Exam TMA4313, MA8801
24.08.23

1 a) $y'(t) = y(t), y(0) = 1$, Euler $y_{n+1} = y_n + h y_n, y_0 = 1$ 1 1

b) $d = B^2 - 4AC = 1^2 - 4 \cdot 0 \cdot 0 = 1 > 0 \to$ hyperbolic 1

c) Yes 1

d) Ex.: sol. to PDE/VF 1

e) $\int_1^3 x^2 dx \approx \frac{3-1}{2}(3^2+1^2) = 10$ 1

f) $\|u-u_h\|_E \leq C \cdot h$, 0, where $u_h \to$ FE sol with mesh h, $\|\cdot\|_E$ energy norm 1

g) (i) Compute element contribution 1
(ii) Add this contribution at correct location in global matrix 1

h) If $a: H \times H \to \mathbb{R}$ is bilinear form, continuous/bounded, coercive 1 and $l: H \to \mathbb{R}$ is continuous/bounded 0 linear form, then the prob. has a unique sol. 1

2 a) $|a(u,v)| \underset{\substack{C\text{-}S \\ C \in L^\infty}}{\leq} \|u'\|_{L^2(I)} \|v'\|_{L^2(I)} + \|c\|_{L^\infty(I)} \|u\|_{L^2(I)} \|v\|_{L^2(I)} \leq$ 1

$\underset{\text{Def } H^1}{\leq} (1 + \|c\|_{L^\infty}) \|u\|_{H^1} \|v\|_{H^1} \leq k \cdot \|u\|_{H^1} \|v\|_{H^1}$ 1

b) To show: $a(u,u) \geq \alpha \|u\|_{H^1}^2 \quad \forall u \in H_0^1$ 1

For $u \in H_0^1$, one has: $a(u,u) \geq \|u'\|_{L^2}^2 + c_0 \|u\|_{L^2}^2 \geq (1-\alpha)\|u'\|_{L^2}^2 + \alpha \|u'\|_{L^2}^2$ $\hookrightarrow \alpha \in \mathbb{R}$ TBD 1
$+ c_0 \|u\|_{L^2}^2 \underset{\text{Hint}}{\geq} 4(1-\alpha)\|u\|_{L^2}^2 + c_0 \|u\|_{L^2}^2 + \alpha \|u'\|_{L^2}^2 \geq$
$\geq (4(1-\alpha) + c_0)\|u\|_{L^2}^2 + \alpha \|u'\|_{L^2}^2 \geq \alpha \|u\|_{H^1}^2$ 1

We want $4(1-\alpha) + c_0 > \alpha$ or $0 < \alpha < \frac{4+c_0}{5}$ (OK since $c_0 > -4$) 1

3 a) Multiply with $v \in H^1$, integrate by parts and get:
$\int_a^b p(x) u'(x) v'(x) dx - [p(x) u'(x) v(x)|_a^b] + \int_a^b q(x) u(x) v(x) dx = \int_a^b f(x) v(x) dx$ 0,5

VF) Find $u \in H^1$ s.t. $\int_a^b p(x) u'(x) v'(x) dx + \int_a^b q(x) u(x) v(x) dx + \beta u(b) v(b) + \beta u(a) v(a) = \int_a^b f(x) v(x) dx \quad \forall v \in H^1$ 0,5

b) Space of pwl fct. on partition 1 $=$ span$(\{\varphi_i\}_{i=0}^N)$ 1, where φ_i are hat fct.

FE) Find $u_h \in V_h$ s.t. $\int_a^b p u_h' v_h' dx + \int_a^b q u_h v_h dx + \beta u_h(b) v_h(b) + \beta u_h(a) v_h(a) = \int_a^b f v_h dx \quad \forall v_h \in V_h$

c) $u_h(x) = \sum_{j=0}^N \xi_j \varphi_j(x)$, ξ_j unknown, φ_j hat fct. 1

d) Insert u_h from c) and take $v_h = \varphi_i$, $i = 0, \ldots, N$ into (FE) to get

$$\sum_{j=0}^{N} \xi_j \int_a^b p\,\varphi_j' \varphi_i' \,dx + \sum_{j=0}^{N} \int_a^b q\,\varphi_j \varphi_i \,dx + \beta \sum_{j=0}^{N} \xi_j \varphi_j(b)\varphi_j(a) + \beta \sum_{j=0}^{N} \xi_j \varphi_j(a)\varphi_i(a) = \int_a^b f \varphi_i \,dx \quad \forall i = 0,\dots,N$$ (2)

Or $S'\xi + M\xi = b$ with $b = \left((f,\varphi_i)_{L^2}\right)_{i=0}^{N} + \begin{pmatrix}\beta\\0\\\vdots\\\beta\end{pmatrix}$ (1)

$\left((p\varphi_j',\varphi_i')_{L^2}\right)_{ij=1}^{N}$, $\left((q\varphi_j,\varphi_i)_{L^2}\right)_{ij=0}^{N}$ (1)

e) GO says $A(u-u_h, v_h) = 0 \ \forall v_h \in V_h$ (0.5)

Hence $A(-u_h,v_h) = A(-u,v_h)$ or $A(\Pi_h u - u_h, v_h) = A(\Pi_h u - u, v_h) \ \forall v_h \in V_h$ (0.5)

f) $\|\Pi_h u - u_h\|_{H^1}^2 \underset{\text{coercive}}{\leq} C \cdot A(\Pi_h u - u_h, \underbrace{\Pi_h u - u_h}_{\in V_h}) \underset{e)}{\leq} C \cdot A(\Pi_h u - u, \Pi_h u - u_h) \underset{\text{continuous}}{\leq} C \|\Pi_h u - u\|_{H^1} \|\Pi_h u - u_h\|_{H^1}$

$\Rightarrow \|\Pi_h u - u_h\|_{H^1} \leq C \cdot \|\Pi_h u - u\|_{H^1}$ (1)

Finally, $\|u - u_h\|_{H^1} \overset{\Delta}{\leq} \|u - \Pi_h u\|_{H^1} + \|\Pi_h u - u_h\|_{H^1} \leq C \cdot \|\Pi_h u - u\|_{H^1}$ (1)

g) $\|u - u_h\|_{H^1} \underset{f)}{\leq} C \|\Pi_h u - u\|_{H^1} \underset{\text{Hint}}{\leq} C \cdot h$ (1)

(4) a) Hom. Dirichlet BC $\Rightarrow$ test = trial = $H_0^1(\Omega)$ (1)

For all t, (VF) Find $u(\cdot,t) \in H_0^1$ s.t. $\begin{cases} (u_t(\cdot,t),v)_{L^2} + (\nabla u(\cdot,t), \nabla v)_{L^2} + (u(\cdot,t),v)_{L^2} = (f,v) \ \forall v \in H_0^1 \\ u(x,0) = u_0(x) \ \forall x \in \Omega \end{cases}$ (1)

b) $V_h^0 = \mathrm{span}\left(\{\varphi_j\}_{j=1}^{N}\right)$ with φ_j hat fct. (1)

(FE) For all t, find $u_h(\cdot,t) \in H_0^1$ s.t. $\begin{cases} (u_{h,t}(\cdot,t),v)_{L^2} + (\nabla u_h(\cdot,t), \nabla v_h)_{L^2} + (u_h(\cdot,t), v_h)_{L^2} = (f,v_h)_{L^2} \ \forall v_h \in V_h \\ u_h(x,0) = \Pi_h u_0(x) \end{cases}$ (1)

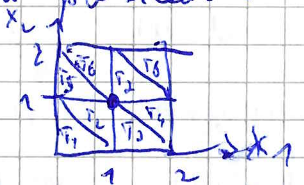
c) See lecture, $M = \left((\varphi_j,\varphi_i)_{L^2}\right)_{ij=1}^{N}$ (1), $S' = \left((\nabla\varphi_j,\nabla\varphi_i)_{L^2}\right)_{ij=1}^{N}$ (1), $F = \left((f,\varphi_i)_{L^2}\right)_{i=1}^{N}$ (1), $\xi(0) = \left(\Pi_h u_0(x_i)\right)_{i=1}^{N}$

(5) a) Hom. Dirichlet BC $\Rightarrow$ trial = test = H_0^1 (1)

(VF) Find $u \in H_0^1$ s.t. $(\nabla u, \nabla v)_{L^2} = (1,v)_{L^2} \ \forall v \in H_0^1$ (see lecture) (1)

b) (FE) Find $u_h \in V_h^0$ s.t. $(\nabla u_h, \nabla v_h)_{L^2} = (1, v_h) \ \forall v_h \in V_h^0$, where (1)

$V_h^0 = \{v_h : \Omega \to \mathbb{R}$ cont and pw linear on partition T_h with $v_h = 0$ on $\partial\Omega\} = \mathrm{span}(\varphi_1)$, where φ_1 hat on node $(1,1)$ (1)



c) $u_h(x) = \xi_1 \varphi_1(x)$, take $v_h = \varphi_1$ in (FE) and get: $\xi_1 \overbrace{\int_\Omega \nabla\varphi_1 \nabla\varphi_1 \,dx}^{S'} = \overbrace{\int_\Omega \varphi_1 \,dx}^{b}$ with (1)

$S' = \int_{T_2} + \int_{T_3} + \int_{T_4} + \int_{T_5} + \int_{T_6} + \int_{T_7}$ (0.5) and $\int_{T_2} = \int_{T_7} = 1$ (0.5), $\int_{T_3} = \int_{T_4} = \int_{T_5} = \int_{T_6} = \frac{1}{2}$ (0.5) $\Rightarrow S' = 4 \cdot \frac{1}{2} + 2 \cdot 1 = 4$ (0.5)

d) $b = 1$ (1) e) $\xi_1 = \frac{1}{4}$ and $u_h(x) = \frac{1}{4}\varphi_1(x)$ (1)

(6) $1 = l_1(\Phi_1) = \Phi_1'(0) = b \Rightarrow b = 1$ (1), $0 = l_2(\Phi_1) = \Phi_1'(1) = 2a + 1 \Rightarrow a = -\frac{1}{2}$ (1), $0 = l_3(\Phi_1) = -\frac{1}{6} + \frac{1}{2} + c \Rightarrow c = -\frac{1}{3}$

$\Rightarrow \Phi_1(x) = -\frac{1}{2}x^2 + x - \frac{1}{3}$ (1)