neumann or Robin BC, general operator instead of the Laplacian, etc.

Further resources:

- $1d$ Poisson FD solver at deepxde
- FD for $1d$ Poisson and heat eq. at unibs.it
- FD for Poisson in $2d$ at youtube.com
- FD for Poisson in $1d$ and 2d at sc.fsu.edu.edu

Applications: Several concepts of stability of a numerical solution exists. In short, they explain why a numerical method is good, or not, for a particular class of problems. Finite difference methods are other popular numerical methods for the approximation of solution to ODE and PDE. Some possible applications are: computational electrodynamics (Yee's scheme), computational fluid dynamics (upwind scheme), etc.