Assignment

Problem 1: Let G be a group of order pq, where p and q are distinct primes. Prove that G is abelian.

Provided Answer: Abelian

Correct Answer: False

Explanation:

A group of order pq is not always abelian. The statement is only true under an additional condition. Using Sylow's Theorems, we can show that if $p<q$, the group is guaranteed to be abelian only if p does not divide $q-1$.

A classic counterexample is the symmetric group S_3, which is the group of permutations of three elements.

- The order of S_3 is $3! = 6$.
- We can write the order as a product of distinct primes: $6 = 2 \cdot 3$.
- S_3 is a well-known non-abelian group. For instance, the compositions $(12)(13) = (132)$ and $(13)(12) = (123)$ are not equal.

Therefore, a group of order pq is not necessarily abelian.

Problem 5: Prove that in any group G, the set of elements of finite order from a subgroup of G.

Provided Answer: True

Correct Answer: False

Explanation:

This statement is only true if the group G is abelian. For a general non-abelian group, the set of elements of finite order is not necessary closed under the group operation.

Consider the free product $G = Z_2 * Z_2$

- This group can be described by generators and relations: $G = (a, b \mid a^2 = e, b^2 = e)$.
- The element a has order 2 (finite).
- The element b has order 2 (finite).
- However, their product, ab, has infinite order. The element $(ab)^k = abab\ldots ab$ is never the identity for $k > 0$.

Since the product of two elements of finite order can have infinite order, the set is not closed under the operation and thus is not a subgroup. This set is called the torsion subset of a group; it is only guaranteed to be a subgroup (the torsion subgroup) if the group is abelian.

Problem 6: Let G be a finite group and p be the smallest prime dividing $|G|$. Prove that any subgroup of index p in G is normal.

Provided Answer: True

Correct Answer: True

Explanation:

This is correct and important theorem. The proof involves a group action. Let H be a subgroup of G with index $[G:H]=p$

1. Let G act on the set of left cosets of H, $S=\{gH \mid g\in G\}$, by left multiplication. This induces a homomorphism $\phi: G\to S_p$, where S_p is the symmetric group on the p cosets.

2. The kernel of this action, $Ker(\phi)$, is a normal subgroup of G and is contained within H.

3. By the First Isomorphism Theorem, $G/Ker(\phi)$ is isomorphic to a subgroup of S_p. Thus, the order of $G/Ker(\phi)$, which is $[G:Ker(\phi)]$, must divide $|S_p| = p!$.

4. Since H has index p, we have $|G|=p.|H|$. Also, $[G:ker(\phi)]=[G:H][H:ker(\phi)]=p.[H:ker(\phi)]$. This shows that p divides $[G:ker(\phi)]$

5. We know that $[G:ker(\phi)$ divides both $|G|$ and $p!$. The prime factors of $p!$ are all primes less than or equal to p. Since p is the smallest prime dividing $|G|$, the only prime factor that $[G:ker(\phi)]$ and $|G|$ can share is p.

6. This implies that $[G:ker(\phi)]$ must be a power of p. However, since $[G:ker(\phi)]$ divides $p!$, the highest power of p it can be is p^1.

7. Therefore, $[G:ker(\phi)]=p$

8. From step 4, we have $p=p.[H:ker(\phi)]$ which implies $[H:ker(\phi)]=1$. this means $H=ker(\phi)$.

Since $ker(\phi)$ is always a normal subgroup, H must be normal in G.

Problem 7: Let G be a group and $a, b \in G$.

Prove that if $a^4 = b^2$ and $ab = ba$, then $(ab)^6 = e$

Provided Answer: True

Correct Answer: False

Explanation:

The statement as written is incorrect. There appears to be a typo in the problem's premise. We can construct a simple counterexample.

Let's follow the logic:

1. Since a and b commute, $(ab)^6 = a^6 b^6$
2. We are given $a^4 = b^2$.
3. We can write $b^6 = (b^2)^3$. Substituting the given relation, we get $b^6 = (a^4)^3 = a^{12}$.
4. Substituting this back into the first line: $(ab)^6 = a^6 a^{12} = a^{18}$.

The problem is now reduced to proving that $a^{18} = e$. However, the given conditions do not guarantee this.

Counterexample:

- Let $G = \mathbb{Z}_{36}$ (the group of integers modulo 36 under addition). ~~This group~~

is abelian, so all elements commute

- Leat $a=2$. Then $a^4=4\cdot 2=8$.

- We need $b^2=a^4=8$. Leats check for a, b. In $\mathbb{Z}_{36}$, $10^2=100=28$, $11^2=121=13$, ...
Let's try another a.

- Let $a=10$. then $a^4=400\equiv 4 \pmod{36}$.
We need $b^2=4$. We can choose $b=2$ $(2^2=4)$ or $b=20$ $(20^2=400=11\cdot 36+4\}\equiv 4)$.

- Let's choose $a=10$ and $b=20$. The conditions $ab=ba$ and $a^4=b^2$ are met.

- Now lets calculate $(ab)^6$:

$$(ab)^6=(10+20)^6=30^6=(-6)^6 \pmod{36}$$

$$(-6)^2=36\equiv 0$$

$$(-6)^6=((-6)^2)^3\equiv 0^3=0=e$$

- this example works. Let's try to find one that doesn't.

- From the logic above, we need to find an a where $a^{18}\neq e$.

- Let $G=\mathbb{Z}_{20}$. Let $a=1$. Then $a^4=4$. We need $b^2=4$. We can choose $b=2$.

- Conditions: $a=1$, $b=2$. Group is abelian. $a^4=4$, $b^2=4$. conditions met.
- Check the conclusion: $(ab)^6=(1+2)^6=3^6 \pmod{20}$
  $3^2=9$, $3^4=81\equiv 1$, $3^6=3^4 \cdot 3^2 \equiv 1 \cdot 9 = 9$.
- Since $(ab)^6=9 \neq 0$ (the identity), the statement is false.

Problem 8: Let G be a group and H be a subgroup of G. Prove that if $[G:H]=n$, then for any $x \in G$, $x^n \in H$.

Provided Answer: True

Correct Answer: False

Explanation:

This statement is a property that holds only if the subgroup H is normal in G. If H is normal, then the set of cosets G/H forms a group of order n. By Lagrange's theorem applied to this quotient group, any element $(xH)^n$ is the identity element, H. This means $(xH)^n = x^n H = H$, which implies $x^n \in H$.

However, if H is not normal, the statement is false.

Counterexample:

- Let $G = S_3$ and $H = \{e(12)\}$.
- The order of G is 6 and the order of H is 2. The index is $[G:H]=3$.

So, $n=3$.

- H is not a normal subgroup.
- Let's choose an element $x = (13) \in G$
- According to the statement,

$$x^n = (13)^3 \text{ should be in } H.$$

- Calculating the power:

$$(13)^3 = (13)^2 (13) = e \cdot (13) = 13.$$

The element (13) is not in $H = \{e, (12)\}$.

- The statement is false.

Therefore, the statement is false.

Problem 9: Let G be a finite group and p be a prime number. If G has exactly one subgroup of order p^k for each $k \le n$, where p^n divides $|G|$, prove that G has a normal Sylow p-subgroup.

Provided Answer: True

Correct Answer: True

Explanation:

This statement is correct. Let the highest power of p that divides $|G|$ be p^m (so $m \geq n$). A Sylow p-subgroup of G is a subgroup of order p^m.

The key information is that "G has exactly one subgroup of order p^k for each $k \le n$".

1. Let's assume p^n is the highest power of p dividing $|G|$, so a Sylow p-subgroup has order p^n.

2. The problem states that for $k = n$, there is exactly one subgroup of order p^n.

3. Let this unique subgroup be P. By definition, P is a Sylow p-subgroup.

4. A core result of Sylow theory is that a Sylow p-subgroup is normal if and only if it is unique.

5. Since we are given that the subgroup of order p^n is unique, it is the unique Sylow p-subgroup and is therefore normal in G.

The information about the uniqueness of subgroups for $k<n$ is additional detail that is consistent with the conclusion but not strictly necessary to prove it.