$$(()) \, \text{II} \, \measuredangle bla \tag{1}$$

Signal detection over an optical channel

$$\pi_0 = \frac{g}{g+1}, \quad g = \pi_0/\pi_1, \quad \lambda_1 = |S_1 + l|^2, \quad \pi_0 = \frac{g}{g+1}, \quad \frac{S_0\pi_0 - S_1\pi_1}{\pi_1 - \pi_0}$$

Given an optical channel, a signal on that channel can be described as a coherent state denoted by $|S\rangle$ where $S \in \mathbb{C}$

Considering a Noise free environment, in order to detect a signal in this type of channel, we use a photon counter receiver. Such a receiver is a direct detection receiver which detects the intensity of the optical and generates a Poisson process, where the rate of the process holds $\lambda = |S|^2$.

Suppose we have two coherent state signals denoted by $|S_0\rangle, |S_1\rangle$ and we would like to distinguish between the two binary hypotheses with the corresponding priori probabilities π_0, π_1 respectively under hypotheses $H = 0, 1$, while holding some transmission cost constraint.

One approach was given by **Kennedy** who proposed adding a constant additional coherent state signal $|l\rangle$ before feeding the signal's input to the receiver. Doing so generates a coherent state $|S + l\rangle$ which the receiver in turn outputs a Poisson process with rate $\lambda_i = |S_i + l|^2$.

An additional approach was given by **Dolinar** who suggested as continuation to Kennedy's design to replace the constant signal with a controlled signal $|l(t)\rangle$ which is chosen adaptively based on the photon arrivals up to that moment, in order to achieve more certainty in the hypothesis choice with time.

The core concept of Dolinar's updated design includes a recursive method in which the posterior probabilities of the two possible hypotheses denoted by

$\pi_1(t), \pi_2(t)$ are updated after each step of time Δ to yield $\pi_1(t+\Delta), \pi_2(t+\Delta)$.

At this point, by making Δ arbitrarily small we can expect the current Poisson process to return:

$$Poisson(\lambda) = \begin{cases} 0 & \text{w.p. } (1 - \lambda_i\Delta) \\ 1 & \text{w.p. } (\lambda_i\Delta) \end{cases}$$

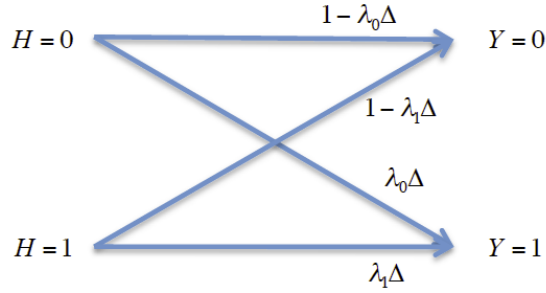Which can be thought of as the following binary channel (Figure 1):

Figure 1: Equivalent binary channel over time Δ

Since we obtained an approximation of a binary channel we may now ask how should $l(t)$ be decided in other to maximize the mutual information of the binary channel.

The Entropy can be calculated as follows:

$$H(Y) = H_b[\Delta(\pi_0\lambda_0 + \pi_1\lambda_1)]$$
$$H(Y|X) = \pi_0 H_b(\Delta\lambda_1) + \pi_1 H_b(\Delta\lambda_1)$$

Which gives us the mutual information of the channel:

$$I(X;Y) = H(Y) - H(Y|X)$$
$$= H_b[\Delta(\pi_0\lambda_0 + \pi_1\lambda_1)]$$
$$- [\pi_0 H_b(\Delta\lambda_1) + \pi_1 H_b(\Delta\lambda_1)]$$

2

Since we want to maximize the mutual information, we shall compare its derivative to zero:

$$\frac{dI(X;Y)}{dl} = \log_2\left[\frac{1 - \Delta[\pi_0\lambda_0 + \pi_1\lambda_1]}{\Delta[\pi_0\lambda_0 + \pi_1\lambda_1]}\right] 2\Delta(\pi_0\sqrt{\lambda_0} + \pi_1\sqrt{\lambda_1})$$

$$- \log_2\left[\frac{1 - \lambda_0\Delta}{\lambda_0\Delta}\right] 2\Delta\pi_0\sqrt{\lambda_0}$$

$$- \log_2\left[\frac{1 - \lambda_1\Delta}{\lambda_1\Delta}\right] 2\Delta\pi_1\sqrt{\lambda_1}$$

$$= 0$$

$$\frac{dI(X;Y)}{dl} = \log_2\left[\frac{1 - \Delta[\pi_0\lambda_0 + \pi_1\lambda_1]}{\Delta[\pi_0\lambda_0 + \pi_1\lambda_1]}\right] 2\Delta(\pi_0\sqrt{\lambda_0} + \pi_1\sqrt{\lambda_1})$$

$$- 2\Delta\left[\log_2\left(\frac{1 - \lambda_0\Delta}{\lambda_0\Delta}\right)\pi_0\sqrt{\lambda_0} + \log_2\left(\frac{1 - \lambda_1\Delta}{\lambda_1\Delta}\right)\pi_1\sqrt{\lambda_1}\right]$$

$$= 0$$

In the optimal case where $l = l^* = \frac{S_0\pi_0 - S_1\pi_1}{\pi_1 - \pi_0}$, we can calculate the following expressions for the λ's:

$$\lambda_i = (l^* + S_i)^2 = (\frac{S_0\pi_0 - S_1\pi_1}{\pi_1 - \pi_0} + S_i)^2 = (\frac{S_0\pi_0 - S_1\pi_1 + S_i(\pi_1 - \pi_0))}{\pi_1 - \pi_0})$$

Which yields:

$$
\begin{aligned}
\overline{\lambda} &= \pi_0\lambda_0 + \pi_1\lambda_1 \\
&= \pi_0(\frac{\pi_1(S_0 - S_1)}{\pi_1 - \pi_0})^2 + \pi_1(\frac{\pi_0(S_0 - S_1)}{\pi_1 - \pi_0})^2 \\
&= (\pi_0\pi_1^2 + \pi_1\pi_0^2)(\frac{(S_0 - S_1)}{\pi_1 - \pi_0})^2
\end{aligned}
$$

Since $\pi_0 + \pi_1 = 1$ we have that:

$$
\begin{aligned}
\pi_0\pi_1^2 + \pi_1\pi_0^2 &= \pi_0\pi_1(1 - \pi_0) + \pi_1\pi_0^2 \\
&= \pi_0\pi_1 - \pi_0\pi_1^2 + \pi_0\pi_1^2 \\
&= \pi_0\pi_1
\end{aligned}
$$

Thus we get:

$$(\pi_0\pi_1^2 + \pi_1\pi_0^2)(\frac{(S_0 - S_1)}{\pi_1 - \pi_0})^2 = \pi_0\pi_1(\frac{(S_0 - S_1)}{\pi_1 - \pi_0})^2$$

Overall we get the following:

$$
= \begin{cases}
\lambda_0 = (\frac{\pi_1(S_0 - S_1)}{\pi_1 - \pi_0})^2 \\
\lambda_1 = (\frac{\pi_0(S_0 - S_1)}{\pi_1 - \pi_0})^2 \\
\overline{\lambda} = \pi_0\pi_1(\frac{(S_0 - S_1)}{\pi_1 - \pi_0})^2
\end{cases}
$$

At this point we can calculate the mutual information in the optimal case:

$$
\begin{aligned}
I &= H(Y) - H(Y|X) \\
&= H_b[\bar{\lambda}\Delta] - \pi_0 H_b(\lambda_0\Delta) - \pi_1 H_b(\lambda_1\Delta) \\
&= \bar{\lambda}log\left[\frac{1}{\bar{\bar{\lambda}}}\right] - \pi_0\lambda_0 log\left[\frac{1}{\lambda_0}\right] - \pi_1\lambda_1 log\left[\frac{1}{\lambda_1}\right] \\
&= \pi_0\pi_1\left(\frac{(S_0-S_1)}{\pi_1-\pi_0}\right)^2 log\left[\frac{1}{\pi_0\pi_1\left(\frac{(S_0-S_1)}{\pi_1-\pi_0}\right)^2}\right] \\
&\quad -\pi_0\left(\frac{\pi_1(S_0-S_1)}{\pi_1-\pi_0}\right)^2 log\left[\frac{1}{\left(\frac{\pi_1(S_0-S_1)}{\pi_1-\pi_0}\right)^2}\right] \\
&\quad -\pi_1\left(\frac{\pi_0(S_0-S_1)}{\pi_1-\pi_0}\right)^2 log\left[\frac{1}{\left(\frac{\pi_0(S_0-S_1)}{\pi_1-\pi_0}\right)^2}\right] \\
&= \pi_0\pi_1\left(\frac{(S_0-S_1)}{\pi_1-\pi_0}\right)^2\left(-\log\pi_0\pi_1\left(\frac{(S_0-S_1)}{\pi_1-\pi_0}\right)^2 + \pi_0\log\left(\frac{\pi_1(S_0-S_1)}{\pi_1-\pi_0}\right)^2 + \pi_1\log\left(\frac{\pi_0(S_0-S_1)}{\pi_1-\pi_0}\right)^2\right)
\end{aligned}
$$