IOQM 2023 Solutions

1. Let n be a positive integer such that $1 \leq n \leq 1000$. Let M_n be the number of elements in the set $X_n = \{\sqrt{4n+1}, \sqrt{4n+2}, \cdots, \sqrt{4n+1000}\}$. Let

$$a = \max\{M_n : 1 \leq n \leq 1000\}, \text{and } b = \min\{M_n : 1 \leq n \leq 1000\}$$

ANS: Find $a - b$

$$b = \lfloor\sqrt{4 \times 1000 + 1000}\rfloor - \lfloor\sqrt{4 \times 1000 + 1}\rfloor = \lfloor\sqrt{5000}\rfloor - \lfloor\sqrt{4001}\rfloor = 70 - 63 = 7$$

$$a = \lfloor\sqrt{4 \times 1 + 1000}\rfloor - \lfloor\sqrt{4 \times 1 + 1}\rfloor = \lfloor\sqrt{1004}\rfloor - \lfloor\sqrt{5}\rfloor = 31 - 2 = 29$$

$$a - b = 29 - 7 = 22$$

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2. Find the number of elements in the set

$$\big\{(a,b) \in \mathbb{N} : 2 \leq a, b \leq 2023, \log_a(b) + 6\log_b(a) = 5\big\}$$

ANS:

$$S = \big\{(a,b) \in \mathbb{N} : 2 \leq a, b \leq 2023, \log_a(b) + 6\log_b(a) = 5\big\}$$
$$\Rightarrow \log_a(b) + 6\log_b(a) = 5$$
$$\Rightarrow \log^2(b) + 6\log^2(a) = 5\log(b)\log(a)$$
$$\Rightarrow \log^2(b) - 3\log(b)\log(a) - 2\log(b)\log(a) + 6\log^2(a) = 0$$
$$\Rightarrow \log(b)(\log(b) - 3\log(a)) - 2\log(a)(\log(b) - 3\log(a)) = 0$$
$$\Rightarrow (\log(b) - 2\log(a))(\log(b) - 3\log(a)) = 0$$
$$\Rightarrow a^2 = b, a^3 = b$$
$$S1 = \{(a,b) : 2 \leq a, b \leq 2023, a^2 = b\} = \{(2,4), (3,9), \ldots.(44,1936)\}$$
$$S2 = \{(a,b) : 2 \leq a, b \leq 2023, a^3 = b\} = \{(2,8), (3,27), \ldots.(12,1728)\}$$
$$|S| = |S1| + |S2| = (44-1) + (12-1) = 56 - 2 = 54$$

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3. Let α and β be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}$$

Find the smallest possible value of β.

ANS:

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}$$

$$\frac{37}{16} > \frac{\beta}{\alpha} > \frac{16}{7}$$

$$2.31 > \frac{\beta}{\alpha} > 2.28$$

$$2.31 > 2.30 > 2.28$$

$$\alpha = 10, \beta = 23$$

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4. Let x, y be positive integers such that

$$x^4 = (x-1)(y^3 - 23) - 1$$

Find the maximum value of x + y #### ANS:

$$\text{Let } z = y^3 = \frac{x^4+1}{(x-1)} + 23$$

$$\Rightarrow z = (x^3 + x^2 + x + 1) + \frac{2}{x-1} + 23$$

Possible values of x = {-1,0,2,3} for which z = {21, 22,40,64}

Only 64 is perfect cube, so (x, y) = (3, 4)

So x + y = 7

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7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.

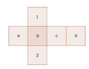

Figure 1: Alt text

ANS: number of distinct dice

$$= |Colours| \times Permutation(a, b, c, d) = 2 \times {}_n\mathrm{P}_k = 2 \times {}_4\mathrm{P}_4 = 2 \times 4! = 48$$

8. Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a 2×7 rectangle using some of these tiles.

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* ab is a prime;
* bc is a prod-

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| # | a | b | c | abc | |:–: |

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10. The sequence $\langle a_n \rangle_{n\geq 0}$ is defined by $a_0 = 1, a_1 = -4$ and $a_{n+2} = -4a_{n+1} - 7a_n,$ for $n \geq 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.

ANS:

$$\{1, -4, 9, -8, -31, ...\}$$
$$a_{n+1}^2 - a_n a_{n+2} = \{?, ?, 7, 49, 343, ...\}$$
$$a_{n+1}^2 - a_n a_{n+2} = 7^{n+1}$$
$$a_{50}^2 - a_{49}a_{51} = 7^{50}$$

Number of positive integer divisors = 51

11. A positive integer m has the property that m^2 is expressed in the format $4n^2 - 5n + 16$ where n is an integer.Find the maximum possible value of |m-n|. #### ANS:

$\Rightarrow m^2 = 4n^2 - 5n + 16 = (2n)^2 - 2.5.2.\frac{1}{4}n + 4^2$

$\Rightarrow m^2 = (2n)^2 - 2.5.2.\frac{1}{4}n + \frac{25}{16} - \frac{25}{16} + 4^2$

$\Rightarrow m^2 = \left(2n - \frac{5}{4}\right)^2 + \frac{231}{16}$

$\Rightarrow (4m)^2 = (8n-5)^2 + 231$

$\Rightarrow (4m)^2 - (8n-5)^2 = 231$

$\Rightarrow (4m - 8n + 5)(4m + 8n - 5) = 231 = 1 \times 231 = 3 \times 77 = 21 \times 11 = 33 \times 7 \Rightarrow 8m = \{232, 80, 32, 40\}$

$\Rightarrow m = \{29, 10, 4, 5\}$

$\Rightarrow m^2 = 4n^2 - 5n + 16$

$\Rightarrow 4n^2 - 5n +$

12. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of P(x) at x=i. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

ANS:

$$P(x) = x^3 + ax^2 + bx + c$$
$$P(1) = p_1 = 1 + a + b + c$$
$$P(2) = p_2 = 8 + 4a + 2b + c$$
$$P(3) = p_3 = 27 + 9a + 3b + c$$
$$\text{either } p1 = p2 = p3 \text{ or } p1 + p2 + p3 = 0$$
$$p1 + p2 + p3 \neq 0 \text{ as } 2 \nmid c \text{ so } 2 \nmid (36 + 14a + 6b + c)$$
$$\Rightarrow p2 - p1 = 7 + 3a + b = 0$$
$$\Rightarrow p3 - p2 = 19 + 5a + b = 0$$
$$\Rightarrow 12 + 2a = 0$$
$$\Rightarrow a = -6$$
$$\Rightarrow b = 11$$
$$\Rightarrow p1 = 1 - 6 + 11 + c = 6 + c$$
$$\Rightarrow p2 = 8 - 24 + 22 + c = 6 + c$$
$$\Rightarrow p0 = c$$
$$\Rightarrow p_2 + 2p_1 - 3p_0 = 6 + c + 2(6 + c) - 3c = 18$$

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13. The ex-radii of a triangle are $10\ \frac{1}{2}$, 12 and 14. If the sides of the triangle are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p + q + r}$.

ANS:

$$r_1 = 10\frac{1}{2} = \frac{\Delta}{s-a}; r_2 = 12 = \frac{\Delta}{s-b}; r_3 = 14 = \frac{\Delta}{s-c}$$
$$\frac{s-a+s-b+s-c}{\Delta} = \frac{s}{\Delta} = \frac{2}{21} + \frac{1}{12} + \frac{1}{14} = \frac{8+7+6}{7.12} = \frac{1}{4}$$
$$s = \frac{\Delta}{4}$$
$$r_1r_2r_3 = s\Delta = 1764$$
$$\Delta^2 = 4.1764$$
$$\Delta = 2.42 = 84$$
$$s = \frac{\Delta}{4} = 21$$

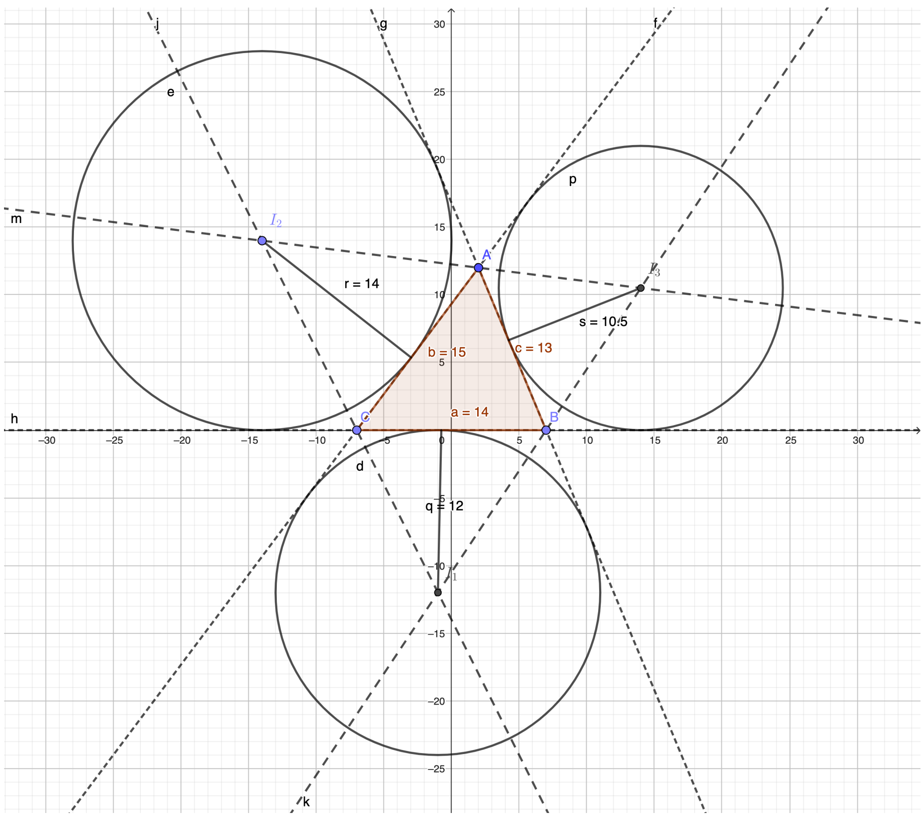

Figure 2: Alt text

$$\frac{21}{2}(s-a) = 12(s-b) = 14(s-c) = \Delta = 84$$

$$a = s - \frac{84 \times 2}{21} = 13$$

$$b = s - \frac{84}{12} = 14$$

$$c = s - \frac{84}{14} = 15$$

$$a + b + c = 13 + 14 + 15 = 42 = p$$

$$a.b.c = 2730 = r$$

$$a.b + b.c + c.a = 587 = q$$

$$\sqrt{p+q+r} = \sqrt{42 + 587 + 2730} \approx 58$$

14. Let ABC be a triangle in the xy plane, where B is at the origin (0,0). Let BC be produced to D such that BC:CD=1:1, CA be produced to E such that CA:AE=1:2 and AB be produced to F such that AB:BF=1:3. Let G(32,24) be the centroid of the triangle

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1. The six sides of a convex hexagon A_1 A_2 A_3 A_4 A_5 A_6 are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_iA_jA_k$, where $1 \leq i < j < k \leq 6$, has at least one red side, find the sum of the squares of the digits of N.

Figure 3: Alt text

ANS: Only non-opposite diagonals are part of triangles $A_iA_jA_k$. So, the opposite diagonals can either be red/blue so, there are $2^3 = 8$ possible ways to color opposite diagonals.

All except two triangles $A_1A_3A_5$ and $A_2A_4A_6$ shares a side with hexagon which is red in color. The triangles $A_1A_3A_5$ and $A_2A_4A_6$, cannot be all blue. So, each of $A_1A_3A_5$ and $A_2A_4A_6$ can have $2^3 - 1 = 7$ possible ways to color.

So N $= 2^3 \times (2^3 - 1) \times (2^3 - 1) = 8 \times 7 \times 7 = 392$

So, sum of square of digits $= 3^2 + 9^2 + 2^2 = 9 + 81 + 4 = 94$

17. Consider the set

$$\mathcal{S} = \{(a, b, c, d, e) : 0 < a < b < c < d < e < 100\}$$

where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of $\mathcal{S}$, find the largest integer less than or equal to D.

ANS:

- If d = 99-1, there are ${}^{98-1}C_3$ ways to select (a, b, c) and 1 way to select e
- If d = 99-2, there are ${}^{98-2}C_3$ ways to select (a, b, c) and 2 ways to select e
- If d = 99-3, there are ${}^{98-3}C_3$ ways to select (a, b, c) and 3 ways to select e
- ...
- If d = 99-95, there are ${}^{98-95}C_3$ ways to select (a, b, c) and 95 ways to select e

Total number of possibilities $= |S| = {}^{99}C_5$

$$D = \frac{\sum_{i=1}^{95} {}^{98-i}C_3 (99 - i) i}{{}^{99}C_5}$$

18. Let $\mathcal{P}$ be a convex polygon with 50 vertices. A set $\mathcal{F}$ of diagonals of $\mathcal{P}$ is said to be minimally friendly if any diagonal $d \in \mathcal{F}$ intersects at most one other diagonal in $\mathcal{F}$ at a point interior to $\mathcal{P}$. Find the largest

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20. For any finite non empty set X of integers, let $\max(X)$ denote the largest element of X and $|X|$ denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

ANS:

$$\max(A) \times |B| = 12; \text{ and } \&|A| \times \max(B) = 11$$

and N can be written as $100a + b$ where a, b are positive integers less than 100 , find $a + b$.

$$\max(A) \times |B| = 12 = 1 \times 12 = 2 \times 6 = 3 \times 4 = 4 \times 3 = 6 \times 2 = 12 \times 1$$

$$|A| \times \max(B) = 11 = 1 \times 11 = 11 \times 11$$

$\max(A) \times \vert B\vert$	$\vert A\vert \times \max(B)$	A	B	$\vert A \times B\vert$	*Remark*
$1X12$	$1X11$	$\{1\}$	$\{\}$	0	Not Possible as you cannot have a set of 11 positive integers maximum less than 11
$12X1$	$11X1$	${}^{11}C_{10}$	$\{1\}$	11	Select 10 elements from 1..11
$< any >$	$11X1$	$\{\}$	$\{\}$	0	Not Possible as you cannot have a set of 11 positive integers maximum less than 11
$12X1$	$1X11$	$\{12\}$	$\{11\}$	1	
$2X6$	$1X11$	$\{2\}$	${}^{10}C_5$	252	Select 5 elements sans 11 from 1 ..10
$6X2$	$1X11$	$\{6\}$	${}^{10}C_1$	10	Select 1 element sans 11 from 1 .. 10
$3X4$	$1X11$	$\{3\}$	${}^{10}C_3$	120	Select 3 elements sans 11 from 1 .. 10
$4X3$	$1X11$	$\{4\}$	$10C\_2$	45	Select 2 elements sans 11 from 1 .. 10
Total				439	

$$439 = 4 \times 100 + 39 = 100a + b$$

$$a + b = 4 + 39 = 43$$

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21 For $n \in \mathbb{N}$, consider non-negative integer-valued functions f on $\{1, 2, \ldots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n}(i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

ANS:

$$\sum_{i=1}^{n}(i + f(i)) = 2023$$

$$\Rightarrow \sum_{i=1}^{n}(i + f(i)) = \sum_{i=1}^{n} i + \sum_{i=1}^{n} f(i) = \frac{n(n+1)}{2} + \sum_{i=1}^{n} f(i) = 2023$$

$$\Rightarrow \sum_{i=1}^{n} f(i) = 2023 - \frac{n(n+1)}{2} \geq 0$$

$f(i) \geq f(j)$ means f is monotonically increasing

$\sum_{i=1}^{n} f(i)$ is the least means, $\frac{n(n+1)}{2}$ is largest possible which makes $n = 63$ as $\frac{63(63+1)}{2} < 2023 < \frac{64(64+1)}{2}$

$$\Rightarrow \sum_{i=1}^{n} f(i) = 2023 - 2016 = 7$$

So now the question is, in how many ways can we partition 7? Also, the partition should be monotonically increasing

$$7 = 7 + 0 = 6 + 1 = 5 + 2 = 5 + 1 + 1 = 4 + 3 = 4 + 2 + 1$$
$$= 4 + 1 + 1 + 1 = 3 + 3 + 1 = 3 + 2 + 2 = 3 + 2 + 1 + 1 = 3 + 1 + 1 + 1 + 1$$
$$= 2 + 2 + 2 + 1 = 2 + 2 + 1 + 1 + 1 = 2 + 1 + 1 + 1 + 1 + 1$$
$$= 1 + 1 + 1 + 1 + 1 + 1 + 1$$

Total Partition = 15