Chapter 7: The wave equation in $1d$ (summary)

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Goal: Briefly study the exact solution of the wave equation and present a numerical discretisation of this PDE.

- Consider the (inhomogeneous) wave equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t) & 0 < x < 1, 0 < t \leq T \\ u(0,t) = u(1,t) = 0 & 0 < t \leq T \\ u(x,0) = u_0(x) & 0 < x < 1 \\ u_t(x,0) = v_0(x) & 0 < x < 1, \end{cases}$$

  where u_0, v_0 and f are given (nice) functions.

  Introducing a new variable for the velocity $v = u_t$, one can rewrite the above wave equation as a system of first order differential equations

$$w_t(x,t) = Aw(x,t) + F(x,t),$$

  with $w(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$, $F(x,t) = \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix}$ and the operator $A = \begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$.

  For the homogeneous wave equation, that is when $f \equiv 0$ in the above PDE, one has conservation of the energy

$$\frac{1}{2} \|u_t(\cdot,t)\|_{L^2}^2 + \frac{1}{2} \|u_x(\cdot,t)\|_{L^2}^2 = \frac{1}{2} \|v_0\|_{L^2}^2 + \frac{1}{2} \|u_0'\|_{L^2}^2 \quad \text{for} \quad 0 \leq t \leq T.$$

- The numerical discretisation of the wave equation is similar to the one for the heat equation:

  1. The VF reads: For each $0 < t \leq T$, find $u(\cdot,t) \in H_0^1$ such that

$$(u_{tt}(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2}$$

     for all test functions $v \in H_0^1$ and with initial conditions $u(x,0) = u_0(x)$, $u_t(x,0) = v_0(x)$.

  2. The FE problem reads: For each $0 < t \leq T$, find $u_h(\cdot,t) \in V_h^0$ such that

$$(u_{h,tt}(\cdot,t), v_h)_{L^2} + (u_{h,x}(\cdot,t), v_{h,x})_{L^2} = (f(\cdot,t), v_h)_{L^2}$$

     for all test functions $v_h \in V_h^0$ and initial conditions $u_h(x,0) = \pi_h u_0(x)$ and $u_{h,t}(x,0) = \pi_h v_0(x)$. As always, $u_h(x,t) = \sum_{j=1}^{m} \zeta_j(t) \varphi_j(x)$.

  3. The linear system of ODEs is given by

$$\begin{aligned} M\dot{\zeta}(t) &= M\eta(t) \\ M\dot{\eta}(t) + S\zeta(t) &= F(t). \end{aligned}$$

     Finally, one obtains a numerical approximation of the solution to this ODE by using the Crank–Nicolson scheme with time step k for instance:

$$\begin{pmatrix} M & -\frac{k}{2}M \\ \frac{k}{2}S & M \end{pmatrix} \begin{pmatrix} \zeta^{(n+1)} \\ \eta^{(n+1)} \end{pmatrix} = \begin{pmatrix} M & \frac{k}{2}M \\ -\frac{k}{2}S & M \end{pmatrix} \begin{pmatrix} \zeta^{(n)} \\ \eta^{(n)} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{k}{2}\left(F(t_{n+1}) + F(t_n)\right) \end{pmatrix}.$$

The Crank–Nicolson scheme preserves a discrete energy (when applied to a homogeneous wave equation). The forward and backward Euler schemes do not have this property.

Further resources:

- wave eq. at wikipedia.org
- wave eq. at brilliant.org
- wave eq. at math.lamar.edu
- wave eq. at chem.libretexts.org

Applications: The wave equation (in $1d$ and higher dimension) is used to model elastic and acoustic phenomena such as: seismic waves, water waves, electromagnetic waves.