Question: Prove that set of rational Number Q is a field? with two binary operations addition and multiplication.

Answer:

claim $Q = \{ \frac{p}{q} \mid p, q \in Z, q \neq 0 \}$ with ordinary addition and multiplication is field

Proof: we verify the field axioms.

1. well-definedness of operations (brief remark)

A rational number is an equivalence class of pairs (p, q) with $q \neq 0$ Under $\frac{p}{q} = \frac{p'}{q'} \Leftrightarrow pq' = p'q$

The usual formulas,

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}, \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

2. closure: If $\frac{p}{q}, \frac{r}{s} \in Q$ (with $q, s \neq 0$) $\}$

then.

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} \in Q, \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs} \in Q$$

Since integers are closed under + and · and $qs \neq 0$

3. Associativity of + and ·

Associativity follows from associativity in Z and the formulas for sum/product of fractions.

4. commutativity of + and ·

From commutativity in 2 :

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} = \frac{rq + ps}{sq} = \frac{r}{s} + \frac{p}{q}$$

5. Identities :

Additive inverses,

For $\frac{p}{q} \in Q$ the additive inverse is $\frac{p}{q} = \frac{-p}{q}$

Since,

$0 = \frac{0}{1}$ For any $\frac{p}{q}$

$$\frac{p}{q} + \frac{0}{1} = \frac{p.1 + 0.q}{q.1} = \frac{p}{1}$$

Multiplicative identity : $1 = \frac{1}{1}$ For any $\frac{p}{q}$

$$\frac{p}{q} \cdot \frac{1}{1} = \frac{p.1}{q.1} = \frac{p}{q}$$

6. Additive inverses:

For $\frac{p}{q} \in Q$ the additive inverse is $-\frac{p}{q} = \frac{-p}{q}$

$$\frac{p}{q} + \frac{-p}{q} = \frac{pq + (-p)q}{q^2} = \frac{0}{q^2} = 0$$

7. Multiplicative inverses

If $\frac{p}{q} \in Q$ and $\frac{p}{q} \neq 0$ then $p \neq 0$

The inverse is $\left(\frac{p}{q}\right)^{-1} = \frac{q}{p}$ and indeed

$$\frac{p}{q} \cdot \frac{q}{p} = \frac{pq}{qp} = 1$$