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Chapter 3: Interpolation and numerical integration (summary)

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Goal: Interpolation: We want to pass a (simple) function through a given set of data points.
Numerical integration: We want to find numerical approximations of integrals $\int_a^b f(x)\,\mathrm{d}x$.

- Let q be a positive integer. Consider an interval $[a,b]$ and a grid of $(q+1)$ distinct points $x_0 = a < x_1 < \ldots < x_q = b$. One defines Lagrange polynomials by

$$\lambda_i(x) = \prod_{j=0, j\neq i}^{q} \frac{x-x_j}{x_i-x_j}$$

for $i = 0,1,\ldots,q$. One then has

$$\mathscr{P}^{(q)}([a,b]) = \operatorname{span}\big(\lambda_0(x), \lambda_1(x), \ldots, \lambda_q(x)\big),$$

where we recall that $\mathscr{P}^{(q)}([a,b])$ denotes the set of polynomials, defined on $[a,b]$, of degree at most q.

- Let $q \in \mathbb{N}$. Consider a continuous function $f\colon [a,b] \to \mathbb{R}$ and $q+1$ distinct interpolation points $\big(x_j, f(x_j)\big)_{j=0}^{q}$ with $a = x_0 < x_1 < \ldots < x_q = b$. A polynomial $\pi_q f \in \mathscr{P}^{(q)}([a,b])$ is a polynomial interpolant for f if

$$\pi_q f(x_j) = f(x_j) \quad \text{for} \quad j = 0,1,2,\ldots,q.$$

Examples of polynomial interpolants: Remembering that $\mathscr{P}^{(q)}([a,b]) = \operatorname{span}(1, x, x^2, \ldots, x^q)$, one gets a polynomial interpolant $\pi_q f$ in the standard basis. Taking $\mathscr{P}^{(q)}([a,b]) = \operatorname{span}(\lambda_0(x), \lambda_1(x), \ldots, \lambda_q(x))$, one gets the Lagrange interpolant $\pi_q f$. These are the same polynomials (seen in a different basis).

- Let m be a positive integer. Consider a uniform partition of an interval $[a,b]$, denoted $\tau_h : x_0 = a < x_1 < \ldots < b = x_{m+1}$ with mesh size $h_j = x_j - x_{j-1}$. Next, define the mesh function $h(x) = h_j$ if $x \in (x_{j-1}, x_j)$ and $j = 1,2,\ldots,m+1$.

We define the hat function $\{\varphi_j\}_{j=0}^{m+1}$ by

$$\varphi_j(x) = \begin{cases} \frac{x-x_{j-1}}{h_j} & \text{for } x_{j-1} \leq x \leq x_j \\ \frac{x-x_{j+1}}{-h_{j+1}} & \text{for } x_j \leq x \leq x_{j+1} \\ 0 & \text{else} \end{cases}$$

for $j = 1,\ldots,m$. The functions $\varphi_0(x)$ and $\varphi_{m+1}(x)$ are defined as half hat functions.

The space of continuous piecewise linear functions on τ_h reads $V_h = \operatorname{span}(\varphi_0, \ldots, \varphi_{m+1})$. As usual, one has $v(x) = \sum_{j=0}^{m+1} \zeta_j \varphi_j(x)$, where $\zeta_j = v(x_j)$, for any $v \in V_h$.

The continuous piecewise linear interpolant of f is defined by

$$\pi_h f(x) = \sum_{j=0}^{m+1} f(x_j)\varphi_j(x) \quad \text{for} \quad x \in [a,b].$$

If $f \in \mathscr{C}^2([a,b])$ (can be relaxed) one has, for instance, the following bounds for the interpolation error for the continuous piecewise linear interpolant on a uniform partition with constant mesh h:

$$\begin{aligned}\|\pi_h f - f\|_{L^p} &\leq C_1 h^2 \|f''\|_{L^p}, \\ \|\pi_h f - f\|_{L^p} &\leq C_2 h \|f'\|_{L^p)}, \\ \|(\pi_h f)' - f'\|_{L^p} &\leq C_3 h \|f''\|_{L^p},\end{aligned}$$

for $p = 1, 2, \infty$.

In case of non-uniform partitions, one uses the mesh function $h(x)$ and gets for instance

$$\|\pi_h f - f\|_{L^p} \leq C \|h^2 f''\|_{L^p}$$

and similarly for the other estimates.

- Let us give 3 classical quadrature rules or quadrature formulas to numerically approximate the integral $\int_a^b f(x)\,\mathrm{d}x$:

  The midpoint rule reads
  $$\int_a^b f(x)\,\mathrm{d}x \approx (b-a) f\left(\frac{a+b}{2}\right).$$
  The trapezoidal rule reads
  $$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{2}\big(f(a)+f(b)\big).$$
  The Simpson rule reads
  $$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

  In practice, one first considers a (uniform) partition of the interval $[a,b]$, $a = x_0 < x_1 < \ldots < x_N = b$, and then apply a quadrature rule (denoted by $QF(x_j, x_{j+1}, f)$ below) on each small subintervals:
  $$\int_a^b f(x)\,\mathrm{d}x = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x)\,\mathrm{d}x \approx \sum_{j=0}^{N-1} QF(x_j, x_{j+1}, f).$$

Further resources:

- Lagrange interpolation on youtube
- Lagrange interpolation on www.dcode.fr
- interpolation at www.maths.lth.se
- Lagrange interpolation at www.phys.libretexts.org
- trapezoidal rule at www.khanacademy.org
- quadrature formulas at tutorial.math.lamar.edu

Applications: Interpolation can be used to find approximation of function values at points where there is no given data. One can also estimate or predict data. Quadrature formulas can be used to approximate solutions to integral equations (for instance in problems coming from electrical circuits).