CUBIC MAP

Introduction

The function chosen is the cubic map:

$$x_{t+1} = rx_t^2(1 - x_t)$$

On *Figure 1* are shown the maps for $r = 1$ (blue), $r = 3$ (green), $r = 5$ (red) and $r = 7$ (purple).

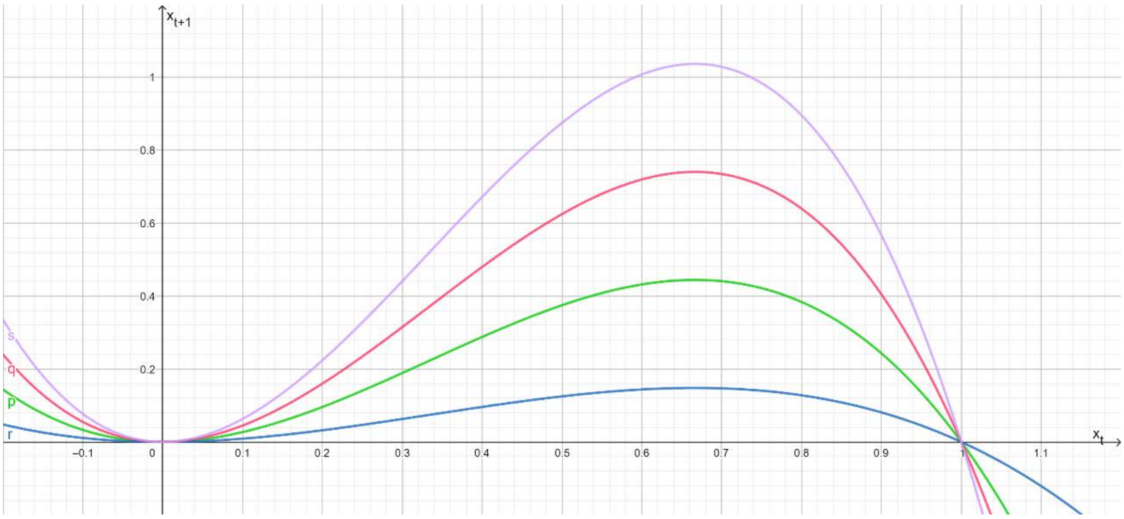

Figure 1

We can see that the function always cuts the x axis on $x = 0$ and $x = 1$.

Fixed points

We calculate the fixed points using the definition of a fixed point:

$$f(x^*) = x^*$$

$$x^* = r(x^*)^2(1 - x^*) = x^*[rx^*(1 - x)]$$

$$x_1^* = 0;\ x_2^* = \frac{r + \sqrt{r^2 - 4r}}{2r};\ x_3^* = \frac{r - \sqrt{r^2 - 4r}}{2r}$$

We can also calculate this graphically for a particular *r*; on *Figure 2*, for instance, we can see the fixed points when $r = 5$.

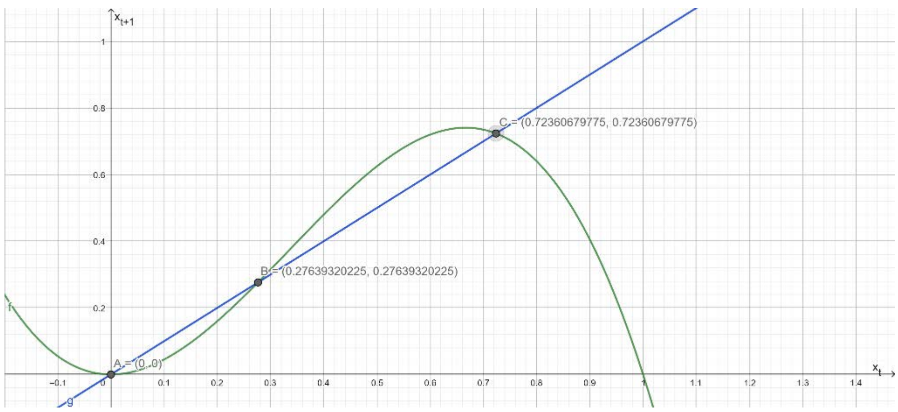

Figure 2

Range of r

To find the lower limit, we just have to find at which value of r the fixed points that we have called x_2^* and x_3^* appear:

$$r^2 - 4r = 0$$

$$r_1 = 0;\ r_2 = 4$$

$r = 0$ is not possible, because then the divider of the fixed points would be 0. Thus, r has to be greater than 4.

To find the upper limit, we first calculate where in the X axis is the maximum of the function. Afterwards, we substitute in $f(x)$ x for that value and equal it to 1, to find at which value of r the maximum x is 1 (since x is a density, that is the maximum value).

$$f'(x) = 0 = r(2x - 3x^2)$$

$$x = \frac{2}{3}$$

$$1 = r\left(\frac{2}{3}\right)^2\left(1 - \frac{2}{3}\right)$$

$$r = 6'75$$

Therefore, $\mathbf{4 < r < 6'75}$. On *Figure 3* are shown the curves of $f(x)$ for $r = 4$ (green) and $r = 6'8$ (blue), along with the diagonal (red).

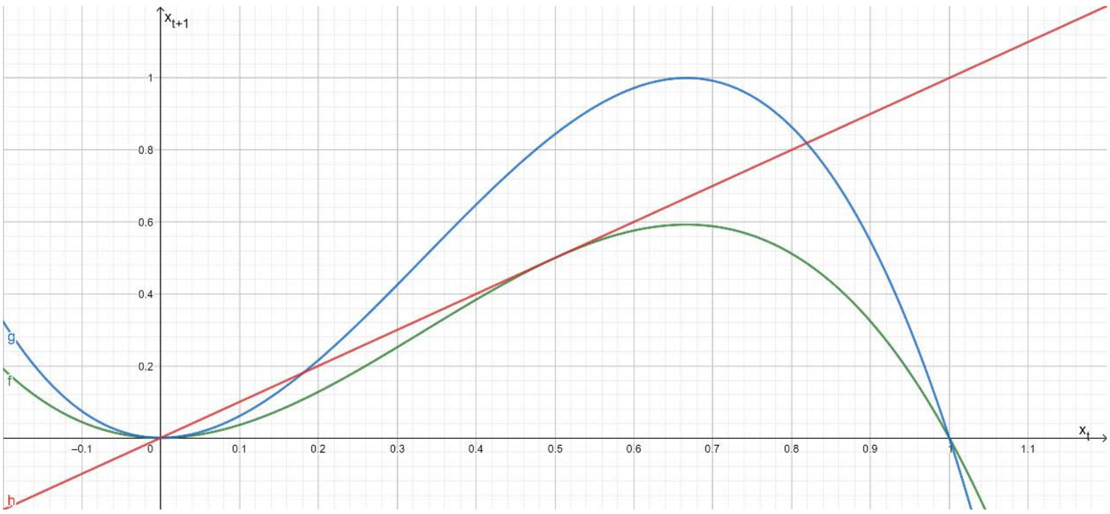

Figure 3

Fixed points stability

A fixed point x^* will be stable if the slope of the tangent to the function at that point is in between -1 and 1:

$$If\ -1 < f'(x^*) < 1, then\ x^*\ is\ stable$$

In our case, the derivative is $f'(x) = r(2x - 3x^2)$. First, let us study the stability of the fixed point $x = 0$:

$$\lambda^{(1)}(x_1^*) = 0$$

Therefore, $x = 0$ is always a stable fixed point.

Now, we will study the stability of our other two fixed points. In this case, substituting is difficult, because the equations we obtained for x_2^* and x_3^* are complicated. Thus, we may take a graphical approach.

We know the fixed points are stable when $-1 < r(2x - 3x^2) < 1$, so we have to find at which points the slope of the tangent at fixed points is -1 and 1. First, let us have a look at what happens when $r = 4$ on *Figure 4*.

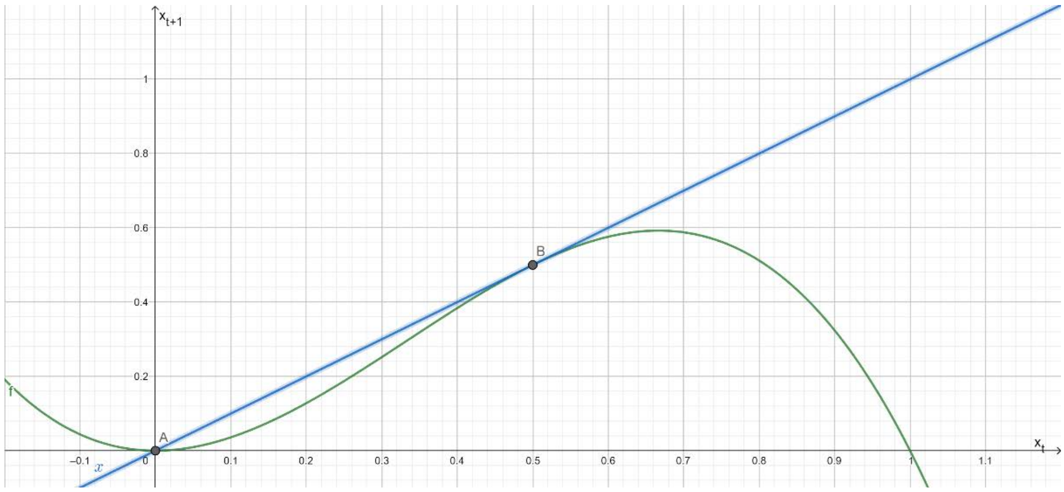

Figure 4

As we saw previously, we have two fixed points: *A*, at $x = 0$, and *B*, at $x = 0'5$. *B* is an unstable fixed point, because the slope is exactly 1. When we increase *r*, to for example 4'2, a new fixed point *C* appears (*Figure 5*).

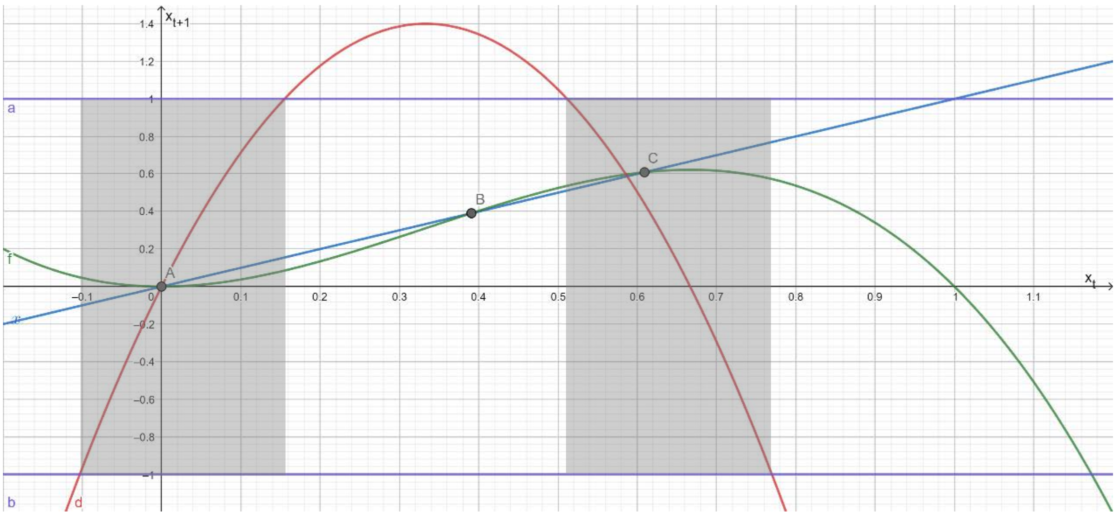

Figure 5

On *Figure 5* we have our function in green, its derivative on red, the function $f(x) = x$ on blue, the lines $y = 1$ and $y = -1$ on purple. Finally, the regions in which fixed points are stable (in which the value of the slope of the tangent to the function is in between -1 and 1). The fixed point *B*, which corresponds to $x_2^* = \frac{r+\sqrt{r^2-4r}}{2r}$, remains unstable, while the fixed point *C*, which corresponds to $x_3^* = \frac{r-\sqrt{r^2-4r}}{2r}$, remains unstable. This kind of bifurcation process is called tangent bifurcation. The initial slope is 1. Afterwards, one point becomes stable and the other unstable.

Finally, we must calculate at which values of *x* this fixed points change stability. In the range of *r* previously calculated, x_3^* (*B*) never becomes stable. On the other side, x_2^* (*C*) becomes unstable when $r = 5'\hat{3}$

Therefore, we have got the following stability:

$$4 < r < 5'\hat{3}: x_1^* \; stable, x_2^* \; unstable, x_3^* \; stable$$

$$5'\hat{3} < r < 6'75: x_1^* \; stable, x_2^* \; unstable, x_3^* \; unstable$$

Of course, this is valid for any number of iterations, since the stability of fixed points remains the same on every iteration.

I have adapted an R script that does the cobweb plot and the plot of the Lyapunov exponent for the logistic equation I found on the internet (https://bayesianbiologist.com/tag/cobweb-plot/) to work on our map. We will continue our fixed points analysis using the cobweb plot.

On *Figure 6* we can see two plots for $r = 4$, with initial conditions $x_0 = 0'4$ and $x_0 = 0'9$. As expected, in both cases the population tends to 0, the only stable fixed point.

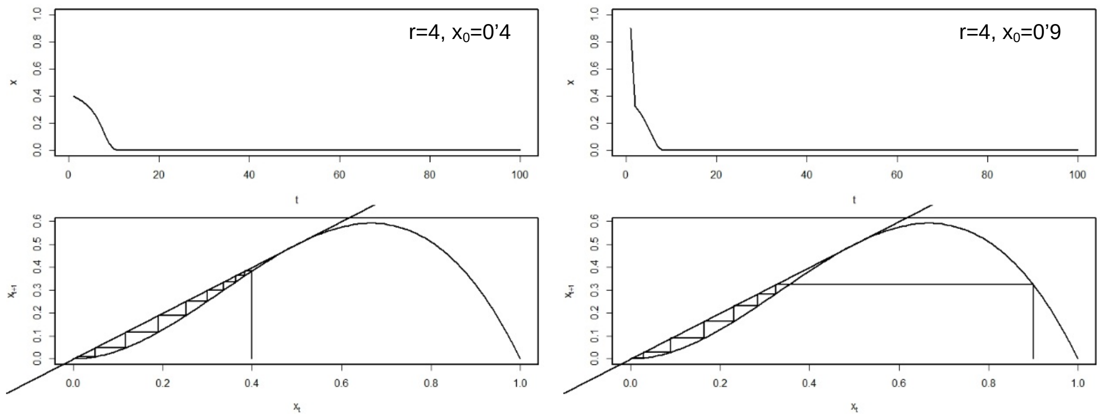

Figure 6

On *Figure 7* are shown four plots: for $r = 4'1$, with initial conditions $x_0 = 0'4$ and $x_0 = 0'5$, and for $r = 5'3$, with initial conditions $x_0 = 0'4$ and $x_0 = 0'98$. In these values there are two stable points; therefore, depending on the initial conditions, the population will tend to 0 or to the other stable point. As can be seen when $x_0 = 0'98$, very high initial values of the population tend to its extinction.

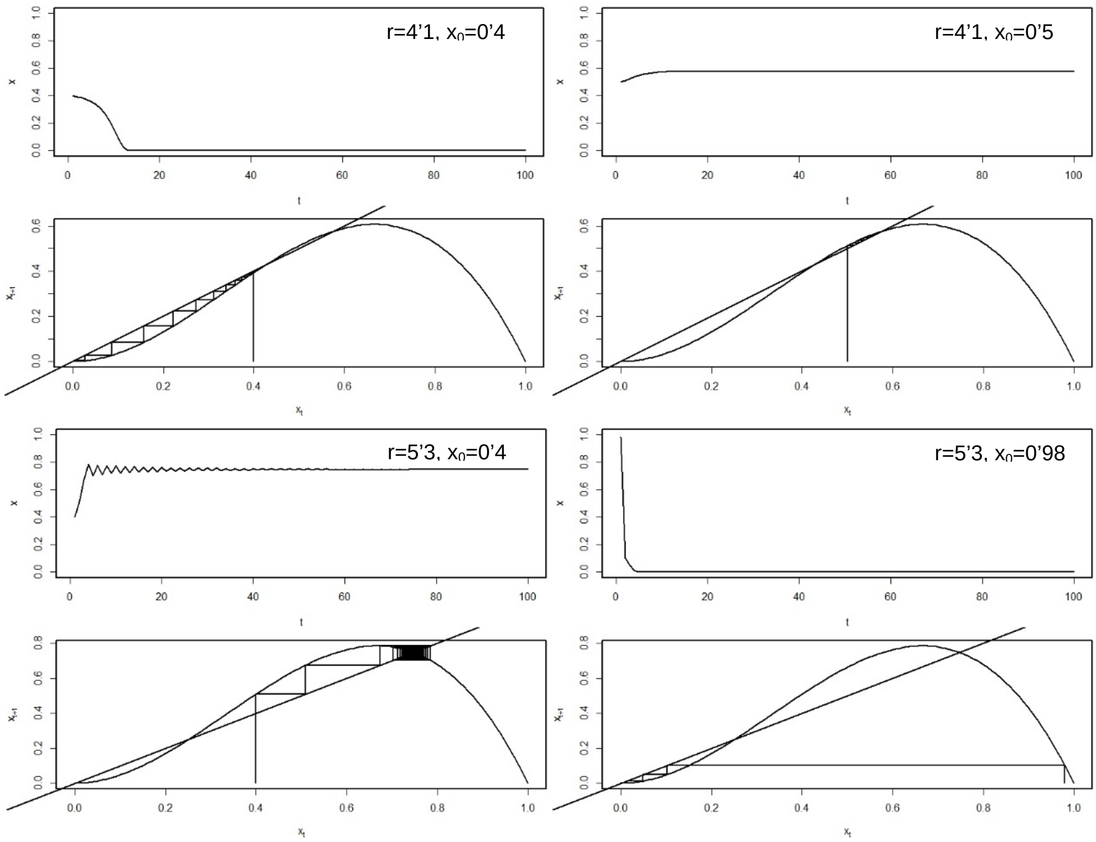

Figure 7

What happens, then, when $r = 5'4$? As we have seen, at this point again there is only one stable fixed point, $x = 0$. The situation now, however, is different than in *Figure 6*. On *Figure 8* are shown the plots for $r = 5'4$, and initial conditions $x_0 = 0'4$ and $x_0 = 0'98$.

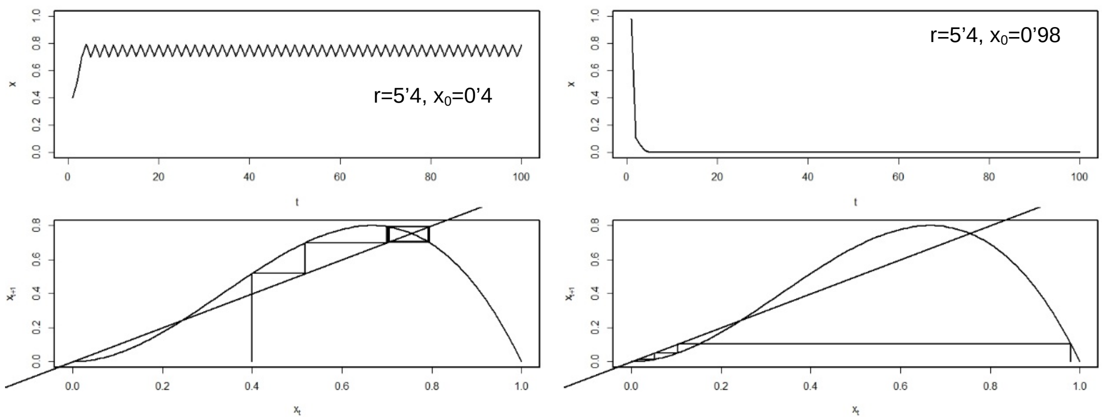

Figure 8

When $x_0 = 0'98$ there is no surprise, the population tends to extinction. When $x_0 = 0'4$, however, we get a new behavior: there is not a fixed point to which tends the population, but rather a cycle of period 2 has been formed. To see what is happening, let us take a look at our function again.

On *Figure 9* are shown the first and second iteration of our function (red and blue, respectively) for $r = 5'3$. The second iteration is:

$$x_{t+2} = f\big(f(x_t)\big) = r[rx^2(1-x)]^2[1-[rx^2(1-x)]]$$

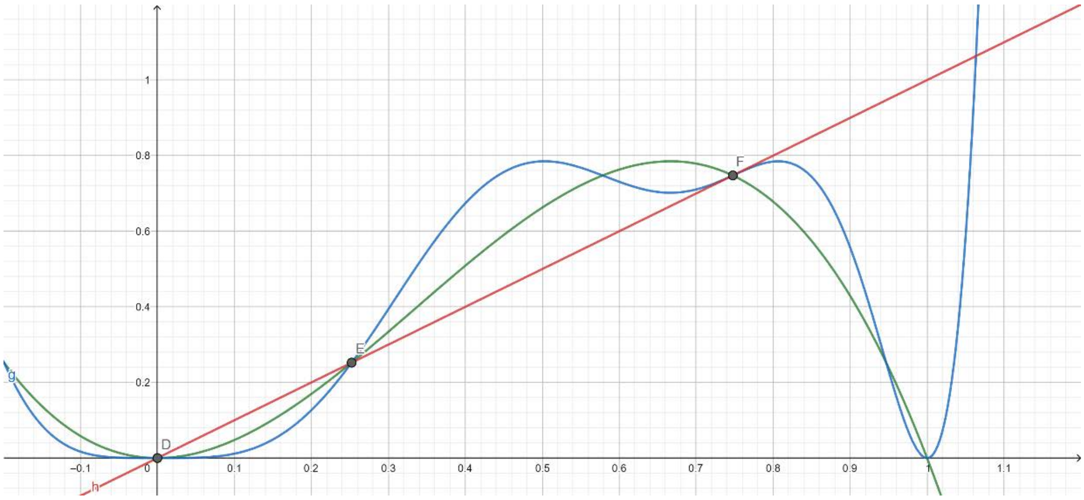

Figure 9

As we saw before, when $r = 5'3$, we have the fixed point *F* that is about to become unstable. On *Figure 10* we can see the same as in *Figure 9*, but when $r = 5'4$. There, we can see a pitchfork bifurcation, in which a originally stable fixed point becomes unstable. Two new periodic points are generated, both stable with the same slope, and that is what we see on *Figure 8*. As *r* increases, those points will too become unstable, generating each two new stable points.

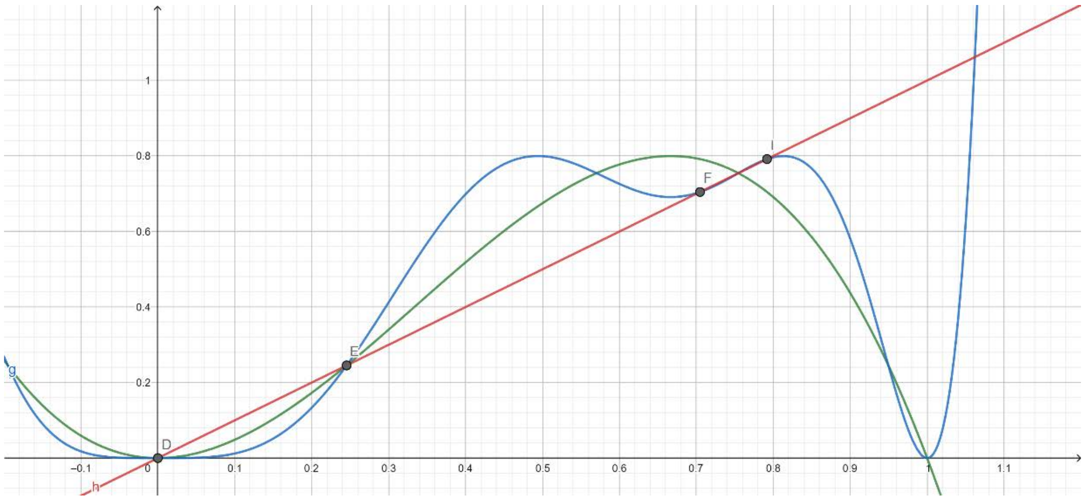

Figure 10

These new points, *F* and *I*, are population values which recur every second generation (that is, points with period 2). A period-2 solution is a pair x_0^*, x_1^* with $f(x_0^*) = x_1^*$ and $f(x_1^*) = x_0^*$, but $x_0^* \neq x_1^*$.

Bifurcation diagram

Finally, we will have a look at the bifurcation diagram. In order to plot it, I've recycled the code from https://www.r-bloggers.com/chaos-bifurcation-diagrams-and-lyapunov-exponents-with-r-2/. The result is shown on *Figure 11*.

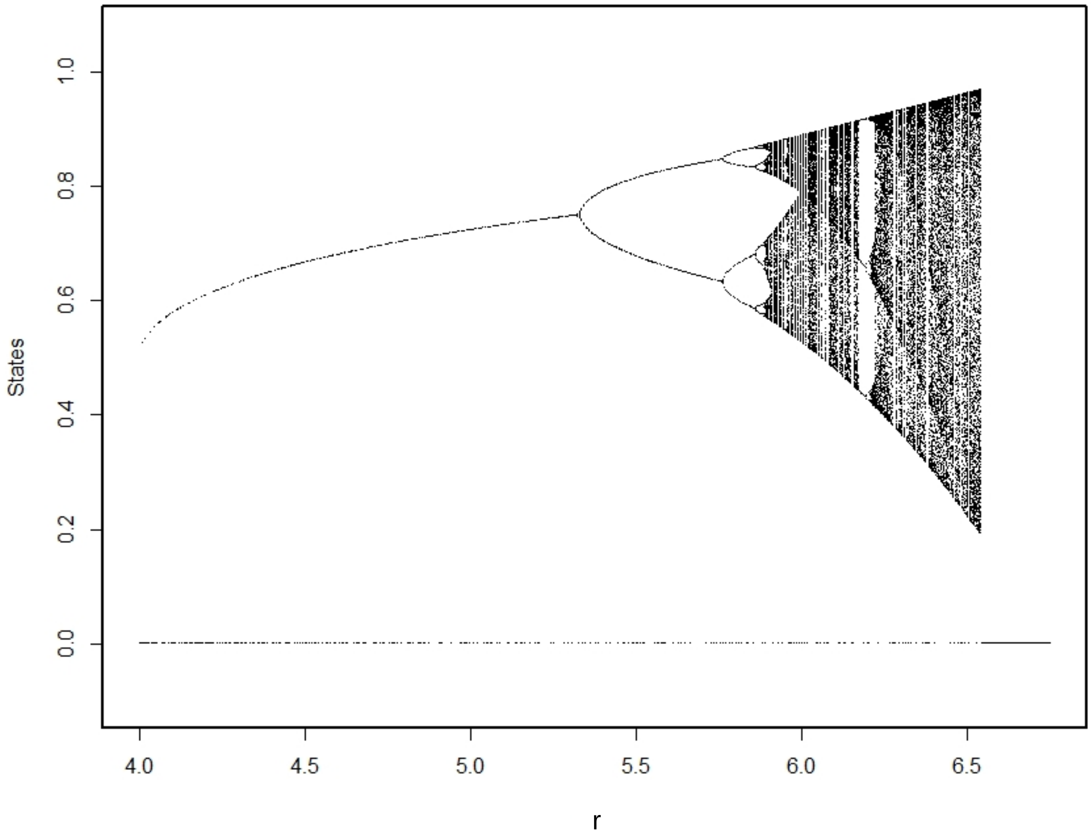

Figure 11

This plot is somewhat similar to the one we studied for the logistic function. As expected, $x = 0$ always remains stable. The other stable point remains stable until $x = 5'\hat{3}$, and then breaks into a 2-period cycle. Somewhere in between 5'5 and 6, this 2-period cycle splits in a 4-period cycle. On *Figure 12* is shown an example of 4-period cycle, with $r = 5'8$ and $x_0 = 0'8$.

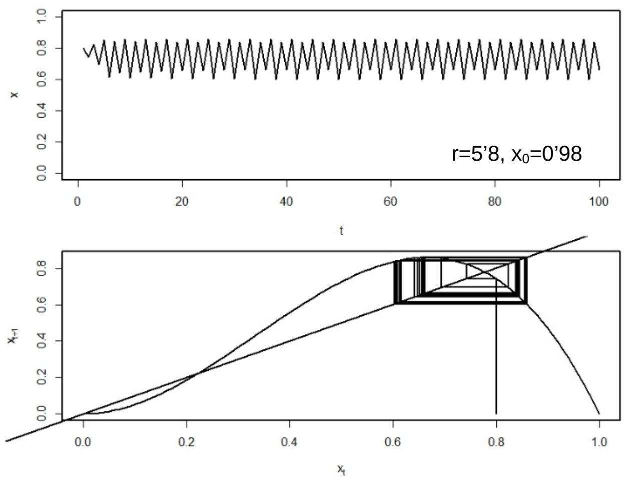

Figure 12

As *r* keeps increasing, new cycles of periods 2^n appear (infinite cascade of periodic orbits). However, we can see on *Figure 11* that somewhere between 6 and 6'5 a period 3 cycle appears. How can this happen? Although this process produces an infinite sequence of cycles with periods 2^n ($n \to \infty$), the "window" of r values wherein any one cycle is stable progressively diminishes (that is, the distance between bifurcations in r-space become progressively smaller as n increases). At a certain critical parameter value, the point of accumulation of period 2^n cycles, all periodic solutions of period 2^n are unstable.

Beyond this point of accumulation there are an infinite number of fixed points with different periodicities, and an infinite number of different periodic cycles. There are also an uncountable number of initial points x_0 which give totally aperiodic (although bounded) trajectories; no matter how long the time series generated by $F(x)$ is run out, the pattern never repeats. Such a situation, where an infinite number of different orbits can occur, depending on very slight initial modifications, is called "**chaotic**".

As the parameter increases beyond the critical value, at first all these cycles have even periods. Although these cycles may in fact be very complicated (having a non-degenerate period of, say, 5.726 points before repeating), they will seem to the casual observer to be rather like a somewhat "noisy" cycle of period 2. As the parameter value continues to increase, there comes a stage at which the first odd period cycle appears. At first, these odd cycles have very long periods, but as the parameter value continues to increase cycles with smaller and smaller odd periods are picked up, until at last the three-point cycle appears, which is what we have discussed before. Afterwards, we have a second chaotic regime.

On *Figure 13* are shown two plots for the second chaotic regime, $r = 6'4$, for $x_0 = 0'8$ and $x_0 = 0'80001$. Even though the change is extremely small, the resulting plots are extremely different; this is what chaotic means.

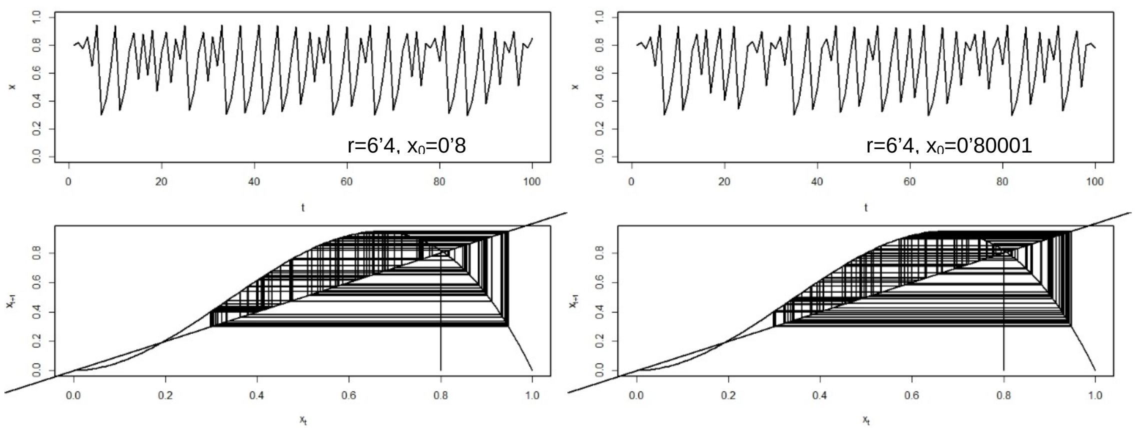

Figure 13