A CLASS OF OPTICAL RECEIVERS USING OPTICAL FEEDBACK

by

Samuel Joseph Dolinar, Jr.

S.B., Massachusetts Institute of Technology
(1973)

S.M., Massachusetts Institute of Technology
(1973)

E.E., Massachusetts Institute of Technology
(1973)

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June, 1976

Signature of Author...
Department of Electrical Engineering and
Computer Science, May 10, 1976

Certified by...
Thesis Supervisor

Accepted by...
Chairman, Departmental Committee on Graduate Students

A CLASS OF OPTICAL RECEIVERS USING OPTICAL FEEDBACK

by

Samuel Joseph Dolinar, Jr.

Submitted to the Department of Electrical Engineering and Computer Science on May 10, 1976, in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

ABSTRACT

An (ideally) implementable receiver structure is proposed for quantum-noise-limited optical communication. A spatial array of ideal photodetectors is used to measure the energy in the sum of the received field and a local oscillator field generated at the receiver, which is modulated as a causal function of the photodetector output via a feedback arrangement. The photodetector output is characterized as a regular point process in space and time with a partially controllable intensity function.

The optimum receiver with this structure is determined by dynamic programming for general signal sets and cost criteria, under the assumption that the received field is described by one of a finite number of quantum coherent states. The photodetectors may also be subject to dark current. Explicit solution of the optimality condition is usually impractical, so a sub-optimum local optimality criterion is also discussed.

A general correspondence is established, for fields with no spatial modulation which also satisfy certain regularity conditions, between the performance attainable within the given receiver structure and that of measurements realizable as contingent sequences of arbitrary measurements performed separately on ordered infinitesimal time-samples of the received field.

Several specific communication problems are considered. For many two-state problems, most notably the binary coherent state detection problem, the optimum structured receiver achieves precisely the same performance as the optimum measurement consistent with quantum mechanics. In addition, exponentially optimum performance can be achieved for an M-state detection problem with pulse-position-modulated (PPM) signals.

THESIS SUPERVISOR: Robert S. Kennedy
TITLE: Professor of Electrical Engineering

ACKNOWLEDGMENT

I wish to express my sincere appreciation to Professor Robert S. Kennedy for his guidance and patience during the course of this work. I have benefited both professionally and personally from my association with him.

I am also grateful to Professors Jeffrey H. Shapiro and Alan S. Willsky for carefully reading the thesis and offering many helpful comments.

Many of my fellow graduate students deserve special thanks for their friendship and companionship that have made my years at M.I.T. enjoyable.

I am thankful to Mrs. Rose Bella, who typed most of the thesis, for her accuracy and perseverance.

This research was largely supported by the National Aeronautics and Space Administration under Grant NGL 22-009-013, and for this I am very grateful.

To My Parents

Samuel and Rita

TABLE OF CONTENTS

LIST OF THEOREMS AND LEMMAS

LIST OF FIGURES AND EXAMPLES

GLOSSARY OF SYMBOLS

This glossary is divided into two parts. The first contains fairly standard mathematical or physical notation, much of which is not defined in the text of the thesis. The second part lists frequently appearing symbols which we have introduced. Section or subsection references are included in parentheses whenever helpful; the reader is also referred to the special section on our space-time point process notation, Section 2.3.3. Omitted are most symbols which occur in only one or two sections and are adequately defined in the text. In particular, most of the specialized notation of Sections 4.5 and 6.5 falls in this category.

(I) Mathematical Notation

Symbol	Meaning
a.e.	almost everywhere (everywhere except on a set of measure zero)
¢	the set of complex numbers
convex U	g(x) is convex U over a convex region A if $\gamma g(x) + (1-\gamma)g(y) \leq g[\gamma x + (1-\gamma)y]$ for all $0 \leq \gamma \leq 1$ and $x,y \varepsilon A$
convex $\wedge$	g(x) is convex $\wedge$ if $-g(x)$ is convex U
δ_{jk}	Kronecker delta, $\delta_{jk}=1$ if $j=k$ and $\delta_{jk}=0$ if $j \neq k$
$\frac{\partial g(\underline{\xi})}{\partial \underline{\xi}}$	the (column) vector of partial derivatives of the scalar function $g(\underline{\xi})$ with respect to the components ξ_j of $\underline{\xi}$
E	expectation; see (II)
ε	set inclusion; see also (II)
i	$\sqrt{-1}$; see also (II)
inf	infimum; greatest lower bound
Im[•]	imaginary part of the complex number •
j	$\sqrt{-1}$; see also (II)

Symbol	Meaning
$\lim_{\Delta \downarrow 0}$	limit as Δ approaches 0 from positive values
$\limsup_{\Delta \downarrow 0} g(\Delta)$	$\lim_{\Delta \downarrow 0} \sup_{0<\delta<\Delta} g(\delta)$
log	natural logarithm
max	maximum, largest element of a set
min	minimum, smallest element of a set
$o(\Delta)$	a function of Δ with the property $\lim_{\Delta \downarrow 0} \frac{o(\Delta)}{\Delta} = 0$
Pr	probability; see (II)
Π	product
QED	end of proof
R	the set of real numbers
R^+	the set of positive real numbers
$Re[\cdot]$	real part of the complex number $\cdot$
sup	supremum, least upper bound
$\sum$	summation
$\cup$	union of sets
$\cdot^T$	transpose of the vector (or matrix) $\cdot$
$\cdot \times \cdot$	cross product of the sets $\cdot$ and $\cdot$
$\sim$	asymptotically equal to or asymptotically proportional to
$\otimes$	tensor product
$\{\cdot\}$	a set
$\{x:\cdot\}$	the set of x such that $\cdot$ is true
$\lvert\cdot\rvert$	absolute value ($\cdot$ real), complex magnitude ($\cdot$ complex), or sum of absolute values of components ($\cdot$ a vector)

Symbol	Meaning
$\equiv$	equals (used in definitions)
$\approx$	approximately equal to
$\cdot^*$	complex conjugate of $\cdot$; an optimum value of $\cdot$
$\subseteq$	set inclusion
$\cdot!$	$\cdot$ factorial
$g:A\to B$	a function g with domain A and range in B
$g(\tau^-)$	$\lim_{\Delta\downarrow 0} g(\tau-\Delta)$
$g(\tau^+)$	$\lim_{\Delta\downarrow 0} g(\tau+\Delta)$
B^N	if B is a set, $B^N = B\mathrm{x}B\mathrm{x}\ldots\mathrm{x}B$ (N times)
$(\cdot,\cdot)$	a Hilbert space inner product
$\lvert\cdot\rangle$	a quantum state (Dirac notation)
$\langle\cdot\lvert\cdot\rangle$	inner product between quantum states
$\langle\cdot\lvert\cdot\rvert\cdot\rangle$	a quantum measurement operator matrix element

(II) Other Thesis Notation

Symbol	Meaning
a	a parameter to be estimated (2.5.1); a temporary constant
$\hat{a}$	an estimate of a (2.5.1)
$\hat{a}(\cdot)$	an estimate of a based on the data $\cdot$ (2.5.1)
$\lvert\alpha_j\rangle$	a quantum coherent state corresponding to the received field over [0,T], under H_j (4.2.2; 4.2.3)
$\lvert\alpha_j^i\rangle_i$	a quantum coherent state corresponding to the received field over $[(i-1)\Delta, i\Delta]$, under H_j (4.2.3)
b	a temporary constant

Symbol	Meaning
B	a set of possible quantum measurement outcomes (4.2.2, 4.4.1)
β	outcome of a quantum measurement (4.2.1, 4.2.2)
β_i	outcome of the ith quantum measurement (4.4.1)
$\underline{\beta}_i$	a set of measurement outcomes from the first i measurements, $\underline{\beta}_i \equiv (\beta_1,\ldots,\beta_i)$ (4.4.1)
c	a temporary constant
$c(a,\hat{a})$	a differential cost function (matrix) (2.5.2)
$\overline{c}(\underline{\xi},\tau)$	a differential average cost increment (5.2)
$\tilde{c}(\underline{\xi},\tau,\vec{r})$	a contribution to $\overline{c}(\underline{\xi},\tau)$ associated with $\vec{r}$ (5.3)
$C(a,\hat{a})$, $C(j,\hat{j})$	a cost function (matrix) denoting the cost of guessing $\hat{a}$ ($\hat{j}$) when in fact a (j) is the true value (2.5.1)
$C(j,\hat{j};\underline{\xi},\tau)$	a time- and state-dependent cost matrix (5.2)
$C_\tau(j,\hat{j};\underline{\xi},\tau)$	$\dfrac{\partial C(j,j;\underline{\xi},\tau)}{\partial\tau}$ (5.2)
$C_i(j,\hat{j};\underline{\xi},\tau)$	$\dfrac{\partial C(j,\hat{j};\underline{\xi},\tau)}{\partial\xi_i}$ (5.2)
$C_1(j,\hat{a})$, $C_1(j,\hat{j})$	an effective cost matrix (3.1.1)
$\overline{C}(\underline{\xi},\tau)$	average cost (cost-to-go) from observations on $[\tau,T]$ with initial probabilities $\underline{\xi}$ at time τ (3.4.1)
$\overline{C}^*(\underline{\xi},\tau)$	an optimum cost-to-go function
$\overline{C}^\ell(\underline{\xi},\tau)$	a cost-to-go function achieved by the feedback function ℓ
$\overline{C}_\Delta(\underline{\xi},\tau)$	a cost-to-go function associated with a Δ-interval or with a subdivision of $[0,T]$ into Δ-intervals

Symbol	Meaning
$\overline{C}^{+}(\underline{\xi},\tau)$	average cost from observations on $[0,\tau]$ with initial probabilities $\underline{\xi}$ at time 0 (5.4)
$\overline{C}'(\underline{\xi},\underline{x},\tau)$	a differential cost increment in the direction $\underline{x}$ from $\underline{\xi}$ (3.4.2)
$\overline{C}''(\underline{\rho},\underline{\xi},\tau)$	a second-order cost difference (3.4.2)
$\overline{C}''(\rho_j,\underline{\xi},\tau)$	$\overline{C}''(\underline{\rho},\underline{\xi},\tau)$ $(\underline{\rho} = [\rho_1,\ldots,\rho_M]^T)$
$\widetilde{C}''(\underline{\rho},\underline{\xi},\tau)$	$\overline{C}''(\underline{\xi},\underline{\rho},\tau)$
C_{max}	an upper bound on the average cost (3A.4)
c	a symmetric a priori cost function (6.3.1a)
γ	used for miscellaneous constants
γ_{jk}, γ^{i}_{jk}	a correlation coefficient between $S_j(\tau)$ and $S_k(\tau)$ (4.2.2, 4.2.3)
dim $(\underline{t})$	the number of point events represented by $\underline{t}$ (if $\underline{t} = (\vec{r}_1,\tau_1),\ldots,(\vec{r}_n,\tau_n)$ then dim $(\underline{t})=n$)
δ	a small positive number
Δ	a small positive time increment, or the length of a small time interval; often $\Delta = \frac{T}{N}$
$\Delta A,\delta A$	small area elements within Σ
$\Delta\tau,\delta\tau$	small subdivisions of $[0,T]$ (used rarely instead of Δ)
Ex, $E[x]$	expectation of x
E_y	expectation over y
$E_{y\|z}$	conditional expectation over y given z
$E[x\|z]$	conditional expectation of x given z; see also (2.3.3)
E_o, E_j, E^{i}_{j}	various signal energies, measured in number of photons

Symbol	Meaning
$E(\tau)$	energy in the difference of binary signals from 0 to τ (6.2.3)
$E(\vec{r},\tau)$	a real optical field (1.1.1)
ε, $\varepsilon(t)$, $\varepsilon(\vec{r},\tau)$	the complex envelope of the received field (1.1.1); see also (I)
ε_j, $\varepsilon_j(t)$, $\varepsilon_j(\vec{r},\tau)$	the deterministic received field (envelope) under H_j
$\varepsilon_o(\vec{r})\ S_j(\tau)$	the received field (envelope) under H_j in the absence of signal-dependent spatial modulation
η_j, η_{ij}, etc.	miscellaneous expressions
$f(\tau)$, $f(\tau:\underline{t})$	a deterministic ratio of a posteriori probabilities associated with the optimum receiver for the binary coherent state problem (6.2.2)
$f(\tau;\xi_1,\xi_2)$	a generalization of $f(\tau)$ (6.3.2b)
$\|\phi\rangle, \|\phi_j\rangle$, $\|\phi_j'\rangle$, $\|\phi_j^k\rangle$	linear combinations of coherent states used in the proofs of the quantum measurement correspondence theorems (4A.2, 4A.4, 5A.2)
$\emptyset$	a measurement outcome corresponding to either no observations or else observations of zero point events during some interval of time
g	miscellaneous functions
h	miscellaneous functions
H_j	the jth hypothesis in the generalized finite-state detection problem (under H_j the received field is $\varepsilon_j(t)$ and the quantum state is $\|\alpha_j\rangle$) (3.1.2)
$\mathcal{H}$, $\mathcal{H}_i$	Hilbert spaces of quantum states (4.2.2, 4.2.3)
i	an index, often refers to a subdivision of [0,T] into Δ-intervals; see also (I)

Symbol	Meaning
I_H	the identity operator in the Hilbert space H
j	an index, usually refers to a quantity associated with H_j
$\hat{j}$	a discrete estimate (decision)
$\hat{j}(\cdot)$	a discrete estimator (decision function) based on $\cdot$
$\hat{j}^*, j^*$	an optimum estimate
$j^*(\underline{\xi})$	an optimum estimate given probability state $\underline{\xi}$ (3.5.2)
$j^*(\underline{\xi},\tau)$	an optimum estimate at time τ given probability state $\underline{\xi}$ (5.2)
$j^*(\underline{\xi},\underline{x},\tau)$	an estimate at time τ which is optimum for $\underline{\xi}$ and for small deviations from $\underline{\xi}$ in the direction $\underline{x}$ (5.2)
$\left.\begin{matrix} j_0, \hat{j}_0, j_1, \hat{j}_1 \\ \hat{j}', j', \hat{j}_0^*, \hat{j}_1^* \end{matrix}\right\}$	miscellaneous estimates
J_0, $J_0(\underline{\xi},\tau)$	a set of optimum a priori guesses
k	an index
K	an index; miscellaneous constants
ℓ	a feedback function
$\ell(t:\underline{t})$	an event-dependent feedback function (2.1, 2.2)
$\ell(t,\underline{\xi})$	a state-dependent feedback function (3.3.4)
ℓ_τ	the restriction of the state-dependent feedback function ℓ to $\Sigma x[\tau,T] \times P$ (3.4.1)
ℓ^Δ	a feedback function associated with a Δ-interval or a subdivision of $[0,T]$ into Δ-intervals
$\lambda(\tau)$	$\frac{\partial E(\tau)}{\partial \tau}$ (6.2.1); also the conditional intensity function for a single detector (1.1.2)
$\lambda_j(t:\underline{t})$	the event-dependent point process intensity function under H_j with feedback ℓ (3.2)

Symbol	Meaning
$\lambda_j(t,\underline{\xi})$	the state-dependent point process intensity function under H_j with feedback ℓ (3.4.2)
$\hat{\lambda}(t:\underline{t})$	the event-dependent (conditional) average intensity function with feedback ℓ (2.3.2, 3.1.3)
$\hat{\lambda}(t,\underline{\xi})$	the state-dependent average intensity function with feedback ℓ (3.4.2)
$\Lambda(\varepsilon,\ell)$	$\|\varepsilon+\ell\|^2$ (2.4)
m	an index
M	the number of states (hypotheses) in the finite-state communication problem model (3.1.2)
$\mu(t)$	a dark current (3.7)
$\mu_j(t:\underline{t})$	a generalized signal- and event-dependent dark current (3.7)
n	an index
N	the number of Δ-intervals in a subdivision of $[0,T]$; also, an index
$o(\Delta)$	see (I)
$o_j(\Delta)$	$o(\Delta)$
$\underline{o}(\Delta)$	a vector function of Δ with the property $\|\underline{o}(\Delta)\| = o(\Delta)$
$p(\tau:\underline{t})$	unconditional probability density function for events at $\underline{t}$ during $[0,\tau]$ (3.2)
$p_j(\tau:\underline{t})$	probability density function for events at $\underline{t}$ during $[0,\tau]$, under H_j (3.2)
P	the set of M-dimensional probability vectors $\underline{\xi}$ (3.3.4)
P_τ, P_i, $\hat{P}$	subsets of P
$P_e^+(\underline{\xi},\tau)$	probability of error cost from observations over $[0,\tau]$ with initial probabilities $\underline{\xi}$ (5.5)

Symbol	Meaning
$P^{+}_{ev}(\tau)$, $P^{-}_{ev}(\tau)$, $P_{ev}(\tau)$	probabilities of observing an even number of counts in $[0,\tau]$ under various hypotheses (6.3.2a)
PPM	pulse-position-modulated (6.4.1)
$Pr[x]$	probability of x
$Pr[x \mid z]$	conditional probability of x given z
$\{Q_\beta\}_{\beta\varepsilon B}$	a set of operators representing an arbitrary quantum measurement (4.2.2)
$\{Q^{i}_{\beta}(\underline{\beta}_{i-1})\}_{\beta\varepsilon B}$	a set of measurement operators chosen as a function of the outcomes $\underline{\beta}_{i-1}$ of prior measurements (4.4.1)
$\{Q^{i}_{\beta}(\underline{\xi})\}_{\beta\varepsilon B}$	a set of measurement operators chosen on the basis of the current probability state $\underline{\xi}$ (4.4.2)
$\{Q^{\tau}_{\beta}(\underline{\xi})\}_{\beta\varepsilon B}$	a set of state-dependent measurement operators associated with the continuous time variable τ (5.9)
$\vec{r}$	a point on the detector aperture area Σ
$\vec{r}_i$	spatial location of the ith point event within Σ; location of the ith detector in an array
$\underline{\vec{r}}$	the set of spatial locations of an arbitrary number of observed point events $\underline{\vec{r}} = (\vec{r}_1,\ldots,\vec{r}_n)$ for some $n \geq 1$, or else $\underline{\vec{r}} = \emptyset$
$\underline{\vec{r}}_n$	the set of spatial locations of a specified number, n, of observed point events, $\underline{\vec{r}}_n = (\vec{r}_1,\ldots,\vec{r}_n)$
$\int_\Sigma d\vec{r}$	integral over the aperture area Σ
R	the set of possible values of the parameter to be estimated, a or j
$\hat{R}$	the range of the estimator function, $\hat{a}(\cdot)$ or $\hat{j}(\cdot)$

Symbol	Meaning
RPP	regular point process
$\underline{\rho}$	a probability vector in P
ρ_j	a component of $\underline{\rho}$
$\underline{\rho}(t:\underline{t})$	the event-dependent version of $\underline{\rho}(t,\underline{\xi})$
$\rho_j(t:\underline{t})$	a component of $\underline{\rho}(t:\underline{t})$
$\underline{\rho}(t,\underline{\xi})$	the potential a posteriori probability vector should an immediate count occur at $t=(\vec{r},\tau)$ given initial probabilities $\underline{\xi}$ at time τ
$\rho_j(t,\underline{\xi})$	a component of $\underline{\rho}(t,\underline{\xi})$
$S_j(\tau)\ \varepsilon_o(\vec{r})$	the received field (envelope) under H_j in the absence of signal-dependent spatial modulation
σ	a dummy variable
Σ	the detector aperture area
$t \equiv(\vec{r},\tau)$ $t' \equiv(\vec{r}',\tau')$ etc.	a space-time point in $\Sigma x[0,T]$
$t_i \equiv (\vec{r}_i,\tau_i)$	the space-time location of the ith event in $\Sigma x[0,T]$
$\underline{t} \equiv (\underline{\vec{r}},\underline{\tau})$	the space-time locations of an arbitrary number of time-ordered events $(\vec{r}_1,\tau_1),\ldots,(\vec{r}_n,\tau_n)$ for some $n \geq 1$ $\vec{r}_i \varepsilon\ \Sigma,\ 0<\tau_1<\ \ldots<\tau_n<T$, or else $\underline{t}=\emptyset$
$\underline{t}_n \equiv (\underline{\vec{r}}_n,\underline{\tau}_n)$	the space-time locations of a specified number, n, of time-ordered events $(\vec{r}_1,\tau_1),\ldots,(\vec{r}_n,\tau_n),\ \vec{r}_i \varepsilon\ \Sigma\ ,\ 0<\tau_1<\ldots<\tau_n<T$.
$\underline{t}'' = (\underline{t},\underline{t}')$	decomposition of the event vector $\underline{t}''$ with respect to events occurring before and after some fixed time τ(3.3.1)
$t_i<t<T$	the time parts of the space-time variable are ordered in the indicated fashion (2.3.3)

Symbol	Meaning
$0<\underline{t}<t<T$	if $\underline{t} = (t_1,\ldots,t_n)$, then $0<\underline{t}<t<T$ means $0<t_1< \ldots<t_n<t<T$
$\int dt$	integral over space and time (spatial integration area is always understood to be all of Σ) (2.3.3)
T	the length of the signaling interval $[0,T]$; see also (I)
τ,τ', etc.	a point in time within the interval $[0,T]$
τ_i	the endpoint of the ith subinterval, $\tau_i = i\Delta$; the occurrence time of the ith point event
$\underline{\tau}$	the ordered occurrence times of an arbitrary number of point events $\underline{\tau} = (\tau_1,\ldots,\tau_n)$ for some $n \geq 1$, $0<\tau_1<\ldots<\tau_n<T$, or else $\underline{\tau} = \emptyset$
$\underline{\tau}_n$	the ordered occurrence times of a specified number, n, of point events, $\underline{\tau} = (\tau_1,\ldots,\tau_n)$, $0<\tau_1<\ldots<\tau_n<T$
$\int d\tau$	integral over time
x	miscellaneous variables
$\underline{x}$	miscellaneous vector variables; a vector specifying the direction of a small deviation from $\underline{\xi}$
x_j	a component of $\underline{x}$
$\underline{\xi}$	a probability vector in P
ξ_j	a component of $\underline{\xi}$
$\underline{\xi}_o$	an a priori probability vector
ξ_j^o	a component of $\underline{\xi}_o$
$\underline{\xi}(\tau:\underline{t})$	the vector of a posteriori probabilities $\underline{\xi}(\tau:\underline{t}) = [\xi_1(\tau:\underline{t}),\ldots,\xi_M(\tau:\underline{t})]^T$ (3.1.3)
$\xi_j(\tau:\underline{t})$	the a posteriori probability of H_j given observed point events at $\underline{t}$ prior to τ (3.1.3)

Symbol	Meaning
$\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o)$	the vector of a posteriori probabilities $\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o)=[\xi_1(\tau:\underline{t};\underline{\xi}_o,\tau_o),\ldots,\xi_M(\tau:\underline{t};\underline{\xi}_o,\tau_o)]^T$
$\xi_j(\tau:t;\underline{\xi}_o,\tau_o)$	the a posteriori probability of H_j given observed point events at $\underline{t}$ during $[\tau_o,\tau]$ and initial probabilities $\underline{\xi}_o$ at time τ_o (3.4.1)
$\underline{\xi}(\underline{\beta}_N)$	the vector of a posteriori probabilities $\underline{\xi}(\underline{\beta}_N) = [\xi_1(\underline{\beta}_N),\ldots,\xi_M(\underline{\beta}_N)]^T$ (4.4.2)
$\xi_j(\underline{\beta}_N)$	the a posteriori probability of H_j given measurement outcomes $\underline{\beta}_N$ (4.4.2)
$\underline{\xi}^i(\beta;\underline{\xi})$	the vector of a posteriori probabilities $\underline{\xi}^i(\beta;\underline{\xi})=[\xi_1^i(\beta;\underline{\xi}),\ldots,\xi_M^i(\beta;\underline{\xi})]^T$ (4.4.2)
$\xi_j^i(\beta;\underline{\xi})$	the a posteriori probability of H_j given measurement outcome β from the ith measurement and initial probabilities $\underline{\xi}$ prior to the ith measurement (4.4.2)
y	miscellaneous variables
z, Z	miscellaneous variables
ω	a sample point of a random process
$\|0\rangle_1\cdots\|0\rangle_N$	the zero-energy coherent state, corresponding to a classical field $S_j(\tau)\varepsilon_o(\vec{r})\equiv 0$
$[0,T]$	the signaling interval
$[\tau,\tau']$	a subinterval of $[0,T]$; see remark at end of (2.3.3)
$[0,T]^*$	the set of all possible space-time event location vectors (2.3.3)
$[0,T]^+$	the domain of $\ell(t:\underline{t})$ (2.3.3)
$\hat{\cdot}$	an estimator; conditional expectation given prior data
$\cdot^*$	complex conjugate of $\cdot$; an optimum value of $\cdot$

CHAPTER I

INTRODUCTION

1.1 BACKGROUND

1.1.1 Two Approaches to the Design of Optical Receivers

In communication systems, receivers are designed to process the available data coming over a channel in response to a message source in a manner which faithfully reproduces the message sent, as measured against some appropriate criterion of performance. In theoretical analyses, we usually allow arbitrary transformations of the received data, even though it may not be readily apparent how to implement all of them. The theoretical receiver which is optimum according to the given performance criterion may be useful for several reasons. It may itself be implementable; if not, it may suggest the structure of good sub-optimum implementable receivers, and in any case it establishes a performance limit against which any proposed receiver may be evaluated.

In optical communication systems, the received data can be represented as an optical field $\vec{E}(\vec{r},\tau) = \mathrm{Re}[\vec{\varepsilon}(\vec{r},\tau)\, e^{-j\, 2\pi f_o \tau}]$ over space and time, when f_o is the optical carrier frequency and $\vec{\varepsilon}(\vec{r},\tau)$ is the low-pass complex envelope of the field.[1] Throughout this thesis it will be assumed that the received field consists of a single polarization component, and all scalar optical fields $E(\vec{r},\tau)$ will be identified by their scalar low-pass complex envelopes $\varepsilon(\vec{r},\tau)$. In addition to contraints of practical implementability there are fundamental constraints imposed by the laws of quantum mechanics on the types of processing of $\varepsilon(\vec{r},\tau)$ which are even theoretically allowable. For instance, any processing which requires measurement of the amplitude and phase of the received field at all space-time points $(\vec{r},\tau)$ cannot be performed precisely because the needed measurements are of incompatible observables.[2]

In view of these considerations, two basic approaches to the design of optical receivers have been developed. The obvious optimum method is to investigate the class of all mathematical receiver models which comply with the fundamental quantum mechanical constraints. Unfortunately this procedure is often not very practical, for two major reasons. It turns out that the quantum communication problem is exceedingly difficult to solve (as compared to the corresponding classical one for which the received field may be measured precisely), and, even when a solution is found, the determination of a physical implementation is an unsolved measurement operator synthesis problem.[3]

The second approach involves positing an implementable receiver structure which is consistent with the quantum constraints, and optimizing performance within this reduced receiver class. A typical rigidly structured receiver has the effect of converting the received field into a particular form of "noisier" data which may then be processed in an arbitrary manner by a classical receiver. The major deficiency of this method is that there is no guarantee that significantly better data cannot be generated from the received field by other allowable quantum measurements which are not realizable within the given structure.

This thesis investigates a certain (ideally) implementable receiver structure which is more flexible than the rigid structures normally considered in that some control may be exercised over the statistics of the noisy data generated by measuring the received field. This condition results in the realization of a much broader class of quantum measurements, but it also increases the analytical complexity to a level which is more comparable with that associated with quantum communication problems than with classical ones. In addition, most of the individual realizable measurements are much more difficult to implement than those represented by conventional receiver structures. The flexibly structured receiver thus represents a compromise between the other two approaches, and its usefulness depends on whether it achieves the most desirable trade-off between attainable performance and ease of implementation and analysis.

1.1.2 Conventional Receiver Structures

Thus far in optical communication, all physically implemented receiver structures have incorporated energy measurement devices known as photodetectors. Although satisfactory models have been developed to account for the bandwidth and noise limitations of real detectors,[4] our interest is in an idealization of these real devices which reflects the essential quantum-noise-limited nature of the energy measurement. It is well known[5] that the output of an ideal photodetector is, conditioned on the (complex envelope of the) incident field $\varepsilon(\vec{r},\tau)$, an inhomogeneous Poisson process with time-varying intensity given by

$$\lambda(\tau) = K \int_{\Sigma} d\vec{r}\,|\varepsilon(\vec{r},\tau)|^2 \tag{1.1}$$

where $\int_{\Sigma} d\vec{r}$ represents an integral over the aperture area Σ of the photodetector and $K = \frac{\eta}{hf_o Z_o}$ is determined by the photon energy hf_o at the carrier frequency, the impedance of free space Z_o, and the quantum efficiency η of the photodetector. We shall assume throughout this thesis that the constant $\sqrt{K}$ is absorbed into the definition of $\varepsilon(\vec{r},\tau)$, and conditional intensities will be written as in (1.1) with $K \equiv 1$.

When many photodetectors are used, with non-overlapping aperture areas Σ_i, the output of the array of photodetectors is, conditioned on the incident field $\varepsilon(\vec{r},\tau)$, a set of independent inhomogeneous Poisson processes with individual intensities determined by (1.1) with Σ replaced by Σ_i.[5] When the individual aperture areas Σ_i are small but together comprise a significant area Σ it is convenient to pass to the limit of an infinite number of arbitrarily small detectors at a continuum of placements $\vec{r}$ within the combined aperture Σ. This limiting device has been referred to as a differential array of ideal photodetectors.[6] We shall use it exclusively in our receiver model presented in Chapter II, because no finite array can accomplish a measurement superior to that performed by the differential array. The output of the differential array of ideal photodetectors is, conditioned on the incident field $\varepsilon(\vec{r},\tau)$, a Poisson process in space and time, with space- and time-varying intensity given by

$$\lambda(\vec{r},\tau) = |\varepsilon(\vec{r},\tau)|^2 \tag{1.2}$$

When the conditioning on the incident field is removed, the resulting unconditional random process has been referred to as a conditional Poisson process[1] or as a doubly stochastic Poisson process.[7] We shall see in the next chapter that this model must be generalized somewhat to adequately represent the receiver structure we consider.

Prior to photodetection, the received field may be subject to linear space-time filtering and/or addition of a local oscillator field. A fairly general optical communication system with this form of structured receiver is shown in Figure 1.1.[8]

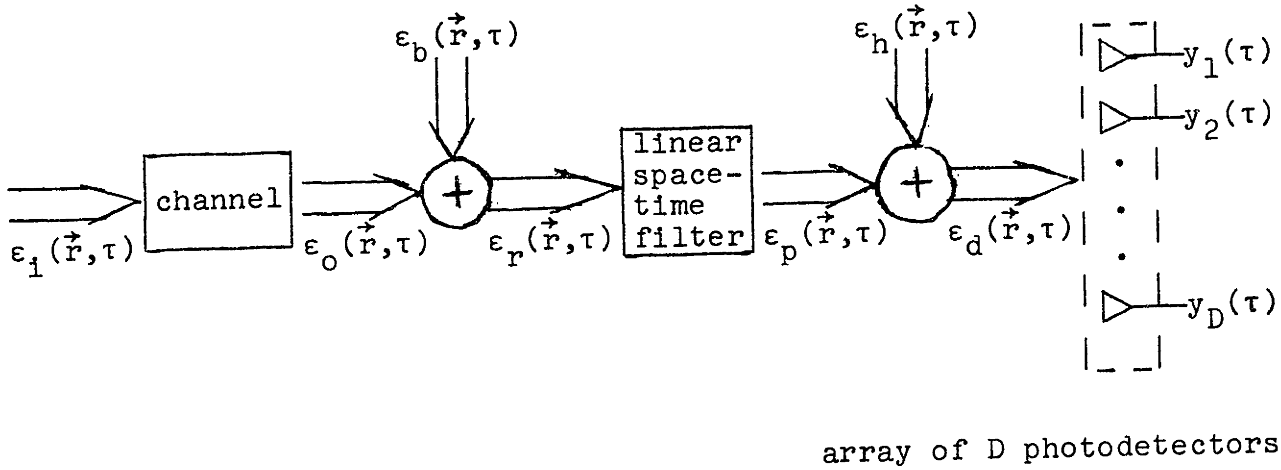

Figure 1.1

The input field, $\varepsilon_i(\cdot)$, modulated according to the message sent, is transformed by the (possibly random) channel into an output field $\varepsilon_o(\cdot)$, which is received in the presence of an additive background noise field $\varepsilon_b(\cdot)$. The receiver may linearly filter $\varepsilon_r(\cdot) = \varepsilon_o(\cdot) + \varepsilon_b(\cdot)$, obtaining $\varepsilon_p(\cdot)$, and add a local oscillator field $\varepsilon_h(\cdot)$ before energy-detecting the result, $\varepsilon_d(\cdot) = \varepsilon_p(\cdot) + \varepsilon_h(\cdot)$, with a spatial array of photodetectors.

If a strong local oscillator field ε_h, with frequency offset greater than the bandwidth of ε_p, is added to ε_p and additional bandpass processing is performed on the photodetector output, the resulting receiver scheme is heterodyne detection.[1] By transforming ε_p in this manner prior to the square-law measurement, the heterodyne receiver is able to effectively accomplish a different type of measurement on ε_p itself. The (classically processible) quantum-noise-corrupted data generated by it consists of D spatial samples $\varepsilon_p(\vec{r}_i,\cdot)$ of ε_p at the positions $\vec{r}_i$ of the D detectors (assuming the detector areas are small) in the presence of D independent additive complex white Gaussian noise processes. When background noise is absent and the channel is non-random, the effective quantum noise level may be reduced and superior performance achieved by using a local oscillator field with zero frequency offset; this method is known as homodyning.[1]

If the local oscillator field is absent, the receiver is referred to as a direct detector.[1] In this case the processible data derived from the field is, conditioned on ε_p, a set of D independent Poisson processes with (time-varying) counting rates $|\varepsilon_p(\vec{r}_i,\tau)|^2$.

1.1.3 Receiver Structure Considered in This Thesis

In this thesis we consider a class of receivers whose structure differs from that shown in Figure 1.1 in one important respect. We allow the choice of local oscillator field amplitude and phase to depend causally on the actual photodetector output. In other words, for every time τ, $\varepsilon_h(\cdot,\tau)$ is permitted to be an arbitrary functional of the photodetector outputs $\{y_i(\sigma), 0\leq \sigma\leq \tau\}$ prior to τ. We shall refer to this class of structured receivers as feedback receivers. It includes in principle all heterodyne, homodyne, and direct detectors. Unlike these rigid structures for which the statistics of the processible data are fixed, the feedback receiver has the power to select from any of an infinite number of classical statistical characterizations induced on the quantum noise, by adjusting the feedback functional. It is possible that a judicious choice will produce significant performance improvement in a given communication problem.

1.2 STATEMENT OF THESIS OBJECTIVE

This thesis is concerned with analyzing the general class of feedback receivers, finding conditions for determining an optimum or a reasonably good sub-optimum feedback field, and comparing the performance of these receivers with that attainable by the optimum quantum measurement and by conventionally structured receivers. Specific communication problems are chosen for consideration on the dual basis of their own intrinsic interest and their solvability with respect to these questions. We expect that the primary useful application of feedback receivers will be in quantum-noise-limited systems, because simpler structures seem to be adequate when performance is relatively insensitive to the exact statistical characterization of the quantum measurement noise.

Because this is intended chiefly as a theoretical presentation no effort is made to discuss the physical implementation of the black box which produces the feedback field. We recognize that this problem is usually nontrivial even for simple feedback functions and therefore emphasize that this factor must be included in the trade-off considerations that determine whether the feedback receiver approach is of practical interest.

1.3 BRIEF OUTLINE AND SUMMARY OF RESULTS

Chapter II presents the statistical model we use for analyzing feedback receivers, along with a discussion of a general Bayesian minimum expected cost objective for evaluating their performance. The feedback receiver data is shown to be a regular point process whose conditional statistics depend on the received field in a manner influenced greatly by the choice of feedback field. The feedback receiver that achieves minimum average cost is determined by simultaneously choosing a classical optimum estimator or decision function based on the point data and a feedback function which optimizes the data statistics, with respect to the given performance measure and statistical characterization of the received field.

In Chapter III the main theoretical results of the feedback receiver analysis are derived. A finite-state received field model is introduced as a simplifying assumption which remains in effect throughout the remainder of the thesis. The feedback function optimization is accomplished by a dynamic programming technique. A backward-time differential equation is derived for the average cost as a function of an arbitrary initial probability state at each time, and the optimum feedback function is constructed backward in time by recursively maximizing the magnitude of this time derivative.

In Chapter IV our attention is restricted to received fields with no spatial modulation. The results of Chapter III are extended to a more general class of contingent quantum measurement sequences, and it is found that under appropriate regularity conditions feedback receivers can achieve the same minimum cost as can arbitrary contingent sequences of general measurements performed consecutively on (time-ordered) infinitesimal time-samples of the received field. The same correspondence is established between a more general class of contingent measurement sequences, for which the natural time-ordering constraint is removed, and a similarly defined class of generalized feedback receiver measurements.

Because solution of the dynamic programming optimality condition is usually impractical, a sub-optimum local optimality criterion is suggested in Chapter V. Under this criterion the feedback field at each time is chosen on the assumption that observations cease a short time later. The objective is to minimize the short-time increment to an average cost rather than the effect of the current-time feedback on an interval measurement. Relatively explicit conditions are derived for determining the incrementally optimum feedback function, and several examples are presented. It is shown that, at least for received fields which are not spatially modulated, the incrementally optimum feedback receiver realizes the analogously defined incrementally optimum contingent sequence of quantum measurements performed on infinitesimal time-samples of the field.

In Chapter VI the theoretical results are applied to some specific communication problems. In several two-state problems, most notably the binary coherent state detection problem with minimum error probability cost criterion, a stronger equivalence than that predicted by the general theory of Chapters IV and V is established between the incrementally optimum feedback receiver and the interval optimum quantum measurement. Explicit solutions have not been obtained for problems with more than two states, but a sub-optimum feedback receiver is shown to approximate the performance of the optimum quantum measurement within a small multiplicative factor for the M-ary coherent state minimum error probability detection problem with pulse-position-modulated (PPM) signals. Finally, a reduction of the complexity of the general dynamic programming interval optimality condition is obtained for M-state problems with signal sets prescribed by arbitrary binary coding of M messages into sequences of time-segments of two deterministic fields and for which the optical detector may be subject to dark current.

In Chapter VII conclusions are drawn and some suggestions are offered for future research.

1.4 RELATION TO PREVIOUS WORK

For a general background, the reader is referred to Pratt[9] or Ross[10] or Gagliardi and Karp[1] for discussions of conventional optical receiver structures, and to Helstrom et al.[2] for presentation of a quantum communication model. There is an extensive body of literature on these subjects, but we shall list only those references which seem most pertinent to our work.

The regular point process model in Chapter II is taken from Rubin,[11] and it is nearly identical to one developed earlier for conditional (doubly stochastic) Poisson processes by Clark.[12] Bayesian communication objectives are considered in standard texts.[13]

The procedure for determining the optimum feedback function in Chapter III is based on Bellman's[14] dynamic programming optimality principle. Other useful references, particularly for analyzing non-deterministic control processes, include Dreyfus[15] and Kushner.[16]

Yuen[17] has derived necessary and sufficient conditions for determining the optimum quantum measurement for all the (finite-state, finite estimator range) problems we consider in Chapters III and IV, but explicit solutions have been obtained just for a few special cases. Our work is more directly related to that of Benioff[18] or Chan[19] or Kaufmann and Chan[20] on contingent quantum measurment sequences.

The incremental optimality criterion presented in Chapter V is similar to that used by Baras and Harger[21] in their study of causal minimum mean square error estimation with a contingent sequence of quantum measurements.

The first example of Chapter VI was inspired by Kennedy's[22] demonstration that near-optimum performance can be achieved for the binary coherent state detection problem by a feedback receiver with data-independent feedback nulling one of the two possible signals. [Indeed, this result provided our original motivation for studying the general class of feedback receivers.] For several of the problems we consider in Chapter VI, optimum quantum measurements have been determined elsewhere[23,24,25].

CHAPTER II

PROBLEM MODEL

2.1 INTRODUCTION

In this chapter we extend the characterization of conventional optical receiver structures to include the more general class of feedback receivers, in the manner briefly indicated in Chapter I. In the next two sections we present the complete statistical model for the data generated by a feedback receiver in response to the received optical field. The output of the differential array of photodetectors is represented as a regular point process in space and time whose intensity function depends on the field ε_p (see Figure 1.1) in a manner which is partially controllable by the choice of feedback functional. A compact notation $\underline{t} \equiv (\underline{\vec{r}}, \underline{\tau})$ is introduced for the vector of space-time photodetection event locations which completely characterizes any sample function of the output of the detector array, and a new symbol, ℓ, is used for the feedback field instead of ε_h, in order to avoid confusion with the particular case of a strong heterodyne field. The feedback receiver is regarded as performing a measurement on ε_p, so the effects of all forward components of the optical system in Figure 1.1 are summarized by the classical statistical characterization of ε_p, which we assume is given as part of the problem specification. We find it convenient to drop the subscript and denote the detected field in the absence of feedback by ε rather than ε_p.

In later sections in this chapter, we describe a general Bayesian average cost functional for measuring the performance of feedback receivers, and we briefly discuss the criterion for choosing an optimum feedback function. In the final section, a short summary of our complete problem model is presented,

2.2 BLOCK DIAGRAM OF A FEEDBACK RECEIVER

In view of the discussion in Chapter I and the introductory remarks in this chapter on our notational simplifications, the class of receivers considered in this thesis can be represented by the following diagram.

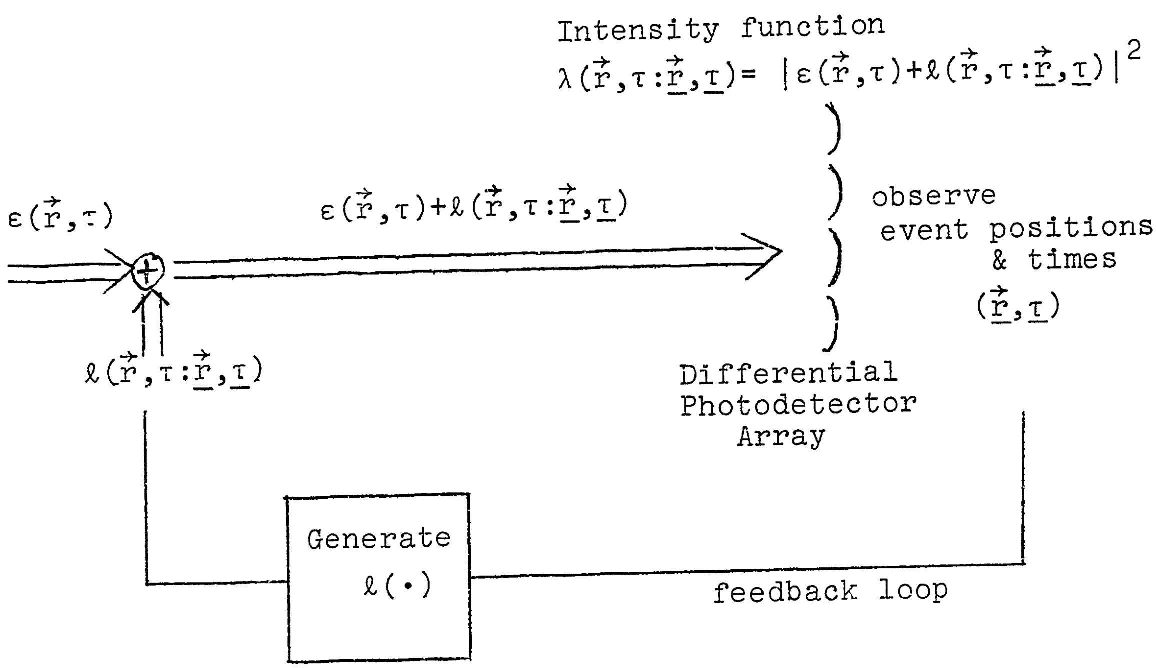

Figure 2.1

A linearly polarized optical space-time field $\varepsilon(\vec{r},\tau)$ is incident at the receiver. As in Chapter I, the principal component of the receiver is a differential array of ideal photodetectors, the output of which is a space-time point process whose intensity is proportional to the squared magnitude of the field impinging on the detectors. But prior to photodetection, the incident field $\varepsilon(\vec{r},\tau)$ is modified by the addition of a local oscillator field $\ell(\vec{r},\tau)$ which may be chosen as a causal function of the actual prior photodetector output, via a feedback arrangement. Henceforth (as well as in the diagram above) this output dependence of the feedback field will be denoted explicitly as $\ell(\vec{r},\tau:\underline{\vec{r}},\underline{\tau})$, where $(\underline{\vec{r}},\underline{\tau})\equiv(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N,\tau_1,\tau_2,\ldots,\tau_N)$ is an arbitrary time-ordered set of positions $\vec{r}_i$ and times τ_i for events occurring prior to time τ, $0<\tau_1<\tau_2<\ldots<\tau_N<\tau$, Thus, the point process intensity function $\lambda(\vec{r},\tau:\underline{\vec{r}},\underline{\tau})$ is also dependent on prior events and is proportional to $|\varepsilon(\vec{r},\tau)+\ell(\vec{r},\tau:\underline{\vec{r}},\underline{\tau})|^2$. This intensity function provides a complete incremental, conditional specification of the output point process:

Given the value of the incident field $\varepsilon(\vec{r},\tau)$ and also the entire history $(\underline{\vec{r}},\underline{\tau})$ of the process prior to τ, the conditional probability that a single event occurs in area ΔA around $\vec{r}$ between times τ and $\tau+\Delta\tau$ is

$$\lambda(\vec{r},\tau:\underline{\vec{r}},\underline{\tau})\Delta A\Delta\tau + o(\Delta A\Delta\tau) \tag{2.1}$$

and the conditional probability of multiple events in $[\tau,\tau+\Delta\tau)$ is $o(\Delta\tau)$. (2.2)

The conditional probability of zero events in $[\tau,\tau+\Delta\tau)$ is of course derivable from these as

$$1 - \int_{\Sigma} \lambda(\vec{r},\tau:\underline{\vec{r}},\underline{\tau})\, d\vec{r}\, \Delta\tau + o(\Delta\tau), \tag{2.3}$$

where $\int_{\Sigma} d\vec{r}$ represents an integral over the aperture area Σ.

This receiver structure is more general than heterodyning, homodyning, and direct detection since it includes all of them as special cases. Furthermore, unlike these other structures, it allows the receiver enormous flexibility to choose among the many different quantum measurements on $\varepsilon(\cdot)$ represented by the selection of different feedback functions $\ell(\cdot)$. Of course there is no a priori guarantee that this class of measurements includes all possible quantum measurements, or at least optimal ones, or even any measurements that perform significantly better than those corresponding to simpler receiver structures. The motivation for studying them is that they correspond to a class of physical receivers, implementable in principle, which are interesting by reason of the qualitative considerations mentioned above. A quantitative measure of the extent to which the performance gap between optimal quantum measurements and simpler receiver structures is bridged by our intermediate receiver class can only be calculated for a few special communication problems, which is not surprising in view of the paucity of problems for which the gap itself is known. Some encouraging evidence of the usefulness of our receiver structure is offered by the examples presented in Chapter VI, for which it is possible to find feedback receivers that achieve optimal or exponentially optimal performance.

2.3 RECEIVED DATA STATISTICS

2.3.1 Conditional Increment Specification

The output point process model described above in (2.1), (2.2), (2.3), is a straightforward generalization of the one presented earlier for simpler receiver structures, based on the idealized observed characteristics of real photodetectors to respond to the magnitude squared of the impinging field in the discrete probabilistic manner indicated. But even though the model is physically motivated and, conversely, we are primarily interested in models which are applicable to physical problems, some care must be exercised to establish a proper mathematical foundation. Except in cases of trivial output independent feedback, the output process is not a conditional Poisson process. Even with all other randomness removed, the future of the process is inextricably bound to its past through the latter's effect on the feedback field $\ell(\cdot)$, so a conditional independent increment Poisson model is invalid for the output process. The natural specification of such a process is in terms of its conditionally dependent increment statistics.

2.3.2 Regularity Assumption

Implicit in our expressions for the conditional increment probabilities is the imposition of a certain regularity property on the point processes so modeled. Some types of point processes, such as those which can have simultaneous events, are immediately excluded. Even so, the model can be stretched to include processes that have no physical counterpart and only tend to make the mathematical analysis more difficult. Therefore, we shall also require that $\lambda(\cdot)$ be "well-behaved," in a sense defined by Rubin[11] for non-spatial point processes and described below for our optical model.

Specifically, observations of point events $\{t_i \equiv (\vec{r}_i,\tau_i)\}_{i=1}^{N}$ are assumed to take place on some finite aperture area Σ and during a finite time period $[0,T]$. The ith event is specified by its locations in time, $\tau_i \epsilon [0,T]$, and inside the aperture area, $\vec{r}_i \epsilon \Sigma$; events are time-ordered, $0<\tau_1<\tau_2<\ldots<\tau_N<T$. _The single symbol t, t_i, t', etc., is used to denote space-time points $(\vec{r},\tau)$, $(\vec{r}_i,\tau_i)$, $(\vec{r}',\tau')$ in $\Sigma x[0,T]$._ For a non-random incident field $\varepsilon(t)$, the process is specified by its conditional increment probabilities, given by (2.1), (2.2), (2.3), where $\lambda(t:\underline{t})$ satisfies the following conditions:

For each $\underline{t}=(t_1,\ldots,t_N)$, $t_i \equiv (\vec{r}_i,\tau_i)$, $\lambda(t:\underline{t})$ is a nonnegative piecewise continuous function for $t \epsilon \Sigma x[\tau_N,T]$. At discontinuity points it is taken to be left continuous (continuous when approached from the right). (2.4)

For each $(\vec{r},\tau) \epsilon \Sigma x[0,T]$, $E\ \lambda(\vec{r},\tau:\underline{t})<\infty$, where the expectation is over all possible location vectors $\underline{t}$ for events occurring prior to τ, $\underline{t}=(\underline{\vec{r}},\underline{\tau})=\{(\vec{r}_i,\tau_i)\}_{i=1}^{N}$ with $\tau_N\leq\tau$. (2.5)

Using assumption (2.4), it is possible to integrate expression (2.3) to yield, for any $0<\tau<\tau'<T$,

$$\Pr\{0 \text{ events in } [\tau,\tau'] \text{ given events at } \underline{t} \text{ prior to } \tau\}$$

$$= \exp\{-\int_{\tau}^{\tau'} d\tau'' \int_{\Sigma} d\vec{r}\,\lambda(\vec{r},\tau'':\underline{t})\}$$

$$\geq \exp[-(\tau'-\tau)A(\Sigma) \sup_{t\varepsilon\Sigma x[\tau,T]} \lambda(t:\underline{t})]$$

$$\geq 1 - K(\tau:\underline{t})\ (\tau'-\tau), \tag{2.6}$$

where

$A(\Sigma)$ is the area of Σ,

$K(\tau:\underline{t}) = A(\Sigma) \sup_{t\varepsilon\Sigma x[\tau,T]} \lambda(t:\underline{t})$

and the last inequality follows from $e^{-x} \geq 1-x$ for all real x. Finally, we assume

$$E\ K(\tau:\underline{t}) < \infty \tag{2.7}$$

and

infinitely many events cannot occur in [0,T]; i.e.,

$$\sum_{n=0}^{\infty} \Pr\{n \text{ events in } [0,T]\} = 1 \tag{2.8}$$

These properties characterize a regular point process (RPP), as defined by Rubin. For random $\varepsilon(\cdot)$, we assume that properties (2.1-2.8) hold for each realization of $\varepsilon(\cdot)$, with all probabilities conditioned on knowledge of $\varepsilon(\cdot)$. The unconditional point process, called a compound RPP, is itself a RPP, with a conditional intensity function $\hat{\lambda}$ satisfying (2.1-2.8):

$$\hat{\lambda}(\vec{r},\tau:\underline{t}) = E[\lambda(\vec{r},\tau:\underline{t}), \text{ given events } \underline{t} \text{ prior to } \tau]. \tag{2.9}$$

2.3.3 Notation Used throughout Thesis for Data Statistics

Having required a very cumbersome notation in order to explain in easily understandable terms the physical model and the defining properties (2.1-2.8) of the mathematical model, we now introduce for later use a set of compact symbols for the key expressions and operations.

Symbol	Meaning
$\tau,\tau',\tau'',\ldots$	a point in time within the interval $[0,T]$
$\vec{r},\vec{r}',\vec{r}'',\ldots$	a point in space within the aperture area Σ
$t,(\vec{r},\tau),t',(\vec{r}',\tau'),\ldots$	a space-time point within $\Sigma x[0,T]$
$t_i=(\vec{r}_i,\tau_i)$	the space-time location of the ith event, $t_i=(\vec{r}_i,\tau_i)\varepsilon\Sigma x[0,T]$
$\underline{t}=(\underline{\vec{r}},\underline{\tau})$ or $\underline{t}_N=(\underline{\vec{r}}_N,\underline{\tau}_N)$	the space-time locations of $N=\dim(\underline{\tau})$ time-ordered events, $(\vec{r}_1,\tau_1),\ldots,(\vec{r}_N,\tau_N)$ $\vec{r}_i\varepsilon\Sigma,\ 0<\tau_1<\tau_2<\ldots<\tau_N<T.$
$\underline{t}=\emptyset$	the vector of space-time event locations is null, i.e., zero events observed in $[0,T]$
$[0,T]^*$	the set of all possible space-time event location vectors, $[0,T]^*=\{\emptyset\}\cup\bigcup_{N=1}^{\infty}\{\underline{t}_N=(\vec{r}_1,\tau_1),\ldots,(\vec{r}_N,\tau_N):\vec{r}_i\varepsilon\Sigma,\ 0<\tau_1<\ldots<\tau_N<T\}$ (dependence on Σ understood)
$[0,T]^+$	the domain of $\lambda(t:\underline{t})$ and of $\ell(t:\underline{t})$, $[0,T]^+=\{(t:\underline{t})=(\vec{r},\tau:\underline{t}_N):(\vec{r},\tau)\varepsilon\Sigma x[0,T],\ \underline{t}_N\varepsilon[0,\tau]^*\}$ (dependence on Σ understood)

Symbol	Meaning
$t_i<t<T$, etc.	the time parts of the space-time variables are ordered in the indicated fashion, i.e., if $t_i=(\vec{r}_i,\tau_i)$ and $t=(\vec{r},\tau)$, then $t_i<t<T$ means $\tau_i<\tau<T$.
$0<\underline{t}<t<T$	if $\underline{t}=(t_1,\ldots,t_N)$, then $0<\underline{t}<t<T$ means $0<t_1<t_2<\ldots<t_N<t<T$.
$P(\cdot\|0<\underline{t}<\tau)$ $E(\cdot\|0<\underline{t}<\tau)$	conditional probability (expectation) given observation of vector $\underline{t}$ of space-time event locations (and no others) over area Σ during time interval $[0,\tau]$.
$f(t)$	if f is any function with domain(f) $=[0,T]$ and $t=(\vec{r},\tau)\varepsilon\Sigma x[0,T]$, then $f(t)$ is understood to mean $f(\tau)$.
$f(t,t',\ldots)$, e.g.,	if f has domain $[0,T]x[0,T]x\ldots$ and $t=(\vec{r},\tau)\varepsilon\Sigma x[0,T]$, $t'=(\vec{r}',\tau')\varepsilon\Sigma x[0,T],\ldots$ then $f(t,t',\ldots)\equiv f(\tau,\tau',\ldots)$.

$$\int_{t_{i-1}}^{t_i}\lambda(\vec{r}',\tau':\underline{t}_{i-1})d\tau' = \int_{\tau_{i-1}}^{\tau_i}\lambda(\vec{r}',\tau':\underline{t}_{i-1})d\tau' \text{ if } t_{i-1}=(\vec{r}_{i-1},\tau_{i-1}) \text{ and } t_i=(\vec{r}_i,\tau_i)$$

$$\int_{t_{i-1}}^{t_i}\lambda(t':\underline{t}_{i-1})dt' = \int_{\Sigma}\int_{t_{i-1}}^{t_i}\lambda(\vec{r}',\tau':\underline{t}_{i-1})d\tau'd\vec{r}'$$

[new definition, not derivable from $f(t,t',\ldots)$]

We remark that throughout the thesis we shall be quite sloppy in our references to time intervals $[0,T]$, $[0,\tau]$, $[\tau,T]$, etc. Although we shall consistently use closed interval notation, the reader will notice that a closed interval interpretation is often not rigorous. As a general rule, most time intervals should be regarded more properly as half-open, half-closed, with the right end open; e.g., $[0,T)$, $[0,\tau)$, $[\tau,T)$.

2.4 CHOICE OF FEEDBACK FIELD

In setting up our mathematical model, we suppressed the most important feature of our problem, namely the fact that the conditional intensity function $\lambda(t;\underline{t})$ is partially controllable through choice of the feedback function ℓ. With the definition

$$\Lambda(\varepsilon,\ell) = |\varepsilon+\ell|^2 \tag{2.10}$$

the conditional intensity is given by

$$\begin{aligned}
\lambda(t:\underline{t}) &= \Lambda[\varepsilon(t),\ell(t:\underline{t})], \ (t:\underline{t})\varepsilon[0,T]^+ (\text{non-random } \varepsilon) \\
\hat{\lambda}(t:\underline{t}) &= E\{\Lambda[\varepsilon(t),\ell(t:\underline{t})]|0<\underline{t}<t\}, \ (\text{random } \varepsilon) \\
& \qquad (t:\underline{t})\varepsilon[0,T]^+
\end{aligned} \tag{2.11}$$

All we require is that the conditional intensity function determined through (2.10), (2.11) by ℓ should specify a regular point process. Any feedback function satisfying this condition is called <u>regular</u>. This definition is independent of ε as long as ε is such that the point process is regular for $\ell=0$. It implies that $\ell(t:\underline{t}_n)$ is piecewise continuous as a function of $t>t_n$ for any fixed $\underline{t}_n\varepsilon[0,T]^*$ and that it does not increase too fast (for fixed t) as additional prior events are observed (i.e., as $n \to \infty$).

For any given regular feedback function $\ell:[0,T]^{+}\to \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers), the statistical connection between the signaling field ε and the observed point process data $\underline{t}$ is entirely determined by (2.10), (2.11) together with the defining properties (2.1)-(2.9). The problem of recovering a message signal imbedded in ε by processing the observations $\underline{t}$ is one that has been studied elsewhere. [For various formulations and applications, see Rubin,[11] Clark[12], Snyder[7].] Often for many problems one can suggest, on a combined basis of simple implementation and obvious relative merit, a particular "good" feedback function for which this analysis can be applied.

We shall be concerned with finding algorithms for determining "good," preferably optimal, feedback functions. So far the notion of a criterion for optimally choosing ℓ has been left very vague. It is necessary to define the communication objective more specifically: the possible message signals, the message statistics, the manner in which the message signals influence the statistics of ε, and a performance criterion which reflects the cost of erring in attempting to reproduce the message on the basis of the received point process data.

Throughout the thesis it will be assumed that the point data processor used is one which is optimum with respect to the given performance criterion for whatever feedback function is employed. An <u>optimum regular feedback function</u> ℓ^{*} is defined as one for which the performance of the corresponding optimal point data processor is best. Thus the optimization over ℓ is the second stage of a dual optimization problem.

One way to solve the dual optimization is by exhaustive enumeration of the optimum performance attainable for each possible feedback function. However, explicit performance results for processing general RPP's are not known, so this technique has little practical utility. It is better to start from scratch and formulate the two optimizations simultaneously. This approach, which is taken here, does not simplify the determination of the optimum processing for each ℓ, but it couples the ℓ-optimization with the performance evaluation in a manner that requires calculation of performance just for the optimal feedback function ℓ^*.

2.5 BAYESIAN MODEL FOR COMMUNICATION OBJECTIVE

2.5.1 General Problem Model

A very general model for Bayesian detection and estimation problems with point process data is the following. Some parameter a, which influences the received field ε, is to be estimated on the basis of observations on the output point process. An estimator function $\hat{a}(\cdot)$ whose domain is $[0,T]^*$ assigns to each possible observation $\underline{t}$ a corresponding estimate $\hat{a}(\underline{t})$ of the parameter a. Associated with guessing $\hat{a}$ when the parameter value is actually a is a (real) cost $C(a,\hat{a})$. The communication objective is to choose the estimator function $\hat{a}(\cdot)$, $\hat{a}(\underline{t})\varepsilon\hat{R}$ for all $\underline{t}$, so as to minimize the average cost

$$\overline{C} = E_{a,\underline{t}}\, C[a,\hat{a}(\underline{t})], \tag{2.12}$$

where $\hat{R}$ is the allowable range of the estimator function and the expectation is over the joint statistics of a and $\underline{t}$. These joint statistics are dependent on the feedback function, and thus so are the optimum estimator function and the corresponding minimum average cost. For any given feedback ℓ, the optimum point data processor $\hat{a}(\cdot)$ is derived classically to minimize the expectation in (2.12) over the particular data statistics induced by ℓ. Our objective is to minimize the set of minimum average costs thus achieved for each ℓ by varying the feedback function.

2.5.2 Examples of Interesting Communication Problems

It might appear from the language used that this model is limited to the very special case of single parameter estimation. Actually it is completely general since the range $\hat{R}$ of $\hat{a}$ and the set R of possible values of a were deliberately left unspecified. These may be discrete integers, real numbers, finite-dimensional real vectors, waveforms defined on [0,T], fields defined on $\Sigma x[0,T]$, etc.; the same problem formulation applies. The only special requirement is the Bayesian one that the expectation in (2.12) over the joint statistics of a and $\underline{t}$ be well defined.

With appropriate selection of the ranges R and $\hat{R}$ and the cost function $C:Rx\hat{R} \to \mathrm{R}$ (R is the set of real numbers), any Bayesian communication problem can be modeled this way. Some important examples are:

(I) Minimum error probability M-ary detection

$$R = \hat{R} = \{1,2,\ldots,M\}$$
$$C(a,\hat{a}) = 1 - \delta_{a\hat{a}}$$

(II) Continuous estimation of a discrete real parameter

$$R = \{x_1,x_2,\ldots,x_M\} \quad , \quad \hat{R} = \mathrm{R}$$
$$C(\cdot,\cdot) \text{ arbitrary}$$

(III) <u>MMSE single parameter estimation</u>

$$R = \hat{R} = R$$

$$C(a,\hat{a}) = (a-\hat{a})^2$$

(IV) <u>Quadratic cost multiple parameter estimation</u>

$$R = \hat{R} = R^n$$

$$C(a,\hat{a}) = (a-\hat{a})^T Q(a-\hat{a}), \quad Q \text{ a non-negative definite symmetric } n\text{x}n \text{ matrix}$$

(V) <u>Interval estimation of a field (with diagonal cost)</u>

$$R = \hat{R} = \text{(appropriately restricted) set of real functions on } \Sigma \text{x}[0,T]$$

$$C(a,\hat{a}) = \int_0^T dt\; c[a(t),\hat{a}(t),t] \quad \left(\text{recall that } \int_0^T dt \equiv \int_\Sigma d\vec{r}\int_0^T d\tau\right)$$

In this case, an alternative to the introduction of the stochastic integral for $C(\cdot,\cdot)$ and consequent worries about the validity of interchanging the order of integration and expectation in (2.12) is to <u>define</u> the performance objective as the minimization of $\int_0^T dt\; E_{a,\underline{t}}\; c[a(t),\; \hat{a}(t:\underline{t}),t]$, where $\hat{a}(\cdot:\underline{t})$ is the estimator given observations $\underline{t}\varepsilon[0,T]^*$. Then $\hat{a}(t:\cdot)$ obviously minimizes $E_{a,\underline{t}}\; c[a(t),\; \hat{a}(t:\underline{t}),t]$, $\underline{t}\varepsilon[0,T]^*$, for each t.

(VI) Causal estimation of a field

Example V should not be confused with the often more interesting case of causal estimation of a field, for which the estimate of $a(\vec{r},\tau)$ must be based only on observations prior to τ instead of during the entire interval $[0,T]$. Since our model is equally valid if the endpoint T is replaced by τ, the causal estimation problem given a fixed feedback function ℓ, can be formulated as a set of independent parameter estimation problems, one for each $\tau\varepsilon[0,T]$:

for each $t=(\vec{r},\tau)\varepsilon\ \Sigma x[0,T]$: $\quad R_t=\hat{R}_t=R$

$$C_t(a,\hat{a}) \equiv c[a,\hat{a},t]$$

$$\text{Minimize } \bar{c}(t) = E_{a,\underline{t}}\ c[a(t),\hat{a}(t:\underline{t}),t] \text{ by choosing } \hat{a}(t:\cdot)$$

as a function of possible observations $\underline{t}\varepsilon[0,\tau]^*$

The function c will be called a differential cost function and $\bar{c}$ the average differential cost. A total interval average cost $\bar{C}$ for the entire set of causal point estimates may be defined as

$$\bar{C} = \int_0^T dt\ \bar{c}(t) \tag{2.13}$$

Then $\hat{a}(\cdot:\cdot)$ minimizes $\bar{C}$ if and only if $\hat{a}(t:\cdot)$ minimizes $\bar{c}(t)$ for each $t\varepsilon\ \Sigma x[0,T]$. If a non-uniform weighting is desired in (2.13), the (non-negative) weighting function may be incorporated in the definition of c without changing the pointwise optimality condition.

The only difference between V and VI is the domain of the observations $\underline{t}$, a distinction not made in the notation which is the same for both. The differential cost terminology introduced above will also apply to the non-causal field estimation problem. A problem in which the estimated parameter is a waveform in time only, $a(\tau)$, $\tau\varepsilon[0,T]$, may be regarded as a special case of the field estimation problem.

2.5.3 Discussion on the Feedback Function Optimization

That V and VI (and also IV in the case of diagonal Q) reduce to pointwise minimizations is due to the fact that each component of the estimator affects only a single corresponding isolatable additive component of the total cost. This property does not carry over to the minimization over ℓ, because $\ell(\vec{r},\tau:\underline{t})$ influences the statistics of all observations between τ and T and thus affects all cost components in cases IV and V and cost components $\bar{c}(\vec{r}',\tau')$, $\vec{r}'\varepsilon\Sigma$, $\tau\leq\tau'\leq T$, in case VI. Therefore it is essential for the formulation of the ℓ-optimization problem to have a performance criterion of total interval cost such as (2.13).

Another way to see the inadequacy of a pointwise ℓ-minimization principle for causal waveform estimation is to consider that the ℓ defined on $[0,\tau]^+$ minimizing $\bar{c}(\vec{r},\tau)$ satisfies a criterion which in general bears no resemblance to that satisfied by the ℓ defined on $[0,\tau']^+$ minimizing $\bar{c}(\vec{r}',\tau')$, $\tau'>\tau$, and so it would be most surprising if the restriction of the latter to $[0,\tau]^+$ equaled the former, except in special cases. A pointwise minimization rule would imply a freedom to choose ℓ on $[0,\tau]^+$ to optimally estimate $a(\vec{r},\tau)$ and then later re-choose ℓ on $[0,\tau']^+$ for the estimation of $a(\vec{r}',\tau')$. Since we are allowed just a single choice of ℓ the performance criterion must reflect a compromise between the estimate costs at $(\vec{r},\tau)$ and $(\vec{r}',\tau')$ (and all other positions and times in $\Sigma x[0,T]$).

2.6 SUMMARY OF MODEL

In concluding this chapter, let us briefly summarize the model that has been constructed.

We shall consider Bayesian communication problems which are specified by the joint statistics of a random field $\varepsilon(t)$ and a (generalized) random parameter a [or random field $a(t)$], the range $\hat{R}$ of the estimator $\hat{a}(\cdot)$ [or of $\hat{a}(t{:}\cdot)$ for each t], and a cost function $C(a,\hat{a})$ [or differential cost function $c(a,\hat{a},t)$]. Available to the receiver are observations $\underline{t}$ of a space-time point process over an aperture area Σ and time interval $[0,T]$ [or, possibly, a variable subset thereof, $\Sigma x[0,\tau]$, for causal estimation], whose conditional statistics given ε and a are determined by the (regular) conditional intensity function $\lambda(t{:}\underline{t})=\Lambda[\varepsilon(t),\ell(t{:}\underline{t})]$, where ℓ is a (feedback) control function chosen by the receiver. The receiver selects an optimum feedback function $\ell^*{:}[0,T]^+\to \mathbb{C}$ and a corresponding optimum estimator function $\hat{a}^*{:}\ [0,T]^*\to\hat{R}$ [or $\hat{a}^*{:}\Sigma x[0,T]x[0,T]^*\to\hat{R}$ (interval estimation) or $\hat{a}^*{:}[0,T]^+\to\hat{R}$ (causal estimation)], which jointly minimize an average cost $\overline{C} = E_{a,\underline{t}}\ C[a,\hat{a}(\underline{t})]$, $\underline{t}\varepsilon[0,T]^*$ [or $\overline{C} = \int_0^T dt\ \overline{c}(t)$ where $\overline{c}\ (\vec{r},\tau) \equiv E_{a,\underline{t}}\ c[a(\vec{r},\tau),\hat{a}(\vec{r},\tau{:}\underline{t}),\vec{r},\tau]$, $\underline{t}\varepsilon[0,T]^*$ (interval estimation) or $\underline{t}\varepsilon[0,\tau]^*$ (causal estimation)].

This general model is proposed because the structure of the optimization procedure is similar for large classes of such problems. This is particularly true with regard to dependence on the cost function C and less true concerning dependence on the statistics of ε and a. Therefore in Chapter III, which discusses the ℓ-optimization, results will be derived for the most general problem specification to which they apply.

CHAPTER III

DETERMINATION OF THE OPTIMUM FEEDBACK FUNCTION

3.1 INTRODUCTION

3.1.1 Finite-State Assumption for the Received Field

The communication problems considered in this chapter and the remainder of the thesis are finite coherent state approximations to the general problems proposed in Chapter II. Instead of permitting the received random field $\varepsilon(t)$ to be arbitrary, we assume that it is with probability one a number of a finite set $\{\varepsilon_j(t),\ t\varepsilon\ \Sigma x[0,T]\}_{j=1}^{M}$ of deterministic fields. Thus for (almost every sample function $\varepsilon_\omega(t)$ there is a $j(\omega)$, $1 \leq j(\omega) \leq M$, such that $\varepsilon_\omega(t) = \varepsilon_{j(\omega)}(t)$.

For a general communication objective as modeled in Chapter II, the average cost is given by

$$\overline{C} = E_{\omega,\underline{t}}\ C[a(\omega),\ \hat{a}(\underline{t})] \tag{3.1a}$$

$$= E_{\underline{t},j}\ C_1[j,\ \hat{a}(\underline{t})] \tag{3.1b}$$

where

$$\begin{aligned} C_1[j,\hat{a}] &\equiv E_{\omega|\underline{t},j}\ C[a(\omega),\ \hat{a}] \\ &= E_{\omega|j}\ C[a(\omega),\ \hat{a}] \end{aligned} \tag{3.2}$$

because $\underline{t}$ and ω are conditionally independent given $j(\omega)=j$ (i.e., the statistics of the point process data are determined by the received field realization, $\varepsilon_j(t)$).

The important observation is that the expectation in (3.2) does not depend on the choice of feedback field ℓ. We see that C_1 is derived from the given cost matrix C by averaging over a priori statistics alone. Therefore the general communication problem formulated in (3.1a) is, according to (3.1b), equivalent to estimation of the discrete random variable j, with cost matrix $C_1[j,\hat{a}]$.

In the equivalent problem, the estimator range $\hat{R}$ is unchanged from the one originally specified, so it is usually inappropriate to interpret $\hat{a}(\underline{t})$ as an estimate of which signal $\varepsilon_j(t)$ is received. Nonetheless it is convenient to describe the equivalent problem in language normally reserved for the M-ary detection problem. In the following, we denote the a priori statistics of j by $\xi_j \equiv \Pr[\varepsilon(\cdot)=\varepsilon_j(\cdot)]$ and drop the subscript notation for the effective cost matrix, and we shall usually assume that the estimator range $\hat{R}$ is finite and use the symbol $\hat{j}$ rather than $\hat{a}$ for the estimator.

We emphasize that the finite-state received field model and the restriction of the estimator range are introduced mainly to simplify the mathematical analysis and that conceptually both approximations can be made arbitrarily fine, although the difficulty of obtaining explicit solutions increases markedly with the number of states. Indeed, our ability to completely solve for the optimum feedback function has been limited to certain two-state problems (see Chapter VI).

However, the broad conceptual outlook can aid our understanding of other aspects of the problem, such as the generality of the quantum measurement correspondence established in Chapter IV.

3.1.2 Equivalent Generalized M-ary Detection Problem

The equivalent (generalized) detection problem is described as follows. Under hypothesis H_j, which is true with a priori probability $\xi_j>0$, the received field is $\varepsilon_j(t)$, a completely nonrandom function on $\Sigma x[0,T]$, $j=1,2,\ldots,M$. We assume for simplicity that for almost all $\tau\varepsilon[0,T]$ not all signal fields $\varepsilon_j(\cdot,\tau)$ are equal (almost everywhere in Σ), because any observations during periods of identical signals are worthless regardless of the feedback used.

The receiver makes observations $\underline{t}$ of a point process over $\Sigma x[0,T]$ whose conditional intensity function, given H_j, is $\lambda_j(t:\underline{t}) \equiv \Lambda[\varepsilon_j(t),\ell(t:\underline{t})]$. The objective is to select an optimum feedback function $\ell=\ell^*$ and an optimum decision function $\hat{j}=j^*$ which jointly minimize the average cost $\overline{C}(\underline{\xi},0) \equiv E_{j,\underline{t}}\, C[j,\hat{j}(\underline{t})]$. Here we have explicitly denoted the dependence of the average cost on $\underline{\xi} \equiv [\xi_1,\xi_2,\ldots,\xi_M]^T$, the (column) vector of a priori probabilities, and on the initial time, $\tau=0$, of the period of observations, but have left implicit the dependence of $\overline{C}$ on the feedback and decision functions $\ell,\hat{j}$, and on the endpoint T of the observation interval. The decision $\hat{j}$ is selected from a finite range $\hat{R}$, not necessarily

$\{1,\ldots,M\}$, and the cost matrix $C(j,\hat{j})$ is arbitrary, possibly derived as an average over a priori statistics of a more naturally defined cost, as in (3.2).

In many of the examples in Chapters V and VI we specialize this general model to the case of a true coherent-state detection problem for which $\hat{R} = \{1,\ldots,M\}$ and the cost criterion is probability of error. For this problem the only randomness in the actual field present at the receiver is due to the receiver's uncertainty about which hypothesis is true. Classically, this problem is singular (assuming $\varepsilon_j(\cdot)\neq\varepsilon_k(\cdot), j\neq k$) because the receiver may simply measure the field $\varepsilon(t), t\varepsilon\ \Sigma x[0,T]$, and infer with probability one the correct hypothesis H_j by determining for which j, $\varepsilon(\cdot)=\varepsilon_j(\cdot)$. The possibility of error is introduced only by the quantum mechanical prohibition of such a precise field measurement. As in quantum communication theory, this is an important special case of the general M-ary detection problem (which we model as an M'-ary coherent state problem with M' states, $M'\neq M$ in general, and a derived cost no longer interpretable as error probability), because it retains completely the flexibility in the selection of a quantum-consistent measurement represented by the feedback function ℓ, while dispensing with any classical field randomness which would complicate even the classical M-ary detection problem and probably tend to obscure the effect of the choice of quantum field measurement.

3.1.3 Concept of System State and State Probability Vector

The value of j may be regarded as the (unchanging) state of the system since it uniquely determines the received field ε_j. Even though the state is constant, the receiver's uncertainty about it is updated as observations occur. The a posteriori probability of state j, given events at $\underline{t}$, $0<\underline{t}<\tau$, is denoted by $\xi_j(\tau:\underline{t})$, and the set of these by the (column) vector $\underline{\xi}(\tau:\underline{t})\equiv[\xi_1(\tau:\underline{t}),\ldots,\xi_M(\tau:\underline{t})]^T$. Of course, the a posteriori probability vector, with no observations, is equal to the a priori probability vector, $\underline{\xi}(0:\emptyset)=\underline{\xi}$. The a posteriori probability vector $\underline{\xi}(\tau:\underline{t})$ is important because it determines the conditional intensity function of the compound regular point process,

$$\hat{\lambda}(\vec{r},\tau:\underline{t}) \equiv E_{j|\underline{t}}[\lambda_j(\vec{r},\tau:\underline{t})|0<\underline{t}<\tau] = \sum_{j=1}^{M}\xi_j(\tau:\underline{t})\lambda_j(\vec{r},\tau:\underline{t}), \quad (3.3)$$

but its major significance is explained in the next subsection.

3.1.4 Optimization by Dynamic Programming

Our procedure for optimizing over feedback functions in this chapter (and over quantum measurements in Chapter IV) is based on the general dynamic programming method developed by Bellman[14] and others. We shall show that the vector $\underline{\xi}(\tau:\underline{t})$ of a posteriori probabilities summarizes the effects of data observed and feedback applied prior to τ insofar as they bear on the specification of the optimum feedback to be used after time τ. Therefore it is possible to determine for any <u>potential</u> realization $\underline{\xi}$ of the probability vector $\underline{\xi}(\tau:\underline{t})$ at time τ the optimum feedback for the remaining time interval $[\tau,T]$ by minimizing an average "cost-to-go" which is dependent just on observations after τ.

This approach is used to recursively construct a solution backward in time from $\tau=T$. At each stage, the feedback field applied prior to τ has not yet been determined, so it is impossible to anticipate for which probability vector $\underline{\xi}$ the optimum future feedback need be computed. Therefore, we must determine solutions for <u>all</u> potential trajectories $\underline{\xi}(\tau:\underline{t})$ of the a posteriori probability vector. It is necessary to obtain the optimum feedback for an entire family of related communication problems in order to solve any particular one of them.

In Section 3.4, a backward-time differential equation is derived for the average cost-to-go from time τ starting in probability state $\underline{\xi}$. In the next section it is shown that the state-dependent optimum feedback function may be determined pointwise by minimizing the time derivative of the average cost-to-go for each $\underline{\xi},\tau$, as a function of the (already determined) feedback for later times.

3.2 EVALUATION OF THE A POSTERIORI PROBABILITY VECTOR

The a posteriori state probability $\xi_j(\tau:\underline{t}_n)$ may be evaluated as ξ_j multiplied by the ratio of the probability density, $p_j(\tau:\underline{t}_n)$, conditioned on the state being j, for observing exactly n events in $\Sigma x[0,\tau]$ at locations $t_1,\ldots,t_n$, to the unconditional probability density, $p(\tau:\underline{t}_n) = \sum_{j=1}^{M} \xi_j p_j(\tau:\underline{t}_n)$, for the same event. These are event occurrence densities for RPP's with conditional intensities $\lambda_j(\cdot)$ and $\hat{\lambda}(\cdot)$ respectively. A general expression for such densities was derived by Rubin [11] for time processes and extended to space-time processes by Fishman [26].

$$p_j(\tau:\underline{t}_n) = \prod_{i=1}^{n} \lambda_j(t_i:\underline{t}_{i-1}) \prod_{i=1}^{n+1} \exp\left[- \int_{\Sigma} d\vec{r} \int_{\tau_{i-1}}^{\tau_i} d\tau' \ \lambda_j(\vec{r},\tau':\underline{t}_{i-1})\right]$$

$$p(\tau:\underline{t}_n) = \prod_{i=1}^{n} \hat{\lambda}(t_i:\underline{t}_{i-1}) \prod_{i=1}^{n+1} \exp\left[- \int_{\Sigma} d\vec{r} \int_{\tau_{i-1}}^{\tau_i} d\tau' \ \hat{\lambda}(\vec{r},\tau':\underline{t}_{i-1})\right] \tag{3.4}$$

where

$$\underline{t}_n = (t_1,\ldots,t_n), \quad \underline{t}_{i-1} = (t_1,\ldots,t_{i-1}), \quad t_i = (\vec{r}_i,\tau_i), \ i=1,\ldots,n$$

$$\tau_0 \equiv 0, \quad \tau_{n+1} \equiv \tau$$

Thus

$$\xi_j(\tau:\underline{t}_n) = \xi_j \prod_{i=1}^{n} \frac{\lambda_j(t_i:\underline{t}_{i-1})}{\hat{\lambda}(t_i:\underline{t}_{i-1})} \prod_{i=1}^{n+1} \exp\left[-\int_{\Sigma} d\vec{r} \int_{\tau_{i-1}}^{\tau_i} d\tau' \left[\lambda_j(\vec{r},\tau':\underline{t}_{i-1}) - \hat{\lambda}(\vec{r},\tau':\underline{t}_{i-1})\right]\right] \tag{3.5}$$

It is also convenient to have a differential form for the a posteriori state probabilities. Differentiating (3.5) with respect to τ, we have for $\tau > \tau_n$

$$\frac{\partial}{\partial\tau}\,\xi_j(\tau:\underline{t}_n) = -\,\xi_j(\tau:\underline{t}_n)\int_\Sigma d\vec{r}\,\left[\lambda_j(\vec{r},\tau:\underline{t}_n) - \hat{\lambda}(\vec{r},\tau:\underline{t}_n)\right]$$

$$\equiv -\int_\Sigma d\vec{r}\,\hat{\lambda}(\vec{r},\tau:\underline{t}_n)\left[\rho_j(\vec{r},\tau:\underline{t}_n) - \xi_j(\tau:\underline{t}_n)\right]\,, \tag{3.6}$$

where

$$\rho_j(\vec{r},\tau:\underline{t}_n) \equiv \xi_j(\tau:\underline{t}_n)\;\frac{\lambda_j(\vec{r},\tau:\underline{t}_n)}{\hat{\lambda}(\vec{r},\tau:\underline{t}_n)} \tag{3.7}$$

At an event time τ_n, the a posteriori state probability is updated according to

$$\xi_j(\tau_n^+:\underline{t}_n) = \xi_j(\tau_n:\underline{t}_{n-1})\;\frac{\lambda_j(\vec{r}_n,\tau_n:\underline{t}_{n-1})}{\hat{\lambda}(\vec{r}_n,\tau_n:\underline{t}_{n-1})} \tag{3.8}$$

$$\equiv \rho_j(\vec{r}_n,\tau_n:\underline{t}_{n-1})$$

The explicit expression (3.5) may be recovered by integrating (3.6) to obtain the smooth change in $\xi_j(\cdot)$ between event times and using (3.8) to calculate the discontinuity at an event time. The initial condition is $\xi_j(0:\emptyset) = \xi_j$.

3.3 STATE SUFFICIENCY PRINCIPLE

Since $\underline{\xi}(\cdot)$ represents everything that can possibly be deduced about the stochastic system state j from the information available to the receiver, it is potentially more useful for the purposes of the detection problem to regard $\underline{\xi}(\cdot)$ as the nonrandom (given observations made by the receiver) dynamic system state. The state sufficiency lemma in this section shows that $\underline{\xi}(\tau:\underline{t})$ summarizes all the information in the a priori probabilities $\underline{\xi}$ and in the observations $\underline{t}$ prior to τ that is pertinent to the evaluation of the decision function or to determination of the feedback field for times following τ. Thus the optimum receiver processes the observations in a manner which accomplishes computation of the updated state $\underline{\xi}(\cdot)$, according to (3.5) or (3.6)-(3.8) or any other equivalent equations for determining $\underline{\xi}(\tau:\underline{t})$.

3.3.1 Event Location Vector Decomposition

We first introduce a conceptual technique which will be useful in the statement and proof of the state sufficiency principle and also in future applications. For any event location vector $\underline{t}''\varepsilon[0,T]^*$ and any time τ, $0<\tau<T$, it is possible to uniquely decompose $\underline{t}''$ into two time-ordered event location vectors, $\underline{t}\,\varepsilon[0,\tau]^*$ and $\underline{t}'\varepsilon[\tau,T]^*$, for events occurring before and after time τ, respectively. The decomposition procedure is obvious: Given $\underline{t}'' = (t_1'',\ldots,t_n'')$, $t_i'' \equiv (\vec{r}_i'',\tau_i'')$, $0\leq\tau_1''<\tau_2''<\cdots<\tau_n''<T$, find $m = \max\limits_{i:\tau_i''<\tau} i$ (if $\tau_i''\geq\tau$ for all i, define m=0).

Then define $\underline{t} \equiv (t_1'',\ldots,t_m'')$ and $\underline{t}' \equiv (t_{m+1}'',\ldots,t_n'')$, with the understanding $\underline{t} \equiv \emptyset$ if m=0 and $\underline{t}' \equiv \emptyset$ if m=n . This rule is obviously reversible: Vectors $(\underline{t},\underline{t}') \in [0,\tau]^* \times [\tau,T]^*$ are simply concatenated to form the single vector $\underline{t}'' \in [0,T]^*$.

Because there is such a natural one-to-one correspondence between them, we will tend to refer to the sets $[0,T]^*$ and $[0,\tau]^* \times [\tau,T]^*$ interchangeably. In fact, if $\underline{t},\underline{t}'$ are related to $\underline{t}''$ by the above decomposition rule, we will write $\underline{t}''=(\underline{t},\underline{t}')$. The decision function $\hat{j}(\underline{t}'')$ defined for $\underline{t}''\in[0,T]^*$ will often be more conveniently interpreted as $\hat{j}(\underline{t},\underline{t}')$ if it is desired to separate the effects of observations before and after τ. The same decomposition can be applied to event locations $\underline{t}''$ influencing the feedback field $\ell(\vec{r}'',\tau'':\underline{t}'')$. Due to the causality condition on ℓ, $\underline{t}''\in[0,\tau'']^*$ instead of $\underline{t}''\in[0,T]^*$ and therefore the decomposition of the event vector $\underline{t}''$ with respect to any given separation time τ is nontrivial only when $\tau''>\tau$; for such $(\vec{r}'',\tau'')\equiv t''$, we will frequently write $\ell(t'':\underline{t}'')$ as $\ell(t'':\underline{t},\underline{t}')$.

3.3.2 Determination of Optimum Future Strategy Given Past Data and Feedback

For each fixed τ, $0<\tau<T$, and any $\underline{t}\in[0,\tau]^*$, it is possible to view $\hat{j}(\underline{t},\cdot)$ and $\ell(\cdot:\underline{t},\cdot)$ as functions from $[\tau,T]^*$ to $\hat{R}$ and $[\tau,T]^+$ to ¢ respectively. Our freedom to independently select values for the feedback function $\ell(t'':\underline{t}'')$ for every $(t'':\underline{t}'')\in[0,T]^+$

and for the decision function $\hat{j}(\underline{t}")$ for every $\underline{t}" \varepsilon [0,T]^*$ is equivalent to a freedom to independently choose for each $\underline{t}\varepsilon[0,\tau]^*$ different functions $\ell(\cdot:\underline{t},\cdot)$ and $\hat{j}(\underline{t},\cdot)$ on $[\tau,T]^+$ and $[\tau,T]^*$ respectively, as well as the feedback function $\ell(\cdot:\cdot)$ on the subdomain $[0,\tau]^+$. This property enables us to calculate the optimum feedback function in two stages. For arbitrary feedback (optimal or not) used prior to τ, it is possible to determine the best feedback function $\ell(\cdot:\underline{t},\cdot)$ to be applied in the future after τ by minimizing a conditional average cost, given events at $\underline{t}$ prior to τ. The optimization may then be completed by minimizing the resulting functional of the "past" feedback.

This idea is stated more precicely in Lemma 3.1 below. Its proof, and proofs of all the remaining lemmas and theorems in this chapter, are given in the Appendix to Chapter III.

<u>Lemma 3.1</u> For any τ, $0<\tau<T$, and any "past" feedback function $\ell(\cdot:\cdot):[0,\tau]^+\to ¢$, the set of "future" decision functions $j^*(\underline{t},\cdot):[\tau,T]^*\to\hat{R}$, $\underline{t}\varepsilon[0,\tau]^*$, and the set of "future" feedback functions $\ell(\cdot:\underline{t},\cdot):[\tau,T]^+\to ¢$, $\underline{t}\varepsilon[0,T]^*$, jointly minimize $\overline{C}(\underline{\xi},0) \equiv E_{j,\underline{t}"}\, C[j,j(\underline{t}")]$ if and only if $j^*(\underline{t},\cdot)$ and $\ell^*(\cdot:\underline{t},\cdot)$ jointly minimize $E_{j,\underline{t}'|\underline{t}}\{C[j,\hat{j}(\underline{t},\underline{t}')]|0<\underline{t}<\tau\}$, for all $\underline{t}\varepsilon[0,\tau]^*$ except possibly for a set of probability zero (with probability measure determined by the "past" feedback $\ell(\cdot:\cdot):[0,\tau]^+\to ¢$) where they may be arbitrary.

Proof: See Appendix (Section 3A.1)

The corresponding minimum value of this conditional expectation, denoted by $E^*_{j,\underline{t}'|\underline{t}}\left[C[j,j^*(\underline{t},\underline{t}')]\,|\,0<\underline{t}<\tau\right]$, is of course an implicit functional of the "past" feedback function $\ell(\cdot:\cdot):[0,\tau]^+\to\cent$. The solution of the original optimization problem (i.e., finding $j^*(\underline{t},\cdot):[\tau,T]^*\to\hat{R},\underline{t}\varepsilon[0,\tau]^*$, $\ell^*(\cdot:\underline{t},\cdot):[\tau,T]^+\to\cent$, $\underline{t}\varepsilon[0,\tau]^*$, and $\ell^*(\cdot:\cdot):[0,\tau]^*\to\cent$ which minimize $\overline{C}(\underline{\xi},0)$) may be completed by determining $\ell^*(\cdot:\cdot):[0,\tau]^+\to\cent$ to minimize $E_{\underline{t}}\,E^*_{j,\underline{t}'|\underline{t}}\,\{C[j,j^*(\underline{t},\underline{t}')]\,|\,0<\underline{t}<\tau\}$.

3.3.3 State Sufficiency Lemma

With the help of Lemma 3.1, we are now ready to justify in the following lemma our previous claim that knowledge of the state vector $\underline{\xi}(\tau:\underline{t})$ at time τ is sufficient for determining optimal strategy after τ.

Lemma 3.2 (State Sufficiency Lemma) For any τ, $0<\tau<T$, and any "past" feedback function $\ell(\cdot:\cdot):[0,\tau]^+\to\cent$, the optimum "future" functions $j^*(\underline{t},\cdot)$ and $\ell^*(\cdot:\underline{t},\cdot)$ specified according to Lemma 3.1 depend on $\underline{\xi},\underline{t}$, and $\ell(\cdot:\cdot):[0,\tau]^+\to\cent$ just through $\underline{\xi}(\tau:\underline{t})$.

Proof: See Appendix (Section 3A.2)

3.3.4 State-Dependent Feedback and Decision Functions

The implication of Lemma 3.2 is that for any $0<\tau<T$ the a posteriori probability vector $\underline{\xi}(\tau:\underline{t})$ summarizes all the

information in the data prior to time τ which is relevant to the determination of the optimum feedback field ℓ^* after time τ. It is natural to specify any feedback function such as ℓ^* which has this type of data dependence in terms of its corresponding <u>state-dependent</u> <u>feedback</u> <u>function</u> $\ell^*[\tau,\underline{\xi}]$ defined on $\Sigma x[0,T]xP$, where P is the set of M-dimensional probability vectors $\underline{\xi}$: $P \equiv \{\underline{\xi} = (\xi_1,\ldots,\xi_M) : 0 \le \xi_j, j=1,\ldots,M, \sum_{j=1}^{M} \xi_j = 1\}$. The correspondence is the natural one

$$\ell^*(\vec{r},\tau:\underline{t}) = \ell^*[\vec{r},\tau,\underline{\xi}(\tau:\underline{t})]. \tag{3.9}$$

We will always use the same symbol (ℓ^*,ℓ,etc.) to denote both the event-dependent feedback function defined on $[0,T]^+$ and its corresponding state-dependent feedback function defined on $\Sigma x[0,T]xP$. State-dependent forms for certain functions of $\ell(t:\underline{t})$ (e.g., $\lambda_j(t:\underline{t}) \longleftrightarrow \lambda_j(t,\underline{\xi})$, $\hat{\lambda}(t:\underline{t}) \longleftrightarrow \hat{\lambda}(t,\underline{\xi})$) will also be introduced whenever convenient.

The optimum decision function $j^*(\underline{t})$ also has the state dependence property. Any such decision function $\hat{j}(\underline{t})$ possesses a corresponding state-dependent decision function $\hat{j}(\underline{\xi}):P\rightarrow\hat{R}$, with

$$\hat{j}(\underline{t}) = \hat{j}[\underline{\xi}(T:\underline{t})] \tag{3.10}$$

In the next two sections a differential equation describing the backward-time propagation of the average cost will be derived. This equation is valid for any feedback and decision functions the receiver may decide to employ, not necessarily the optimum ones, as long as they have state-dependent form. Therefore we now restrict our attention to feedback and decision functions with the state dependence property. Lemma 3.2 guarantees that the best state-dependent feedback and decision functions, though optimum over a smaller class, are also optimum overall.

3.4 BACKWARD-TIME COST PROPAGATION EQUATION

3.4.1 Family of Related Detection Problems

As described in the introduction to this chapter, the dynamic programming optimality principle calls for minimization of an average "cost-to-go" from time τ for all possible values of the unknown initial probability state at τ. This requires the simultaneous solution of an entire class of related detection problems, which is defined more precisely in this section.

Consider a family of M-ary detection problems associated with the one initially defined above, one for each $\underline{\xi}\varepsilon P$ and each subinterval $[\tau,T]$, $0<\tau<T$, whose a priori probabilities at time τ are $\{\xi_j\}_{j=1}^{M}$ and for which the signals $\{\varepsilon_j(t)\}_{j=1}^{M}$ are replaced by their restrictions to $\Sigma x[\tau,T]$ and the state-dependent feedback function ℓ is replaced by its restriction ℓ_τ to $\Sigma x[\tau,T]xP$. The cost function, $C(j,\hat{j})$, is the same as for the $[0,T]$ problem, and so is the state-dependent decision rule $\hat{j}(\cdot)$.

It is convenient to define for each $[\tau,T]$ problem the a posteriori probability vector $\underline{\xi}(\tau':\underline{t};\underline{\xi},\tau)$, $\tau<\underline{t}<\tau'<T$, where $\xi_j(\tau':\underline{t};\underline{\xi},\tau)$ denotes the conditional probability of j given data $\underline{t}$ between times τ and τ' and a priori probabilities $\boldsymbol{\xi}$ at time τ. Clearly, $\underline{\xi}(\tau':\underline{t};\underline{\xi},\tau)$ satisfies the same differential state propagation equations (3.6)-(3.8) (restricted to $\tau'\varepsilon[\tau,T]$) as $\underline{\xi}(\tau':\underline{t})\equiv\underline{\xi}(\tau':\underline{t};\underline{\xi},0)$, but with initial condition $\underline{\xi}(\tau:\emptyset;\underline{\xi},\tau)=\underline{\xi}$.

The objective is to choose ℓ_τ and $\hat{j}$ to minimize the average cost

$$\overline{C}(\underline{\xi},\tau) \equiv E_{j,\underline{t}}\; C[j,\hat{j}(\underline{\xi}(T:\underline{t};\underline{\xi},\tau))] \quad , \quad \underline{t}\;\varepsilon[\tau,T]^{*} \qquad (3.11)$$

In other words, $\overline{C}(\cdot,\tau)$ is what the average cost for the original $[0,T]$ detection problem would be if the observations started at time τ instead of time 0, viewed as a function of the a priori probabilities at the respective initial times.

The properties of the optimum feedback and decision functions derived in Lemmas 3.1 and 3.2 for the $[0,T]$ problem are obviously equally valid for all $[\tau,T]$ problems.

In the next lemma it is shown that the average cost $\overline{C}(\underline{\xi},\tau)$ for an arbitrary $[\tau,T]$ problem may be calcualted by averaging the average costs $\overline{C}(\underline{\xi}',\tau+\Delta)$ for any $[\tau+\Delta,T]$ problem, $\Delta>0$ (i.e., a problem for which the same state-dependent decision and feedback functions, the latter appropriately time-restricted, are used, but observations begin at the later time $\tau+\Delta$) over a set of possible initial states $\underline{\xi}'$. For small enough Δ it is possible to obtain an approximate evaluation of this expectation and the result leads to a partial differential equation (in $\underline{\xi}$ and τ) for the average costs $\overline{C}(\underline{\xi},\tau)$.

<u>Lemma 3.3</u> For any $0<\tau<\tau+\Delta<T$ and $\underline{\xi}\varepsilon P$,

$$\overline{C}(\underline{\xi},\tau) = E_{\underline{t}}\; \overline{C}[\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\; \tau+\Delta] \quad , \quad \tau<\underline{t}<\tau+\Delta \qquad (3.12)$$

<u>Proof</u>: See Appendix (Section 3A.3)

This lemma confirms our previous claim and clarifies the procedure for averaging over initial states. Specifically, $\overline{C}(\underline{\xi},\tau)$ is given by

$$\overline{C}(\underline{\xi},\tau) = E_{\underline{\xi}'} \quad \overline{C}(\underline{\xi}',\tau+\Delta) \tag{3.13}$$

where $\underline{\xi}'$ is a random state vector whose probability measure is determined from that of $\underline{t}$ via the transformation

$$\underline{\xi}' = \underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau) \tag{3.14}$$

3.4.2 Partial Differential Equation for the Average Cost

Now we let $\Delta \downarrow 0$ in (3.12) and calculate the right hand side within $o(\Delta)$. From (2.2), the probability of two or more events ($\dim \underline{t} \geq 2$) in $[\tau,\tau+\Delta]$ is $o(\Delta)$. It is shown in the Appendix (Lemma 3A.1 in Section 3A.4) that the average cost $\overline{C}(\underline{\xi},\tau+\Delta)$ is uniformly bounded, $\overline{C}(\underline{\xi},\tau+\Delta) \leq C_{max} < \infty$ for all $\underline{\xi} \varepsilon P$, $\tau+\Delta \; \varepsilon [0,T]$, and so the expectation in (3.12) may be evaluated within $o(\Delta)$ by averaging just over $\underline{t}$ with $\dim \underline{t} \leq 1$.

When $\dim \underline{t}=0$ (i.e., $\underline{t}=\emptyset$, no events observed between τ and $\tau+\Delta$), $\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau)$ is evaluated from (3.6) as

$$\underline{\xi}(\tau+\Delta:\emptyset;\underline{\xi},\tau) = \underline{\xi} - \Delta \int_{\Sigma} d\vec{r} \; \hat{\lambda}(\vec{r},\tau,\underline{\xi})[\underline{\rho}(\vec{r},\tau,\underline{\xi}) - \underline{\xi}] + \underline{o}(\Delta) \tag{3.15}$$

where $\underline{o}(\Delta) \equiv [o_1(\Delta),\ldots,o_M(\Delta)]^T$ has the defining property $\lim_{\Delta\to 0} \frac{o_j(\Delta)}{\Delta} = 0$ for all j, and $\hat{\lambda}(\cdot)$ and $\underline{\rho}(\cdot)$ are the state-dependent functions corresponding to their event-dependent versions introduced earlier. Specifically, we define

$$\lambda_j(t,\underline{\xi}) = \Lambda[\varepsilon_j(t),\ \ell(t,\underline{\xi})] \qquad j=1,\ldots,M \tag{3.16}$$

$$\hat{\lambda}(t,\underline{\xi}) = \sum_{j=1}^{M} \xi_j\ \lambda_j(t,\underline{\xi}) \tag{3.17}$$

$$\rho_j(t,\underline{\xi}) = \xi_j\ \frac{\lambda_j(t,\underline{\xi})}{\hat{\lambda}(t,\underline{\xi})}\ ,\quad \underline{\rho}(t,\underline{\xi}) \equiv [\rho_1(t,\underline{\xi}),\cdots,\rho_M(t,\underline{\xi})]^T \tag{3.18}$$

From (2.3), the case dim $\underline{t}=0$ occurs with probability

$$\Pr(\underline{t}=\emptyset) = 1-\Delta \int_{\Sigma} d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi}) + o(\Delta) \tag{3.19}$$

When dim $\underline{t}=1$ (i.e., $\underline{t}=t_1=(\vec{r}_1,\tau_1)$, $\tau<\tau_1<\tau+\Delta$, one event observed at position $\vec{r}_1 \varepsilon \Sigma$ and time τ_1 between τ and $\tau+\Delta$), $\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau)$ is evaluated from (3.6)-(3.8) as

$$\underline{\xi}(\tau+\Delta:t_1;\underline{\xi},\tau) = \underline{\rho}(t_1,\underline{\xi}) + \frac{\underline{o}(\Delta)}{\Delta} \tag{3.20}$$

From (2.1), the probability density for single events is given by

$$\Pr[1 \text{ event in area } \Delta A \text{ around } \vec{r}_1,\ \text{time } \Delta\tau \text{ around } \tau_1]$$
$$= \hat{\lambda}(\vec{r}_1,\tau_1,\underline{\xi})\ \Delta A\ \Delta\tau + o(\Delta A\ \Delta\tau) \tag{3.21}$$

In order to obtain the equations (3.15), (3.19), (3.20), (3.21) in state-dependent form from the event-dependent expressions (3.6)-(3.8), (2.1), (2.3), it was necessary to observe that $\underline{\xi}(\tau:\emptyset;\underline{\xi},\tau) = \underline{\xi}$.

Use of these equations to evaluate the right side of (3.12) for small Δ leads to the cost propagation equation. Before stating this result we introduce a useful definition. For any $\tau \in [0,T]$ and any $\underline{\xi},\Delta,\underline{x}$, $\underline{o}(\Delta)$ with $\underline{\xi}\in P$ and $\underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta) \in P$, let

$$\overline{C}'(\underline{\xi},\underline{x},\Delta,\underline{o}(\Delta),\tau) \equiv \frac{\overline{C}(\underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta),\tau) - \overline{C}(\underline{\xi},\tau)}{\Delta} \qquad (3.22)$$

If $\lim\limits_{\Delta\downarrow 0} \overline{C}'(\underline{\xi},\underline{x},\Delta,\underline{o}(\Delta),\tau)$ exists, we shall denote it by

$$\overline{C}'(\underline{\xi},\underline{x},\tau) \equiv \lim_{\Delta\downarrow 0} \frac{\overline{C}(\underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta),\tau) - \overline{C}(\underline{\xi},\tau)}{\Delta} \qquad (3.23)$$

If $\overline{C}(\cdot)$ is actually differentiable with respect to $\underline{\xi}$, $\overline{C}'(\cdot)$ may be evaluated as

$$\overline{C}'(\underline{\xi},\underline{x},\tau) = \underline{x}^T \frac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\underline{\xi}} \qquad (3.24)$$

where $\dfrac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\underline{\xi}} \equiv \left[\dfrac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\xi_1}, \ldots, \dfrac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\xi_M} \right]^T$ and

$\dfrac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\xi_j}$ is the partial derivative of $\overline{C}(\underline{\xi},\tau)$ with respect to ξ_j, $j=1,\ldots,M$.

Lemma 3.6 in the next section shows that the one-sided directional derivatives in (3.23) always exist as long as a uniformly optimum decision function is used.

<u>Theorem 3.1</u> (Cost Propagation Equation). Assume that the limit in (3.23) exists for all $\underline{\xi},\underline{x},\underline{o}(\Delta)$ such that $\underline{\xi}\varepsilon P$, $\underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta)\varepsilon P$ and $\lim\limits_{\Delta\to 0} \dfrac{|\underline{o}(\Delta)|}{\Delta} = 0$. Then

(a) The average cost for the $[\tau,T]$ problem considered as a function of the a priori probabilities may be computed from the average cost function for the $[\tau+\Delta,T]$ problem as

$$\overline{C}(\underline{\xi},\tau) = \overline{C}(\underline{\xi},\tau+\Delta) + \Delta \left\{ \int_{\Sigma} d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \left\{ \overline{C}[\underline{\rho}(\vec{r},\tau,\underline{\xi}),\ \tau+\Delta] - \overline{C}(\underline{\xi},\tau+\Delta) \right\} \right.$$

$$\left. + \overline{C}'[\underline{\xi}, -\int_{\Sigma} d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi})(\underline{\rho}(\vec{r},\tau,\xi)-\underline{\xi}),\ \tau+\Delta] \right\} + o(\Delta)$$

(3.25)

(b) If $\dfrac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\underline{\xi}}$ exists, then $\overline{C}(\underline{\xi},\tau)$ is time-differentiable from the left at τ, i.e.,

$$\lim_{\Delta\downarrow 0} \frac{\overline{C}(\underline{\xi},\tau-\Delta)-C(\underline{\xi},\tau)}{(-\Delta)} \equiv \frac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\tau^{-}} \tag{3.26}$$

exists and may be evaluated as

$$-\frac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\tau^{-}} = \int d\vec{r}\ \hat{\lambda}(\vec{r},\tau^{-},\underline{\xi})\ \overline{C}''[\underline{\rho}(\vec{r},\tau^{-},\underline{\xi}),\underline{\xi},\tau] \tag{3.27}$$

where

$$\overline{C}''[\underline{\rho},\underline{\xi},\tau] \equiv \overline{C}(\underline{\rho},\tau) - \overline{C}(\underline{\xi},\tau) - (\underline{\rho}-\underline{\xi})^T \frac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\underline{\xi}} \tag{3.28}$$

and

$$\hat{\lambda}(\vec{r},\tau^-,\underline{\xi}) = \lim_{\Delta\downarrow 0} \hat{\lambda}(\vec{r},\tau-\Delta,\underline{\xi}) \tag{3.29}$$

$$\underline{\rho}(\vec{r},\tau^-,\underline{\xi}) = \lim_{\Delta\downarrow 0} \underline{\rho}(\vec{r},\tau-\Delta,\underline{\xi})$$

(c) If $\dfrac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\underline{\xi}}$ exists and $\lim_{\Delta\downarrow 0} \overline{C}''(\underline{\rho},\underline{\xi},\tau+\Delta) \equiv \overline{C}''(\underline{\rho},\underline{\xi},\tau^+)$ exists for all $\underline{\rho}\varepsilon P$, then $\overline{C}(\underline{\xi},\tau)$ is also time-differentiable from the right at τ, i.e.,

$$\lim_{\Delta\downarrow 0} \frac{\overline{C}(\underline{\xi},\tau)-\overline{C}(\underline{\xi},\tau+\Delta)}{(-\Delta)} \equiv \frac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\tau^+} \tag{3.30}$$

exists and equals

$$-\frac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\tau^+} = \int d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi})\ \overline{C}''[\underline{\rho}(\vec{r},\tau,\underline{\xi}),\ \underline{\xi},\ \tau^+] \tag{3.31}$$

(d) If the conditions of (c) are satisfied and furthermore $\overline{C}''(\underline{\rho},\underline{\xi},\tau) = \overline{C}''(\underline{\rho},\underline{\xi},\tau^+)$, $\hat{\lambda}(\vec{r},\tau^-,\underline{\xi}) = \hat{\lambda}(\vec{r},\tau,\underline{\xi})$, $\underline{\rho}(\vec{r},\tau^-,\underline{\xi}) = \underline{\rho}(\vec{r},\tau,\underline{\xi})$, then $\overline{C}(\underline{\xi},\tau)$ is time-differentiable at τ and

$$-\frac{\partial\overline{C}(\underline{\xi},\tau)}{\partial\tau} = \int d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi})\ \overline{C}''[\underline{\rho}(\vec{r},\tau,\underline{\xi}),\ \underline{\xi},\tau] \tag{3.32}$$

<u>Proof</u>: See Appendix (Section 3A.4)

3.5 OPTIMIZATION OF THE FEEDBACK FUNCTION

3.5.1 Simultaneous Optimality Property

In the preceding section we introduced a family of detection problems closely related to the one we initially considered and then derived a cost propagation equation which interrelates the average costs for all of these problems whenever the same (appropriately time-restricted) state-dependent feedback and decision functions are used. This equation will be helpful in solving the original optimization problem only if it can be shown that the same feedback and decision functions minimize the average costs for all problems simultaneously. Otherwise an equation such as (3.12) could in general only relate an optimal cost for the $[\tau,T]$ problem to a non-optimal one for the $[\tau+\Delta,T]$ problem or vice-versa, and further analysis would be required to relate optimal costs for both problems. In this section Lemma 3.4 will demonstrate that the desired condition is valid for our family of problems. After that, the basic optimization procedure is clear. The optimum state-dependent decision function $j^*(\underline{\xi})$ is the same as the one which minimizes the a priori guess performance $\overline{C}(\underline{\xi},T) = \sum_{j=1}^{M} \xi_j C[j,\hat{j}(\underline{\xi})]$ (average cost with no observations). The optimum state-dependent feedback function $\ell^*(\vec{r},\tau,\underline{\xi})$ may be found in the following manner. Suppose $\ell^*(\cdot,\tau',\cdot)$ has already been determined for $\tau' \geq \tau$. The average cost for the $[\tau-\Delta,T]$ problem, $\overline{C}(\underline{\xi},\tau-\Delta)$, may be obtained from the cost propagation equation as $\overline{C}(\underline{\xi},\tau)$ plus an increment which depends on $\{\ell(\cdot,\tau',\cdot),\ \tau-\Delta\leq\tau'<\tau\}$ and on $\overline{C}(\cdot,\tau)$. Since the lemma guarantees that the feedback function which minimizes $\overline{C}(\underline{\xi},\tau-\Delta)$ equals the already determined $\ell^*(\cdot,\tau',\cdot)$ for $\tau\leq\tau'\leq T$, the solution for the $[\tau-\Delta,T]$ problem may be completed by optimizing just the cost increment over $\{\ell(\cdot,\tau',\cdot),\tau-\Delta\leq\tau'<\tau\}$, treating the optimal $\overline{C}(\cdot,\tau)$ as known. In this way it is possible to construct the optimum state-dependent feedback function $\ell(\vec{r},\tau,\underline{\xi})$ backward in time from $\tau=T$ by successively optimizing the cost increments appearing in the cost propagation equation. A more precise statement of this optimization method is given in Theorems 3.2, 3.3, and 3.4.

<u>Lemma 3.4</u> Let $\hat{j}$ be any state-dependent decision function and ℓ^*: $\Sigma x[0,T]xP\rightarrow ¢$ a state-dependent feedback function. For any $\tau\varepsilon[0,T]$ denote the restriction of ℓ^* to $\Sigma x[\tau,T]xP$ by ℓ^*_τ. Fix $\underline{\xi}_o\varepsilon P$ and $\tau_o\varepsilon[0,T]$. Then $\ell^*_{\tau_o}$ minimizes $\overline{C}(\underline{\xi}_o,\tau_o)$ (given $\hat{j}$) if and only if:

> for every $\tau\varepsilon[\tau_o,T]$, ℓ^*_τ minimizes $\overline{C}(\underline{\xi},\tau)$ for all $\underline{\xi}$ in some subset P_τ of P with the property $Pr[\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o)\varepsilon P_\tau]=1$, where the probability measure on $\underline{t}$ is that induced by ℓ^*.

<u>Proof</u>: See Appendix (Section 3A.5)

3.5.2 Optimum Decision Function

Before presenting the remaining theorems on the feedback function optimization, we shall first discuss the considerably simpler determination of an optimum state-dependent decision function. The next lemma re-iterates our previous statement that this is accomplished by simply optimizing the a priori guess performance, and it demonstrates that this minimization problem is really trivial. The difficulty, of course, in implementing the solution is in the computation of the a posteriori probability vector at time T which is needed to evaluate the event-dependent decision function.

Two versions of the optimality principle are given in Lemma 3.5. Part a refers to decision functions which simultaneously minimize $\overline{C}(\underline{\xi},\tau)$ for all $\underline{\xi}\varepsilon P$ and all $\tau\varepsilon[0,T]$; such decision functions will be called uniformly optimum for the class of problems considered. Part b pertains to decision functions minimizing $\overline{C}(\underline{\xi}_o,\tau_o)$ for some fixed $\underline{\xi}_o\varepsilon P$ and $\tau_o\varepsilon[0,T]$; such decision functions will also be simultaneously optimum for all reachable states $\underline{\xi}$ at all times after τ_o.

Lemma 3.5 For any feedback function ℓ, the optimum state-dependent decision function $j^*(\underline{\xi})$ is chosen to minimize $\overline{C}(\underline{\xi},T) = \sum_{j=1}^{M} \xi_j \, C[j,\hat{j}(\underline{\xi})]$, at least for reachable states $\underline{\xi}$; i.e.,

a. $j^*(\cdot)$ minimizes $\overline{C}(\underline{\xi},\tau)$ for all $\underline{\xi}\varepsilon P$ and all $\tau\varepsilon[0,T]$ if and only if for every $\underline{\xi}\varepsilon P$,

$$\sum_{j=1}^{M} \xi_j \; C[j,j^*(\underline{\xi})] \leq \sum_{j=1}^{M} \xi_j \; C[j,\hat{j}(\underline{\xi})] \quad \text{for all } \hat{j}(\underline{\xi})\varepsilon \; \hat{R} \qquad (3.33)$$

b. Let $j^*(\cdot)$ be any state-dependent decision function and let $\hat{P}\subseteq P$ be the set of states $\underline{\xi}$ for which $j^*(\underline{\xi})$ satisfies (3.33). Then $j^*(\cdot)$ minimizes $\overline{C}(\underline{\xi}_o,\tau_o)$ if and only if

$$\Pr[\underline{\xi}(T:\underline{t};\underline{\xi}_o,\tau_o)\varepsilon \; \hat{P}] = 1 \qquad (3.34)$$

where the probability measure on $\underline{t}$ is that induced by ℓ. In other words (3.33) must be satisfied for all states which are reachable at time T with nonvanishing probability from state $\underline{\xi}_o$ at time τ_o.

Proof: See Appendix (Section 3A.6).

The conditions for a uniformly optimum decision function are not significantly more stringent than the ones for any particular optimum. On the other hand, uniformly optimum decision functions guarantee certain desired properties such as the differentiability discussed in the next lemma, which facilitate the consideration of the feedback function optimization problem. For these reasons, we shall assume from now on that our optimum decision functions are uniformly optimum. This assumption is not essential in the sense that basically the same results could be stated and proved without it, but the language and techniques would be much more cumbersome without adding any insight.

For the feedback optimization problem, we shall maintain the dichotomic approach of considering both uniformly and non-uniformly optimal feedback functions, because the distinction between those two is potentially non-trivial.

Lemma 3.6. Let $\overline{C}^*(\cdot,\cdot)$ be the average cost function associated with any state-dependent regular feedback function ℓ and a uniformly optimum state-dependent decision function j^*. Then

$$\lim_{\Delta\downarrow 0} \frac{\overline{C}^*[\underline{\xi}+\Delta\underline{x} + \underline{o}(\Delta),\tau\,] - C^*[\underline{\xi},\tau]}{\Delta} \equiv \overline{C}^{*\prime}(\underline{\xi},\underline{x},\tau) \text{ exists}$$

for all $\underline{\xi},\underline{x}$, $\underline{o}(\Delta)$ such that $\underline{\xi}\varepsilon P$, $\underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta)\varepsilon P$, and $\lim\limits_{\Delta\to 0} \dfrac{|\underline{o}(\Delta)|}{\Delta} = 0$

Proof: See Appendix (Section 3A.7).

3.5.3 Feedback Function Optimality Condition

We are now ready to present the main theorems of this chapter, on the feedback function c[illegible]mization.

Theorem 3.2a Let $j^*(\underline{\xi})$ be a uniformly optimum state-dependent decision function. A uniformly optimum feedback function

$\ell^*(\vec{r},\tau,\underline{\xi})$ can be constructed backward in time from $\tau=T$ by the following recursive relation:

for every $\underline{\xi}\varepsilon P$, $\tau\varepsilon[0,T]$,

$\ell^*(\cdot,\tau^-,\underline{\xi})$ is the complex function of $\vec{r}$ which minimizes the term proportional to Δ on the right side of (3.25) (with $\tau+\Delta$ replaced by τ and τ replaced by τ^-), as a function of the (previously determined) optimum future feedback function $\ell_\tau^*(\cdot,\cdot,\cdot)$ and the optimum decision function $j^*(\cdot)$.

Proof: See Appendix (Section 3A.9)

Condition (3.35) is necessary and sufficient: (1) if it is not possible to satisfy it by choosing (finite) feedback $\ell^*(\cdot)$ for any τ or $\underline{\xi}$, then minimum average costs $\overline{C}(\underline{\xi},\tau)$ cannot be achieved for all $\underline{\xi}\varepsilon P$, $\tau\varepsilon[0,T]$ by any admissible (i.e., regular) feedback function; (2) if any state-dependent feedback function $\ell^*(\cdot,\cdot,\cdot)$ can be found which solves (3.35) for all $\underline{\xi}\varepsilon P$, $\tau\varepsilon[0,T]$, then the average costs $\overline{C}^*(\underline{\xi},\tau)$ achieved by ℓ^*, j^* are all minimal; and (3) if it is possible to simultaneously minimize $\overline{C}(\underline{\xi},\tau)$ for all $\underline{\xi}\varepsilon P$, $\tau\varepsilon[0,T]$ by the choice of a regular state-dependent feedback function $\ell^*(\cdot,\cdot,\cdot)$, then $\ell^*(\cdot,\cdot,\cdot)$ solves (3.35).

When the differentiability and continuity conditions of Theorem 3.1d are satisfied, (3.35) is equivalent to:

for every $\underline{\xi}\varepsilon P$, $\tau\varepsilon[0,T]$, and (almost every) $\vec{r}\varepsilon\Sigma$

[illegible] is the smallest number which minimizes [illegible] the integrand on the right side of [illegible]

Referring to [illegible] and to the definitions [illegible] write this condition more explicitly as [illegible] if and only if

$$[illegible]$$

$$[illegible]$$

for all [illegible]

where we have used the notation [illegible]

Theorem 3.2a specifies conditions for a simultaneous minimization of the average costs for all $[\tau, T]$ problems with arbitrary a priori probabilities. If we are interested in obtaining minimum average cost for just one particular problem, say [illegible], then (3.35) is still a sufficiency condition but it is too restrictive. It is only necessary that (3.35) be satisfied for those $\underline{\xi}$ which are "reachable" at time τ, starting from $\underline{\xi}_o$ at time τ_o. This idea is expressed more precisely in Theorem 3.2b.

$$\ell^*(\vec{r},\tau,\underline{\xi}) \text{ is the complex number which minimizes the integrand on the right side of (3.32)} \tag{3.36}$$

Referring to (3.32) and to the definitions of $\hat{\lambda},\underline{\rho}$, we can write this condition more explicitly as :

$\ell^*(\vec{r},\tau,\underline{\xi}) = Z^* \varepsilon$ ₵ if and only if:

$$\sum_{i=1}^{M} \xi_i \, \Lambda[\varepsilon_i(\vec{r},\tau),\, Z^*] \, \overline{C}''\left[\frac{\xi_j \Lambda[\varepsilon_j(\vec{r},\tau),Z^*]}{\sum_{i=1}^{M} \xi_i \Lambda[\varepsilon_i(\vec{r},\tau),Z^*]},\, \underline{\xi},\, \tau\right]$$

$$\leq \sum_{i=1}^{M} \xi_i \Lambda[\varepsilon_i(\vec{r},\tau),\, Z] \, \overline{C}''\left[\frac{\xi_j \Lambda[\varepsilon_j(\vec{r},\tau),Z]}{\sum_{i=1}^{M} \xi_i \Lambda[\varepsilon_i(\vec{r},\tau),Z]},\, \underline{\xi},\tau\right] \tag{3.37}$$

for all $Z\varepsilon$₵

where we have used the notation $\overline{C}''[\rho_j,\underline{\xi},\tau] \equiv \overline{C}''[\underline{\rho},\underline{\xi},\tau]$ if $\underline{\rho} = [\rho_1,\, \rho_2,\ldots,\rho_M]^T$.

Theorem 3.2a specifies conditions for a simultaneous minimization of the average costs for all $[\tau,T]$ problems with arbitrary a priori probabilities. If we are interested in obtaining minimum average cost for just one particular problem, say $\overline{C}(\underline{\xi}_o,\tau_o)$, then (3.35) is still a sufficiency condition but it is too restrictive. It is only necessary that (3.35) be satisfied for those $\underline{\xi}$ which are "reachable" at time τ, starting from $\underline{\xi}_o$ at time τ_o. This idea is expressed more precisely in Theorem 3.2b.

Theorem 3.2b Let j^* be a uniformly optimum state-dependent decision function. Given any state-dependent feedback function ℓ, define for every $\tau\varepsilon[0,T]$ the subset $P_\tau \subseteq P$ of probability vectors $\underline{\xi}$ for which (3.35) is satisfied by $\ell(\cdot,\tau^-,\underline{\xi})$. Let L^* be the collection of feedback functions ℓ which have the property $\Pr[\underline{\xi}(\tau^-:\underline{t};\underline{\xi}_o,\tau_o)\ \varepsilon P_\tau] = 1$ for all $\tau\varepsilon[\tau_o,T]$, where the probability measure on $\underline{t}$ is that induced by ℓ. Then L^* is precisely the set of feedback functions which minimize $\overline{C}(\underline{\xi}_o,\tau_o)$; i.e., any $\ell\ \varepsilon\ L^*$ is optimal, no ℓ outside L^* is optimal, and no minimum $\overline{C}(\underline{\xi}_o,\tau_o)$ exists if and only if L^* is empty.

Proof: See Appendix (Section 3A.9)

Even though Theorem 3.2b shows that for any fixed problem the optimality condition(3.35) need not be solved for non-reachable states, this exemption does not greatly simplify the backward-time constructive algorithm for generating the optimum feedback function, because at any time τ the determination of the set of reachable states requires knowledge of the (as yet undetermined) values of the feedback function for times prior to τ. It can be utilized to advantage over Theorem 3.2a when there are regions of state-space which are obviously non-reachable for optimum decisions (e.g., $\{\underline{\xi}\varepsilon P: \max_j \xi_j < \max_j \xi_j^o\}$ for a probability of error cost) or when a solution can be guessed and the theorem is simply used to validate it.

3.5.4 Possible Non-Existence of Optimum Regular Feedback

The possible non-existence of a minimum is due to our insistence that it actually be achieved by a regular feedback function. Since $\overline{C}$ is bounded below, it is always reasonable to try to achieve or at least approach performance equal to $\inf_{\hat{j},\ell} \overline{C}(\underline{\xi}_o,\tau_o)$ where the inf is taken just over regular feedback functions ℓ. The following theorem shows that whenever $\inf_{\hat{j},\ell} \overline{C}(\underline{\xi}_o,\tau_o)$ cannot be precisely realized by any regular feedback function satisfying (3.35), it is possible to approach optimum performance with a sequence of feedback functions which "almost" satisfy (3.35). The question of whether performance superior to $\inf_{\hat{j},\ell} \overline{C}(\underline{\xi}_o,\tau_o)$ may be achieved by taking the inf over non-regular feedback functions is one that we do not consider.

<u>Theorem 3.3</u> Let j^* be a uniformly optimum state-dependent decision function. For any K>0 define $¢_K$ to be the compact region in the complex plane $¢_K \equiv \{Z\varepsilon¢ : |Z|\leq K\}$. Modify (3.35), obtaining $(3.35)_K$, by replacing "complex-valued regular feedback function ℓ" with "K-bounded complex-valued regular feedback function ℓ" where the latter refers to functions with range $¢_K$ instead of $¢$. Then

a. There always exists a solution $\ell^K(\cdot,\tau^-,\underline{\xi})$ to $(3.35)_K$ for all $\tau\varepsilon[0,T]$ and all $\underline{\xi}\varepsilon P$. If $\overline{C}^K(\underline{\xi},\tau)$ is the average cost function associated with any such solution (and decision function j^*), then $\bar{C}^K(\underline{\xi},\tau) \downarrow \inf_{\hat{j},\ell} \bar{C}(\underline{\xi},\tau)$ as $K \to \infty$, for all $\underline{\xi}\varepsilon P$ and all $\tau\varepsilon[0,T]$.

b. If $\ell(\cdot,\tau^-,\underline{\xi})$ satisfies $(3.35)_K$ for $\underline{\xi}\varepsilon P^K_\tau$, $\tau\varepsilon[0,T]$, and L^K is the collection of feedback functions ℓ with the property $Pr[\underline{\xi}(\tau^-:\underline{t};\underline{\xi}_o,\tau_o)\ \varepsilon P^K_\tau] = 1$, all $\tau\varepsilon[\tau_o,T]$, and $\bar{C}^K(\underline{\xi}_o,\tau_o)$ is the average cost associated with j^* and any $\ell\varepsilon L^K$, then L^K is nonempty and $\bar{C}^K(\underline{\xi}_o,\tau_o) \downarrow \inf_{\hat{j},\ell} \bar{C}(\underline{\xi}_o,\tau_o)$.

Proof: See Appendix (Section 3A.10)

3.5.5 Discrete-Time Approximation

Although (3.35) was described as a constructive algorithm for evaluating the optimum feedback, it is not clear whether that pointwise condition can be used to build an optimum function over any nonzero length time interval $[\tau-\Delta,\tau]$, $\Delta>0$. Condition (3.35) is constructive in roughly the same sense as a (backward-time) Volterra integral equation, specifying the current value of a function as a functional of all of its future values. One simple way to approximate the solution of such an equation is to constrain ℓ to be a step function, constant over intervals of length Δ. The next theorem shows that a piecewise constant feedback function determined by a discrete version of (3.35) achieves performance which approaches minimum cost as $\Delta\downarrow 0$.

Theorem 3.4 Let N be a large integer, $\Delta=\frac{T}{N}$, and ℓ^{Δ} a piecewise constant feedback function: $\ell^{\Delta}(\vec{r},\tau,\underline{\xi}) = \ell_i(\vec{r},\underline{\xi})$, if $(i-1)\Delta \leq \tau < i\Delta$, $1 \leq i \leq N$. Let $\overline{C}^{\Delta}(\underline{\xi},\tau)$ denote the cost associated with ℓ^{Δ}.

a.(i) Suppose ℓ^{Δ} satisfies (3.35) for all $\underline{\xi}\varepsilon P$ and all $\tau=i\Delta$, $1 \leq i \leq N$. Then $\lim\limits_{\Delta\downarrow 0} \overline{C}^{\Delta}(\underline{\xi},\tau) = \inf\limits_{\hat{J},\ell} \overline{C}(\underline{\xi},\tau)$ for all $\underline{\xi}\varepsilon P$ and $\tau\varepsilon[0,T]$.

a.(ii) Suppose $\ell^{\Delta}\equiv\ell^{\Delta,K}$ satisfies $(3.35)_K$ for all $\underline{\xi}\varepsilon P$ and all $\tau=i\Delta$, $1 \leq i \leq N$. Then $\ell^{\Delta,K}$ achieves cost $\overline{C}^{\Delta,K}(\underline{\xi},\tau)$ with $\lim\limits_{K\to\infty} \lim\limits_{\Delta\downarrow 0} \overline{C}^{\Delta,K}(\underline{\xi},\tau) = \inf\limits_{\hat{J},\ell} \overline{C}(\underline{\xi},\tau)$ for all $\underline{\xi}\varepsilon P$, $\tau\varepsilon[0,T]$.

b.(i) Suppose, for each $\tau=i\Delta$, ℓ^{Δ} satisfies (3.35) for $\underline{\xi}\varepsilon P_i$, with $\Pr[\underline{\xi}((i-1)\Delta:\underline{t};\underline{\xi}_o,\tau_o)\varepsilon P_i] = 1$ for all $i > \tau_o/\Delta$, where the probability measure on $\underline{t}$ is that induced by ℓ^{Δ}. Then $\lim\limits_{\Delta\downarrow 0} \overline{C}^{\Delta}(\underline{\xi}_o,\tau_o) = \inf\limits_{\hat{J},\ell} \overline{C}(\underline{\xi}_o,\tau_o)$.

b.(ii) Suppose, for each $\tau=i\Delta$, $\ell^{\Delta}\equiv\ell^{\Delta,K}$ satisfies $(3.35)_K$ for $\underline{\xi}\varepsilon P_i^K$, with $\Pr[\underline{\xi}((i-1)\Delta:\underline{t};\underline{\xi}_o,\tau_o)\varepsilon P_i^K] = 1$ for all $i > \tau_o/\Delta$, where the probability measure on $\underline{t}$ is that induced by $\ell^{\Delta,K}$. Then $\ell^{\Delta,K}$ achieves cost $\overline{C}^{\Delta,K}(\underline{\xi}_o,\tau_o)$ with $\lim\limits_{K\to\infty} \lim\limits_{\Delta\downarrow 0} \overline{C}^{\Delta,K}(\underline{\xi}_o,\tau_o) = \inf\limits_{\hat{J},\ell} \overline{C}(\underline{\xi}_o,\tau_o)$.

Proof: See Appendix (Section 3A.11)

3.6 DISCUSSION OF THE OPTIMALITY CONDITION

For the remainder of this chapter we will assume for simplicity and concreteness that the minimum cost $\bar{C}^*(\underline{\xi},\tau)$ exists and is differentiable with respect to $\underline{\xi}$ and τ. Then, from (3.32) and (3.36),

$$-\frac{\partial \bar{C}^*(\underline{\xi},\tau)}{\partial\tau} = \int d\vec{r}\ \hat{\lambda}^*(\vec{r},\tau,\underline{\xi})\ \bar{C}^{*''}[\underline{\rho}^*(\vec{r},\tau,\underline{\xi}),\ \underline{\xi},\ \tau] \tag{3.38}$$

and

$\ell^*(\vec{r},\tau,\underline{\xi})$ is chosen to simultaneously adjust

$$\hat{\lambda}^*(\vec{r},\tau,\underline{\xi}) = \sum_{j=1}^{M} \xi_j\ \Lambda[\varepsilon_j(\vec{r},\tau),\ \ell^*(\vec{r},\tau,\underline{\xi})]$$

and

$$\rho_j^*(\vec{r},\tau,\underline{\xi}) = \xi_j\ \frac{\Lambda[\varepsilon_j(\vec{r},\tau),\ \ell^*(\vec{r},\tau,\underline{\xi})]}{\hat{\lambda}^*(\vec{r},\tau,\underline{\xi})} = \xi_j\ \frac{\lambda_j^*(\vec{r},\tau,\underline{\xi})}{\hat{\lambda}^*(\vec{r},\tau,\underline{\xi})}\ ,\quad j=1,\ldots,M \tag{3.39}$$

so as to minimize the product $\hat{\lambda}^*(\vec{r},\tau,\underline{\xi})\ \bar{C}^{*''}[\underline{\rho}^*(\vec{r},\tau,\underline{\xi}),\underline{\xi},\tau]$.

Equation (3.38) is similar to the standard Hamilton-Jacobi-Bellman equation [27] which simultaneously determines the backward-time cost propagation and the optimum control when the state vector $\underline{\xi}(\tau)$ is deterministic. Extensions have been made to random states, especially those influenced by Wiener-type increment processes.[16] In our problem, $\underline{\xi}(\tau:\underline{t})$ is determined by a Poisson-type increment process. As we saw in (3.8), an abrupt change of state occurs at each event time, in contrast with the continuous state evolution that characterizes the deterministic and Wiener noise problems. This fact complicates the standard Hamilton-Jacobi-Bellman partial differential equation by turning it into a partial differential - difference equation. [See Dreyfus[15].] In our problem the integration over $\vec{r}$ produces further complexity.

We show in the Appendix (Section 3A.8) that $\bar{C}^*$ is a convex $\cap$ function of $\underline{\xi}$ for all τ, so $\bar{C}^{*''}$ is always nonpositive. Thus, optimality requires the maximization of the product of non-negative functions $\hat{\lambda}^*(\vec{r},\tau,\underline{\xi})\,|\bar{C}^{*''}[\underline{\rho}^*(\vec{r},\tau,\underline{\xi}), \underline{\xi}, \tau]|$. Writing

$$\bar{C}^{*''}(\underline{\rho},\underline{\xi},\tau) = \bar{C}^*(\underline{\rho},\tau) - \bar{C}^*(\underline{\xi},\tau) - (\underline{\rho} - \underline{\xi})^T \frac{\partial \bar{C}^*(\underline{\xi},\tau)}{\partial\, \underline{\xi}} \tag{3.40}$$

we see that the second factor represents the deviation of $\bar{C}^*$ from linearity between the a priori probability vector $\underline{\xi}$ and the potential a posteriori probability vector $\underline{\rho}^*(\vec{r},\tau,\underline{\xi})$ should an immediate point process event occur at $(\vec{r},\tau)$. [See Figure 3.1, which depicts a one-dimensional projection along the direction of $\underline{\rho} - \underline{\xi}$.]

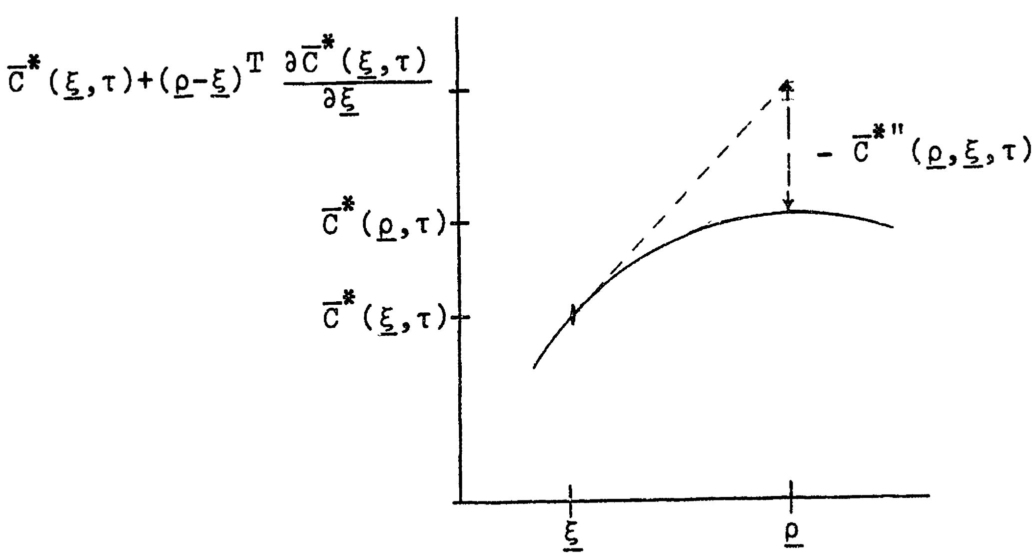

Figure 3.1

To make $|\bar{C}^{*''}(\underline{\rho},\underline{\xi},\tau)|$ large, one can in general select an optimum direction for $\underline{\rho} - \underline{\xi}$ and then try to make $|\underline{\rho} - \underline{\xi}| = \sum_{j=1}^{M} \xi_j |\lambda_j/\hat{\lambda} - 1|$ as large as possible. This corresponds to our intuition that ℓ should be chosen in a manner that maximizes the separation of the point process intensities λ_j under the various hypotheses H_j, $j=1,\ldots,M$. But all the hypotheses are not weighted equally, and the optimal biases must be determined from $\bar{C}^{*''}$. Adding to the complexity is the other multiplicative factor $\hat{\lambda}$. For fixed separation of the hypotheses given a potential count, the average count rate $\hat{\lambda}$ should be maximized. But this can only be accomplished in a feedback receiver by making $|\ell|$ arbitrarily large, which would eventually eliminate any separation of the hypotheses under potential counts by forcing $|\lambda_j/\hat{\lambda} - 1| \to 0$ for all j. The best trade-off among all these factors is accomplished by choosing ℓ to maximize $\hat{\lambda}|\bar{C}^{*''}(\underline{\rho},\underline{\xi},\tau)|$.

Another point that needs to be discussed is the exact sense in which maximization of $\hat{\lambda}|\bar{C}^{*''}|$ "solves" our original problem. We started with one minimization problem and apparently have succeeded only in converting it into another undetermined minimization problem. But, whereas the original problem required a very complicated minimization over feedback <u>functions</u> $\ell:[0,T]^+ \to \mathbb{C}$, the pointwise minimization of $\hat{\lambda}\bar{C}^{*''}$ for each $(\vec{r},\tau,\underline{\xi})$ involves a function of a single complex variable ℓ. The second minimization problem is trivial compared to the first, and it may usually be solved by setting a derivative equal to zero. An example of this procedure is given in Section 6.5.

The reduction of the original minimization problem to a pointwise minimization is gained only at the expense of having to solve it for an extra dimension of points. Even though in any application it is only necessary to evaluate the feedback function at the particular probability vector $\underline{\xi}(\tau : \underline{t})$ actually realized at time τ, the dynamic programming optimality condition requires pre-determination of an optimum strategy to cover every potential (reachable) state $\underline{\xi}$.

The next two chapters investigate different optimality criteria. In Chapter IV we consider a broader class of receivers consistent with quantum mechanics and show that optimum performance within this class may often be matched by a feedback receiver. In Chapter V we determine a considerably simpler optimality condition for feedback receivers that are not chosen to minimize a target cost for observations ceasing at a fixed time T, as assumed in Chapter III, but rather are designed recursively in time, small Δ-interval by small Δ-interval, to assemble maximum information given the past and potential observations just in the single Δ- interval instead of all the remaining time until T. The optimality condition derived from this criterion does not suffer from the extra dimensionality of the [0,T] interval optimality condition (3.35). Finally, in Chapter VI, we will consider some examples of communication problems for which optimum or near-optimum feedback functions can be determined.

3.7 GENERALIZATION OF MODEL TO INCLUDE DARK CURRENT

Before closing this chapter, we note a trivial extension of the problem model for which all the theorems remain valid. Real optical detectors are affected by a dark current $\mu(t)$ which adds to the counting intensity. We generalize this by allowing the conditional intensity functions $\lambda_j(t:\underline{t})$ to be of the form

$$\lambda_j(t:\underline{t}) = \Lambda[\varepsilon_j(t), \ell(t:\underline{t})] + \mu_j(t:\underline{t}) \qquad (3.41)$$

where $\mu_j:[0,T]^* \to \mathbf{R}^+$, $j=1,\ldots,M$, are arbitrary conditional intensity functions. Since none of the results in this chapter depended on the form of Λ(or on the fact that Λ was independent of j), they are all applicable with appropriate substitutions for λ_j, ρ_j, $\hat{\lambda}$ determined by (3.41).

CHAPTER IV

QUANTUM MEASUREMENTS REALIZED BY FEEDBACK RECEIVERS

4.1 INTRODUCTION

In previous chapters we briefly alluded to a certain structural similarity between the problem of determining the optimum feedback function for a feedback receiver and that of determining the best possible measurement from the class of all physically allowable measurements consistent with quantum theory. In each case the receiver is not free to examine the received field noiselessly in all its detail but instead must be content to extract from it a set of noisy data which is statistically, but never deterministically, dependent on the actual received state. A seemingly similar situation arises in the analysis of non-ideal classical receivers which also introduce noise in the measurement process. But there the noise statistics are known (at least in principle) and the receiver can only be designed to optimally process the particular set of noisy data with the given statistics. On the other hand, quantum noise is different because, though the receiver is constrained by the uncertainty principle to introduce a certain nonzero "amount" of noise, the exact statistics of the noise are not fixed but instead are determined by the particular quantum measurement being performed. The quantum receiver has the same unrestricted freedom of the classical receiver to arbitrarily process whatever noisy data it gets, but it has an additional level of control, absent in the classical receiver, over the conditional probability measure tying the data to the received signal.

A feedback receiver with a specified feedback function ℓ generates from the received signal a particular type of data, point process data $\underline{t}$, whose conditional statistics are determined by ℓ, and it has the freedom to process $\underline{t}$ to realize an arbitrary estimator function $\hat{a}(\underline{t})$. A feedback receiver with the power to select its feedback function retains the unrestricted data processing capability of the classical receiver but also has an extra level of control over the conditional statistics of the data similar to the one that characterizes the quantum communication problem.

However, the quantum measurement problem has a much larger class of admissible controls. If comparisons between the two are to be made not only in the method and difficulty of solution but also with respect to the solutions themselves, it is necessary to characterize the subclass of quantum measurements which can be realized by feedback receivers.

At first glance this subclass may seem negligibly small. After all, quantum measurements can generate arbitrary types of data, not necessarily the point data to which all feedback receivers are restricted. However, the exact form of the data is often irrelevant. Performance is only dependent on the statistics induced by the measurement on one particular function of the data, the optimum estimator $\hat{a}$, and is unaffected by the intrinsic physical significance or names attached to the measurement outcomes which result in a given estimate.

We shall regard two measurement classes as equivalent for a given family of problems if the minimum costs attainable by measurements within each class are always equal. In Theorem 4.2 we establish that feedback receiver measurements are equivalent, for problems with no spatial field modulation which also satisfy certain regularity conditions, to contingent sequences of arbitrary quantum measurements performed on infinitesimally small time-samples of the received field separately. This result is motivated by the theorem preceding it, which compares the performance of feedback receivers with that of a single arbitrary quantum measurement when the optical field is present for an infinitesimally short period of time. The final theorem of the chapter demonstrates the equivalence of classes of generalized contingent measurement sequences and generalized feedback receiver measurements, which are free of the constraint to measure the time-samples of the received field in their natural time-ordering.

Before presenting these theorems, we describe in the next section a quantum communication model for our problems. We shall not attempt to justify the validity of the model; the reader is referred to Helstrom _et_ _al_.[2]

4.2 QUANTUM COMMUNICATION PROBLEM MODEL

4.2.1 Bayesian Communication Objective

We will consider general Bayesian communication problems as modeled in Chapter II. Each problem is specified by a (generalized) random parameter a with range R, an estimator range $\hat{R}$ (which is assumed to be finite, as in Chapter III), and a real-valued cost function C with domain $R\times\hat{R}$, where $C(a,\hat{j})$ represents the relative cost associated with guessing $\hat{j}$ when a is the true value of the random parameter. The communication objective is to simultaneously choose: (i) an allowable measurement M, which generates from the optical signal a set of data $\{\beta\epsilon B_M\}$ whose conditional probability measure given a is dependent on M; and (ii) an estimator function $\hat{j}(\beta)$ with domain B_M and range $\hat{R}$, in order to minimize the expected value of the cost

$$\overline{C} = E_{a,\beta}\, C[a,\hat{j}(\beta)] \tag{4.1}$$

For allowable feedback receiver measurements, the data space is always $[0,T]^*$ and the conditional probability density for $\underline{t}\epsilon[0,T]^*$ is fully specified by the feedback function ℓ, according to equation (3.4). An allowable quantum measurement can be represented by a set of Hilbert space operators.[28]

4.2.2 Quantum Communication Model for the Received Field

The finite-state model of Chapter III is assumed for the (classical) received field, which is, under hypothesis H_j, a deterministic space-time function $\varepsilon_j(t)$. It is further assumed in this chapter that there is no spatial modulation; i.e., that

$$\varepsilon_j(\vec{r},\tau) = S_j(\tau)\ \varepsilon_o(\vec{r}) \qquad j=1,\ldots,M \tag{4.2}$$

with $\varepsilon_o(\vec{r})$ normalized such that

$$\int_\Sigma d\vec{r}\ |\varepsilon_o(\vec{r})|^2 = 1 \tag{4.3}$$

[We rely on this simplification in proving our theorems, but we have found no reason to believe that similar measurement correspondences are not valid for more general signals.]

Our quantum communication model for this problem is constructed as follows. There is an M-dimensional Hilbert space $\mathcal{H}$ spanned by M linearly independent states $|\alpha_j\rangle$, $j=1,\ldots,M$, called <u>coherent states</u>[29], which are characterized by the inner products

$$(|\alpha_j\rangle,\ |\alpha_k\rangle) \equiv \langle\alpha_j|\alpha_k\rangle = e^{-\frac{1}{2}[E_j+E_k-2\gamma_{jk}\sqrt{E_jE_k}]} \tag{4.4}$$

where

$$E_j = \int_0^T d\tau |S_j(\tau)|^2 \quad , \quad E_k = \int_0^T d\tau |S_k(\tau)|^2$$

$$\gamma_{jk}\sqrt{E_jE_k} = \int_0^\tau d\tau\ S_j^*(\tau)\ S_k(\tau) \quad , \quad |\gamma_{jk}| \leq 1 \tag{4.5}$$

It is assumed that the coherent state $|\alpha_j\rangle$ furnishes the correct quantum mechanical description of the received field under hypothesis H_j. The classical field $\varepsilon_j(t) = S_j(\tau)\varepsilon_o(\vec{r})$ can be interpreted as the (complex envelope of the) received field that would be measurable in the absence of quantum noise, if the quantum state is $|\alpha_j\rangle$. This model is consistent with the semi-classical one developed in the previous three chapters for feedback receivers, because a photodetection measurement performed on any coherent state $|\alpha_j\rangle$ produces a space-time Poisson process with intensity $|\varepsilon_j(t)|^2$. [5]

An arbitrary quantum measurement to be performed on the field can be represented[28] by an index set B and a set of nonnegative definite operators $\{Q_\beta\}_{\beta\varepsilon B}$ on $\mathcal{H}$, with $\sum_{\beta\varepsilon B} Q_\beta = I_{\mathcal{H}}$, the identity operator on $\mathcal{H}$. The data generated by the measurement is one of the possible values of $\beta\varepsilon B$, and the conditional probability of obtaining outcome β when the system is in one of the coherent states $|\alpha_j\rangle$ is given by

$$\Pr\left(\beta \,\middle|\, |\alpha_j\rangle \right) = \left(|\alpha_j\rangle, Q_\beta|\alpha_j\rangle \right) = \langle\alpha_j|Q_\beta|\alpha_j\rangle \tag{4.6}$$

As in Chapter III the average cost $\overline{C}$ in (4.1) is more conveniently re-written as an expected value over β and j,

$$\overline{C} = \sum_{j=1}^{M} \sum_{\beta\varepsilon B} \xi_j \langle\alpha_j|Q_\beta|\alpha_j\rangle \quad C_1[j,\hat{j}(\beta)], \tag{4.7}$$

where $\{\xi_j\}$ are the a priori probabilities

$$\xi_j = \Pr\,[\varepsilon(\cdot)=\varepsilon_j(\cdot)] = \Pr\,[\text{state of received field is } |\alpha_j\rangle] \tag{4.8}$$

and the effective cost matrix C_1 is derived independently of the measurement $\{Q_\beta\}$ by averaging the original cost matrix C over a priori statistics,

$$C_1[j,\hat{j}] = E_{a|j}\, C(a,\hat{j}) \tag{4.9}$$

For the remainder of the chapter we shall deal exclusively with the effective cost matrix and therefore shall drop the subscript notation for convenience.

4.2.3 Subdivision of the Signaling Interval

A slightly different quantum communication model for the received field, which is equivalent to the one presented in the last subsection, is more directly useful for analyzing the class of contingent measurement sequences to which feedback receivers correspond. This model is based on a subdivision of the signaling interval $[0,T]$ into N subintervals $[\tau_{i-1}, \tau_i] \equiv [(i-1)\Delta, i\Delta]$, $i=1,\ldots,N$, of length $\Delta=\frac{T}{N}$. The theorems in this chapter will be derived for the limit as $\Delta\downarrow 0$, so N may be regarded as a large integer.

The previous model is applicable to the received field over each subinterval $[\tau_{i-1}, \tau_i]$, which is therefore assumed to correspond to one of M coherent states $|\alpha_j^i\rangle_i$ in a Hilbert space $\mathcal{H}_i$, characterized by the inner products

$$(|\alpha_j^i\rangle_i, |\alpha_k^i\rangle_i) \equiv {}_i\langle\alpha_j^i|\alpha_k^i\rangle_i \equiv e^{-\frac{1}{2}[E_j^i+E_k^i-\,2\gamma_{jk}^i\sqrt{E_j^iE_k^i}]} \tag{4.10}$$

where

$$E_j^i = \int_{\tau_{i-1}}^{\tau_i} d\tau |S_j(\tau)|^2 \quad , \quad E_k^i = \int_{\tau_{i-1}}^{\tau_i} d\tau |S_k(\tau)|^2 \tag{4.11}$$

$$\gamma_{jk}^i \sqrt{E_j^i E_k^i} = \int_{\tau_{i-1}}^{\tau_i} d\tau \; S_j^*(\tau) \; S_k(\tau) \quad , \quad |\gamma_{jk}^i| \leq 1$$

From (4.11) we see that the signal energy E_j^i in the ith sub-interval is proportional to Δ, for small Δ. Specifically, defining for any fixed time $\tau\varepsilon[0,T]$ the index $i(\Delta)$ of the subinterval to which it belongs, $\tau\varepsilon[\tau_{i(\Delta)-1}, \tau_{i(\Delta)}]$, we can write

$$\begin{aligned} E_j^{i(\Delta)} &= |S_j(\tau)|^2\Delta + o(\Delta) \\ \gamma_{jk}^{i(\Delta)} \sqrt{E_j^{i(\Delta)} \; E_k^{i(\Delta)}} &= S_j^*(\tau) \; S_k(\tau)\Delta + o(\Delta) \end{aligned} \tag{4.12}$$

Each of the M possible signals $\varepsilon_j(\cdot)$ is regarded as corresponding to one of the tensor product coherent states

$$|\alpha_j\rangle = |\alpha_j^1\rangle_1 |\alpha_j^2\rangle_2 \cdots |\alpha_j^N\rangle_N \tag{4.13}$$

in the M^N-dimensional tensor product $\mathcal{H}$ of the N Hilbert spaces $\mathcal{H}_i$. The inner products between the possible signal states in the tensor product space $\mathcal{H}$ are

$$(|\alpha_j\rangle, |\alpha_k\rangle) \equiv \langle\alpha_j|\alpha_k\rangle \equiv \prod_{i=1}^{N} {}_i\langle\alpha_j^i|\alpha_k^i\rangle_i \tag{4.14}$$

Using (4.10)and (4.11) to calculate the rightmost expression in (4.14) we obtain the right side of (4.4). This verifies that the two formulations are equivalent.

Arbitrary quantum measurements are represented as before, and the average cost associated with a measurement $\{Q_\beta\}$ and estimator $\hat{j}(\beta)$ is given by

$$\overline{C} = \sum_{j=1}^{M} \sum_{\beta \varepsilon B} \xi_j \langle \alpha_j | Q_\beta | \alpha_j \rangle \; C[j, \hat{j}(\beta)] \tag{4.15}$$

where $\{Q_\beta\}$ are arbitrary nonnegative definite operators <u>on the product space H</u> summing to the identity I_H.

4.3 A SINGLE SHORT-INTERVAL MEASUREMENT

The subdivision of the signaling interval lends a useful conceptual interpretation to receiver behavior. Any receiver can be regarded as extracting information from each of the N Δ-intervals, either separately or simultaneously. Even though receiver strategy is certainly dependent on the interrelations among the information contained in all the Δ-intervals, it is instructive to try to determine just how much information can be obtained from a single Δ-interval measurement. Therefore in this section we shall examine a case in which the entire signaling interval is small, with $[0,T]$ replaced by $[T-\Delta,T]$, $\Delta\downarrow 0$. Not much information can be expected from observations on this interval, so achievable performance will not differ much from the a priori guess performance. It will be shown that, whenever there is a uniquely optimum a priori guess, the improvement over guess performance achieved by the best quantum measurement can be asymptotically matched by that of the optimum feedback receiver in the limit as $\Delta\downarrow 0$. In the next section this result will be extended to obtain a correspondence theorem for a sequence of such small-interval measurements.

The quantum communication model for the small signaling interval problem has the system in one of the M coherent states $|\alpha_j^N>_N$, $j=1,\ldots,M$, specified by (4.10) with $i=N$. Let $\{Q^N_{\beta,\Delta}\}$ represent the optimum quantum measurement and $\hat{j}^*(\beta)$ the optimum decision function, and denote by $\overline{C}^*_\Delta$ the corresponding average cost

$$\overline{C}^*_\Delta = \sum_{j=1}^{M} \sum_{\beta\varepsilon B} \xi_j \; {}_N\langle\alpha_j^N|Q^N_{\beta,\Delta}|\alpha_j^N\rangle_N \quad C[j,\hat{j}^*(\beta)] \tag{4.16}$$

Now let $\overline{C}^\ell_\Delta$ denote the average cost achieved when a feedback receiver with (data-independent) feedback) $\ell(\vec{r},\tau)$ is used in place of the optimum quantum measurement, in conjunction with an estimator function $\hat{j}(n)$ of the number n of photodetection events observed during $[T-\Delta,T]$. Conditioned on the jth signal, n is a Poisson random variable with mean $\int_{T-\Delta}^{T} dt|\varepsilon_j(t)+\ell(t)|^2$, so

$$\overline{C}^\ell_\Delta = \sum_{j=1}^{M} \sum_{n=0}^{\infty} \frac{\xi_j}{n!}\left[\int_{T-\Delta}^{T} dt|\varepsilon_j(t)+\ell(t)|^2\right]^n \exp\left(-\int_{T-\Delta}^{T} dt|\varepsilon_j(t)+\ell(t)|^2\right) \; C[j,\hat{j}(n)] \tag{4.17}$$

The following theorem shows that by appropriate choice of ℓ and $\hat{j}$ the achievable feedback receiver cost $\overline{C}^\ell_\Delta$ can be reduced almost to the optimum performance $\overline{C}^*_\Delta$. [Note: the Δ-dependence of $\hat{j}^*$, $\hat{j}$, and ℓ is suppressed in our notation.]

<u>Theorem 4.1.a</u> Suppose there is an unambiguous best a priori guess; i.e., suppose that for some $\hat{j}_o \varepsilon \hat{R}$,

$$\overline{C}_0^* \equiv \sum_{j=1}^{M} \xi_j \; C(j,\hat{j}_o) < \sum_{j=1}^{M} \xi_j \; C(j,\hat{j}) \text{ for all } \hat{j}\varepsilon\hat{R} - \{\hat{j}_o\} \tag{4.18}$$

Define for $\hat{j}\varepsilon\hat{R}$,

$$\eta_{\hat{j}} = \sum_{j=1}^{M} \xi_j |S_j(T^-)|^2 [C(j,\hat{j}_o) - C(j,\hat{j})] - \frac{\left| \sum_{j=1}^{M} \xi_j S_j(T^-)[C(j,\hat{j}_o) - C(j,\hat{j})] \right|^2}{\sum_{j=1}^{M} \xi_j [C(j,\hat{j}_o) - C(j,\hat{j})]} \tag{4.19}$$

and let $\hat{j}_1$ denote the value of $\hat{j}\varepsilon\hat{R}$ which maximizes $\eta_{\hat{j}}$. Then the average cost $\overline{C}_\Delta^*$ achieved by the optimum short-interval quantum measurement is differentiable at $\Delta=0$, with

$$\lim_{\Delta \downarrow 0} \frac{\overline{C}_0 - \overline{C}_\Delta^*}{\Delta} = \eta_{\hat{j}_1} \tag{4.20}$$

Furthermore, for appropriate choice of feedback level $\ell(\vec{r},\tau)$, specifically,

$$-\ell(\vec{r},\tau) = \frac{\sum_{j=1}^{M} \xi_j \; S_j(\tau)\varepsilon_o(\vec{r})[C(j,\hat{j}_o) - C(j,\hat{j}_1)]}{\sum_{j=1}^{M} \xi_j [C(j,\hat{j}_o) - C(j,\hat{j}_1)]} \tag{4.21}$$

this optimum differential cost can be achieved by a feedback receiver; i.e.,

$$\lim_{\Delta \downarrow 0} \frac{\overline{C}_0^* - \overline{C}_\Delta^\ell}{\Delta} = \eta_{\hat{j}_1} \tag{4.22}$$

Proof: See Appendix (Section 4A.2)

When two or more a priori guesses are equally good, the optimum cost $\overline{C}_\Delta^*$ is no longer differentiable at $\Delta=0$. This peculiarity has been noted in several of the symmetric quantum communication problems (e.g., M-ary detection of equally likely orthogonal signals) for which optimum performance expressions can be evaluated.[23,25] In these cases $\overline{C}_0^* - \overline{C}_\Delta^*$ has been found to be proportional to $\sqrt{\Delta}$ rather than Δ, for small Δ. There is no hope of matching such performance with any particular feedback receiver because any feedback level ℓ yields a cost increment which is proportional to Δ, for small Δ. However, it turns out that in all these cases feedback receiver performance can be improved by letting $|\ell|$ be very large, and the infimum of feedback receiver costs, $\inf_\ell \overline{C}_\Delta^\ell$, has the $\sqrt{\Delta}$-dependence.

Part b of Theorem 4.1 confirms these claims. It is shown that both optimal cost increments are proportional to $\sqrt{\Delta}$, and by construction it is demonstrated that the proportionality factors differ by no more than $\frac{1}{e} \approx 0.368$.

Theorem 4.1.b Let $J_o \subseteq \hat{R}$ denote the set of best a priori guesses; i.e., for any $\hat{j}_o \in J_o$,

$$\overline{C}_0^* \equiv \sum_{j=1}^{M} \xi_j \; C(j,\hat{j}_o) < \sum_{j=1}^{M} \xi_j \; C(j,\hat{j}) \text{ for all } \hat{j}\varepsilon \; \hat{R} - J_o \tag{4.23}$$

Define for $\hat{j},\hat{j}'\varepsilon \; J_o$,

$$\eta_{\hat{j}\hat{j}'} \equiv \sum_{j=1}^{M} \xi_j \; S_j(T^-)[C(j,\hat{j}')-C(j,\hat{j})] \tag{4.24}$$

and let $(\hat{j}_o,\hat{j}_1)$ denote the pair $(\hat{j},\hat{j}')\varepsilon \; J_o x J_o$ which maximizes $|\eta_{\hat{j}\hat{j}'}|$.

Then

$$|\eta_{\hat{j}_o\hat{j}_1}|\sqrt{\Delta} + o(\sqrt{\Delta}) \leq \overline{C}_0^* - \overline{C}_\Delta^* \leq 2|\eta_{\hat{j}_o\hat{j}_1}|\sqrt{\Delta} + o(\sqrt{\Delta}) \tag{4.25}$$

and

$$\overline{C}_0^* - \inf_{\ell} \overline{C}_\Delta^\ell \geq \frac{2}{e} \; |\eta_{\hat{j}_o\hat{j}_1}|\sqrt{\Delta} + \; o(\sqrt{\Delta}) \tag{4.26}$$

<u>Proof:</u> See Appendix (Section 4A.2)

4.4 CONTINGENT SEQUENCES OF SHORT-INTERVAL QUANTUM MEASUREMENTS

4.4.1 Definition of the Class of Contingent Measurement Sequences

In this section we consider an extension of the previous result to time intervals which are not vanishingly small. We have demonstrated in Theorem 4.1 that the measurement accomplished by photodetecting the received field as modified by the addition of an adjustable feedback level can achieve near-optimal performance for small signaling intervals. Because our proof of that theorem relied heavily on the vanishingly small energy of the coherent states, we have not been able to directly extend it to larger intervals. However, since we know that the correspondence established in Theorem 4.1 is applicable to each of the N Δ-intervals which comprise [0,T], it is natural for us to investigate the sense in which these individual results can be combined.

The first thing we notice is that the correspondence is between feedback receiver measurements and general quantum measurements performed on each Δ-interval separately. Thus we might logically expect to be able to approximate the performance of the optimum sequence of such measurements with a feedback receiver. However, the measurements class consisting of <u>specified</u> sequences of small Δ-interval measurements is not very general, and so the result would not be noteworthy. Fortunately, we can identify a much broader realizable class of measurements by considering the degree of flexibility in choice of feedback level that is available to us during each

Δ-interval. For each set of point process data that could be observed in prior intervals an independent choice of feedback level is permitted. This observation leads us to consider contingent sequences of small Δ-interval quantum measurements, each chosen as a function of prior measurement outcomes.

Specifically, letting β_i denote the outcome of the measurement on the ith Δ-interval, we consider measurement sequences $\{Q^i_\beta(\underline{\beta}_{i-1})\}_{\beta\varepsilon B}$, $i=1,\ldots,N$, where $\underline{\beta}_{i-1} \equiv [\beta_1,\ldots,\beta_{i-1}]$ and $\underline{\beta}_0 \equiv \emptyset$. We assume for simplicity that the set B of measurement outcomes is finite and identical for all Δ-intervals, and that it is independent of the size of Δ. [The last two conditions can always be met by re-naming the data, provided there is a finite bound on the number of possible outcomes as Δ is varied.]

The ith measurement is performed just on the received field during the ith Δ-interval, and so for each $\underline{\beta}_{i-1} \varepsilon\ B^{i-1} \equiv \underbrace{BxBx\ldots\ldots xB}_{(i-1)\text{ times}}$, $\{Q^i_\beta(\underline{\beta}_{i-1})\}_{\beta\varepsilon B}$ are nonnegative definite operators on $\mathcal{H}_i$ summing to the identity $I_{\mathcal{H}_i}$. The probability of obtaining outcome β during the ith Δ-interval, when the signal is represented by the coherent state $|\alpha_j\rangle = |\alpha^1_j\rangle,\ldots|\alpha^N_j\rangle_N$ and $\underline{\beta}_{i-1}$ is the vector of prior measurement outcomes, is given by

$$\Pr\left(\beta \,\Big|\, |\alpha_j\rangle,\ \underline{\beta}_{i-1}\right) = {}_i\langle\alpha^i_j|Q^i_\beta(\underline{\beta}_{i-1})|\alpha^i_j\rangle_i \qquad (4.27)$$

The [0,T] interval estimate is calculated on the basis of the sequence of measurement outcomes $\underline{\beta}_N$, which occurs with conditional probability

$$\Pr(\underline{\beta}_N \,\big|\, |\alpha_j\rangle) = \prod_{i=1}^{N} {}_i\langle\alpha_j^i|Q_{\beta_i}^i(\underline{\beta}_{i-1})|\alpha_j^i\rangle_i \qquad (4.28)$$

Comparing (4.28) with (4.15), we see that the class of measurements we are considering are those that can be represented in the tensor product Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$ as

$\{Q_{\beta_1}^1(\emptyset) \otimes \cdots \otimes Q_{\beta_N}^N(\underline{\beta}_{N-1})\}_{\underline{\beta}_N \varepsilon B^N}$.

The average cost which results from using the contingent measurement sequence $\{Q_\beta^i(\underline{\beta}_{i-1})\}$ in conjunction with an estimator $\hat{j}(\underline{\beta}_N)$ is

$$\overline{C} = \sum_{j=1}^{M} \sum_{\underline{\beta}_N \varepsilon B^N} \xi_j \prod_{i=1}^{N} {}_i\langle\alpha_j^i|Q_{\beta_i}^i(\underline{\beta}_{i-1})|\alpha_j^i\rangle_i \quad C[j,\hat{j}(\underline{\beta}_N)] \qquad (4.29)$$

We hasten to point out that this measurement class is not completely general either. For instance, in Section 4.5 we look at a wider class of contingent sequential measurements obtained by lifting the fixed time-ordering constraint on the N Δ-intervals. The larger question of what types of <u>[0,T] interval</u> measurements can be realized by contingent sequences of Δ-interval measurements is a current research topic in quantum communication theory (see, for example, Chan[19]) and one that we do not consider.

4.4.2 Determination of the Optimum Contingent Measurement Sequence

We remark that the correspondence between feedback receiver measurements and general time-ordered contingent sequences does not follow trivially from Theorem 4.1, because from (4.29) the measurement objective during all Δ-intervals except the last is not to minimize an average cost such as (4.16) but rather to optimize the a priori conditions presented to the receiver at the start of the next Δ-interval so as to achieve minimum cost at the fixed end time T. A dynamic programming approach to the minimization of (4.29), analogous to the one used for analyzing feedback receivers, is appropriate. The vector $\underline{\xi}(\underline{\beta}_{i-1})$ of a posteriori probabilities (with M components indexed by j) given all measurement outcomes during the first (i-1) Δ-intervals summarizes all the information in the prior data bearing on the measurement choices for the ith and following Δ-intervals and on the evaluation of the estimate $\hat{j}$. Therefore we may re-write the minimization problem (4.29) in state-dependent form as

$$\overline{C} = \sum_{j} \sum_{\underline{\beta}_N} \xi_j \prod_{i=1}^{N} {}_i\langle\alpha_j^i|Q^i_{\beta_i}(\underline{\xi}(\underline{\beta}_{i-1}))|\alpha_j^i\rangle_i \quad C[j,\hat{j}(\underline{\xi}(\underline{\beta}_N))] \tag{4.30}$$

The a posteriori probabilities are evaluated from (4.28) as

$$\xi_j(\underline{\beta}_m) = \frac{\xi_j \prod_{i=1}^{m} {}_i\langle\alpha_j^i|Q_{\beta_i}^i(\underline{\xi}(\underline{\beta}_{i-1}))|\alpha_j^i\rangle_i}{\sum_k \xi_k \prod_{i=1}^{m} {}_i\langle\alpha_k^i|Q_{\beta_i}^i(\underline{\xi}(\underline{\beta}_{i-1}))|\alpha_k^i\rangle_i} \tag{4.31}$$

Letting $\overline{C}(\underline{\xi},\tau_i)$ denote the average cost-to-go from time $\tau_i = i\Delta$ starting in probability state $\underline{\xi}$, we obtain the following lemma on the determination of the optimum measurement sequence.

<u>Lemma 4.1</u>

a. Let $\overline{C}(\underline{\xi},\tau_i)$ be the average cost-to-go achieved by an arbitrary state-dependent contingent measurement sequence, and assume that $\overline{C}(\underline{\xi},\tau_i)$ is everywhere differentiable with respect to $\underline{\xi}$. Then $\overline{C}(\underline{\xi},\tau_i)$ is determined by the following backward-time difference equation

$$\overline{C}(\underline{\xi},\tau_{i-1}) = E_\beta\, \overline{C}[\underline{\xi}^i(\beta;\underline{\xi}),\ \tau_i] \tag{4.32a}$$

$$= \overline{C}(\underline{\xi},\tau_i) + \sum_\beta \sum_j \xi_j\ {}_i\langle\alpha_j^i|Q_\beta^i(\underline{\xi})|\alpha_j^i\rangle_i\ \overline{C}''[\underline{\xi}^i(\beta;\underline{\xi}),\underline{\xi},\tau_i] \tag{4.32b}$$

where

$$\overline{C}''[\underline{\rho},\underline{\xi},\tau_i] \equiv \overline{C}(\underline{\rho},\tau_i) - \overline{C}(\underline{\xi},\tau_i) - (\underline{\rho}-\underline{\xi})^T \frac{\partial \overline{C}(\underline{\xi},\tau_i)}{\partial \underline{\xi}} \tag{4.33}$$

and $\underline{\xi}^i(\beta;\underline{\xi})$ is the a posteriori probability vector after measuring outcome β during the ith interval given initial probabilities $\underline{\xi}$,

$$\xi_j^i(\beta;\underline{\xi}) = \frac{\xi_j \; {}_i\langle\alpha_j^i|Q_\beta^i(\underline{\xi})|\alpha_j^i\rangle_i}{\sum_k \xi_k \; {}_i\langle\alpha_k^i|Q_\beta^i(\underline{\xi})|\alpha_k^i\rangle_i} \tag{4.34}$$

b. The optimum contingent measurement sequence is constructed backward in time by minimizing for each i the second term on the right side of (4.32b) over $\{Q_\beta^i(\underline{\xi})\}$ in terms of the already computed optimum cost-to-go $\overline{C}(\underline{\xi},\tau_i) = \overline{C}^*(\underline{\xi},\tau_i)$.

Proof: See Appendix (Section 4A.3)

4.4.3 Comparison with Optimum Feedback Receiver Performance

While the cost propagation equation and optimality condition contained in Lemma 4.1 appear very similar to the corresponding ones, (3.32) and (3.36), for the feedback function, we have not yet exploited the small Δ assumption. Lemma 4.1 is applicable to contingent sequential measurements on arbitrary subdivisions of [0,T] into subintervals $[\tau_{i-1},\tau_i]$, with τ_i not necessarily equal to $i\Delta$, as long as it is understood that $|\alpha_j^i\rangle_i$ refers to the coherent-state representation of the received field during the ith subinterval under hypothesis H_j and that the total number of possible states and measurement outcomes is finite.

When Δ is small, some of the measurement outcomes β will cause only a small change in the probability vector $\underline{\xi}^i(\beta;\underline{\xi})$, as did the feedback receiver data "0 events in $[\tau_{i-1},\tau_i]$", and these outcomes are overwhelmingly probable. If $|\underline{\xi}^i(\beta;\underline{\xi})-\underline{\xi}|\sim\Delta$ then by the Taylor theorem $\overline{C}''[\underline{\xi}^i(\beta;\underline{\xi}),\underline{\xi},\tau_i] = o(\Delta)$ and such β can be ignored in an $o(\Delta)$ approximation of the expression (4.32b). [Unfortunately it can also happen that $|\underline{\xi}^i(\beta;\underline{\xi}) - \underline{\xi}|\sim\sqrt{\Delta}$ and, as in Theorem 4.1, this case can cause difficulty, but we circumvent it by assuming that the magnitude of the second derivative of $\overline{C}(\underline{\xi},\tau_i)$ is not too large.] The remaining measurement outcomes cause an abrupt change in the probability vector, analogous to that due to feedback receiver data "1 event in $[\tau_{i-1},\tau_i]$", but these occur with small probability proportional to Δ and their contribution to the cost increment in (4.32b) can be matched within $o(\Delta)$ by a feedback receiver. We then conclude by iterating the difference equation (4.32b) that the average cost achieved by the optimum feedback receiver over the interval $[0,T]$ differs from that of the optimum contingent measurement sequence by $\frac{o(\Delta)}{\Delta}$, which goes to zero as the measurement subintervals become arbitrarily small.

This result is the subject of the following theorem. In it we make several simplifying assumptions which are required in our proof but which are probably not all necessary for the correspondence to be valid.

<u>Theorem 4.2</u> We assume the following conditions on the communication problem considered in this chapter, for an arbitrary size Δ of the measurement subintervals:

(i) the signals have been displaced so that one of them, say j=1, is in the zero-energy state, $|\alpha_1^1\rangle_1 \ldots |\alpha_1^N\rangle_N = |0\rangle_1 \ldots |0\rangle_N$

(ii) the optimum contingent measurement sequence $\{Q^i_{\beta,\Delta}(\underline{\xi})\}_{\beta\varepsilon B}$ exists for all reachable states $\underline{\xi}$ (i.e., for $\underline{\xi}\varepsilon P_{\tau_i}$ with $\Pr[\underline{\xi}(\underline{\beta}_{i-1})\varepsilon P_{\tau_i}]=1$) and achieves average cost-to-go $\overline{C}^*_\Delta(\underline{\xi},\tau_i)$.

(iii) the ground-state matrix elements of the individual optimum measurement operators, ${}_i\langle 0|Q^i_{\beta,\Delta}(\underline{\xi})|0\rangle_i$, are differentiable with respect to Δ(at $\Delta=0$) for all reachable states $\underline{\xi}$.

(iv) the optimum feedback function $\ell(t,\underline{\xi})$ exists for all reachable states $\underline{\xi}$ and achieves average cost-to-go $\overline{C}^\ell(\underline{\xi},\tau)$

(v) for all reachable states $\underline{\xi}$, $\overline{C}^*_\Delta(\underline{\xi},\tau_i)$ is twice differentiable with respect to $\underline{\xi}$, and

$$2\sum_{j=1}^{M}\sum_{k=1}^{M} \xi_j\xi_k \left|S_j(\tau_{i-1})-\hat{S}(\tau_{i-1})\right|\cdot\left|\frac{\partial^2\overline{C}^*_\Delta(\underline{\xi},\tau_i)}{\partial\xi_j\partial\xi_k}\right|\cdot\left|S_k(\tau_{i-1})-\hat{S}(\tau_{i-1})\right|$$

$$\leq \int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau_{i-1},\underline{\xi})\left|\overline{C}^{*\prime\prime}_\Delta[\underline{\rho}(\vec{r},\tau_{i-1},\underline{\xi}),\ \underline{\xi},\tau_i]\right| \tag{4.35}$$

where $\hat{\lambda}(t,\underline{\xi})$ and $\underline{\rho}(t,\underline{\xi})$ are evaluated from (3.17) and (3.18) with feedback $\ell(t,\underline{\xi})$ specified by (iv) and

$$\hat{S}(\tau) \equiv \sum_{j=1}^{M} \xi_j\ S_j(\tau) \tag{4.36}$$

then

$$\lim_{\Delta\downarrow 0}\ \overline{C}^*_\Delta(\underline{\xi}_o,0) = \overline{C}^\ell(\underline{\xi}_o,0) \tag{4.37}$$

The reason for the first assumption is to establish a handy reference signal; by Lemma 4A.1 in the Appendix, it is not at all restrictive. The existence conditions (ii) and (iv) can probably be weakened to include cases in which the optimal cost can only be arbitrarily closely approximated, as in Theorem 3.3. The main purpose of the third assumption is to insure that the index β always corresponds to the same physical measurement outcome as Δ is varied. Otherwise, even though the measurements corresponding to operator <u>sets</u> $\{Q^i_{\beta,\Delta}(\underline{\xi})\}_{\beta\varepsilon B}$, $\{Q^i_{\beta,\Delta'}(\underline{\xi})\}_{\beta\varepsilon B}$ might be nearly identical for Δ' close to Δ, a permutation of the indices β may be required for the same to be true of the individual operators $Q^i_{\beta,\Delta}(\underline{\xi})$, $Q^i_{\beta,\Delta'}(\underline{\xi})$. The differentiability assumption is somewhat stronger than necessary to guarantee a consistent permutation, and we think it should be possible, as was done in Theorem 4.1, to prove a similar theorem for which this property follows as a consequence from seemingly weaker assumptions. The condition on the magnitude of the second derivative is somewhat analogous to the question of the existence of an unambiguous best a priori guess for the infinitesimal interval problem in Theorem 4.1, because the second derivative of the a priori guess performance is either zero or infinite, the latter occurring at points where two or more co-optimum estimates exist. We do not know whether there is an analog to Theorem 4.1b which establishes a similar measurement correspondence for the interval problem when the second derivative is too large.

4.5 SCRAMBLED-TIME CONTINGENT MEASUREMENT SEQUENCES

In the last section we demonstrated that, at least under the conditions assumed in Theorem 4.2, a feedback receiver can extract as much information affecting an interval estimate from any arbitrarily short time-sample of the received field as can general quantum measurements, as long as the available strategies for performing sequences of such measurements are equally flexible. A feedback receiver performs a particular quantum measurement on each small time-sample of the field determined by the feedback level at that time, which is chosen as a function of the outcomes of previous measurements <u>on the preceding time-samples</u>. Our definition of the contingent quantum measurement sequence in the last section afforded it the same freedom to base its measurement at any time on the results of measurements on preceding time-samples. In quantum communication theory the received field from 0 to T is usually regarded as being enclosed in a large lossless cavity, and a completely general measurement is performed at a single instant of time on the entire field within the cavity [2]. Such a measurement may or may not be representable as a contingent sequence of measurements performed on separate "parts" of the field within the cavity corresponding to the received field at the detector during small subintervals of time $[\tau_{i-1},\tau_i]$. In the general measurement perspective, the time-ordering of these cavity field "parts" is irrelevant and there is no reason to limit our attention to sequences of measurements performed consecutively on $[\tau_0,\tau_1]$, $[\tau_1,\tau_2],\ldots,[\tau_{N-1},\tau_N]$. For instance, it might be possible to design a better measurement sequence performed successively on parts of the cavity field corresponding to the time-reversed sequence of aperture field samples during the intervals $[\tau_{N-1},\tau_N],\ldots,[\tau_0, \tau_1]$.

Thus it is natural to consider generalizing the class of contingent measurement sequences to include sequences for which the order of measurement of the N time-samples of the field is not pre-determined, but rather is chosen along with the measurement as a function of prior data. Theorem 4.3 in this section demonstrates that the optimum performance of this class of generalized contingent quantum measurement sequences can be matched (under conditions analogous to the ones in Theorem 4.2) by correspondingly generalized feedback receivers which are likewise not constrained to measure the received field in the natural time order.

As in the last section, measurements are performed on the N Δ-intervals separately. The first measurement $\{Q^1_\beta(\emptyset)\}_{\beta\varepsilon B}$, based on no prior observations, is made on an arbitrarily chosen $i_1(\emptyset)$th subinterval, not necessarily the first. The next measurement, $\{Q^2_\beta(\beta_1)\}_{\beta\varepsilon B}$, and the Δ-interval on which it is performed, $i_2(\beta_1)$, are selected on the basis of the outcome β_1 of the first measurement. The construction of the measurement sequence is continued in this manner until all N Δ-intervals have been examined. There are always two control parameters, the next measured subinterval, $i_m(\underline{\beta}_{m-1})$, and the next measurement, $\{Q^m_\beta(\underline{\beta}_{m-1})\}_{\beta\epsilon B}$, to be chosen at each stage of the sequential measurement as a function of the outcomes $\underline{\beta}_{m-1} = (\beta_1,\ldots,\beta_{m-1})$ of prior measurements.

The conditional probability of obtaining the sequence of outcomes $\underline{\beta}_N$ when measurements $\{Q^m_\beta(\underline{\beta}_{m-1})\}$ are performed on the sequence of Δ-intervals $\{i_1(\emptyset),\ i_2(\beta_1),\ldots,i_N(\underline{\beta}_{N-1})\}$ is given by

$$\Pr\,(\underline{\beta}_N \mid |\alpha_j>\,)$$
$$= \prod_{m=1}^{N} {}_{i_m(\underline{\beta}_{m-1})}< \alpha_j^{i_m(\underline{\beta}_{m-1})} \left| Q^m_{\beta_m}(\underline{\beta}_{m-1}) \right| \alpha_j^{i_m(\underline{\beta}_{m-1})} >_{i_m(\underline{\beta}_{m-1})} \tag{4.38}$$

and the average cost for the interval communication problem is

$$\bar{C} = \sum_{j=1}^{M} \sum_{\underline{\beta}_N \epsilon B^N} \xi_j \Pr(\underline{\beta}_N \mid |\alpha_j>)\ C[j,\hat{j}(\underline{\beta}_N)] \tag{4.39}$$

We use a dynamic programming approach to the minimization of the average cost in (4.39). For this problem the a posteriori probability vector $\underline{\xi}$ does not summarize all the pertinent information gained from the previously chosen measurements, because it is also necessary to know which subintervals have already been measured. It is not difficult to show that a sufficient state is obtained by coupling $\underline{\xi}$ with the set $J \subseteq \{1,\ldots,N\}$ of Δ-intervals which remain unmeasured at the current time.

The updating of the state due to the mth measurement $\{Q_\beta^m(\underline{\xi},J)\}$ on the $i_m(\underline{\xi},J)$th Δ-interval ($i_m(\underline{\xi},J) \in J$) is determined by

$$\xi_j^m(\beta;\underline{\xi},J) = \frac{\xi_j \;\; {}_{i_m(\underline{\xi},J)}\langle \alpha_j^{i_m(\underline{\xi},J)} \left| Q_\beta^m(\underline{\xi},J) \right| \alpha_j^{i_m(\underline{\xi},J)} \rangle_{i_m(\underline{\xi},J)}}{\sum_k \xi_k \;\; {}_{i_m(\underline{\xi},J)}\langle \alpha_k^{i_m(\underline{\xi},J)} \left| Q_\beta^m(\underline{\xi},J) \right| \alpha_k^{i_m(\underline{\xi},J)} \rangle_{i_m(\underline{\xi},J)}} \tag{4.40}$$

$$J_m(\beta;\underline{\xi},J) = J - \{i_m(\underline{\xi},J)\} \tag{4.41}$$

The average cost-to-go $\overline{C}(\underline{\xi},J,m)$ starting in state $(\underline{\xi},J)$ after the first m measurements propagates backward according to the difference equation

$$\begin{aligned} \overline{C}(\underline{\xi},J,m-1) = {} & \overline{C}(\underline{\xi},J - \{i_m(\underline{\xi},J)\},\, m) \\ & + \sum_\beta \sum_j \xi_j \;\; {}_{i_m(\underline{\xi},J)}\langle \alpha_j^{i_m(\underline{\xi},J)} \left| Q_\beta^m(\underline{\xi},J) \right| \alpha_j^{i_m(\underline{\xi},J)} \rangle_{i_m(\underline{\xi},J)} \\ & \cdot \overline{C}''[\underline{\xi}^m(\beta;\underline{\xi},J),\, \underline{\xi},\, J - \{i_m(\underline{\xi},J)\},m] \end{aligned} \tag{4.42}$$

where $\overline{C}''(\cdot)$ is defined by

$$\overline{C}''(\underline{\rho},\underline{\xi},J,m) = \overline{C}(\underline{\rho},J,m) - \overline{C}(\underline{\xi},J,m) - (\underline{\rho}-\underline{\xi})^T \frac{\partial \overline{C}(\underline{\xi},J,m)}{\partial \underline{\xi}} \tag{4.43}$$

The optimum scrambled-time contingent measurement sequence is determined recursively in the following manner. Given that the state-dependent functions $i_{m+1},\ldots,i_N$, $\{Q_\beta^{m+1}\},\ldots,\{Q_\beta^N\}$, have already been determined, the optimum mth measurement still unmeasured.

$\{Q^m_\beta(\underline{\xi},J)\}_{\beta\varepsilon B}$ is chosen to minimize the second term on the right side of (4.42) for any given value of $i_m(\underline{\xi},J)$. The resulting expression for $\overline{C}(\underline{\xi},J,m-1)$ is then minimized over $i_m(\underline{\xi},J)\ \varepsilon\ J$. We omit a detailed justification for these results because they are exactly analogous to Lemma 4.1.

Corresponding to the class of generalized contingent measurement sequences we propose a class of generalized feedback receivers which also have the ability to scramble the time-order in which they look for photodetection events. Initially an index i_1 is selected along with a feedback field $\ell^1(\vec{r},\tau)$ which is added to the received field over the time interval $\tau\ \varepsilon\ [\tau_{i_1-1}, \tau_{i_1}]$. A photodetection measurement is performed on the combined field, producing point process data $\underline{t}^1\ \varepsilon\ [\tau_{i_1-1}, \tau_{i_1}]^*$. Next an index $i_2(\underline{t}^1)$ and a feedback field $\ell^2(\vec{r},\tau:\underline{t}^1)$ are chosen on the basis of the data $\underline{t}^1$ from the first subinterval, and the sum of the received field and the feedback field is photodetected over the interval $[\tau_{i_2(\underline{t}^1)-1}, \tau_{i_2(\underline{t}^1)}]$. Given the data $(\underline{t}^1, \underline{t}^2,\ldots,\underline{t}^{m-1})$ from the first (m-1) such measurements, an arbitrary feedback field $\ell^m(\vec{r},\tau:\underline{t}^1,\ldots,\underline{t}^{m-1})$ is selected and added to the received field, and the result is photodetected over a Δ-interval, $i_m(\underline{t}^1,\ldots,\underline{t}^{m-1})$, arbitrarily chosen from the ones that are still unmeasured.

The couple $(\underline{\xi},J)$ introduced above is a sufficient state, which is updated due to the mth measurement according to (4.41) and

$$\xi_j^m(\underline{t}^m = \emptyset;\underline{\xi},J) = \xi_j - \Delta \int_\Sigma d\vec{r}\ \hat{\lambda}^m(\vec{r},\underline{\xi},J)[\rho_j^m(r,\underline{\xi},J)-\xi_j] + o(\Delta) \tag{4.44}$$

$$\xi_j^m(\underline{t}^m = t = (\vec{r},\tau);\ \underline{\xi},J) = \rho_j^m(\vec{r},\underline{\xi},J) + \frac{o(\Delta)}{\Delta} \tag{4.45}$$

where

$$\hat{\lambda}^m(\vec{r},\underline{\xi},J) \equiv \sum_{j=1}^{M} \xi_j\ \lambda_j^m(\vec{r},\underline{\xi},J)$$

$$\lambda_j^m(\vec{r},\underline{\xi},J) \equiv \left|\varepsilon_j(\vec{r},\tau_{i_m(\underline{\xi},J)-1}) + \ell^m(\vec{r},\tau_{i_m(\underline{\xi},J)-1},\underline{\xi},J)\right|^2 \tag{4.46}$$

$$\rho_j^m(\vec{r},\underline{\xi},J) \equiv \xi_j\ \frac{\lambda_j^m(\vec{r},\underline{\xi},J)}{\hat{\lambda}^m(\vec{r},\underline{\xi},J)} \tag{4.47}$$

and $\ell^m(\vec{r},\tau,\underline{\xi},J)$ is the state-dependent feedback function for the mth measurement. Update equations for data $\underline{t}^m$ with $\dim(\underline{t}^m) \geq 2$ are not necessary because the probability of these events is $o(\Delta)$.

The average cost-to-go $\bar{C}(\underline{\xi},J,m)$ starting in state $(\underline{\xi},J)$ after the first m measurements propagates backward according to the difference equation

$$\bar{C}(\underline{\xi},J,m-1) = \bar{C}(\underline{\xi},J - \{i_m(\underline{\xi},J)\},\ m) \tag{4.48}$$

$$+\ \Delta \int_\Sigma d\vec{r}\ \hat{\lambda}^m(\vec{r},\underline{\xi},J)\ \bar{C}''[\underline{\rho}^m(\vec{r},\underline{\xi},J),\ \underline{\xi},\ J - \{i_m(\underline{\xi},J)\},\ m] + o(\Delta)$$

where $\bar{C}''(\cdot)$ is defined by (4.43) (assuming that $\bar{C}(\cdot)$ is differentiable with respect to $\underline{\xi}$).

A near-optimum scrambled-time feedback function $\ell^m(t,\underline{\xi},J)$ for the mth measurement is determined recursively by minimizing for any given value of $i_m(\underline{\xi},J)$ the integrand in (4.48). The resulting expression for $\overline{C}(\underline{\xi},J,m-1)$ is minimized over $i_m(\underline{\xi},J)$. The scrambled-time feedback receiver specified in this manner is optimum for the $[0,T]$ interval problem in the limit $\Delta\downarrow 0$.

Because of the similarity in the sequential measurement structures and our earlier results for small-interval measurements, it is not surprising that the following analog of Theorem 4.2 exists for scrambled-time measurement sequences.

<u>Theorem 4.3</u> Assume the conditions in Theorem 4.2; i.e., for an arbitrary size Δ of the measurement subintervals:

(i) $|\alpha_1^1\rangle_1 \ldots |\alpha_1^N\rangle_N \;=\; |0\rangle_1 \ldots |0\rangle_N$

(ii) the optimum contingent measurement sequence $\{Q^m_{\beta,\Delta}(\underline{\xi},J)\}_{\beta\varepsilon B}$ and measurement order $i_m(\underline{\xi},J)$ exist for all reachable states $(\underline{\xi},J)$, achieving average cost-to-go $\overline{C}^*_\Delta(\underline{\xi},J,m)$

(iii) $\left\{{}_{i_m(\underline{\xi},J)}\langle 0|Q_{\beta,\Delta}^{i_m(\underline{\xi},J)}(\underline{\xi},J)|0\rangle_{i_m(\underline{\xi},J)}\right\}_{\beta\varepsilon B}$ are differentiable with respect to Δ (at $\Delta=0$) for all reachable $(\underline{\xi},J)$

(iv) the optimum feedback measurement sequence $\ell^m(t,\underline{\xi},J)$ and measurement order $i_m(\underline{\xi},J)$ exist for all reachable $(\underline{\xi},J)$ and achieve average cost-to-go $\overline{C}^\ell_\Delta(\underline{\xi},J,m)$

(v) for all reachable $(\underline{\xi},J)$, $\overline{C}_{\Delta}^{*}(\underline{\xi},J,m)$ is twice differentiable with respect to $\underline{\xi}$, and

$$2\sum_{j=1}^{M}\sum_{k=1}^{M}\xi_j\xi_k\left|S_j(\tau_{i_m(\underline{\xi},J)-1})-\hat{S}(\tau_{i_m(\underline{\xi},J)-1})\right|\cdot\left|\frac{\partial^2\overline{C}_{\Delta}^{*}(\underline{\xi},J,m)}{\partial\xi_j\partial\xi_k}\right|$$

$$\cdot\left|S_k(\tau_{i_m(\underline{\xi},J)-1})-\hat{S}(\tau_{i_m(\underline{\xi},J)-1})\right|$$

$$\leq \int_{\Sigma} d\vec{r}\ \hat{\lambda}^m(\vec{r},\underline{\xi},J)\ \left|\overline{C}^{*''}[\underline{\rho}^m(\vec{r},\underline{\xi},J),\underline{\xi},m]\right| \tag{4.49}$$

where $\hat{\lambda}^m(\cdot)$ and $\underline{\rho}^m(\cdot)$ are evaluated from (4.46) and (4.47) with feedback $\ell^m(\cdot)$ specified by (iv). Then

$$\lim_{\Delta\downarrow 0}\overline{C}_{\Delta}^{*}(\underline{\xi}_o,0) = \lim_{\Delta\downarrow 0}\overline{C}_{\Delta}^{\ell}(\underline{\xi}_o,0) \tag{4.50}$$

Proof: The proof is identical to the proof of Theorem 4.2, because the extra state-vector component, J, evolves according to (4.41) in a manner independent of the quantum measurement $\{Q_\beta\}$ or feedback level ℓ. An alternative method for proving this theorem is by applying Theorem 4.2 to each of the N! $(N = \frac{T}{\Delta})$ time-ordered measurement sequence problems corresponding to the N! different signal sets obtained by permuting the N small time-samples of the original signals. We can conclude that the measurement equivalence expressed in Theorem 4.2 holds for all permutations, in particular for the optimum one.

Scrambled-time measurement sequences are not discussed elsewhere in the thesis. Clearly, the generalized feedback receiver is even more difficult to implement than ordinary feedback receivers. The purpose of this section is to illustrate that some potentially more general quantum measurements may also be realized with the use of energy measurement devices.

CHAPTER V

INCREMENTAL OPTIMALITY

5.1 INTRODUCTION

In Chapter III our objective was to minimize an average cost functional $\bar{C}$ of the feedback ℓ over a fixed time interval $[0,T]$. This turned out to be an enormously complicated problem. We found that in order to solve any particular communication problem on $[0,T]$ with a priori probabilities $\underline{\xi}_o$ it was necessary to simultaneously solve an associated family of problems on $[\tau,T]$, $0\leq\tau\leq T$, with essentially arbitrary a priori probabilities $\underline{\xi}$. Adding to this generally unwanted burden is the non-explicit form of the backward-time algorithm (3.35), which requires performance evaluation for all these problems in order to generate the optimum feedback function. Such complexities are characteristic of dynamic programming solutions to stochastic control problems.

We have been unsuccessful in finding approaches more explicit than dynamic programming for our particular optimization problem, because of the far-reaching (and nonlinear) manner in which the feedback influences the information content of the point process observations. The worth of an observation at time τ, as it contributes to distinguishing the possible hypotheses according to the given cost criterion, depends not only on the feedback level at time τ but also on the feedback levels at all times in $[0,T]$, past and future. We managed to isolate the effects of past feedback on the value of current data by showing that they could be summarized by knowledge of the a posteriori message probability vector it produces. Future feedback levels influence the effective information content of current observations in a different way. Although for a given current state probability vector the statistics of current data are dependent just on current feedback, an optimum choice of current data statistics cannot be made independently; one must consider its joint statistics with future data. These two observations directly led to the backward-time recursive algorithm for determining the optimum feedback function.

Even though there is no apparent way to obtain a simpler solution to our original optimization problem, the above discussion suggests a sub-optimum approach which results in more explicit and more tractable answers. In this chapter we shall artificially sever the coupling between current data and future feedback by <u>requiring</u> that current data statistics be chosen independently of potential outcomes of future observations. For any given current state probability vector $\underline{\xi}$ at time τ, we will assume that the current feedback level is chosen to minimize an average cost increment functional $\overline{c}(\underline{\xi},\tau)$ of current data only. This assumption leads to an optimality condition in which the feedback optimization and performance evaluation are separated, and which may be solved by the receiver recursively in forward (real) time, in parallel with its updating of the a posteriori probability vector and only for the single current probability vector that is actually realized at τ. There is no need to anticipate all possible current state vectors and pre-calculate an overly extensive feedback function applicable in all contingencies.

Above we tended to make rather loose references to "current data at time τ". Because point process events occur with vanishing probability at any precise instant of time, it is necessary to consider observations in a small nonzero length interval around τ as the current data at τ. This remark is the basis for the more formal definition of our incremental optimality criterion in the following section.

5.2 DEFINITION OF THE INCREMENTAL OPTIMALITY CRITERION

We consider the worth of observations made during the small time interval $[\tau,\tau+\Delta]$ when the a priori probabilities $\underline{\xi}$ at time τ are known. It is postulated that this worth can be measured by the (negative of the) increment to an average cost function of the type used in Chapter II due to observations during $[\tau,\tau+\Delta]$. This average cost can be written as an expected value of a specified cost matrix $C(j,\hat{j})$, $j\varepsilon\{1,\ldots,M\}$, $\hat{j}\varepsilon\hat{R}$ ($\hat{R}$ finite), which denotes the cost of estimating $\hat{j}$ when the signal is j. As discussed in Section 3.1.1, $\hat{j}$ need not be interpreted as an estimate of j.

It is useful in this chapter to allow the cost matrix to vary with the time τ at which it is evaluated and also with the probability state $\underline{\xi}$ in effect at that time. We replace $C(j,\hat{j})$ with $C(j,\hat{j};\underline{\xi},\tau)$and assume that $C(j,j;\underline{\xi},\tau)$ is differentiable with respect to $\underline{\xi}$ and τ for each $j,\hat{j}$. The time dependence is a natural generalization because previously it has been assumed that all costs are determined at the fixed end time T. On the other hand, the somewhat artificial probability state dependence is introduced just to broaden the application of the results of this chapter to some potentially useful incremental optimality performance measures which are more conveniently expressed as a direct function of the distribution of a posteriori probabilities than as an expected cost associated with an estimation objective. An example is the maximum mutual information criterion considered in Section 5.7.1.

The average cost $\overline{C}_0(\underline{\xi},\tau)$ given no observations is the a priori guess performance

$$\begin{aligned}\overline{C}_0(\underline{\xi},\tau) &= \min_{\hat{j}\varepsilon\hat{R}} \quad E_j \; C[j,\hat{j};\underline{\xi},\tau] \\ &= \min_{\hat{j}\varepsilon\hat{R}} \sum_{j=1}^{M} \xi_j \; C[j,j;\underline{\xi},\tau]\end{aligned} \tag{5.1}$$

or, letting $j^*(\underline{\xi},\tau)$ denote a minimizing value of $\hat{j}$ in (5.1),

$$\overline{C}_0(\underline{\xi},\tau) = \sum_{j=1}^{M} \xi_j \; C[j,j^*(\underline{\xi},\tau);\; \underline{\xi},\tau] \tag{5.2}$$

The average cost $\overline{C}_\Delta(\underline{\xi},\tau)$ given observations just during $[\tau,\tau+\Delta]$ is given by an expectation of the cost matrix at time $\tau+\Delta$ over the hypotheses and the point data jointly,

$$\overline{C}_\Delta(\underline{\xi},\tau) = \min_{\hat{j}:[\tau,\tau+\Delta]^*\to\hat{R}} \; E_{j,\underline{t}} \; C[j,\hat{j}(\underline{t});\; \underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\; \tau+\Delta] \tag{5.3}$$

Lemmas 3.1 and 3.2 are directly applicable, implying that the optimal estimator function $\hat{j}(\underline{t})$ depends on $\underline{t}$ just through the updated a posteriori probability vector $\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau)$. Thus

$$\begin{aligned}\overline{C}_\Delta(\underline{\xi},\tau) &= \min_{\hat{j}:P\to\hat{R}} \; E_{j,\underline{t}} \; C[j,\hat{j}(\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau));\; \underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\; \tau+\Delta] \\ &= \min_{\hat{j}:P\to\hat{R}} \; E_{\underline{t}} \sum_{j=1}^{M} \xi_j(\tau+\Delta:\underline{t};\underline{\xi},\tau) \; C[j,\hat{j}(\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau));\; \underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\tau+\Delta] \\ &= E_{\underline{t}} \sum_{j=1}^{M} \xi_j(\tau+\Delta:\underline{t};\underline{\xi},\tau) \; C[j,j^*(\underline{\xi}(\tau+\Delta;\underline{t};\underline{\xi},\tau),\; \tau+\Delta);\; \underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\tau+\Delta]\end{aligned} \tag{5.4}$$

where $j^*(\cdot,\tau+\Delta)$ is defined by (5.1) and (5.2) with τ replaced by $\tau+\Delta$.

As in Chapter III, within $o(\Delta)$ the only probable observations in $[\tau,\tau+\Delta]$ are no events ($\underline{t}=\emptyset$) and one event ($\underline{t}=t_1$ for some $t_1 = (\vec{r}_1,\tau_1) \varepsilon \Sigma x[\tau,\tau+\Delta]$). The first occurs with probability

$$\Pr[\underline{t}=\emptyset] = 1 - \Delta \int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi}) + o(\Delta) \tag{5.5}$$

and the second with probability density

$$\begin{aligned} &\Pr[1 \text{ event in area } \delta A \text{ around } \vec{r}_1, \text{ time } \delta\tau \text{ around } \tau_1] \\ &= \hat{\lambda}(\vec{r}_1,\tau_1,\underline{\xi})\ \delta A\ \delta\tau + o(\delta A\ \delta\tau) \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \hat{\lambda}(t,\underline{\xi}) &\equiv \sum_{j=1}^{M} \xi_j\ \lambda_j(t,\underline{\xi}) \\ \lambda_j(t,\underline{\xi}) &\equiv \Lambda[\varepsilon_j(t),\ \ell(t,\underline{\xi})] \end{aligned} \tag{5.7}$$

and $\ell(t,\underline{\xi}) = \ell(\vec{r},\tau,\underline{\xi})$ is the feedback used at time τ.

When $\underline{t}=\emptyset$ the a posteriori probability vector is updated according to equation (3.6)

$$\underline{\xi}(\tau+\Delta:\underline{t}=\emptyset;\underline{\xi},\tau) = \underline{\xi} - \Delta \int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi})[\underline{\rho}(\vec{r},\tau,\underline{\xi}) - \underline{\xi}] + o(\Delta) \tag{5.8}$$

where

$$\rho_j(t,\underline{\xi}) \equiv \xi_j \frac{\lambda_j(t,\underline{\xi})}{\hat{\lambda}(t,\underline{\xi})} \tag{5.9}$$

When one event occurs the change in the a posteriori probability vector is abrupt, as in (3.8),

$$\underline{\xi}(\tau+\Delta:t_1;\underline{\xi},\tau) = \underline{\rho}(t_1,\underline{\xi}) + \frac{o(\Delta)}{\Delta} \tag{5.10}$$

We assume that Δ is small enough that there is a decision j^* which is simultaneously optimum for $\underline{\xi}(\tau+\Delta:\underline{t}=\emptyset;\underline{\xi},\tau)$, $\tau+\Delta$ and $\underline{\xi},\tau$; i.e.,

$$j^*(\underline{\xi}(\tau+\Delta:\underline{t}=\emptyset;\underline{\xi},\tau),\ \tau+\Delta) = j^*(\underline{\xi},\tau) = j^* \tag{5.11}$$

The proof of Lemma 3.6 together with equation (5.8) guanantees that this is always possible (within $o(\Delta)$ error) for finite estimator range $\hat{R}$ and a cost matrix $C[j,\hat{j};\underline{\xi},\tau]$ which is independent of $\underline{\xi}$ and τ. The extension to differentiable time- and state-dependent cost matrices is not difficult. However, one precautionary remark is in order. As discussed in the proof of Lemma 3.6, when two or more estimates j^* are optimum for $\underline{\xi}$, the right side of (5.11) is not selected arbitrarily from these. Rather, it minimizes within this collection for small enough Δ, the guess performance $\overline{C}_0(\underline{\xi}+\Delta\underline{x},\tau)$ for a priori probabilities deviating minutely from $\underline{\xi}$ along the tangent direction specified by (5.8),

$$\underline{x} \equiv \underline{x}(\underline{\xi},\tau) \equiv -\int_{\Sigma} d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi})\ [\underline{\rho}(\vec{r},\tau,\underline{\xi}) - \underline{\xi}] \tag{5.12}$$

Denoting by $j^*(\underline{\xi},\underline{x},\tau)$ a (still not necessarily unique) decision which is simultaneously optimum for $\underline{\xi}$ and $\underline{\xi}+\Delta\underline{x}$, $\Delta\downarrow 0$, we see that (5.11) is more properly replaced by

$$j^*(\underline{\xi}(\tau+\Delta:\ \underline{t}=\emptyset;\underline{\xi},\tau),\ \tau+\Delta) = j^*(\underline{\xi},\underline{x}(\underline{\xi},\tau),\ \tau) \tag{5.13}$$

Substituting (5.5)-(5.10) into (5.4) and noting (5.13) and (5.2), we find that

$$\begin{aligned}\overline{C}_\Delta(\underline{\xi},\tau) &= \overline{C}_0(\underline{\xi},\tau)\\ &+ \Delta \int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \Big\{ C[j,j^*(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau);\ \underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau] \\ &\qquad - C[j,j^*(\underline{\xi},\underline{x}(\underline{\xi},\tau),\tau);\underline{\xi},\tau]\Big\}\\ &+ \Delta \sum_{j=1}^{M} \xi_j \quad C_\tau[j,j^*(\underline{\xi},\underline{x}(\underline{\xi},\tau),\tau);\ \underline{\xi},\tau]\\ &+ \Delta \sum_{j=1}^{M} \xi_j \sum_{i=1}^{M} x_i(\underline{\xi},\tau)\ C_i[j,j^*(\underline{\xi},\underline{x}(\underline{\xi},\tau),\tau);\ \underline{\xi},\tau] \quad + o(\Delta)\end{aligned} \tag{5.14}$$

where C_τ and C_i are the partial derivatives

$$\begin{aligned} C_\tau[j,\hat{j};\underline{\xi},\tau] &\equiv \frac{\partial}{\partial\tau}\ C[j,\hat{j};\underline{\xi},\tau] \\ C_i[j,\hat{j};\underline{\xi},\tau] &\equiv \frac{\partial}{\partial\xi_i}\ C[j,\hat{j};\underline{\xi},\tau] \end{aligned} \tag{5.15}$$

Letting

$$\overline{c}(\underline{\xi},\tau) \equiv \lim_{\Delta\downarrow 0}\ [\overline{C}_\Delta(\underline{\xi},\tau) - \overline{C}_0(\underline{\xi},\tau)]\ \frac{1}{\Delta}$$

which exists and can be evaluated from (5.14) as a function of $\ell(\cdot,\tau,\underline{\xi})$, we now <u>define</u> the <u>incrementally optimum feedback $\ell^*(\cdot,\tau,\underline{\xi})$ for probability state $\underline{\xi}$ at time τ</u> as the function of $\vec{r}\varepsilon\Sigma$ which minimizes $\overline{c}(\underline{\xi},\tau)$.

5.3 DETERMINATION OF THE INCREMENTALLY OPTIMUM FEEDBACK

We now re-state the evaluation of $\overline{c}(\underline{\xi},\tau)$ in (5.14) as a theorem.

Theorem 5.1 Let $C[j,\hat{j};\underline{\xi},\tau]$, $j\varepsilon\{1,\ldots,M\}$, $\hat{j}\varepsilon\hat{R}$, $\underline{\xi}\varepsilon P$, $\tau\varepsilon[0,T]$, $\hat{R}$ finite, be a time- and state-dependent cost matrix which is everywhere differentiable with respect to $\underline{\xi}$ and τ and which determines for every $\underline{\xi},\tau$ the cost increments $\overline{C}_\Delta(\underline{\xi},\tau)-\overline{C}_0(\underline{\xi},\tau)$ due to observations during $[\tau,\tau+\Delta]$ according to equations (5.1) and (5.3). Then the incrementally optimum feedback function ℓ^* minimizing (5.16) satisfies:

a. For every $\underline{\xi},\tau$ such that the optimum decision $j^*(\underline{\xi},\tau)$ is unique:

$$\overline{c}(\underline{\xi},\tau) = \int d\vec{r}\ \tilde{c}(\underline{\xi},\tau,\vec{r}) + \sum_{j=1}^{M} \xi_j\ C_\tau[j,j^*(\underline{\xi},\tau);\underline{\xi},\tau] \tag{5.17}$$

where

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) \equiv \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \Big\{C[j,j^*(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau);\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau] - C[j,j^*(\underline{\xi},\tau);\underline{\xi},\tau]\Big\}$$
$$- \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \sum_{i=1}^{M} \xi_j (\rho_i(\vec{r},\tau,\underline{\xi})-\xi_i)\ C_i[j,j^*(\underline{\xi},\tau);\underline{\xi},\tau] \tag{5.18}$$

and for (almost) every $\vec{r}\ \varepsilon\ \Sigma$,

$$\ell^*(\vec{r},\tau,\underline{\xi}) \text{ minimizes } \tilde{c}(\underline{\xi},\tau,\vec{r}) \tag{5.19}$$

b. For $\underline{\xi},\tau$ which allow multiple optimum estimates $j^*(\underline{\xi},\underline{x},\tau)$, not equal for all directions $\underline{x}$:

$\ell^*(\cdot,\tau,\underline{\xi}) : \Sigma \to ₵$ minimizes

$$\begin{aligned}&\int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \Big\{C[j,j^*(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau);\ \underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau] \\ &\qquad\qquad - C[j,j^*(\underline{\xi},\underline{x}(\underline{\xi},\tau),\tau);\underline{\xi},\tau]\Big\} \\ &+ \sum_{j=1}^{M} \xi_j\ C_\tau[j,j^*(\underline{\xi},\underline{x}(\underline{\xi},\tau),\tau);\underline{\xi},\tau] \\ &+ \int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \sum_{i=1}^{M} \xi_j(\rho_i(\vec{r},\tau,\underline{\xi})-\xi_i)\ C_i[j,j^*(\underline{\xi},\underline{x}(\underline{\xi},\tau),\tau);\underline{\xi},\tau]\end{aligned} \tag{5.20}$$

Proof: Part (b) is just (5.14) re-stated. The simplification in part (a) is due to the condition $j^*(\underline{\xi},\underline{x},\tau) = j^*(\underline{\xi},\tau)$ for all $\underline{x}$. It follows from (5.14) that $\bar{c}(\underline{\xi},\tau)$ can be written as an integral over $\vec{r}$ of $\tilde{c}(\underline{\xi},\tau,\vec{r})$, which may be minimized pointwise for each $\vec{r}$, plus the second term in (5.17) which is independent of feedback. The pointwise (over $\vec{r}$) optimization procedure fails in part (b) because $j^*(\underline{\xi},\underline{x}(\underline{\xi},\tau),\tau)$ is a functional of $\ell(\cdot,\tau,\underline{\xi})$ over its entire domain Σ.

QED

The conditions (5.19), (5.20) for incrementally optimum feedback are much more explicit than the corresponding conditions (3.35), (3.37) derived in Chapter III. In (5.18) we see most clearly that the right hand side may be trivially evaluated from the given cost matrix $C[\cdot]$ for any mean intensity $\hat{\lambda}(\vec{r},\tau,\underline{\xi})$ and potential a posteriori probability vector $\underline{\rho}(\vec{r},\tau,\underline{\xi})$. The incrementally optimum feedback $\ell^*(\vec{r},\tau,\underline{\xi})$ is chosen to simultaneously adjust $\hat{\lambda}(\vec{r},\tau,\underline{\xi})$ and $\underline{\rho}(\vec{r},\tau,\underline{\xi})$ (according to (5.7), (5.9)) so as to minimize this explicit function of $\hat{\lambda}(\vec{r},\tau,\underline{\xi})$ and $\underline{\rho}(\vec{r},\tau,\underline{\xi})$. In contrast to this, condition (3.37) for the optimum feedback requires adjustment of $\hat{\lambda}(\vec{r},\tau,\underline{\xi})$ and $\underline{\rho}(\vec{r},\tau,\underline{\xi})$ to minimize a non-explicit function of $\underline{\rho}(\vec{r},\tau,\underline{\xi})$, namely $\hat{\lambda}(\vec{r},\tau,\underline{\xi})\overline{C}''[\underline{\rho}(\vec{r},\tau,\underline{\xi}),\underline{\xi},\tau]$, where $\overline{C}''[\cdot]$ is derived from the backward-time recursive performance calculation.

This major simplification provides one reason for considering feedback functions which are incrementally optimum but generally non-optimum according to the criterion used in Chapter III. In the next section we try to indicate the nature of the approximation that underlies our sub-optimum scheme.

5.4 INTERVAL OBSERVATIONS WITH INCREMENTALLY OPTIMUM FEEDBACK

Now we consider the application of our incremental optimality principle to observations over $[0,T]$, with a priori probabilities $\underline{\xi}_0$ at time 0. Even though we have defined the criterion for choosing the incrementally optimum feedback for arbitrary $\underline{\xi},\tau$, the receiver for the $[0,T]$ communication problem need only determine it for one value of $\underline{\xi}$ at each τ, namely $\underline{\xi}(\tau:\underline{t};\underline{\xi}_0,0)$, the a posteriori probability vector at time τ given the actual observed point data $\underline{t}$ prior to τ. Since $\underline{\xi}(\tau:\underline{t};\underline{\xi}_0,0)$ is updated recursively in forward time according to equations (3.6), (3.8) as the data comes in, the receiver knows at the same time the particular $\underline{\xi}$ for which $\overline{c}(\underline{\xi},\tau)$ must be minimized.

It is also possible to assign an interval performance measure in terms of the incremental cost, which can likewise be defined recursively in forward time. For each τ, we let $\overline{C}^{+}(\underline{\xi}_0,\tau)$ denote the average cost determined by the cost matrix $C(j,\hat{j};\underline{\xi},\tau)$ in effect at time τ and observations over the interval $[0,\tau]$. The notation distinguishes $\overline{C}^{+}(\cdot,\tau)$ from the average cost function $\overline{C}(\cdot,\tau)$ introduced in Chapter III, which results from observations over the remaining time interval $[\tau,T]$.

$$\overline{C}^{+}(\underline{\xi}_0,\tau) \equiv \min_{\hat{j}:[0,\tau]^{*}\to\hat{R}} E_{\underline{t}',j}\, C[j,\hat{j}(\underline{t}');\underline{\xi}(\tau:\underline{t}';\underline{\xi}_0,0),\tau],\quad \underline{t}'\in[0,\tau]^{*} \tag{5.21}$$

Using Lemma 3.2 and the definition in (5.1), this can be written

$$\overline{C}^{+}(\underline{\xi}_o,\tau) = E_{\underline{t}'}\ \overline{C}_0[\underline{\xi}(\tau:\underline{t}';\underline{\xi}_o,0),\tau], \qquad \underline{t}'\varepsilon[0,\tau]^{*} \tag{5.22}$$

Similarly, if the observation interval is extended a small amount to $[0,\tau+\Delta]$, we obtain the average cost

$$\begin{aligned}\overline{C}^{+}&(\underline{\xi}_o,\tau+\Delta)\\ &= \min_{\hat{j}:[0,\tau+\Delta]^{*}\to\hat{R}} E_{\underline{t}'}\ E_{\underline{t}|\underline{t}'}\ E_{j|\underline{t}',\underline{t}}\ C[j,\hat{j}(\underline{t}',\underline{t});\underline{\xi}(\tau+\Delta:\underline{t}',\underline{t};\underline{\xi}_o,0],\\ &\qquad\qquad\qquad\qquad \underline{t}'\varepsilon[0,\tau]^{*},\ \underline{t}\varepsilon[\tau,\tau+\Delta]^{*}\\ &= E_{\underline{t}'}\ \overline{C}_{\Delta}[\underline{\xi}(\tau:\underline{t}';\underline{\xi}_o,0),\tau]\end{aligned} \tag{5.23}$$

The second equality in (5.23) follows from equation (5.4), along with some familiar results from Chapter III, namely Lemma 3.2, the relation $\underline{\xi}(\tau+\Delta:\underline{t}',\underline{t};\underline{\xi}_o,0) = \underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi}(\tau:\underline{t}';\underline{\xi}_o,0),\tau)$, and the fact that the conditional statistics of $\underline{t}$ in (5.23) are the same as the unconditional statistics of $\underline{t}$ in (5.4) with a priori probabilities $\underline{\xi}(\tau:\underline{t}';\underline{\xi}_o,0)$. Therefore we have finally

$$\frac{1}{\Delta}[\overline{C}^{+}(\underline{\xi}_o,\tau+\Delta) - \overline{C}^{+}(\underline{\xi}_o,\tau)] = E_{\underline{t}'}\ \frac{1}{\Delta}\left\{\overline{C}_{\Delta}[\underline{\xi}(\tau:\underline{t}';\underline{\xi}_o,0),\tau] - \overline{C}_0[\underline{\xi}(\tau:\underline{t}';\underline{\xi}_o,0),\tau]\right\} \tag{5.24}$$

Taking the limit as $\Delta \downarrow 0$ we can evaluate the forward time derivative of $\overline{C}^+$ as

$$\frac{\partial \overline{C}^+(\underline{\xi}_o,\tau)}{\partial \tau^+} = E_{\underline{t}'} \quad \overline{c}[\underline{\xi}(\tau:\underline{t}';\underline{\xi}_o,0),\tau] \qquad \underline{t}'\varepsilon[0,\tau]^* \tag{5.25}$$

Interchanging the limit and expectation operations is justified by an argument similar to that used in part (II) of the proof of Lemma 3.6, as long as the cost matrix derivatives C_i, C_τ are continuous.

Equation (5.25) offers a useful interpretation of the incremental optimality criterion, which calls for $\overline{c}[\underline{\xi}(\tau:\underline{t}';\underline{\xi}_o,0),\tau]$ to be minimized for each $\underline{t}'$. Thus the incrementally optimum feedback at time τ is chosen to minimize (within $o(\Delta)$) the increment to the <u>interval</u> cost $\overline{C}^+$ due to observations between τ and $\tau+\Delta$, <u>given the prior data</u> $\underline{t}'$. One might be tempted to conclude by induction on a sequence of small Δ-intervals that the incrementally optimum feedback therefore achieves interval optimum cost $\overline{C}^+(\underline{\xi}_o,T)$, but this is not true in general. The cost increment in (5.25) depends on the feedback function in two ways. The state-dependent differential cost $\overline{c}(\underline{\xi},\tau)$ is a function just of the current state-dependent feedback $\ell(\vec{r},\tau,\underline{\xi})$ and is independent of all feedback levels chosen prior to τ. The <u>interval</u> cost increment is obtained by averaging the differential cost $\overline{c}(\underline{\xi},\tau)$ over all possible states $\underline{\xi}$, with the probability measure on $\underline{\xi}$ determined by the feedback levels chosen prior to τ. If the optimum differential cost $\bar{c}(\underline{\xi},\tau)$ were constant over (reachable states) $\underline{\xi}$, then this expectation would be independent of the past feedback and indeed the incrementally optimum feedback would achieve optimum interval cost. To the extent that it varies with $\underline{\xi}$, it is possible that previous feedback levels may have been more wisely chosen had their selection criterion anticipated their effect on the expectation of $\bar{c}[\underline{\xi}(\tau:\underline{t}';\underline{\xi}_0,0),\tau]$.

In general it is not possible to specify a feedback level at the current time τ which simultaneously minimizes the current state-dependent differential cost $\bar{c}(\underline{\xi},\tau)$ and optimizes the probability state statistics for all future times $\tau' > \tau$. The dynamic programming interval optimality condition of Chapter III determines the best compromise by taking into account both current and future effects of the current feedback level. However, we have seen that this method requires pre-determination of an optimum strategy to cover all potential realizations of the observed point process. Under the considerably simpler incremental optimality criterion, it is only necessary to calculate the (incrementally) optimum feedback for the particular point process realization that is actually observed. The usefulness of the incrementally optimum feedback receiver rests upon the validity of a basic approximation, that the feedback at time τ may be selected to globally minimize the state-dependent differential cost increment $\bar{c}(\underline{\xi},\tau)$, without consideration of its effect on the probability state statistics which influence future _interval_ cost increments $E_{\underline{t}}\bar{c}[\underline{\xi}(\tau':\underline{t};\underline{\xi},\tau),\tau']$, $\underline{t} \in [\tau,\tau']^*$, for $\tau'>\tau$.

It is possible that for some $\underline{\xi},\tau$, the incrementally optimum feedback at $\underline{\xi},\tau$ does not exist. This does not represent a dilemma for the interval problem as long as the state $\underline{\xi}$ is not reachable at time τ. This observation prompts us to make the following definition. If for each $\tau \in [0,T]$ $\ell^*(\cdot,\tau,\underline{\xi})$ is incrementally optimum for $\underline{\xi}\in P_\tau \subseteq P$, and $Pr[\underline{\xi}(\tau:\underline{t};\underline{\xi}_0,0) \in P_\tau] = 1$, where the probability measure on $\underline{t}$ is that determined by ℓ^*, then we say that ℓ^* is _incrementally optimum for the [0,T] interval problem with a priori probabilities_ $\underline{\xi}_0$.

5.5 EXAMPLE: MINIMUM ERROR PROBABILITY DETECTION

We now focus attention on an important example in order to illustrate the ideas introduced in this chapter. The communication objective is to determine which signal was sent, with minimum probability of error. The cost matrix for this problem given interval observations is $C(j,\hat{j}) = \begin{cases} 0, & j=\hat{j} \\ 1, & j\neq\hat{j} \end{cases}$, and we consider the incrementally optimum feedback receiver defined by a small interval cost matrix of the same form. Note that this is not the only or even necessarily the best increment cost matrix that can be used for approximating the interval minimum error probability objective. The incrementally optimum feedback reciever specified by this increment cost matrix will work well under the conditions discussed in the last section. In this case the evolving interval cost $\overline{C}^{+}(\underline{\xi}_0,\tau)$ is the probability of error $P_e^{+}(\underline{\xi}_0,\tau)$ if observations beginning at time 0 were to suddenly be terminated (and an immediate decision forced) at time τ instead of the target end time T. If this observation interval is extended a small amount to $[0,\tau+\Delta]$ we can attain a smaller error probability $P_e^{+}(\underline{\xi}_0,\tau+\Delta)$. The incrementally optimum receiver attempts to achieve near-optimum $P_e^{+}(\underline{\xi}_0,\tau+\Delta)$ by minimizing it (withing $o(\Delta)$), not over the entire past and current feedback, but rather just as a function of current feedback on the assumption that past feedback already selected by the same procedure to obtain near-optimum $P_e^{+}(\underline{\xi}_0,\tau)$ will continue to work well.

Example 5.1

Let $\hat{R} = \{1,\ldots,M\}$ and $C(j,\hat{j}:\underline{\xi},\tau) = 1 - \delta_{j\hat{j}}$, where $\delta_{j\hat{j}}$ is the Kronecker delta function. Then (5.18) reduces to

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) = -\hat{\lambda}(\vec{r},\tau,\underline{\xi})\left[\rho_{j^*(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau)}(\vec{r},\tau,\underline{\xi}) - \rho_{j^*(\underline{\xi},\tau)}(\vec{r},\tau,\underline{\xi})\right] \tag{5.26}$$

For notational brevity we will write $j^*(\underline{\xi},\tau) \equiv j_o$ and $j^*(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau) \equiv j_1$, where j_1 (but not j_o) is understood to depend on $\ell(\vec{r},\tau,\underline{\xi})$. These indices are chosen to minimize $\sum_{j=1}^{M} \xi_j \; C[j,\hat{j};\underline{\xi},\tau]$ and $\sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \; C[j,\hat{j};\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau]$; i.e., $\hat{j}=j_o$ maximizes $\xi_{\hat{j}}$ (uniquely if (5.18) is applicable) and $\hat{j}=j_1$ maximizes $\rho_{\hat{j}}(\vec{r},\tau,\underline{\xi})$. Using (5.9) we can re-write (5.26) as

$$\begin{aligned}\tilde{c}(\underline{\xi},\tau,\vec{r}) &= \xi_{j_o} \; \lambda_{j_o}(\vec{r},\tau,\underline{\xi}) - \xi_{j_1} \; \lambda_{j_1}(\vec{r},\tau,\underline{\xi}) \\ &= \xi_{j_o}\left|\varepsilon_{j_o}(\vec{r},\tau) + \ell(\vec{r},\tau,\underline{\xi})\right|^2 - \xi_{j_1}\left|\varepsilon_{j_1}(\vec{r},\tau) + \ell(\vec{r},\tau,\underline{\xi})\right|^2\end{aligned} \tag{5.27}$$

We will minimize (5.27) over $\ell(\vec{r},\tau,\underline{\xi})$ by temporarily treating j_1 as an arbitrary fixed parameter $\hat{j}$ and minimizing for each $\hat{j} \; \varepsilon \hat{R}$ the remaining function of $\ell(\vec{r},\tau,\underline{\xi})$. That function is a (complex) quadratic form (except when $\hat{j}=j_o$) with positive coefficient of the $|\ell(\vec{r},\tau,\underline{\xi})|^2$ term. Thus it has a unique minimum determined by setting the derivatives of (5.27) with respect to the real and imaginary parts of $\ell(\vec{r},\tau,\underline{\xi})$ equal to zero, namely

$$\ell_{\hat{j}}(\vec{r},\tau,\underline{\xi}) = -\frac{\xi_{j_o}\varepsilon_{j_o}(\vec{r},\tau)-\xi_{\hat{j}}\ \varepsilon_{\hat{j}}(\vec{r},\tau)}{\xi_{j_o}-\xi_{\hat{j}}}\ , \quad \hat{j} \neq j_o \tag{5.28}$$

At this point we do not know if any of the solutions in (5.28) are consistent with the constraint that j_1 is a specified function of $\ell(\vec{r},\tau,\underline{\xi})$, so we will test them by the following calculations (in which all function arguments $\vec{r},\tau,\underline{\xi}$ are dropped for brevity). For $\ell_{\hat{j}}$ given by (5.28), $\tilde{c}$ in (5.27) is evaluated as

$$\begin{aligned}\tilde{c}_{\hat{j}} &= \xi_{j_o}\left[\frac{\xi_{\hat{j}}}{\xi_{j_o}-\xi_{\hat{j}}}\right]^2 |\varepsilon_{j_o}-\varepsilon_{\hat{j}}|^2 - \xi_{\hat{j}}\left[\frac{\xi_{j_o}}{\xi_{j_o}-\xi_{\hat{j}}}\right]^2 |\varepsilon_{j_o}-\varepsilon_{\hat{j}}|^2 \\ &= -\frac{\xi_{j_o}\ \xi_{\hat{j}}}{\xi_{j_o}-\xi_{\hat{j}}}\ |\varepsilon_{j_o}-\varepsilon_{\hat{j}}|^2\end{aligned} \tag{5.29}$$

We let $\hat{j}^*$ minimize $\tilde{c}_{\hat{j}}$ and evaluate $\underline{\rho}$ for the feedback $\ell_{\hat{j}^*}$:

$$\hat{\lambda}\rho_j = \xi_j\ \lambda_j = \begin{cases} \xi_{j_o}\xi^2_{\hat{j}*}\ |\varepsilon_{j_o}-\varepsilon_{\hat{j}*}|^2\left[\dfrac{1}{\xi_{j_o}-\xi_{\hat{j}*}}\right]^2, & j=j_o \\ \xi^2_{j_o}\xi_{\hat{j}*}\ |\varepsilon_{j_o}-\varepsilon_{\hat{j}*}|^2\left[\dfrac{1}{\xi_{j_o}-\xi_{\hat{j}*}}\right]^2, & j=\hat{j}^* \\ \xi_j\,|(\xi_{j_o}-\xi_{\hat{j}*})(\varepsilon_j-\varepsilon_{j_o})+\xi_{\hat{j}*}(\varepsilon_{\hat{j}*}-\varepsilon_{j_o})|^2 \cdot \left[\dfrac{1}{\xi_{j_o}-\xi_{\hat{j}*}}\right]^2, & j\neq j_o,\hat{j}^* \end{cases} \tag{5.30}$$

We need to show that $\hat{\lambda}\rho_j$ in (5.30) is maximized by $j=\hat{j}^*$.

Obviously, $\hat{\lambda}\rho_{\hat{j}^*} > \hat{\lambda}\rho_{j_o}$ because $\frac{\hat{\lambda}\rho_{\hat{j}^*}}{\hat{\lambda}\rho_{j_o}} = \xi_{j_o}/\xi_{\hat{j}^*} > 1$.

Comparison with other j is a bit more difficult. From (5.30) we have for $j \neq j_o, \hat{j}^*$,

$$
\begin{aligned}
|\xi_{j_o}-\xi_{\hat{j}^*}|^2\,\hat{\lambda}\rho_j &\leq \xi_j\Big[(\xi_{j_o}-\xi_{\hat{j}^*})^2|\varepsilon_j-\varepsilon_{j_o}|^2 + \xi_{\hat{j}^*}^2|\varepsilon_{\hat{j}^*}-\varepsilon_{j_o}|^2 \\
&\quad + 2(\xi_{j_o}-\xi_{\hat{j}^*})\xi_{\hat{j}^*}|\varepsilon_j-\varepsilon_{j_o}|\cdot|\varepsilon_{\hat{j}^*}-\varepsilon_{j_o}|\Big] \\
&\leq \xi_j|\varepsilon_{\hat{j}^*}-\varepsilon_{j_o}|^2\Big[(\xi_{j_o}-\xi_{\hat{j}^*})\xi_{\hat{j}^*}\frac{\xi_{j_o}-\xi_j}{\xi_j} + \xi_{\hat{j}^*}^2 \\
&\quad + 2\xi_{\hat{j}^*}\sqrt{(\xi_{j_o}-\xi_{\hat{j}^*})(\xi_{j_o}-\xi_j)\xi_{\hat{j}^*}/\xi_j}\Big] \\
&= \xi_{\hat{j}^*}|\varepsilon_{\hat{j}^*}-\varepsilon_{j_o}|^2\Big[\sqrt{(\xi_{j_o}-\xi_{\hat{j}^*})(\xi_{j_o}-\xi_j)} + \sqrt{\xi_{\hat{j}^*}\,\xi_{j_o}}\Big]^2 \\
&\leq \xi_{\hat{j}^*}|\varepsilon_{\hat{j}^*}-\varepsilon_{j_o}|^2\,\xi_{j_o}^2 \qquad (5.31) \\
&= [\xi_{j_o}-\xi_{\hat{j}^*}]^2\,\hat{\lambda}\rho_{\hat{j}^*}\,,
\end{aligned}
$$

where the second inequality results from the assumption that $\hat{j}^*$ minimizes (5.29) and the third is the Schwarz inequality.

Thus we have found that the index j_1 and feedback level $\ell(\vec{r},\tau,\underline{\xi})$ which simultaneously minimize (5.27) when treated as independent variables also satisfy the constraint $j^*(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau)=j_1$. Therefore this solution is optimum. As seen in (5.29) it always results in a strictly negative cost increment $\tilde{c}(\underline{\xi},\tau,\vec{r})$, unless all the possible signals are identical ($\varepsilon_j(\vec{r},\tau) = \varepsilon_{j_o}(\vec{r},\tau)$ for all j with $\xi_j>0$).

We have still not solved condition (5.20) which is applicable whenever two (or more) a priori guesses are both optimum. But in this case the feedback $\ell(\cdot,\tau,\underline{\xi})$ can be chosen to make $\bar{c}(\underline{\xi},\tau)$ arbitrarily large. This is shown as follows. Let j_o,j_1 be two optimum estimates for $\underline{\xi}$ (i.e., $\xi_{j_o}=\xi_{j_1}\geq\xi_j$, all j) not satisfying $\varepsilon_{j_o}(\cdot,\tau) = \varepsilon_{j_1}(\cdot,\tau)$ almost everywhere. [If the signal fields are equal for all co-optimum indices, the corresponding hypotheses can be effectively combined into a single one and the results above for an unambiguous best a priori guess applied to this M'-ary problem, M'<M, for this particular $\underline{\xi},\tau$.] Take $\ell(\vec{r},\tau,\underline{\xi}) = K(\vec{r})[\varepsilon_{j_o}(\vec{r},\tau)-\varepsilon_{j_1}(\vec{r},\tau)]$, and use the (generally sub-optimum) decisions j_o and j_1 in place of $j^*(\underline{\xi},\underline{x}(\underline{\xi};\tau),\tau)$ and $j^*(\underline{\rho}(\vec{r},\underline{\xi},\tau),\tau)$, respectively, in (5.14).
Then if $K(\vec{r})|\varepsilon_{j_o}(\vec{r},\tau)-\varepsilon_{j_1}(\vec{r},\tau)|$ is large compared to $|\varepsilon_j(\vec{r},\tau)|$ for all j,

$$\hat{\lambda}(\vec{r},\tau,\underline{\xi}) \approx K^2(\vec{r})|\varepsilon_{j_o}(\vec{r},\tau)- \varepsilon_{j_1}(\vec{r},\tau)|^2 \tag{5.32}$$

and

$$\rho_j(\vec{r},\tau,\underline{\xi}) \approx \xi_j \left\{1 + \frac{2}{K(\vec{r})} \operatorname{Re}\left[\frac{\varepsilon_j(\vec{r},\tau) - \sum_{i=1}^{M}\xi_i\varepsilon_i(\vec{r},\tau)}{\varepsilon_{j_o}(\vec{r},\tau) - \varepsilon_{j_1}(\vec{r},\tau)}\right]\right\}$$

so from (5.14),

$$\overline{C}_0(\underline{\xi},\tau) - \overline{C}_\Delta(\underline{\xi},\tau) \approx \Delta\int d\vec{r}\; 2K(\vec{r})|\varepsilon_{j_o}(\vec{r},\tau) - \varepsilon_{j_1}(\vec{r},\tau)|^2 + o(\Delta) \tag{5.33}$$

The term proportional to Δ can be made arbitrarily large by increasing $K(\cdot)$. The use of optimal decisions in place of j_o and j_1 can only increase the cost increment in (5.33).

Summarizing our results, for any τ and all $\underline{\xi}$ such that for some j_o, $\xi_{j_o} > \xi_j$ for all $j \neq j_o$, the optimum feedback $\ell^*(\vec{r},\tau,\underline{\xi})$ as a function of $\vec{r}$ is given by

$$\ell^*(\vec{r},\tau,\underline{\xi}) = -\frac{\xi_{j_o}\varepsilon_{j_o}(\vec{r},\tau) - \xi_{j_1(\vec{r})}\varepsilon_{j_1(\vec{r})}(\vec{r},\tau)}{\xi_{j_o} - \xi_{j_1(\vec{r})}} \tag{5.34}$$

where

$$j = j_1(\vec{r}) \text{ maximizes } \frac{\xi_{j_o}\xi_j}{\xi_{j_o}-\xi_j}\left|\varepsilon_{j_o}(\vec{r},\tau) - \varepsilon_j(\vec{r},\tau)\right|^2 \tag{5.35}$$

The optimum cost increment $\overline{c}(\underline{\xi},\tau)$ is given by

$$\overline{c}(\underline{\xi},\tau) = -\int_\Sigma d\vec{r}\; \frac{\xi_{j_o}\xi_{j_1(\vec{r})}}{\xi_{j_o}-\xi_{j_1(\vec{r})}}\left|\varepsilon_{j_o}(\vec{r},\tau) - \varepsilon_{j_1(\vec{r})}(\vec{r},\tau)\right|^2 \tag{5.36}$$

For any $\underline{\xi}$ with two or more simultaneously maximal components, the magnitude of the cost increment can be made arbitrarily large and the incrementally optimum feedback does not exist.

We shall continue the analysis of this example, especially for the binary (M=2) case, in Chapter VI. It is demonstrated there that the incrementally optimum feedback receiver is actually interval optimal under the error probability cost criterion for binary and quasi-binary coherent state problems, but not for the general M-ary problem.

5.6 GENERAL SOLUTION OF THE INCREMENTAL OPTIMALITY CONDITION

In this section we extend the solution technique used in the previous example to obtain a completely general explicit solution of (5.19), (5.20) in Theorem 5.1 for a state- and time-independent cost matrix.

Theorem 5.2

Let $C(j,\hat{j};\underline{\xi},\tau)$, $j \in \{1,\dots,M\}$, $\hat{j}\in\hat{R}$ ($\hat{R}$ finite), be an increment cost matrix which is independent of $\underline{\xi}$ and τ; i.e., $C_{\underline{\xi}}(j,\hat{j};\underline{\xi},\tau) \equiv C_\tau(j,\hat{j};\underline{\xi},\tau) \equiv 0$. For any $\underline{\xi},\tau$ let $J_o(\underline{\xi},\tau)$ denote the set of optimum a priori guesses $j^*(\underline{\xi},\tau)$. For every $\hat{j}\in\hat{R}$, $\hat{j}_o \in J_o(\underline{\xi},\tau)$ define

$$a_{\hat{j}\hat{j}_o}(\tau,\underline{\xi}) \equiv \sum_{j=1}^{M} \xi_j \left[C(j,\hat{j};\underline{\xi},\tau) - C(j,\hat{j}_o,\underline{\xi},\tau)\right] \geq 0$$

$$b_{\hat{j}\hat{j}_o}(\vec{r},\tau,\underline{\xi}) \equiv \sum_{j=1}^{M} \xi_j \; \varepsilon_j(\vec{r},\tau)\left[C(j,\hat{j};\underline{\xi},\tau) - C(j,\hat{j}_o;\underline{\xi},\tau)\right] \qquad (5.37)$$

$$d_{\hat{j}\hat{j}_o}(\vec{r},\tau,\underline{\xi}) \equiv \sum_{j=1}^{M} \xi_j \left|\varepsilon_j(\vec{r},\tau)\right|^2 \left[C(j,\hat{j};\underline{\xi},\tau) - C(j,\hat{j}_o;\underline{\xi},\tau)\right]$$

Let $\eta_{\hat{j}\hat{j}_o}(\underline{\xi},\tau,\vec{r})$ be given by

$$\eta_{\hat{j}\hat{j}_o}(\underline{\xi},\tau,\vec{r}) = \begin{cases} \left[d_{\hat{j}\hat{j}_o}(\vec{r},\tau,\underline{\xi}) - \dfrac{\left|b_{\hat{j}\hat{j}_o}(\vec{r},\tau,\underline{\xi})\right|^2}{a_{\hat{j}\hat{j}_o}(\tau,\underline{\xi})}\right], & \text{if } a_{\hat{j}\hat{j}_o}(\tau,\underline{\xi}) \neq 0 \qquad (5.38a) \\ d_{\hat{j}\hat{j}_o}(\vec{r},\tau,\underline{\xi}), & \text{if } a_{\hat{j}\hat{j}_o}(\tau,\underline{\xi}) = 0 \text{ and } b_{\hat{j}\hat{j}_o}(\cdot,\tau,\underline{\xi})=0 \text{ (a.e)} \qquad (5.38b) \\ -\infty, & \text{otherwise} \qquad (5.38c) \end{cases}$$

Let $(\hat{j},\hat{j}_o) = (j_1,j_o) \in \hat{R} \times J_o(\underline{\xi},\tau)$ minimize $\eta_{\hat{j}\hat{j}_o}(\underline{\xi},\tau,\vec{r})$ in (5.38), as a function of $\vec{r}$.

Then an incrementally optimum feedback $\ell^*(\cdot,\tau,\underline{\xi})$ exists at $\underline{\xi},\tau$ if and only if $\eta_{j_1 j_o}(\underline{\xi},\tau,\vec{r}) > -\infty$ (almost everywhere in Σ); in this case ℓ^* is given by

$$\ell^*(\vec{r},\tau,\underline{\xi}) = \begin{cases} \dfrac{-b_{j_1 j_o}(\vec{r},\tau,\underline{\xi})}{a_{j_1 j_o}(\tau,\underline{\xi})} & \text{, if } a_{j_1 j_o}(\tau,\underline{\xi}) \neq 0 \qquad (5.39a) \\ \text{arbitrary} & \text{, if } a_{j_1 j_o}(\tau,\underline{\xi}) = 0 \qquad (5.39b) \end{cases}$$

and the corresponding differential cost increment is

$$\overline{c}(\xi,\tau) = \int_\Sigma d\vec{r}\ \eta_{j_1 j_o}(\underline{\xi},\tau,\vec{r}) \tag{5.40}$$

When $\eta_{j_1 j_o}(\underline{\xi},\tau,\vec{r}) = -\infty$ on a nonzero area, $|\overline{c}(\underline{\xi},\tau)|$ may be made arbitrarily large by taking

$$\ell(\vec{r},\tau,\underline{\xi}) = -K(\vec{r},\tau,\underline{\xi})\, b_{j_1 j_o}(\vec{r},\tau,\underline{\xi}) \tag{5.41}$$

and letting $K(\cdot,\tau,\underline{\xi}) \uparrow \infty$.

Proof: See Appendix (Section 5A.1)

The application of Theorem 5.2 is described as follows. When there is a uniquely optimum a priori guess, $J_o(\underline{\xi},\tau) = \{j_o\}$, then $a_{\hat{j}\hat{j}_o}(\tau,\underline{\xi}) \neq 0$ except for $\hat{j}=\hat{j}_o=j_o$ and the existence of the incrementally optimum feedback is guaranteed (because $b_{j_o j_o}(\cdot,\tau,\underline{\xi}) \equiv 0$). The solution is determined by minimizing (5.38a) over $\hat{j} \neq \hat{j}_o$ and using (5.39a), (5.40) if that minimum is negative or (5.39b), $\overline{c}(\underline{\xi},\tau)=0$ otherwise. When there is no unambiguous best a priori guess, there is usually no incrementally optimum feedback at $\underline{\xi},\tau$, unless the signal fields $\varepsilon_j(\cdot,\tau)$ happen to satisfy $b_{\hat{j}\hat{j}_o}(\cdot,\tau,\underline{\xi})=0$ (almost everywhere in Σ) for all co-optimum decision pairs $(\hat{j},\hat{j}_o)\varepsilon\ J_o(\underline{\xi},\tau)\times J_o(\underline{\xi},\tau)$. In this latter instance, the incrementally optimum feedback exists and is determined by (5.38a), (5.38b), (5.39a), (5.39b), (5.40).

Theorem 5.2 is important because it specifies the conditions for existence of the incremental optimum. Even though an explicit solution is obtained, in applications it is often easier and more meaningful to work with Theorem 5.1 directly. The next two problems are further examples of useful applications of Theorem 5.1.

5.7 OTHER EXAMPLES

5.7.1 Maximum Mutual Information

It was pointed out earlier that it is not necessarily prudent to use an incremental cost matrix identical to the one which specifies the interval average cost whose minimization is our true objective. When this was assumed in Example 5.1, we found that the optimum feedback at any particular time was chosen to maximally separate just two of the M possible hypotheses, the one currently most likely and the one potentially most likely, should an immediate count occur. This choice of feedback might distinguish the other (M-2) hypotheses from these two and from each other very poorly. While this type of information is irrelevant to the incremental error probability criterion, it certainly affects the interval error.

Considerations such as these prompt us to propose an "all-purpose" incremental cost criterion which calls for extraction of maximum information from each succeeding $[\tau,\tau+\Delta]$ interval, defined in a manner independent of the $[0,T]$ interval cost objective. A natural definition is the information-theoretic one. Let $\overline{I}^{+}(\underline{\xi}_o,\tau)$ denote the average mutual information between the possible signals $\{\varepsilon_1(\cdot),\ldots,\varepsilon_M(\cdot)\}$ (with a priori probabilities $\underline{\xi}_o \equiv [\xi_1^o,\ldots,\xi_M^o]^T$) and the point process events $\{\underline{t}\varepsilon[0,\tau]^*\}$ observed during $[0,\tau]$.

$$\bar{I}^+(\underline{\xi}_o,\tau) = E_{\underline{t},j} \log \left[\frac{\xi_j(\tau:\underline{t};\underline{\xi}_o,0)}{\xi_j^o}\right]$$

$$= E_{\underline{t}} \sum_{j=1}^{M} \xi_j(\tau:\underline{t};\underline{\xi}_o,0) \log \xi_j(\tau:\underline{t};\underline{\xi}_o,0) - \sum_{j=1}^{M} \xi_j^o \log \xi_j^o$$

$$= - E_{\underline{t}}\ H_o[\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,0)] + H_o(\underline{\xi}_o) \tag{5.42}$$

where

$$H_o(\underline{\xi}) \equiv - \sum_{j=1}^{M} \xi_j \log \xi_j \tag{5.43}$$

By comparing (5.42) with (5.22) and (5.1), we see that the "cost," $-\bar{I}^+(\underline{\xi}_o,\tau)$, may be represented by an increment cost matrix of the form

$$C(j,\hat{j};\underline{\xi},\tau) = -\log \xi_j \tag{5.44}$$

where the estimator variable $\hat{j}$ is superfluous; i.e., we may take $\hat{R} = \{\hat{j}\}$.

Example 5.2 Let $\hat{R} = \{\hat{j}\}$ and $C(j,\hat{j};\underline{\xi},\tau) = -\log \xi_j$. Then (5.18) reduces to

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) = \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \left[\log \xi_j - \log \rho_j(\vec{r},\tau,\underline{\xi})\right]$$

$$= \sum_{j=1}^{M} \xi_j \lambda_j(\vec{r},\tau,\underline{\xi}) \left[\log \hat{\lambda}(\vec{r},\tau,\underline{\xi}) - \log \lambda_j(\vec{r},\tau,\underline{\xi})\right]$$

$$= \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \log \hat{\lambda}(\vec{r},\tau,\underline{\xi}) - \widehat{\lambda(\vec{r},\tau,\underline{\xi}) \log \lambda(\vec{r},\tau,\underline{\xi})}, \tag{5.45}$$

where

$$\widehat{\lambda \log \lambda} \equiv \sum_{j=1}^{M} \xi_j \; \lambda_j \; \log \lambda_j \qquad (5.46)$$

Because $\hat{R}$ consists of a single element, (5.19) is always applicable, and therefore the incrementally optimum feedback for all $\underline{\xi},\tau$ is determined by minimizing the right hand side of (5.45).

We point out that when the incrementally optimum feedback is calculated in real time for a $[0,T]$ interval problem, (5.45) is minimized at any time just for $\underline{\xi} = \underline{\xi}(\tau:\underline{t};\underline{\xi}_o,0)$, where $\underline{t}$ denotes the prior observations. Thus the operation $\hat{\cdot}$ in (5.46) is interpreted as conditional expectation given prior data $\underline{t}$ just as it was in the Chapter III definition of $\hat{\lambda}$.

5.7.2 Minimum Mean Square Error Estimation

One problem for which the incremental optimality idea arises naturally is the filtering problem. A real random process $y(\tau)$ modulates the received field $\varepsilon(\vec{r},\tau)$ and it is desired to obtain a causal estimate $\hat{y}(\tau:\underline{t})$ based on the prior data $\underline{t}\ \varepsilon[0,\tau]^*$ which minimizes the mean square error,

$$\overline{e}(\tau) = E_{\underline{t},y}[y(\tau) - \hat{y}(\tau:\underline{t})]^2 \qquad \underline{t}\varepsilon[0,\tau]^* \qquad (5.47)$$

As specified above, this problem is not well posed relative to the Chapter III interval optimality criterion. We are trying to simultaneously estimate many random variables, each with its own independent error measure $\bar{e}(\tau)$, $\tau\varepsilon[0,T]$. There is no guarantee that the interval optimum feedback ℓ which minimizes $\bar{e}(\tau)$ will also minimize $\bar{e}(\tau')$ for any $\tau'\neq\tau$. So we need to assume a relative weighting of the error costs, such as $\bar{e} = \int_0^T d\tau\, \bar{e}(\tau)$, in order to determine a single feedback function for the interval $[0,T]$.

Instead of considering this hopelessly complex minimization, we might realistically substitute the incremental optimality principle, which allows us to choose the current feedback $\ell(\cdot,\tau,\cdot)$ to minimize the current error, $\bar{e}(\tau)$, based on prior observations $\underline{t}\varepsilon[0,\tau]^*$, without worrying whether past feedback, chosen previously according to the same rule, might have been selected more judiciously to generate prior data $\underline{t}\varepsilon[0,\tau]^*$ which would be more helpful for the current estimate.

In order to apply Theorem 5.1 or Theorem 5.2 we must restrict the estimater $\hat{y}$ to a finite range $\hat{R}$, and we must approximate the possible signals fields $\varepsilon(t)$ with a finite set $\{\varepsilon_j(t)\}_{j=1}^M$. As in Section 3.1.1 we may replace the mean square error cost function with the average of a discrete cost matrix,

$$\bar{e}(\tau) = E_{y(\tau),\underline{t}}\, [y(\tau) - \hat{y}(\tau:\underline{t})]^2 \tag{5.48}$$

$$= E_{\underline{t}} \sum_{j=1}^{M} \xi_j\, C[j,\hat{y}(\tau:\underline{t})],$$

where

$$\xi_j = \Pr\, [\varepsilon(t) = \varepsilon_j(t)] \tag{5.49a}$$

and

$$C(j,\hat{y}) = E_{y(\tau)|j}\,[y(\tau) - \hat{y}]^2 \tag{5.49b}$$

are determined from the a priori statistics of $y(\tau)$.

Example 5.3 Let the a priori probabilities and cost matrix be as given in (5.49). We assume a finite but closely spaced estimator range $\hat{R}$. In such problems the solution in Theorem 5.2 is not really very explicit because the discrete minimization of (5.38) is just as complicated as the original required minimization over complex numbers in Theorem 5.1. Therefore we will utilize Theorem 5.1 directly. For $\underline{\xi},\tau$ such that (5.19) is applicable, (5.18) reduces to

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) = \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \left[\hat{y}(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau) - \hat{y}(\underline{\xi},\tau)\right] \cdot \left[\hat{y}(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau) + \hat{y}(\underline{\xi},\tau) - 2\, m_j(\tau)\right] \tag{5.50}$$

where

$$m_j(\tau) = E_{y(\tau)|j}\,[y(\tau)] \tag{5.51}$$

Assuming a very closely spaced estimator range, the optimum estimates are very nearly the conditional means,

$$\hat{y}(\underline{\xi},\tau) \approx \sum_{j=1}^{M} \xi_j m_j(\tau)$$

and (5.52)

$$\hat{y}(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau) \approx \sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi})\, m_j(\tau)$$

Thus

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) \approx -\hat{\lambda}(\vec{r},\tau,\underline{\xi}) \left\{ \sum_{j=1}^{M} \left[\rho_j(\vec{r},\tau,\underline{\xi}) - \xi_j\right] m_j(\tau) \right\}^2$$

$$= -\frac{1}{\hat{\lambda}(\vec{r},\tau,\underline{\xi})} \left\{ \sum_{j=1}^{M} \xi_j m_j(\tau) \left[\lambda_j(\vec{r},\tau,\underline{\xi}) - \hat{\lambda}(\vec{r},\tau,\underline{\xi})\right]\right\}^2$$

$$= -\frac{[\widehat{y(\tau)\lambda(\vec{r},\tau,\underline{\xi})} - \widehat{y(\tau)}\,\hat{\lambda}(\vec{r},\tau,\underline{\xi})]^2}{\hat{\lambda}(\vec{r},\tau,\underline{\xi})}, \qquad (5.53)$$

where

$$\widehat{y(\tau)} \equiv \sum_{j=1}^{M} \xi_j m_j(t) \qquad = \sum_{j=1}^{M} \xi_j \; E_{y(\tau)|j}[y(\tau)]$$

and (5.54)

$$\widehat{y(\tau)\lambda(\vec{r},\tau,\underline{\xi})} \equiv \sum_{j=1}^{M} \xi_j m_j(\tau)\; \lambda_j(\vec{r},\tau,\underline{\xi})$$

A necessary conditon for incremental optimality may be obtained by setting the derivatives of the right side of (5.53) with respect to the real and imaginary parts of $\ell(\vec{r},\tau,\underline{\xi})$ equal to zero.

5.8 INCREMENTAL OPTIMALITY CONDITION FOR CONTINUOUS ESTIMATOR RANGE

In the last two examples the discrete estimator range approximation has been somewhat unnatural or inconvenient. There is an analog of Theorem 5.1 for continuous estimators.

Theorem 5.3 Let $\hat{R}$ be a continuous estimator range and $C(j,\hat{a};\underline{\xi},\tau)$ a cost matrix with the property that

$$\overline{C}_0(\underline{\xi},\tau) \equiv \min_{\hat{a}\varepsilon\hat{R}} \sum_{j=1}^{M} \xi_j \; C[j,\hat{a};\underline{\xi},\tau] \tag{5.55}$$

is everywhere differentiable with respect to $\underline{\xi}$ and τ. (This condition cannot usually be satisfied when the estimator range is finite.) Then

$$\overline{c}(\underline{\xi},\tau) = \int d\vec{r} \;\; \tilde{c}(\underline{\xi},\tau,\vec{r}) + \frac{\partial\overline{C}_0(\underline{\xi},\tau)}{\partial\tau} \tag{5.56}$$

where

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) = \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \left[\overline{C}_0(\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau) - \overline{C}_0(\underline{\xi},\tau) - (\rho(\vec{r},\tau,\underline{\xi})-\underline{\xi})^T \frac{\partial\overline{C}_0(\underline{\xi},\tau)}{\partial\underline{\xi}}\right] \tag{5.57}$$

and for $\underline{\xi},\tau$ the incrementally optimum feedback $\ell^*(\vec{r},\tau,\underline{\xi})$ is chosen to minimize $\tilde{c}(\underline{\xi},\tau,\vec{r})$ for (almost) every $\vec{r}\;\varepsilon\;\Sigma$.

Proof: From (5.4) and (5.5)-(5.10),

$$\overline{C}_\Delta(\underline{\xi},\tau) = E_{\underline{t}}\ \overline{C}_0[\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\ \tau+\Delta] \qquad \underline{t}\varepsilon[\tau,\tau+\Delta]^*$$

$$= \left[1-\Delta\int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi})\right]\left[\overline{C}_0(\underline{\xi},\tau)+\Delta\ \frac{\partial\overline{C}_0(\underline{\xi},\tau)}{\partial\tau}\right.$$

$$\left. - \int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi})\left[\rho(\vec{r},\tau,\underline{\xi})-\ \underline{\xi}\right]^T \frac{\partial\overline{C}_0(\underline{\xi},\tau)}{\partial\underline{\xi}}\right]$$

$$+ \Delta\ \int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi})\ \overline{C}_0\left[\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau\right] \ + o(\Delta)$$

$$= \overline{C}_0(\underline{\xi},\tau) + \Delta\ \int_\Sigma d\vec{r}\ \tilde{c}(\underline{\xi},\tau,\vec{r}) + \Delta\ \frac{\partial\overline{C}_0(\underline{\xi},\tau)}{\partial\tau} + o(\Delta) \qquad (5.58)$$

Therefore $\overline{c}(\underline{\xi},\tau) \equiv \lim_{\Delta\downarrow 0}\ [\overline{C}_\Delta(\underline{\xi},\tau)-\overline{C}_0(\underline{\xi},\tau)]\ \frac{1}{\Delta}$ exists and is given by (5.56).

QED

We show briefly how Examples 5.2 and 5.3 could have been analyzed with Theorem 5.3. For the maximum mutual information criterion we take $\hat{R} = P$, the set of M-dimensional probability vectors. The cost matrix is $C(j,\hat{\underline{\xi}}) = -\log\hat{\xi}_j$, $j\varepsilon\{1,\ldots,M\}$, $\hat{\underline{\xi}}\ \varepsilon\ P$. Then for any a priori probability vector $\underline{\xi}$, $\sum_{j=1}^{M}\xi_j\ C(j,\hat{\underline{\xi}})$ is a convex U function of $\hat{\xi}$ and is minimized by $\hat{\underline{\xi}} = \underline{\xi}$. Thus, $\overline{C}_0(\underline{\xi},\tau) = -\sum_{j=1}^{M}\xi_j\ \log\xi_j$, which correctly models the problem in Example 5.2. Using this a priori guess performance in (5.57), we have, since $\frac{\partial\overline{C}_0(\underline{\xi},\tau)}{\partial\xi_j} = -\log\xi_j - 1$,

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) = \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \left[\log \rho_j(\vec{r},\tau,\underline{\xi}) - \log \xi_j\right]$$
$$= \sum_{j=1}^{M} \xi_j \ \lambda_j(\vec{r},\tau,\underline{\xi}) \left[\log \hat{\lambda}(\vec{r},\tau,\underline{\xi}) - \log \lambda_j(\vec{r},\tau,\underline{\xi})\right] \quad (5.59)$$

which is the same as (5.45).

For the estimation problem we know that the MMSE estimate of y(t) is the conditional mean (given the probability state $\underline{\xi}$)

$$\hat{y}(\underline{\xi},\tau) = \sum_{j=1}^{M} \xi_j \ m_j(\tau) \quad (5.60)$$

where $m_j(\tau)$ is defined in (5.51). Thus,

$$\overline{C}_0(\underline{\xi},\tau) = E\ y^2(\tau) - \left[\sum_{j=1}^{M} \xi_j \ m_j(\tau)\right]^2 , \quad (5.61)$$

and

$$\frac{\partial \overline{C}_0(\underline{\xi},\tau)}{\partial \xi_i} = -\ 2\left[\sum_{j=1}^{M} \xi_j \ m_j(\tau)\right] m_i(\tau) \quad (5.62)$$

so (5.57) is evaluated as

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) = -\hat{\lambda}(\vec{r},\tau,\underline{\xi}) \left\{ \left[\sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \ m_j(\tau)\right]^2 - \left[\sum_{j=1}^{M} \xi_j \ m_j(\tau)\right]^2 - 2\left[\sum_{j=1}^{M} \xi_j \ m_j(\tau)\right] \sum_{i=1}^{M} \left[\rho_i(\vec{r},\tau,\underline{\xi}) - \xi_i\right] m_i(\tau) \right\}$$

$$= -\hat{\lambda}(\vec{r},\tau,\underline{\xi}) \left[\sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \ m_j(\tau) - \sum_{j=1}^{M} \xi_j \ m_j(\tau)\right]^2$$

$$= -\frac{1}{\hat{\lambda}(\vec{r},\tau,\underline{\xi})} \left[\sum_{j=1}^{M} \xi_j \lambda_j(\vec{r},\tau,\underline{\xi}) \ m_j(\tau) - \sum_{j=1}^{M} \xi_j m_j(\tau) \ \hat{\lambda}(\vec{r},\tau,\underline{\xi})\right]^2 \quad (5.63)$$

which is the same as (5.53).

5.9 QUANTUM MEASUREMENT CORRESPONDENCE

We close this chapter by showing an equivalence between the incrementally optimum feedback receiver and the incrementally optimum contingent sequence of arbitrary quantum measurements performed on separate infinitesimally short time-samples of the received field, at least for the case of no signal-dependent spatial modulation. Our definition of the incremental optimality criterion for general quantum measurement sequences is analogous to the one presented in Sections 5.2 and 5.4 for feedback receivers. The objective of the measurement $\{Q_\beta^\tau(\underline{\xi})\}_{\beta\varepsilon B}$ chosen at time τ for a priori probability state $\underline{\xi}$ is to maximally enhance the worth of observations beginning at τ and ending at $\tau+\Delta$, $\Delta\downarrow 0$. A contingent measurement sequence $\{Q_\beta^\tau(\underline{\xi})\}_{\beta\varepsilon B}$, $\tau\varepsilon[0,T]$, is said to be incrementally optimum for a $[0,T]$ interval problem if for every τ it satisfies the pointwise optimality condition for all reachable states $\underline{\xi}$. We remark that, while the pointwise incremental optimality criterion specifies a set of measurements operators $\{Q_\beta^\tau(\underline{\xi})\}_{\beta\varepsilon B}$ for each $\tau\varepsilon[0,T]$, interval performance is obtained as a limit of the performance achieved by finite sequences of such measurements, and indeed this is the sense in which the reachability question must be resolved in our definition of incremental optimality for the interval problem. This limiting procedure is more precisely detailed in Theorem 5.4 below.

The cost increment due to a measurement on the small interval $[\tau,\tau+\Delta]$ can be calculated for each $\underline{\xi}$, τ by applying

Theorem 4.1 with $[T-\Delta, T]$ replaced by $[\tau,\tau+\Delta]$ and the coherent states $|\alpha_j^N\rangle_N$ replaced by the coherent states $|\alpha_j^\tau\rangle_\tau$ corresponding to the received field $S_j(\tau)\varepsilon_o(\vec{r})$ over the time interval $[\tau,\tau+\Delta]$. If there is an unambiguous best a priori guess given the probability vector $\underline{\xi}$, then the differential cost increment $\overline{c}(\xi,\tau)$ is finite and can be matched by a feedback receiver. By Theorem 5.2 the only other condition under which the incrementally optimum feedback can exist for $\underline{\xi}$, τ is $\sum_{j=1}^{M} \xi_j S_j(\tau) [C(j,\hat{j}) - C(j,\hat{j}_o)] = 0$ for all co-optimum a priori guesses $\hat{j}$, $\hat{j}_o$. This case was ignored in Theorem 4.1 (it corresponds to a zero coefficient of $\sqrt{\Delta}$ in the cost increment expressions in Theorem 4.1b), but the incrementally optimum measurement correspondence remains valid. Finally, even though Theorem 4.1 only implies that the differential cost <u>increments</u> $\overline{c}(\underline{\xi},\tau)$ are equal as a function of the current state $\underline{\xi}$, the equivalence also applies to the interval costs $\overline{C}^{+}(\underline{\xi}_o,T)$ achieved by the two incrementally optimum measurements, which are specified by averaging $\overline{c}(\underline{\xi},\tau)$ over possible values of $\underline{\xi}$ at time τ, as in (5.25). The following theorem summarizes these observations and concludes the chapter.

<u>Theorem 5.4</u>. Assume that the cost matrix $C(j,\hat{j})$ is time- and state-independent. Assume that the received field under H_j has the form $S_j(\tau)\,\varepsilon_o(\vec{r})$. Assume that the incrementally optimum feedback $\ell(t,\underline{\xi})$ exists for the $[0,T]$ interval problem and achieves interval cost $C^{+\ell}(\underline{\xi}_o,T) = \overline{C}^{\ell}(\underline{\xi}_o,0)$. Then the incrementally optimum quantum measurement sequence $\{Q_\beta^\tau(\underline{\xi})\}_{\beta\varepsilon B}$ exists

[illegible] interval problem. For any [illegible] [illegible] and [illegible] the [illegible]

the average interval cost achieved by the [illegible]

ent sequence [illegible], where [illegible]

measurement is performed in the [illegible]

[illegible] Then

$$\lim_{[illegible]} [illegible]$$

Proof: See Appendix Section (A. [illegible])

We remark that this theorem is free of the [illegible] assumptions that we required in order to prove the [illegible] measurement correspondence theorems in Chapter [illegible].

for the [0,T] interval problem. For any positive integer N, let $\Delta = T/N$ and $\tau_i = i\Delta$, $i = 0,1,\ldots,N$, and let $\overline{C}^*_\Delta(\underline{\xi}_o,0)$ denote the average interval cost achieved by the contingent measurement sequence $\{Q_\beta^{\tau_{i-1}}(\underline{\xi})\}_{\beta\varepsilon B}$, $i = 1,\ldots,N$, where the ith measurement is performed on the received field over the interval $[\tau_{i-1},\tau_i]$. Then

$$\lim_{\Delta\downarrow 0} \overline{C}^*_\Delta(\underline{\xi}_o,0) = \overline{C}^\ell(\underline{\xi}_o,0) \tag{5.64}$$

Proof: See Appendix (Section 5A.2)

We remark that this theorem is free of the unnatural assumptions that we required in order to prove the interval measurement correspondence theorems in Chapter IV.

CHAPTER VI

SOME APPLICATIONS OF THE THEORY

6.1 INTRODUCTION

In this chapter we consider the application of our theoretical results of previous chapters to certain specific communication problems. This is not intended as a compendium of all possible interesting examples because we quickly discover that the analysis is very complicated even for the simplest cases.

The first example we study is the binary coherent state detection problem with an error probability cost criterion. This selection is motivated by Kennedy's[22] demonstration that the performance of the optimum interval quantum measurement for this problem may be approximated within a multiplicative factor of 2 by that of a feedback receiver using data-independent feedback nulling one of the two possible signals. Even for this minimally complex two-state problem we have not been able to construct the interval optimum feedback function directly from the optimality condition of Chapter III, but we have luckily discovered the solution by other means. It is possible to conduct a thorough analysis, including an explicit performance evaluation, for the incrementally optimum feedback receiver derived in Example 5.1, and we find that it achieves exactly the same error probability as the optimum interval quantum measurement. This result implies for this problem two equivalences, between the incrementally optimum and interval optimum feedback receivers and between the optimum interval quantum measurement and the optimum contingent sequence of small time-sample measurements, which are not predicted by the theory of previous chapters. The first equivalence is fortuitous, and we present a three-state counterexample later in the chapter. The question of the generality of the second equivalence is one that we leave to the quantum communication theorists.

In Section 6.3 we examine the two-state problem under other cost criteria. We demonstrate later in the chapter that the minimum error probability feedback function satisfies a necessary condition derived from the Chapter III interval optimality condition for any cost matrix resulting in an a priori guess performance which is a symmetric function of the a priori probabilities. This observation prompts us to evaluate the performance of the minimum error probability receiver under arbitrary symmetric cost criteria. A by-product of this calculation is the discovery that the optimum quantum measurement for the two-state MMSE estimation problem is also realized by this same feedback function. We show that the minimum error probability receiver is generally non-optimum for two-state problems with asymmetric a priori costs, but our counterexample is chosen both to illustrate another instance in which the incrementally optimum and interval optimum receivers are equivalent and to motivate our consideration of the particular classes of M-state problems studied in the next two sections.

In Section 6.4 we consider the M-ary coherent state detection problem with pulse-position-modulated (PPM) signals and error probability cost. We find that the incrementally optimum receiver is generally non-optimum, but we propose a near-optimum receiver (suggested by the behavior of the optimum binary receiver) which approximates the error probability of the optimum quantum measurement within a multiplicative factor of 2.23.

In the final section we analyze the Chapter III interval optimality condition for generalized PPM signal sets, for which an arbitrary binary coding of the M messages into sequences of time-segments of two deterministic fields is allowed. The minimization over complex variables is reduced to one over positive real numbers, and an implicit equation specifying the latter solution is determined as a necessary condition by setting the derivative of $\hat{\lambda}\overline{C}''$ equal to zero. Illustration of the usefulness of this result is provided by applying it to some of the two-state problems analyzed earlier without consideration of the interval optimality condition. The result is also extended to problems for which the optical detector array is subject to dark current.

6.2 THE MINIMUM ERROR PROBABILITY DETECTION PROBLEM WITH BINARY COHERENT SIGNALS

6.2.1 Incrementally Optimum Feedback Function

Let us continue the analysis of Example 5.1 for the case of just two possible signals (M=2). The incrementally optimum (state-dependent) feedback function $\ell^*(\vec{r},\tau,\xi_1,\xi_2)$ exists for all probability vectors $\underline{\xi} = [\xi_1,\xi_2]^T$ with $\xi_1 \neq \xi_2$ and is determined by (5.34) as a linear combination of the signal $\varepsilon_{j_o}(\vec{r},\tau)$ that is currently most probable and the signal $\varepsilon_{j_1(\vec{r})}(\vec{r},\tau)$ that is potentially most probable should an immediate count occur. In the binary case we see from (5.35) that the potentially most probable index $j_1(\vec{r})=j_1$ is always the opposite of the currently most probable index j_o. In other words, whenever $j_o=1$ (i.e., $\xi_1>\xi_2$) then $j_1=2$, and whenever $j_o=2$ $(\xi_2>\xi_1)$ then $j_1=1$. Therefore the incrementally optimum ℓ^* in (5.34) can be written as

$$\ell^*(\vec{r},\tau,\underline{\xi}) = - \frac{\xi_1\varepsilon_1(\vec{r},\tau) - \xi_2\varepsilon_2(\vec{r},\tau)}{\xi_1 - \xi_2} \quad , \quad \xi_1 \neq \xi_2 \tag{6.1}$$

and from (5.36) the optimum cost increment $\overline{c}(\underline{\xi},\tau)$ is

$$\overline{c}(\underline{\xi},\tau) = - \frac{\xi_1\xi_2}{|\xi_1-\xi_2|} \lambda(\tau) \tag{6.2}$$

where we have made the definition

$$\lambda(\tau) \equiv \int_\Sigma d\vec{r} \, |\varepsilon_1(\vec{r},\tau) - \varepsilon_2(\vec{r},\tau)|^2 \tag{6.3}$$

As pointed out in Chapter V, in any implementation of the incrementally optimum feedback receiver, it is only necessary to evaluate $\ell^*(\cdot,\tau,\underline{\xi})$ at one particular probability vector realization, $\underline{\xi} = \underline{\xi}(\tau:\underline{t};\underline{\xi}_o,0)$, given the actual observed data $\underline{t}\varepsilon[0,\tau]^*$. However, in order to obtain the interval performance, we are required (as in (5.25)) to average the cost increment expression in (6.2) over all possible realizations. With this in mind we now devote some attention to the statistics of $\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,0)$. For simplicity we shall suppress the dependence of $\underline{\xi}(\cdot)$ on the a priori conditions $\underline{\xi}_o=[\xi_1^o,\xi_2^o]^T$ and $\tau_o=0$, and write $\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,0) = \underline{\xi}(\tau:\underline{t})$.

6.2.2 Data Statistics Induced by the Incrementally Optimum Feedback

Since $\underline{\xi}$ is a two-dimensional probability vector, it is completely specified by the ratio of its components, $\frac{\xi_1}{\xi_2}$. Equation (3.6) and (3.8) may be applied to calculate the smooth evolution of the a posteriori probability ratio $\frac{\xi_1(\tau:\underline{t})}{\xi_2(\tau:\underline{t})}$ between counts and the abrupt change in probability ratio at an event time.

$$\frac{\partial}{\partial\tau} \log \frac{\xi_1(\tau:\underline{t}_n)}{\xi_2(\tau:\underline{t}_n)} = \int_\Sigma d\vec{r}\left(\lambda_2[\vec{r},\tau,\underline{\xi}(\tau:\underline{t}_n)]-\lambda_1[\vec{r},\tau,\underline{\xi}(\tau:\underline{t}_n)]\right), \tau>\tau_n \tag{6.4}$$

$$\frac{\xi_1(\tau_n^+:\underline{t}_n)}{\xi_2(\tau_n^+:\underline{t}_n)} = \frac{\xi_1(\tau_n:\underline{t}_{n-1})}{\xi_2(\tau_n:\underline{t}_{n-1})} \; \frac{\lambda_1[\vec{r}_n,\tau_n,\underline{\xi}(\tau_n:\underline{t}_{n-1})]}{\lambda_2[\vec{r}_n,\tau_n,\underline{\xi}(\tau_n:\underline{t}_{n-1})]} \tag{6.5}$$

These expressions for the probability ratio are simpler than the corresponding ones for either component because they involve the intensities $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ and not the average intensity $\hat{\lambda}(\cdot) = \xi_1(\cdot)\lambda_1(\cdot) + \xi_2(\cdot)\,\lambda_2(\cdot)$.

From (5.30) we can calculate the intensity difference $\lambda_2(\cdot) - \lambda_1(\cdot)$ and the intensity ratio $\frac{\lambda_1(\cdot)}{\lambda_2(\cdot)}$ when the incrementally optimum feedback specified in (6.1) is used

$$\int_\Sigma d\vec{r}[\lambda_2(\vec{r},\tau,\underline{\xi}) - \lambda_1(\vec{r},\tau,\underline{\xi})] = \lambda(\tau)\,\frac{1}{\xi_1-\xi_2} = \lambda(\tau)\,\frac{\xi_1/\xi_2 + 1}{\xi_1/\xi_2 - 1} \tag{6.6}$$

$$\frac{\lambda_1(\vec{r},\tau,\underline{\xi})}{\lambda_2(\vec{r},\tau,\underline{\xi})} = \left(\xi_2/\xi_1\right)^2 \tag{6.7}$$

Inserting (6.6) and (6.7) into the propagation equations (6.4) and (6.5) for the a posteriori probability ratio, we have

$$\frac{\partial}{\partial\tau}\log\frac{\xi_1(\tau:\underline{t}_n)}{\xi_2(\tau:\underline{t}_n)} = \lambda(\tau)\,\frac{\dfrac{\xi_1(\tau:\underline{t}_n)}{\xi_2(\tau:\underline{t}_n)} + 1}{\dfrac{\xi_1(\tau:\underline{t}_n)}{\xi_2(\tau:\underline{t}_n)} - 1} \qquad \tau>\tau_n \tag{6.8}$$

and

$$\frac{\xi_1(\tau_n^+:\underline{t}_n)}{\xi_2(\tau_n^+:\underline{t}_n)} = \frac{\xi_1(\tau_n:\underline{t}_{n-1})}{\xi_2(\tau_n:\underline{t}_{n-1})}\left[\frac{\xi_2(\tau_n:\underline{t}_{n-1})}{\xi_1(\tau_n:\underline{t}_{n-1})}\right]^2 = \frac{\xi_2(\tau_n:\underline{t}_{n-1})}{\xi_1(\tau_n:\underline{t}_{n-1})} \tag{6.9}$$

From (6.9) we see immediately that the occurrence of a count causes an exact inversion of the a posteriori probability ratio. The magnitude of the logarithm of the ratio, $\left|\log \frac{\xi_1(\cdot)}{\xi_2(\cdot)}\right|$ is unaffected by the occurrence of counts. Defining

$$f(\tau:\underline{t}_n) = \exp \left|\log \frac{\xi_1(\tau:\underline{t}_n)}{\xi_2(\tau:\underline{t}_n)}\right| = \begin{cases} \dfrac{\xi_1(\tau:\underline{t}_n)}{\xi_2(\tau:\underline{t}_n)} & \text{if } \xi_1(\tau:\underline{t}_n) \geq \xi_2(\tau:\underline{t}_n) \\[2ex] \dfrac{\xi_2(\tau:\underline{t}_n)}{\xi_1(\tau:\underline{t}_n)} & \text{if } \xi_2(\tau:\underline{t}_n) \geq \xi_2(\tau:\underline{t}_n) \end{cases} \tag{6.10}$$

we have observed that $f(\cdot)$ is continuous at each event occurrence time; between event times it evolves according to equation (6.8). For $\tau > \tau_n$

$$\frac{\partial}{\partial\tau} \log f(\tau:\underline{t}_n) = \begin{cases} \lambda(\tau) \dfrac{f(\tau:\underline{t}_n)+1}{f(\tau:\underline{t}_n)-1} & , \text{ if } \xi_1(\tau:\underline{t}_n) > \xi_2(\tau:\underline{t}_n) \quad (6.11a) \\[3ex] -\lambda(\tau) \dfrac{\dfrac{1}{f(\tau:\underline{t}_n)} + 1}{\dfrac{1}{f(\tau:\underline{t}_n)} - 1} & , \text{ if } \xi_2(\tau:\underline{t}_n) > \xi_1(\tau:\underline{t}_n) \quad (6.11b) \end{cases}$$

Algebraic manipulation of the expression in (6.11b) reduces it to the one in (6.11a), so in all cases $\frac{\partial}{\partial\tau} \log f(\tau:\underline{t}_n)$ can be evaluated by (6.11a). This is a first-order ordinary differential equation that can be explicitly solved because it is separable.

Noting that $\frac{f-1}{f(f+1)} = \frac{2}{f+1} - \frac{1}{f}$, we obtain the solution

$$\log \frac{[f(\tau:\underline{t}_n)+1]^2}{f(\tau:\underline{t}_n)} = \int_{\tau_n}^{\tau} \lambda(\sigma)d\sigma + \log \frac{[f(\tau_n^+:\underline{t}_n)+1]^2}{f(\tau_n^+:\underline{t}_n)}, \quad \tau>\tau_n \tag{6.12}$$

Because of the continuity of $f(\cdot)$ at event times, the initial condition term on the right side of (6.12) may be evaluated by integrating an equation like (6.11a) between τ_{n-1} and τ_n,

$$\log \frac{[f(\tau_n^+:\underline{t}_n)+1]^2}{f(\tau_n^+:\underline{t}_n)} = \log \frac{[f(\tau_n:\underline{t}_{n-1})+1]^2}{f(\tau_n:\underline{t}_{n-1})}$$

$$= \int_{\tau_{n-1}}^{\tau_n} \lambda(\sigma)d\sigma + \log \frac{[f(\tau_{n-1}^+:\underline{t}_{n-1})+1]^2}{f(\tau_{n-1}^+:\underline{t}_{n-1})} \tag{6.13}$$

Iterating this procedure, we find that

$$\log \frac{[f(\tau:\underline{t}_n)+1]^2}{f(\tau:\underline{t}_n)} = \int_{\tau_n}^{\tau} \lambda(\sigma)d\sigma + \int_{\tau_{n-1}}^{\tau_n} \lambda(\sigma)d\sigma + \ldots + \int_{0}^{\tau_1} \lambda(\sigma)d\sigma$$

$$+ \log \frac{[f(0:\emptyset)+1]^2}{f(0:\emptyset)}$$

$$= \int_0^{\tau} \lambda(\sigma)d\sigma - \log \xi_1^o \xi_2^o \tag{6.14}$$

where the initial condition at $\tau=0$ results from the relation between a priori probabilities,

$$\frac{(\frac{\xi_1}{\xi_2}+1)^2}{\frac{\xi_1}{\xi_2}} = \frac{(\frac{\xi_2}{\xi_1}+1)^2}{\frac{\xi_2}{\xi_1}} = \frac{1}{\xi_1\xi_2} \tag{6.15}$$

6.2.3 Calculation of Interval Optimum Performance for the Incrementally Optimum Feedback Receiver

We observe that

$$\frac{f(\cdot)}{[f(\cdot)+1]^2} = \frac{f(\cdot)}{f(\cdot)+1} \cdot \frac{1}{f(\cdot)+1} = \xi_1(\cdot)\xi_2(\cdot) \tag{6.16}$$

for both cases $f(\cdot) = \dfrac{\xi_1(\cdot)}{\xi_2(\cdot)}$ and $f(\cdot) = \dfrac{\xi_2(\cdot)}{\xi_1(\cdot)}$. Therefore (6.14) reduces to

$$\xi_1(\tau:\underline{t}_n)\xi_2(\tau:\underline{t}_n) = \xi_+(\tau:\underline{t}_n)\xi_-(\tau:\underline{t}_n) = \xi_1^o\xi_2^o \; e^{-E(\tau)} \tag{6.17}$$

where we have made the additional definitions

$$E(\tau) = \int_0^\tau \lambda(\sigma)d\sigma \tag{6.18}$$

and

$$\xi_-(\tau:\underline{t}_n) = \min\,[\xi_1(\tau:\underline{t}_n),\ \xi_2(\tau:\underline{t}_n)] \tag{6.19}$$

$$\xi_+(\tau:\underline{t}_n) = \max\,[\xi_1(\tau:\underline{t}_n),\ \xi_2(\tau:\underline{t}_n)] = 1 - \xi_-(\tau:\underline{t}_n)$$

Solving the quadratic equation (6.17) for $\xi_-(\tau:\underline{t}_n)$, we have

$$\xi_-(\tau:\underline{t}_n) = \frac{1}{2}[1 - \sqrt{1-4\xi_1^o\xi_2^o\ e^{-E(\tau)}}] \tag{6.20}$$

[The second root is excluded by the requirement from (6.19) that $\xi_-(\cdot) \leq \frac{1}{2}$.]

Thus $\xi_-(\tau:\underline{t}_n)$ is non-random, decreasing deterministically with time τ from its a priori value $\frac{1}{2}[1-\sqrt{1-4\xi_1^o\xi_2^o}] = \min(\xi_1^o,\xi_2^o)$ toward zero as the energy in the signal difference, $E(\tau)=\int_0^\tau \lambda(\sigma)d\sigma$, increases. The interval average cost (probability of error) $\overline{C}^+(\underline{\xi}_o,T) \equiv P_e^+(\underline{\xi}_o,T)$ given observations on the interval $[0,T]$ may be evaluated as the (trivial) expectation of $\xi_-(T:\underline{t}_n)$,

$$P_e^+(\underline{\xi}_o,T) = E_{\underline{t}}\ \xi_-(T:\underline{t}) \qquad \underline{t}\varepsilon[0,T]^*$$

$$= \frac{1}{2}\ [1 - \sqrt{1-4\xi_1^o\xi_2^o\ e^{-E(T)}}\ ] \tag{6.21}$$

We can deduce from Theorem 5.4 that this performance matches that of the incrementally optimum contingent sequence of small-interval quantum measurements. Moreover, Helstrom[23] has calculated an explicit expression for the performance of the optimum unrestricted (interval) quantum measurement for the binary coherent state detection problem, and it is equivalent to (6.21). Therefore, the feedback receiver employing feedback ℓ^* given by (6.1) is a realization of the optimum quantum measurement for this problem. Immediately we can infer that the incrementally optimum feedback function ℓ^* must also be the interval optimum feedback because no feedback receiver can out-perform the optimum interval quantum measurement. It is also possible to verify this conclusion independently by applying the optimality condition from Chapter III, as simplified in Section 6.5.

6.2.4 Optimum Receiver Structure and Behavior

We now discuss an (ideal) implementation of our optimum receiver for the binary coherent state detection problem. First we re-write the expression (6.1) for the optimum state-dependent feedback function in event-dependent form, using the relation $\left|\log \frac{\xi_1(\cdot)}{\xi_2(\cdot)}\right| = \log f(\cdot)$, and re-arranging terms for convenience.

$$\ell^*(\vec{r},\tau:\underline{t}_n) = -\frac{1}{2}\;\frac{[\xi_1(\tau:\underline{t}_n)-\xi_2(\tau:\underline{t}_n)][\varepsilon_1(\vec{r},\tau)+\varepsilon_2(\vec{r},\tau)]}{\xi_1(\tau:\underline{t}_n)-\xi_2(\tau:\underline{t}_n)}$$

$$-\frac{1}{2}\;\frac{[\xi_1(\tau:\underline{t}_n)+\xi_2(\tau:\underline{t}_n)][\varepsilon_1(\vec{r},\tau)-\varepsilon_2(\vec{r},\tau)]}{\xi_1(\tau:\underline{t}_n)-\xi_2(\tau:\underline{t}_n)}$$

$$= \begin{cases} \ell_1(\vec{r},\tau) \equiv -\dfrac{\varepsilon_1(\vec{r},\tau)+\varepsilon_2(\vec{r},\tau)}{2} - \dfrac{\varepsilon_1(\vec{r},\tau)-\varepsilon_2(\vec{r},\tau)}{2}\;\dfrac{f(\tau)+1}{f(\tau)-1}, & \text{if } \xi_1(\tau:\underline{t}_n)>\xi_2(\tau:\underline{t}_n) \quad (6.22a) \\[2ex] \ell_2(\vec{r},\tau) \equiv -\dfrac{\varepsilon_1(\vec{r},\tau)+\varepsilon_2(\vec{r},\tau)}{2} + \dfrac{\varepsilon_1(\vec{r},\tau)-\varepsilon_2(\vec{r},\tau)}{2}\;\dfrac{f(\tau)+1}{f(\tau)-1}, & \text{if } \xi_2(\tau:\underline{t}_n)>\xi_1(\tau:\underline{t}_n) \end{cases}$$

(6.22b)

In (6.22) we have dropped the event vector argument from $f(\cdot)$ because we discovered in (6.14) that $f(\tau:\underline{t}_n)$ was a deterministic time function, $f(\tau)$, increasing with τ independently of observed events $\underline{t}_n$.

Next we observe that the sign of $\log \frac{\xi_1(\tau:\underline{t}_n)}{\xi_2(\tau:\underline{t}_n)}$ is preserved as τ varies between event times, because the right side of (6.8) always has the same sign as $\log \frac{\xi_1(\tau:\underline{t}_n)}{\xi_2(\tau:\underline{t}_n)}$. We have already found that at each event time $\log \frac{\xi_1(\tau:\underline{t}_n)}{\xi_2(\tau:\underline{t}_n)}$ changes sign, as the probability ratio is updated to its reciprocal according to (6.9). Therefore we can conclude that the optimum feedback switches back and forth between the two deterministic field envelopes $\ell_1(\vec{r},\tau)$ and $\ell_2(\vec{r},\tau)$ at each event time and never between event times. Initialization at $\tau=0$ is determined by whether $\xi_1^o>\xi_2^o$ or $\xi_2^o>\xi_1^o$.

It is convenient to think of constructing a receiver to implement ℓ^* in two stages, corresponding to the two terms in (6.22). The first term, subtracting the unweighted average signal $\frac{\varepsilon_1+\varepsilon_2}{2}$, has the effect of converting the two possible signals from ε_1 and ε_2 to plus and minus half the difference signal, $\pm \frac{\varepsilon_1-\varepsilon_2}{2}$. This term is data-independent and may be implemented separately ahead of any observations. Alternatively, the transmitter may eliminate the need for this stage by using an antipodal signal set.

The second term in (6.22) shifts the indistinguishable (by a photodetector) antipodal signals $\pm \frac{\varepsilon_1-\varepsilon_2}{2}$ to $(\frac{f+1}{f-1} \pm 1) \frac{\varepsilon_1-\varepsilon_2}{2}$

[or $(\frac{f+1}{f-1} \pm 1)\frac{\varepsilon_2-\varepsilon_1}{2}$]. This positioning allows optimal separation of the binary signals by a photodetector, subject to the constraint that the difference, $\varepsilon_1-\varepsilon_2$, is fixed. It is intuitive that the optimum signal set should have common phase, because the separation in the point process intensity functions can always be improved by matching phases. [If $\lambda_1 \equiv |\varepsilon_1'|^2$ is the smaller intensity, the larger intensity $\lambda_2 \equiv |\varepsilon_2'|^2$ can be increased without changing λ_1 to its maximum value $(|\varepsilon_1'|+|\varepsilon_2'-\varepsilon_1'|^2)$, subject to the constraint of fixed $\varepsilon_2'-\varepsilon_1'$, by taking ε_1 to have the same phase as $\varepsilon_2'-\varepsilon_1'$.] The optimum feedback receiver accomplishes this relative signal positioning because the coefficients $\frac{1}{2}(\frac{f+1}{f-1} \pm 1)$ of the difference signal are both positive.

The optimum signal positioning varies with time through $f(\tau)$ and also with the observation of point events. From (6.22) we can find that the signal producing the larger intensity, $(\frac{f+1}{f-1} + 1)(\pm \frac{\varepsilon_1-\varepsilon_2}{2})$, is always the one that is currently regarded as less probable, and the more probable signal, $(\frac{f+1}{f-1} - 1)(\pm \frac{\varepsilon_1-\varepsilon_2}{2})$, produces the smaller intensity. This explains our earlier observation from (6.8) that the a posteriori probability of the more probable hypothesis always increases in the absence of counts. On the other hand, the occurrence of each photodetection event constitutes negative evidence against the hypothesis considered more probable just prior to the count. In fact, since the a posteriori probability ratio is inverted according to (6.9), the evidence is so negative that each count causes the receiver to change its mind about which hypothesis is more likely and to be exactly as confident in its new guess as it was prior to the decision change.

Since, from (6.22), feedback $\ell_j(\vec{r},\tau)$ is applied whenever H_j is currently considered more probable, ℓ_j can be regarded as a negative testing function for the signal ε_j, in the context of this problem. As time goes on, $f(\tau)$ increases monotonically with $E(\tau)$ from the a priori probability ratio, max $(\xi_1^o/\xi_2^o, \xi_2^o/\xi_1^o)$, at $\tau=0$. Thus, as $E(\tau)$ increases, the feedback signal $\ell_j(\vec{r},\tau)$ tends toward nulling the incoming signal if that signal is $\varepsilon_j(\vec{r},\tau)$ and creating the difference signal $\pm[\varepsilon_1(\vec{r},\tau)-\varepsilon_2(\vec{r},\tau)]$ otherwise. The negative test provided by ℓ_j on ε_j becomes more and more conclusive as the hypothesized intensity nears zero. If ℓ_j were to null ε_j exactly, the occurrence of a single count would completely eliminate H_j from contention as a possible hypothesis. The tendency of ℓ_j toward nulling ε_j is the basic reason that each count can be significant enough to cause an inversion of the a posteriori probability ratio, even when that ratio becomes very large (or very small) as the energy $E(\tau)$ in the signal difference accumulates with time.

This does not represent an unstable situation, however, because as the larger a posteriori probability ξ_+ increases it becomes less likely that the currently available information will ever be refuted, and therefore photodetection events and the corresponding decision changes they cause become less frequent as the receiver zeroes in on determining which signal was actually sent. In fact, as shown in the Appendix (Section 6A.1), there is with probability one a final event in the limiting case of an infinite signaling interval ($T=\infty$), even when the total energy in the signal difference over the infinite interval, $E(\infty)$, is also infinite.

A block diagram of the optimum receiver is given in Figure 6.1. Because the a posteriori probabilities depend so trivially on the observed point data, it is convenient to regard the receiver as making tentative intermediate decisions $j^*(\tau:\underline{t})$, $\underline{t}\varepsilon[0,\tau]^*$, at time τ, based on observations during $[0,\tau]$, even though the only one that matters is the decision $j^*(T:\underline{t})$, $\underline{t}\varepsilon[0,T]^*$, at the target end time T. Initially, $j^*(0:\emptyset) = 1$ if $\xi_1^o>\xi_2^o$ or $j^*(0:\emptyset) = 2$ if $\xi_2^o>\xi_1^o$; after that j^* alternates between 1 and 2 with each count. The decision $j^*(\tau:\underline{t}_n)$ depends not at all on the space-time locations $(\vec{r}_1,\tau_1),\ldots,(\vec{r}_n,\tau_n)$, of the events but only on whether the total number n of events is even or odd. The evolving optimum decision can therefore be implemented by a photon counter attached to a two-state switch which changes position with each count.

It is necessary to generate two local oscillator fields with complex envelopes $-\dfrac{\varepsilon_1(\vec{r},\tau)+\varepsilon_2(\vec{r},\tau)}{2}$ and $\dfrac{\varepsilon_1(\vec{r},\tau)-\varepsilon_2(\vec{r},\tau)}{2}\;\dfrac{f(\tau)+1}{f(\tau)-1}$, where $f(\tau)$ is the (deterministic) solution of (6.14) which is greater than unity.

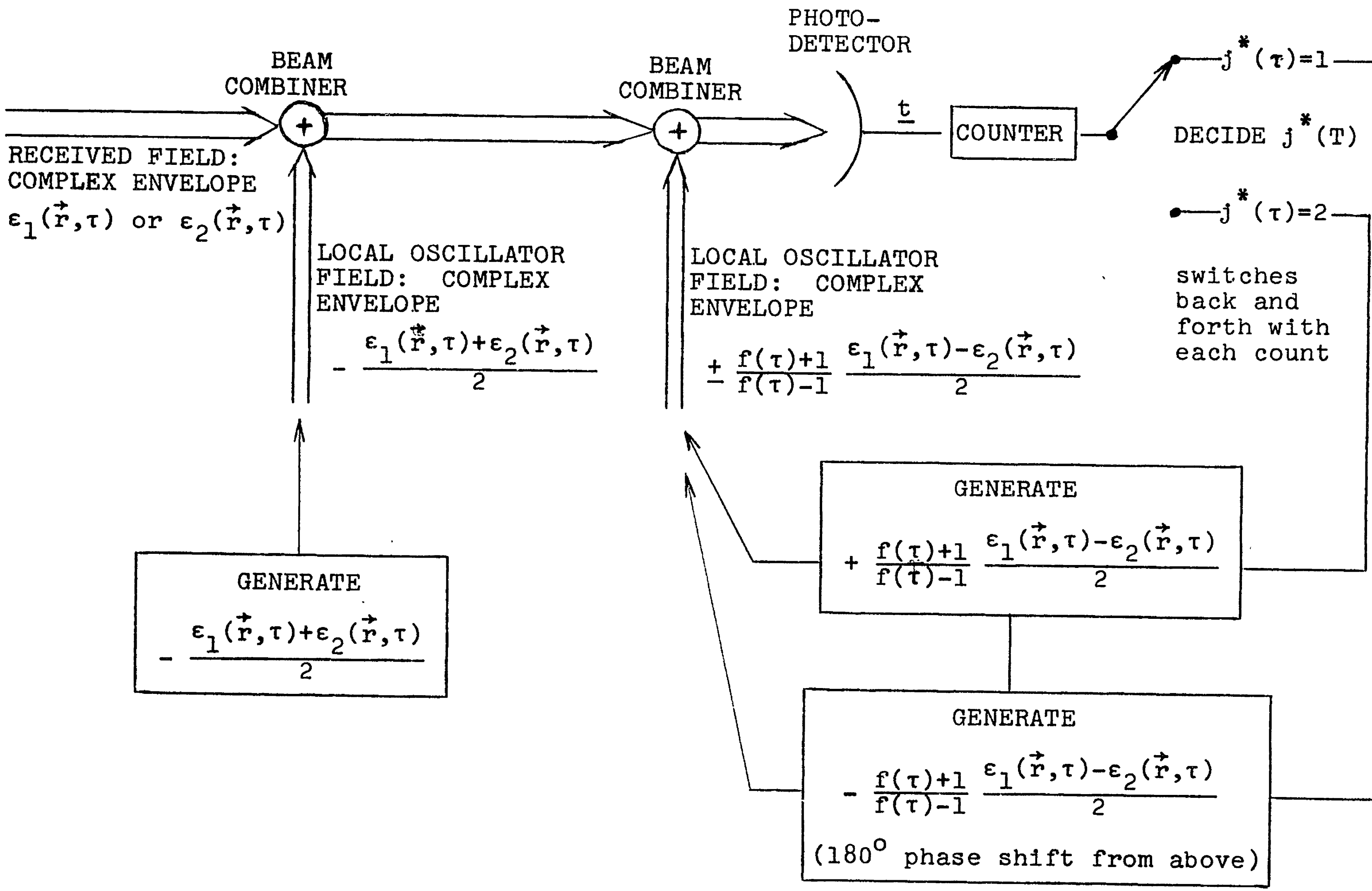

Figure 6.1 Optimum Receiver for the Binary Coherent State Detection Problem

$$\frac{f(\tau)+1}{f(\tau)-1} = \left[1 - 4\xi_1^o\xi_2^o \; e^{-E(\tau)}\right]^{-\frac{1}{2}} \tag{6.23}$$

The first one simply centers the signal set and is added ahead of any observations. The second is either added or subtracted depending on which hypothesis is currently considered more probable. The change of sign occurs with each count in coincidence with the corresponding decision change, and it can be implemented by a 180^o phase shifter controlled by the same switch.

6.2.5 Possible Singularities

In determining the incrementally optimum feedback function in (6.1) we had to make the assumption that the probability vector components were unequal. Indeed, the feedback specified in (6.1) blows up as ξ_1 and ξ_2 approach $\frac{1}{2}$. Fortunately, this singularity causes no essential difficulty for the optimum receiver. The a posteriori probability components are never closer together than are the a priori probabilities, regardless of the data on which they are based. Therefore, as long as the signals are not equally likely a priori, an optimum regular feedback function exists for the interval problem and is given by (6.22).

The case $\xi_1^o=\xi_2^o= \frac{1}{2}$ presents a singularity only at the initial time $\tau=0$. An unbounded initial feedback is called for, because as we saw in Chapter V it is important when the best a priori guess is ambiguous to register an immediate count even though it carries minuscule information differentiating the two possible signals, which are both swamped by the feedback.

For any time Δ, it takes an arbitrarily small amount of distinguishing information during $[0,\Delta]$ to create non-singular a priori conditions for the detection problem on the remaining time interval $[\Delta,T]$. From continuity considerations we would expect that as $\Delta \downarrow 0$ the choice of feedback during $[0,\Delta]$ is relatively unimportant and, as long as the optimum feedback is used after time Δ, optimum performance for the $[0,T]$ problem should be approached.

It is easily shown that this intuitive observation is true. For simplicity take $\ell = -\varepsilon_1$ during $[0,\Delta]$. Then at time Δ the a posteriori probabilities are determined by

$$\left.\begin{aligned} \xi_1(\Delta:\emptyset) &= \frac{1}{1+e^{-E(\Delta)}} \\ \xi_2(\Delta:\emptyset) &= \frac{e^{-E(\Delta)}}{1+e^{-E(\Delta)}} \end{aligned}\right\}$$

$$\left.\begin{aligned} \xi_1(\Delta:\underline{t}_n) &= 0 \\ \xi_2(\Delta:\underline{t}_n) &= 1 \end{aligned}\right\} \quad n \geq 1 \tag{6.24}$$

As long as $E(\Delta) > 0$ the a priori probabilities for the $[\Delta,T]$ problem are always unequal so the optimum feedback receiver can be used after Δ. [If there is no energy in the signal difference until some time $\tau_1 > 0$ (i.e., $\tau_1 = \min\{\tau\varepsilon[0,T]: E(\tau+\Delta) > 0, \forall\Delta>0\}$), the detection problem may be effectively begun at time $\tau=\tau_1$ instead of $\tau=0$.]

The performance of the receiver using feedback $\ell = -\varepsilon_1$ during $[0,\Delta]$ and optimum feedback during $[\Delta,T]$ can be calculated as an expectation of the optimum error probability expression (6.21) as applied to the $[\Delta,T]$ problem over the statistics of the two possible probability vectors given in (6.24),

$$P_e(\Delta) = E_{\underline{t}}\ \frac{1}{2}\left[1-\sqrt{1-4\xi_1(\Delta:\underline{t})\xi_2(\Delta:\underline{t})e^{-[E(T)-E(\Delta)]}}\right] \qquad \underline{t}\varepsilon[0,\Delta]^*$$

$$= \Pr(\underline{t}=\emptyset)\ \frac{1}{2}\left[1-\sqrt{1-4e^{-E(T)}/[1+e^{-E(\Delta)}]^2}\right] + \Pr(\underline{t}\neq\emptyset)\cdot 0$$

$$= \frac{1}{2}\left[1+e^{-E(\Delta)}\right]\frac{1}{2}\left[1-\sqrt{1-4e^{-E(T)}/[1+e^{-E(\Delta)}]^2}\right] \qquad (6.25)$$

Since $\lambda(\cdot) \equiv \int_\Sigma d\vec{r}\,|\varepsilon_1(\vec{r},\cdot) - \varepsilon_2(\vec{r},\cdot)|^2$ is bounded, $\lim_{\Delta\downarrow 0} E(\Delta) = 0$ and therefore

$$\lim_{\Delta\downarrow 0} P_e(\Delta) = \frac{1}{2}\left[1-\sqrt{1-e^{-E(T)}}\right] \qquad (6.26)$$

which is the error probability achieved by the optimum quantum measurement for the case of equiprobable signals.

6.3 THE BINARY COHERENT SIGNAL PROBLEM WITH OTHER COST CRITERIA

6.3.1 Symmetric A Priori Cost Functions

6.3.1a Performance Achieved by the Minimum Error Probability Receiver

Although we have only demonstrated that the feedback function specified by (6.1) or (6.22) is optimum for a minimum error probability cost criterion, it is reasonable to consider its application to other binary coherent signal problems with different cost criteria. It turns out that the performance of this receiver can be evaluated for many useful cost functions as trivially as was the error probability.

To see this we write an arbitrary average cost given observations on [0,T] in terms of the a priori guess performance

$$\overline{C}^{+}(\underline{\xi}_{o},T) \equiv \overline{C}(\underline{\xi}_{o},0) = E_{\underline{t}}\ \overline{C}[\underline{\xi}(T:\underline{t};\underline{\xi}_{o},0),T] \qquad \underline{t}\varepsilon[0,T]^{*} \tag{6.27}$$

For the feedback function which minimizes error probability, the a posteriori probability components at time T are always $\frac{f(T)}{f(T)+1}$ and $\frac{1}{f(T)+1}$; which component is larger is determined by the parity of the number of counts during [0,T]. In many cases the a priori guess performance $\overline{C}(\xi_1,\xi_2,T) \equiv \xi_1 C[1,j^{*}(\xi_1,\xi_2)] + \xi_2 C[2,j^{*}(\xi_1,\xi_2)]$ is a symmetric function of ξ_1 and ξ_2, or equivalently a function of the product $\xi_1\xi_2$

$$\overline{C}(\xi_1,\xi_2,T) = \overline{C}(\xi_2,\xi_1,T) = c(\xi_1\xi_2) \tag{6.28}$$

[Whenever the left equality is satisfied, the function C is determined by

$C(x) = \overline{C}[\frac{1}{2}(1+\sqrt{1-4x}), \frac{1}{2}(1-\sqrt{1-4x}),T] = \overline{C}[\frac{1}{2}(1-\sqrt{1-4x}),\frac{1}{2}(1+\sqrt{1-4x}),T]$, $0 \le x \le \frac{1}{4}$.] For cost functions that satisfy (6.28) the identification of the larger probability component in (6.27) is irrelevant and the expectation is evaluated as

$$\overline{C}^{+}(\underline{\xi}_o,T) = C\left[\frac{f(T)}{(f(T)+1)^2}\right] = C\left[\xi_1^o\xi_2^o \; e^{-E(T)}\right] \qquad (6.29)$$

6.3.1b Example: Optimum Performance for the MMSE Estimation Problem with Binary Coherent Signals

By using the result (6.29) we can demonstrate that the minimum error probability receiver is also the optimum receiver for MMSE estimation of binary coherent signals (i.e., estimation of a binary parameter associated with the signals). For this problem the cost function is

$$\left.\begin{aligned} C(1,\hat{a}) &= (\hat{a}-a_1)^2 \\ C(2,\hat{a}) &= (\hat{a}-a_2)^2 \end{aligned}\right\} a_1 \varepsilon R,\ a_2 \varepsilon R,\ \hat{a}\varepsilon\hat{R}=R \qquad (6.30)$$

and the a priori guess performance is calculated as

$$\overline{C}(\xi_1,\xi_2,T) = \min_{\hat{a}} [\xi_1(\hat{a}-a_1)^2 + \xi_2(\hat{a}-a_2)^2]$$

$$= \xi_1(\xi_1 a_1+\xi_2 a_2-a_1)^2 + \xi_2(\xi_1 a_1+\xi_2 a_2-a_2)^2$$

$$= \xi_1\xi_2 \; (a_2-a_1)^2 \tag{6.31}$$

Therefore the mean square error achieved by the minimum error probability receiver (i.e., a receiver with feedback given by (6.1) and MMSE estimator $\hat{a}(\underline{\xi})=\xi_1 a_1+\xi_2 a_2$) with observations during [0,T] is

$$\overline{C}^+(\underline{\xi}_o,T) = \xi_1^o\xi_2^o \; (a_2-a_1)^2 \; e^{-E(T)} \tag{6.32}$$

Personick[24] has determined the optimum quantum measurement for this problem when the a priori probabilities are equal, and its performance is also given by (6.32) (with $\xi_1^o=\xi_2^o = \frac{1}{2}$).

6.3.2 <u>Arbitrary Cost Functions</u>

6.3.2a <u>Performance Achieved by the Minimum Error Probability Receiver (Generally Non-Optimum)</u>

When the symmetry condition (6.28) is not satisfied it is possible to evaluate the expectation in (6.27) by first computing the probabilities of even and odd numbers of counts during [0,T]. We will calculate the conditional probabilities $P_{ev}^+(\tau)$, $P_{ev}^-(\tau)$, for observing an even number of counts during

$[0,\tau]$ given, respectively, the a priori more probable and less probable hypotheses. From (6.22), under the more probable hypothesis the counting rate is $\left(\frac{f+1}{f-1} - 1\right)^2 \frac{\lambda}{4}$ if the number of counts observed so far is even and $\left(\frac{f+1}{f-1} + 1\right)^2 \frac{\lambda}{4}$ if that number is odd. Conditicned on this hypothesis, the parity of the number of counts is thus a two-state continuous-time Markov process whose state occupation probabilities can be determined from a differential equation,

$$\frac{dP^+_{ev}(\tau)}{d\tau} + \left(\frac{f(\tau)+1}{f(\tau)-1} - 1\right)^2 \frac{\lambda(\tau)}{4} P^+_{ev}(\tau)$$

$$= \left(\frac{f(\tau)+1}{f(\tau)-1} + 1\right)^2 \frac{\lambda(\tau)}{4} \left[1-P^+_{ev}(\tau)\right] \tag{6.33}$$

A particular solution of (6.33) is $\frac{f(\tau)}{f(\tau)+1}$ and the homogeneous solution is proportional to $\frac{f(\tau)}{f(\tau)+1} \frac{1}{f(\tau)-1}$, so the solution that satisfies the initial condition $P^+_{ev}(0)=1$ is

$$P^+_{ev}(\tau) = \frac{f(\tau)}{f(\tau)+1} \left[1 + \frac{f(0)-1}{f(0)} \frac{1}{f(\tau)-1}\right] \tag{6.34}$$

In a similar manner the probability of obtaining an even number of counts under the a priori less likely hypothesis is determined as

$$P^-_{ev}(\tau) = \frac{1}{f(\tau)+1} \left[1 + (f(0)-1) \frac{f(\tau)}{f(\tau)-1}\right] \tag{6.35}$$

Since $1-P^{+}_{ev}(\tau)$ and $P^{-}_{ev}(\tau)$ are interpreted as the conditional error probabilities given the more and less likely hypotheses, respectively, we see that they differ from their expected value, $P^{+}_{e}(\underline{\xi}_o,\tau) = \dfrac{1}{f(\tau)+1}$, by $-\left[\dfrac{f(0)-1}{f(0)}\right] \dfrac{f(\tau)}{f(\tau)+1} \; \dfrac{1}{f(\tau)-1}$ and $+\left[f(0)-1\right] \dfrac{f(\tau)}{f(\tau)+1} \; \dfrac{1}{f(\tau)-1}$ respectively.

Defining $\xi^o_+ = \max(\xi^o_1,\xi^o_2)$ and $\xi^o_- = \min(\xi^o_1,\xi^o_2)$ and noting that $f(0) = \xi^o_+/\xi^o_-$, we can calculate the unconditional probability $P_{ev}(\tau)$ of observing an even number of counts in $[0,\tau]$.

$$P_{ev}(\tau) = \frac{\xi^o_+ f(\tau)+\xi^o_-}{f(\tau)+1} + 2(\xi^o_+-\xi^o_-)\,\frac{f(\tau)}{f(\tau)+1}\;\frac{1}{f(\tau)-1}$$

$$= \frac{\xi^o_+ f(\tau)-\xi^o_-}{f(\tau)-1} \tag{6.36}$$

Also,

$$1 - P_{ev}(\tau) = \frac{\xi^o_- f(\tau)-\xi^o_+}{f(\tau)-1} \tag{6.37}$$

Using (6.36) and (6.37) to evaluate the expectation in (6.27) we arrive at an explicit expression for the average interval cost in terms of the a priori guess performance,

$$\bar{C}^{+}(\underline{\xi}_o,T) = \frac{\xi^o_1 f(T)-\xi^o_2}{f(T)-1}\;\bar{C}\left[\frac{f(T)}{f(T)+1}\,,\;\frac{1}{f(T)+1}\,,\;T\right]$$

$$+\;\frac{\xi^o_2 f(T)-\xi^o_1}{f(T)-1}\;\bar{C}\left[\frac{1}{f(T)+1}\,,\;\frac{f(T)}{f(T)+1}\,,\;T\right] \tag{6.38}$$

6.3.2b Example: Optimum Receiver for a Binary Problem Equivalent to a Trivial M-ary Problem with PPM Signals

We now demonstrate that the minimum error probability receiver is not optimum for all binary coherent state problems by applying the result of (6.38) to the cost matrix

$$\left.\begin{aligned} &C(1,1) = 0 \\ &C(1,2) = C(2,1) = 1 \\ &C(2,2) = \gamma,\ 0<\gamma<1 \end{aligned}\right\} \quad \hat{R} = \{1,2\} \tag{6.39}$$

This example is of limited independent practical utility, but analysis of it motivates our study of the M-ary problems considered in Section 6.4.

The a priori guess performance for the cost matrix (6.39) is

$$\overline{C}\,(\xi_1,\xi_2,T) = \min[\xi_1+\xi_2\gamma,\ \xi_2] = \xi_2\gamma + \min\,[\xi_1,\xi_2(1-\gamma)], \tag{6.40}$$

and the interval cost using the minimum error probability receiver is calculated from (6.38) as

$$\overline{C}^+(\underline{\xi}_o,T) = \begin{cases} \xi_2^o & ,\ \text{if } f(T) \le \dfrac{1}{1-\gamma} \\[2ex] \xi_2^o\gamma + \dfrac{1}{f(T)+1}\left[1 - \gamma\,\dfrac{\xi_1^o\, f(T)-\xi_2^o}{f(T)-1}\right], & \text{if } f(T) \ge \dfrac{1}{1-\gamma} \end{cases} \tag{6.41}$$

The average cost $\overline{C}^{+*}(\underline{\xi}_o,T)$ achieved for this problem by the optimum quantum measurement can be calculated by the same technique used by Helstrom for the minimum error probability,

$$\overline{C}^{+*}(\underline{\xi}_o,T) = \xi_2^o\gamma + \frac{1-\xi_2^o\gamma}{2}\left[1-\sqrt{1-\frac{4\xi_1^o\xi_2^o(1-\gamma)e^{-E(T)}}{(1-\xi_2^o\gamma)^2}}\right], \qquad (6.42)$$

and in general $\overline{C}^{+}(\underline{\xi}_o,T)$ and $\overline{C}^{+*}(\underline{\xi}_o,T)$ are not equal.

It can also be shown that the minimum error probability receiver is not in general the interval optimum feedback receiver for this problem, nor is it incrementally optimum. The first claim is supported by the following counterexample. Let $\xi_2^o = \frac{3}{5}$, $\xi_1^o = \frac{2}{5}$, $\gamma = \frac{1}{2}$. Then $f(\tau)$ is initially $\frac{3}{2}$ and increases monotomically with $E(\tau)$. For energies $E(T)$ which are less than $E[f^{-1}(\frac{1}{1-\gamma})] = E[f^{-1}(2)]$, there is no improvement in $\overline{C}^{+}(\underline{\xi}_o,T)$ over the a priori guess performance ξ_2^o. A feedback receiver which performs better for these energies is one that nulls ε_1. It achieves average cost

$$\overline{C}_1^{+}(\underline{\xi}_o,T) = \xi_2^o\left\{e^{-E(T)} + \frac{1}{2}\left[1-e^{-E(T)}\right]\right\} = \xi_2^o\left[1-\frac{1}{2}[1-e^{-E(T)}]\right] < \xi_2^o \qquad (6.43)$$

The incrementally optimum feedback for this problem can be most conveniently specified by treating an equivalent M-ary detection problem with error probability cost. Let $\underline{\xi}'$ be any M-dimensional probability vector satisfying

$$\xi_1' = \xi_1^o \quad , \quad \xi_2' \geq \xi_j' \quad , \quad 2 \leq j \leq M,$$

and

$$\frac{\xi_2'}{1-\xi_1'} = 1 - \gamma \tag{6.44}$$

and let the M-ary signal set $\{\varepsilon_j'(\vec{r},\tau)\}_{j=1}^{M}$ satisfy

$$\varepsilon_1'(\vec{r},\tau) = \varepsilon_1(\vec{r},\tau)$$

$$\varepsilon_j'(\vec{r},\tau) = \varepsilon_2(\vec{r},\tau) \quad , \quad 2 \leq j \leq M \tag{6.45}$$

Since the last (M-1) signals are indistinguishable, the optimum receiver always chooses H_1 or H_2, and thus $\{H_3,\ldots,H_M\}$ may be regarded as dummy hypotheses although they do contribute to the error probability.

The error probability for the M-ary problem is expressed as

$$1 - P_e^+(\underline{\xi}',T) = E_{\underline{t}} \sum_{j=1}^{M} \xi_j(T:\underline{t};\underline{\xi}',0)\ \delta_{jj^*[\underline{\xi}(T:\underline{t};\underline{\xi}',0)]} \quad , \ \underline{t}\varepsilon[0,T]^*$$

$$= E_{\underline{t}}\ \xi_{j^*[\underline{\xi}(T:\underline{t};\underline{\xi}',0)]}(T:\underline{t};\underline{\xi}',0) \quad , \quad \underline{t}\varepsilon[0,T]^* \tag{6.46}$$

where the optimum decision function $j^*(\underline{\xi})$ maximizes ξ_j. Since the signals $\{\varepsilon_2',\ldots,\varepsilon_M'\}$ are identical, their relative a posteriori probabilities are unaffected by observations; i.e.,

$$\frac{\xi_j(T:\underline{t};\underline{\xi}',0)}{\xi_2(T:\underline{t};\underline{\xi}',0)} = \frac{\xi_j'}{\xi_2'} \leq 1 \quad \text{for all } j\geq 2 \text{ and all } \underline{t}\varepsilon[0,T]^* \tag{6.47}$$

Therefore

$$\xi_{j^*[\underline{\xi}(T:\underline{t};\underline{\xi}',0)]}(T:\underline{t};\underline{\xi}',0) = \max\,[\xi_1(T:\underline{t};\underline{\xi}',0),\ \xi_2(T:\underline{t};\underline{\xi}',0)] \tag{6.48}$$

Summing (6.47) over $j\geq 2$ also yields

$$\frac{\xi_2(T:\underline{t};\underline{\xi}',0)}{1-\xi_1(T:\underline{t};\underline{\xi}',0)} = \frac{\xi_2'}{1-\xi_1'} = 1-\gamma \tag{6.49}$$

Thus the error probability expression (6.46) reduces to

$$P_e^+(\underline{\xi}',T) = E_{\underline{t}}\left\{1-\max\left[\xi_1(T:\underline{t};\underline{\xi}',0),\ (1-\gamma)(1-\xi_1(T:\underline{t};\underline{\xi}',0))\right]\right\}$$

$$= E_{\underline{t}}\min\left[1-\xi_1(T:\underline{t};\underline{\xi}',0),\ \xi_1(T:\underline{t};\underline{\xi}',0)+\gamma(1-\xi_1(T:\underline{t};\underline{\xi}',0))\right]$$

$$= E_{\underline{t}}\,\overline{C}\left[\xi_1(T:\underline{t};\underline{\xi}',0),\ 1-\xi_1(T:\underline{t};\underline{\xi}',0),\ T\right] \tag{6.50}$$

where $\overline{C}(\underline{\xi},T)$ is the a priori guess performance derived in (6.40) from the binary cost matrix (6.39).

Since the statistics of the data $\underline{t}$ are identical for the two problems as long as the same feedback is used, we have shown that the binary problem with cost matrix (6.39) is equivalent to the M-ary minimum error probability detection problem specified by (6.44), (6.45). The incrementally optimum feedback for the latter problem has already been calculated in (5.34). In this case, as in the binary minimum error probability problem, the indices j_o and j_1 in (5.34) are both selected from {1,2} and are always opposite. Therefore the incrementally optimum state-dependent feedback function for the M-ary problem is

$$-\ell_M^*(\vec{r},\tau,\xi_1,\ldots,\xi_M) = \frac{\xi_1\varepsilon_1(\vec{r},\tau) - \xi_2\varepsilon_2(\vec{r},\tau)}{\xi_1 - \xi_2} \tag{6.51}$$

At first glance equation (6.51) appears identical with the expression (6.1) for the incrementally optimum binary feedback, but it is not. The reason is that the state probabilities ξ_1, ξ_2 appearing on the right side of (6.51) do not sum to unity as they do in (6.1). In other words the binary optimum receiver would replace ξ_2 in (6.51) with $\xi_2+\ldots+\xi_M$.

It is possible to ignore the state probabilities $\{\xi_3,\ldots,\xi_M\}$ and compute the a posteriori probability ratio $\dfrac{\xi_1(\tau:\underline{t};\underline{\xi}',0)}{\xi_2(\tau:\underline{t};\underline{\xi}',0)}$ given feedback ℓ_M^* in the same way that we did for the binary problem. We find that

$$\left|\log \frac{\xi_1(\tau:\underline{t};\underline{\xi}',0)}{\xi_2(\tau:\underline{t};\underline{\xi}',0)}\right| = \log\ f[\tau;\xi_1',\xi_2'] \tag{6.52}$$

independent of the data $\underline{t}\varepsilon[0,\tau]^*$, where $f(\cdot)$ is a deterministic time function specified by a generalization of (6.14) to include binary a priori probabilities which do not necessarily sum to unity.

$$\frac{[f(\tau:\xi_1',\xi_2')+1]^2}{f(\tau:\xi_1',\xi_2')} = \frac{(\xi_1'+\xi_2')^2}{\xi_1'\xi_2'} \; e^{+E(\tau)} \tag{6.53}$$

We note that $f(\tau:\xi_1',\xi_2')$ depends on ξ_1',ξ_2' just through their ratio ξ_1'/ξ_2'. In comparison, the binary optimum receiver applied to the M-ary problem results in

$$\left| \log \frac{(1-\gamma)\xi_1(\tau:\underline{t};\underline{\xi}',0)}{\xi_2(\tau:\underline{t};\underline{\xi}',0)} \right| \equiv \log \frac{\xi_1(\tau:\underline{t};\underline{\xi}',0)}{\sum_{j=2}^{M}\xi_j(\tau:\underline{t};\underline{\xi}',0)}$$

$$= f(\tau:\xi_1',\ \xi_2'+\ldots+\xi_M') \tag{6.54}$$

We can calculate the M-ary error probability for the receiver employing feedback ℓ_M^* by using the expressions (6.34), (6.35) which remain valid for the probabilities $P_{ev}^{\pm}(\tau)$ of recording an even number of counts during $[0,\tau]$ conditioned on the pair of hypotheses H_1 and H_2, provided that we replace $f(\tau)$ with $f(\tau:\xi_1',\xi_2')$ given by (6.53). The unconditional probability $P_{ev}(\tau)$ is derived from these by averaging over all the hypotheses, using the facts that the data statistics are identical under $H_2,\ldots,H_M$ and that $\sum_{j=2}^{M}\xi_j' = 1-\xi_1' = \frac{\xi_2'}{1-\gamma}$.

$$P_{ev}(T) = \begin{cases} \dfrac{\xi_1' f(T:\xi_1',\xi_2') - \xi_2'}{f(T:\xi_1',\xi_2') - 1} \quad \dfrac{f(T:\xi_1',\xi_2') + \frac{1}{1-\gamma}}{f(T:\xi_1',\xi_2') + 1} & \xi_1' > \xi_2' \\[2ex] \dfrac{\xi_2' f(T:\xi_1',\xi_2') - \xi_1'}{f(T:\xi_1',\xi_2') - 1} \quad \dfrac{\frac{f(T:\xi_1',\xi_2')}{1-\gamma} + 1}{f(T:\xi_1',\xi_2') + 1} & \xi_2' > \xi_1' \end{cases} \qquad \text{(6.[illegible])}$$

and the opposite relations hold for $P_{od}(T)$.

Given an even number of counts and $\xi_1' > \xi_2'$, the a posteriori probabilities satisfy $\frac{\xi_1''}{\xi_2''} = [illegible]$, implying $\frac{\xi_1''}{1-\xi_1''} = \gamma/(1-\gamma)$. Similarly, when $\xi_1' > \xi_2'$ and an odd number of counts occur, $\frac{1-\xi_1''}{\xi_1''} = \frac{\gamma}{1-\gamma}$, and the opposite relations hold when $\xi_2' > \xi_1'$. The interval average cost can be obtained by averaging the a priori guess performance function $\overline{\pi}(\xi_1'', 1-\xi_1'', \pi)$ from (6.40) over these possibilities.

$$\overline{\pi}(\xi', T) = \frac{\xi_1' f - \xi_2'}{f-1}\,\frac{f+\frac{1}{1-\gamma}}{f+1}\left\{\frac{\gamma(1-\gamma)}{f+\frac{1}{1-\gamma}} + \min\left[\frac{f}{f+\frac{1}{1-\gamma}}, \frac{\frac{1}{1-\gamma}}{f+\frac{1}{1-\gamma}}\right]\right\}$$

$$+ \frac{\xi_2' f - \xi_1'}{f-1}\,\frac{\frac{f}{1-\gamma}+1}{f+1}\left\{\frac{f\gamma(1-\gamma)}{\frac{f}{1-\gamma}+1} + \min\left[\frac{1}{\frac{f}{1-\gamma}+1}, \frac{f}{\frac{f}{1-\gamma}+1}\right]\right\}$$

$$= \frac{\xi_1'+\xi_2'}{f+1} + \frac{\gamma}{1-\gamma}\,\xi_2' = \frac{\xi_1'+\xi_2'}{f+1} + \gamma(1-\xi_1') \qquad (6.56)$$

where $f \equiv f(T:\xi_1',\xi_2')$ from (6.53), $f \geq 1$.

Identifying ξ_1' with ξ_1^0 and ξ_2' with $\xi_2^0(1-\gamma)$ in (6.42),

$$P_{ev}(\tau) = \begin{cases} \dfrac{\xi_1' f(\tau:\xi_1',\xi_2') - \xi_2'}{f(\tau:\xi_1',\xi_2')-1} \quad \dfrac{f(\tau:\xi_1',\xi_2') + \frac{1}{1-\gamma}}{f(\tau:\xi_1',\xi_2') + 1} & \text{if } \xi_1' > \xi_2' \\ \dfrac{\xi_2' f(\tau:\xi_1',\xi_2') - \xi_1'}{f(\tau:\xi_1',\xi_2')-1} \quad \dfrac{\frac{f(\tau:\xi_1',\xi_2')}{1-\gamma} + 1}{f(\tau:\xi_1',\xi_2') + 1} & \text{if } \xi_2' > \xi_1' \end{cases} \tag{6.55}$$

and the opposite relations hold for $1-P_{ev}(\tau)$.

Given an even number of counts and $\xi_1' > \xi_2'$, the a posteriori probabilities satisfy $\dfrac{\xi_1(\cdot)}{\xi_2(\cdot)} = f(\cdot)$, implying $\dfrac{\xi_1(\cdot)}{1-\xi_1(\cdot)} = f(\cdot)(1-\gamma)$. Similarly, when $\xi_1' > \xi_2'$ and an odd number of counts occur, $\dfrac{1-\xi_1(\cdot)}{\xi_1(\cdot)} = \dfrac{f(\cdot)}{1-\gamma}$, and the opposite relations hold when $\xi_2' > \xi_1'$. The interval average cost may be obtained by averaging the a priori guess performance function $\overline{C}[\xi_1(\cdot),\ 1-\xi_1(\cdot),\ T]$ from (6.40) over these possibilities.

$$\overline{C}^+(\underline{\xi}',T) = \frac{\xi_1' f - \xi_2'}{f-1}\ \frac{f+\frac{1}{1-\gamma}}{f+1}\left\{\frac{\gamma/(1-\gamma)}{f+\frac{1}{1-\gamma}} + \min\left[\frac{f}{f+\frac{1}{1-\gamma}}\ ,\ \frac{1}{f+\frac{1}{1-\gamma}}\right]\right\}$$

$$+\ \frac{\xi_2' f - \xi_1'}{f-1}\ \frac{\frac{f}{1-\gamma}+1}{f+1}\left\{\frac{f\gamma/(1-\gamma)}{\frac{f}{1-\gamma}+1} + \min\left[\frac{1}{\frac{f}{1-\gamma}+1}\ ,\ \frac{f}{\frac{f}{1-\gamma}+1}\right]\right\}$$

$$= \frac{\xi_1'+\xi_2'}{f+1} + \frac{\gamma}{1-\gamma}\ \xi_2' \ = \ \frac{\xi_1'+\xi_2'}{f+1} + \gamma(1-\xi_1') \tag{6.56}$$

where $f \equiv f(T:\xi_1',\xi_2')$ from (6.53), $f \geq 1$.

Identifying ξ_1' with ξ_1^o and ξ_2' with $\xi_2^o(1-\gamma)$ in (6.42), and solving (6.53) for $\frac{1}{f+1}$, we see that the incrementally optimum feedback ℓ_M^* achieves exactly the same performance as the optimum quantum measurement for this problem.

6.4 M-ARY PPM SIGNAL SET PROBLEMS

6.4.1 Motivation for Studying

The result obtained in the previous example leads us to consider a class of M-ary signal sets which is not as trivial as the type required there but not completely general either. We assume that the signals are pulse-position-modulated (PPM). This means that the signaling interval $[0,T]$ is divided into M sections of length $\frac{T}{M}$, and in each section only one of the possible signals $\{\varepsilon_j(\vec{r},\tau)\}_{j=1}^{M}$ is nonzero.

The choice of this signal set is inspired by our observation that the conditions of the previous example are satisfied in each $\frac{T}{M}$ - length subinterval (provided of course that the hypotheses are appropriately re-numbered each time). Therefore, the optimum receiver derived there can be applied during any subinterval to achieve minimum error probability at the end of that subinterval. In particular this implies that the minimum error probability receiver for the $[0,T]$ interval uses feedback ℓ_M^* determined by the state-dependent specification (6.51) during the last subinterval $[(1-\frac{1}{M})T,T]$. It does not follow from this, however, that ℓ_M^* is optimum on $[0,T]$. For instance, the objective of the feedback selection during the next-to-last subinterval $[(1-\frac{2}{M})T,(1-\frac{1}{M})T]$ is not to minimize the error probability at time $(1-\frac{1}{M})T$ but rather to optimize the statistics of the a priori conditions presented to the receiver at $(1-\frac{1}{M})T$ which initialize the remaining time interval $[(1-\frac{1}{M})T,T]$ problem.

6.4.2 Non-Optimality of the Incrementally Optimum Receiver for Minimum Error Probability Detection

It is true that ℓ_M^* is incrementally optimum under the [0,T] minimum error probability criterion for PPM signals. It is tempting to plow ahead with a tedious performance calculation for this feedback receiver in the hope that whatever accident or design caused the equivalence between incremental and interval optimality for the binary problems studied above will have the same effect for the M-ary PPM problem. Unfortunately it is much easier than that to resolve this question because the answer is negative.

To see this we consider a ternary (M=3) problem with very large, equal signal energies $(\exp[-\int_0^T d\tau \int_\Sigma d\vec{r}\,|\varepsilon_j(\vec{r},\tau)|^2] \equiv e^{-E_o} \to 0$, $j=1,2,3)$. We assume equal a priori probabilities $\xi_1^o=\xi_2^o=\xi_3^o = \frac{1}{3}$ (or, more rigorously, $\xi_1^o= \frac{1}{3} + \delta, \xi_2^o= \frac{1}{3},\ \xi_3^o= \frac{1}{3} - \delta,\ \delta\downarrow 0$) and that the hypotheses are numbered so that $\varepsilon_j\neq 0$ in the jth subinterval $[(j-1)\frac{T}{3},\ j\ \frac{T}{3}]$, $j=1,2,3$. The optimum quantum measurement for this problem achieves asymptotic error probability $\frac{1}{2}\ e^{-2E_o}$.

One of the ways in which the incrementally optimum feedback receiver can make a mistake occurs when odd numbers of counts are observed in each of the three subintervals. As we have seen, an odd number of counts tends to deny the initially most probable hypothesis in favor of one(s) corresponding to the opposite signal. The observations in the first subinterval thus deny H_1 and create an a posteriori probability ratio at time $\frac{T}{3}$ of $\xi_1:\xi_2:\xi_3$ = 1:f:f, where f is given by (6.53) with $\xi_1'=\xi_2'$ and $E(T)=E_o$ as defined above. The odd number of counts in the next subinterval deny H_2 in favor of H_3 and create an a posteriori probability ratio at time $\frac{2T}{3}$ of 1:1:f, because the relative probability of H_1 and H_3 remains unchanged through this interval. Finally, the odd number of counts in the last subinterval strongly denies the initially (at $\frac{2T}{3}$) very likely H_3 but cannot distinguish H_1 and H_2; so the a posteriori probabilities at time T are in f_2:f_2:1 proportion, where f_2 is given by (6.53) with $\xi_1'/\xi_2' = f$ and $E(T)=E_o$(or alternatively by $\xi_1'/\xi_2' = 1$ and $E(T)=2E_o$).

Thus when E_o is large the sequence of three consecutive odd counts essentially eliminates the possibility that H_3 is true by downgrading its a posteriori probability by a factor $\frac{1}{f_2} \sim \frac{1}{4}e^{-2E_o}$. But it provides very poor resolution between H_1 and H_2; the receiver must simply flip a coin and make an error half the time. Conditioned on H_1 or H_2, this undesirable situation of three odd counts occurs with small, but not optimally small, probability. If, say, H_1 is true, the first odd count denying H_1 is not likely and occurs with asymptotic probability $\frac{1}{f} \sim \frac{1}{4}\, e^{-E_o}$. But the second and third odd counts are expected because these confirm H_1 and their conditional probabilities approach unity. Likewise under H_2 the asymptotic probability of three odd counts is $\frac{1}{4}\, e^{-E_o}$. Thus we have shown that the asymptotic error probability for the incrementally optimum feedback receiver is at least $\frac{1}{12}\, e^{-E_o}$, which differs from that of the optimum interval measurement by a factor of 2 in the exponent.

We can show that the three consecutive odd count situtation represents a worst case and that the error probability exponent for the incrementally optimum receiver is indeed E_o. But this would not be very significant because there are simpler known receivers which already accomplish this. A direct detector can positively identify the signal if it ever gets a count and so it only makes an error with probability $\frac{2}{3}$ (in general, $\frac{M-1}{M}$) when zero counts are observed. This happens with probability e^{-E_o} under any hypothesis, so the error probability of the direct detector is

$$P_e = \frac{M-1}{M} \; e^{-E_o} \tag{6.57}$$

Similarly, it can be shown that the error exponent for a homodyne receiver is also E_o.

The basic reason that the direct detector and the incrementally optimum feedback receiver are exponentially non-optimum is that they assemble information from each subinterval individually, without an effective overall strategy. Data which tends to deny the signal that is actually present can occur with probability proportional to e^{-E_o}, so it is also necessary to utilize the probable data denying the wrong hypotheses to help find the right one. It is possible for the incrementally optimum receiver to realize the

$2E_o$ exponent in the binary case because any information denying one hypothesis automatically confirms the other. In fact, although the signal testing strategy employed by the simple direct detector is still exponentially inferior even for binary signals, we mentioned in Section 6.1 that an adaptation of it, a feedback receiver using data-independent feedback nulling one of the signals, also achieves the $2E_o$ exponent. Neither of these receivers is exponentially optimum in the M-ary case because there is ambiguity as to which of the remaining (M-1) signals is confirmed by a particular denial, and it takes a sound overall strategy to guarantee for every signal the assembly of one subinterval's worth of confirming evidence from denials of all the remaining (M-1) signals.

In the next example we demonstrate the existence of a feedback receiver that achieves the $2E_o$ exponent for the M-ary (PPM) problem.

6.4.3 A Near-Optimum Feedback Receiver for the Minimum Error Probability Detection Problem

We assume that the signals are numbered such that ε_j is nonzero in the jth interval. For now we will also assume that the signals have equal energy E_o and equal a priori probabilities $\xi_j = \frac{1}{M}$.

The operation of the near-optimum receiver is depicted in the following abbreviated tree diagram. Only one stage (out of M) is shown because it is a recursive definition.

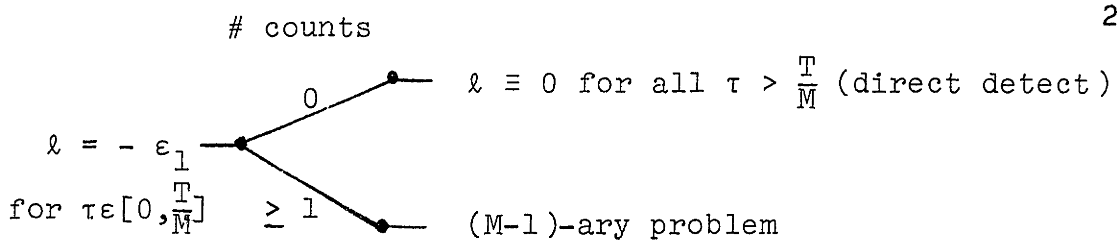

Figure 6.2

In the first subinterval we test for the presence of the first signal by nulling it. If no counts are observed, that tends to confirm H_1 becuase any other hypothesis would produce counts with probability $1-e^{-E_o}$. When this happens, the receiver direct detects for the rest of the observation interval $[\frac{T}{M},T]$ and continues to believe H_1 unless a count positively identifying one of the other hypotheses is recorded. If one of the hypotheses $\{H_2,\ldots,H_M\}$ is true, it has to escape detection in both the first subinterval and its own jth subinterval for an error to occur. This results in an error exponent of $2E_o$ along this branch.

If at least one count is observed in the first subinterval, H_1 is completely eliminated as a possibility for the rest of the problem, because this event is impossible under H_1. On the other hand, no information has yet been obtained to distinguish the other (M-1) signals, so at this point we are left with the (M-1)-ary version of the same problem and can continue. In the second subinterval ε_2 is nulled. If no counts occur, the receiver direct detects for the remaining time interval; otherwise it goes on and tests ε_3, and so forth.

The tree can end with the optimum binary receiver, but it is more convenient computationally and not very sacrificial in performance to use the binary receiver specified by the same tree. This binary receiver cannot make an error if H_1 is true, and given H_2 it errs only if no counts are recorded in either subinterval. Both of these events have probability e^{-E_o}, so the error probability of the binary receiver is

$$P_2 = \frac{1}{2} e^{-2E_o} \tag{6.58}$$

The error probability for the M-ary receiver can be calculated recursively. If H_1 is true there is no possibility of error. Under any of the other hypotheses, errors can occur in two ways. If no counts were observed in the first subinterval the receiver will continue to believe H_1 until it receives a count in some future subinterval. The correct hypothesis can only escape detection if no counts are observed in its own subinterval also. If one or more counts are observed in the first subinterval, the error probability is that for the (M-1)-ary version of this problem. Defining P_M, $P_{M-1}, \ldots, P_2$ to be the error probabilities achieved by the receiver in Figure 6.2 for the M-ary, (M-1)-ary,..., binary problems respectively, we can write

$$P_M = \frac{M-1}{M} \left[e^{-E_o} e^{-E_o} + (1-e^{-E_o}) P_{M-1} \right]$$

or

$$MP_M = (M-1)\, e^{-2E_o} + (1-e^{-E_o})(M-1)\, P_{M-1}$$

$$= (M-1)\, e^{-2E_o} + (1-e^{-E_o}) \left[(M-2) e^{-2E_o} + (1-e^{-E_o})(M-2) P_{M-2} \right]$$

$$= \ldots = e^{-2E_o} \sum_{j=0}^{M-3} (M-1-j)(1-e^{-E_o})^j + (1-e^{-E_o})^{M-2}\, 2\, P_2$$

$$= e^{-2E_o} \sum_{j=0}^{M-2} (M-1-j)(1-e^{-E_o})^j \qquad (6.59)$$

The sum in (6.59) can be evaluated in closed form, yielding

$$MP_M = (1-e^{-E_o})^M + Me^{-E_o} - 1 \qquad (6.60)$$

We want to compare this performance with that of the optimum quantum measurement which has been calculated by Liu[25],

$$MP_M^* = \frac{M-1}{M} \left[\sqrt{1+(M-1)e^{-E_o}} - \sqrt{1-e^{-E_o}} \right]^2 \qquad (6.61)$$

First we examine expressions (6.60) and (6.61) in two limiting cases. For a very high information rate ($\log M \gg 1$), we obtain

$$P_M \approx P_M^* \approx e^{-E_o} \qquad (6.62)$$

as long as the signal energies are not also increased; specifically, the requirement is $Me^{-E_o} \gg 1$. In this limit the error probability exponent is only E_o and the performance of the simple direct detector in (6.57) is just as good. This result evidences the inability of even the optimum receiver to extract confirming information about any particular signal from denials of all the rest when the signal set becomes infinite.

In the opposite limit of high signal energy relative to the information rate ($Me^{-E_o} << 1$), we obtain an error probability exponent of $2E_o$ for both P_M and P_M^*.

$$P_M \approx \frac{M-1}{2} e^{-2E_o}$$
$$P_M^* \approx \frac{M-1}{4} e^{-2E_o} \qquad (6.63)$$

We see that the conditional signal nulling receiver is non-optimum in this limit by just a multiplicative factor of 2 in the error probability.

This small deviation from optimality in the limiting case $Me^{-E_o} << 1$ is indicative of the extent to which the conditional signal nuller is non-optimum for any combination of M and E_o. Specifically, defining

$$\kappa \equiv \sup_{M,E_o} \frac{P_M}{P_M^*} \qquad (6.64)$$

we can show analytically that $\kappa \leq 3\sqrt{\frac{3}{2e}} \approx 2.23$ (see Appendix, Section 6A.2) and believe from hand calculations that the actual worst multiplicative deviation is approximately $\kappa \approx 2.14$, occurring when M and E_o simultaneously become arbitrarily large, with $M\, e^{-E_o} \approx 1$.

Although the optimum performance for the PPM problem is not known when the signal energies or a priori probabilities are unequal, it is still interesting to consider the application of the conditional signal nuller to the more general problem. We define for any M-dimensional probability vector $\underline{\xi}=(\xi_1,\ldots,\xi_M)^T$ and any signal set $\{\varepsilon_j(\vec{r},\tau)\}_{j=1}^{M}$ with energies $\{E_j\}_{j=1}^{M}$ the sequence of error probabilities $P_M(\underline{\xi})$, $P_{M-1}(\underline{\xi}),\ldots,P_2(\underline{\xi})$ as follows: $P_K(\underline{\xi})$ is the error probability achieved by the conditional signal nuller for a K-ary PPM problem with signal energies $\{E_j\}_{j=M-K+1}^{M}$ and a priori probabilities $\left\{\dfrac{\xi_j}{\sum\limits_{i=M-K+1}^{M}\xi_i}\right\}_{j=M-K+1}^{M}$

As before we assume that the signals are numbered so that ε_j is nonzero only in the jth subinterval.

Conditioned on H_1 errors are impossible, just as for the symmetric problem. Given $H_j \neq H_1$, an error can occur on the top branch of the tree in Figure 6.2 with probability $e^{-E_1}\, e^{-E_j}$. On the bottom branch, which is taken with probability $(\xi_2+\ldots+\xi_M)(1-e^{-E_1})$, the error probability after the first stage is $P_{M-1}(\underline{\xi})$ because H_1 has been eliminated and the other hypothesis probabilities have not changed relative to each other. Thus

$$P_M(\underline{\xi}) = \sum_{j=2}^{M} \xi_j \, e^{-E_1} e^{-E_j} + \sum_{j=2}^{M} \xi_j (1-e^{-E_1}) \, P_{M-1}(\underline{\xi})$$

$$= (1-e^{-E_1})(\sum_{j=2}^{M} \xi_j) \, P_{M-1}(\underline{\xi}) + e^{-E_1} \sum_{j=2}^{M} \xi_j \, e^{-E_j}$$

$$= (1-e^{-E_1})(1-e^{-E_2})(\sum_{j=3}^{M} \xi_j) \, P_{M-2}(\underline{\xi}) + (1-e^{-E_1})e^{-E_2} \sum_{j=3}^{M} \xi_j \, e^{-E_j}$$

$$+ e^{-E_1} \sum_{j=2}^{M} \xi_j \, e^{-E_j} \tag{6.65}$$

We notice that when this expression is iterated $P_K(\underline{\xi})$ is always multiplied by $\sum_{j=M-K+1}^{M} \xi_j$, which cancels the normalizing factor in the definition of the a priori probabilities for the K-ary problem.

We shall not write out the result of this iteration for unequal energies. We can see that for large energies the error probability exponent is $\min_{i \neq j} (E_i+E_j)$. In the case of equal energies $E_j=E_o$ but unequal a priori probabilities the required sum is almost the same as before. The result is

$$P_M(\underline{\xi}) = \xi_2 e^{-2E_o} + \xi_3 e^{-2E_o}\left[1+(1-e^{-E_o})\right] + \xi_4 e^{-2E_o}\left[1+(1-e^{-E_o})+(1-e^{-E_o})^2\right]$$

$$+ \ldots + \xi_M e^{-2E_o}\left[1+\ldots+(1-e^{-E_o})^{M-2}\right]$$

$$= e^{-E_o}\left[1- \sum_{j=1}^{M} \xi_j (1-e^{-E_o})^{j-1}\right] \tag{6.66}$$

which reduces to (6.60) when $\xi_j = \frac{1}{M}$ because
$\sum_{j=1}^{M} (1-e^{-E_o})^{j-1} = e^{E_o}[1-(1-e^{-E_o})^M]$.

We observe that the (negative) coefficient of ξ_j in (6.66) decreases in magnitude with increasing j. Thus it is advantageous for the transmitter to assign ε_1 to the most probable hypothesis, ε_2 to the next most probable, and so forth. Our conclusion is that the conditional signal nuller possesses a basic asymmetry that allows it to detect earlier signals slightly better than later ones. The optimum (interval) quantum measurement, on the other hand, should yield performance which is independent of the time-ordering of the signals, because physically the problems seem identical. It is not reasonable to wonder, however, whether the time asymmetry exhibited by the conditional signal nuller is inherent to all forward-time measurement sequences. This question we leave unanswered.

6.5 M-STATE PROBLEMS WITH GENERALIZED PPM SIGNALS

So far in this chapter we have presented and analyzed several examples of good or optimum feedback receivers for some specialized communication problems. None of these, however, have been derived from the optimality condition obtained in Chapter III. In general it is extremely difficult to determine explicit solutions from that condition. We have found that, even when we already know the answer, proving optimality is not easy.

In this section we consider a class of problems for which some small progress can be made toward simplifying the optimality condition. The cost function is arbitrary, except that we assume that the optimum average cost is a differentiable function of the a priori probabilities, implying that the optimality condition (3.37) is applicable. We restrict attention to somewhat generalized PPM signal sets, requiring that at any time $\tau \,\varepsilon[0,T]$ all M signals $\{\varepsilon_j(\cdot,\tau)\}_{j=1}^{M}$ are chosen from a binary set $\{\varepsilon_o(\cdot,\tau),\varepsilon(\cdot,\tau)\}$. These reduce to PPM signals when $\varepsilon_o=0$ and ε equals ε_j on the jth subinterval, $j=1,\ldots,M$, and when it is further assumed that the sets of times for which the various signals equal ε(the "on"-times) are consecutive disjoint connected intervals. Under the generalized model an arbitrary number of the M signals can take on either value ε_o or ε at every instant of time. From a slightly different and perhaps more useful point of view, we see that the generalized model allows an arbitrary binary coding of the M messages into sequences of time-segments of the two coherent signals ε, ε_o.

The problem is specified by knowledge of which signals are "on" and which signals are "off" at each time τ. We define the set $J(\tau)$ of "on" signals and the set $J_o(\tau)$ of "off" signals at time τ by

$$\varepsilon_j(\cdot,\tau) = \begin{cases} \varepsilon(\cdot,\tau), & j \varepsilon J(\tau) \\ \varepsilon_o(\cdot,\tau), & j \varepsilon J_o(\tau) \end{cases} \tag{6.67}$$

The regularity conditions of Chapter II prohibit wild fluctuations in $J(\tau)$, $J_o(\tau)$ as functions of τ. Specifically, the set of times τ for which $J(\tau)$ or $J_o(\tau)$ takes on any particular one of its 2^M possible values is a finite union of (half-open) intervals.

6.5.1 Derivation of the Optimum Phase and a Necessary Condition on the Magnitude of the Optimum Feedback Function

For this class of problems it is convenient to define the state-dependent feedback function ℓ in terms of the following transformation

$$\beta(t,\underline{\xi}) = \frac{\varepsilon(t) + \ell(t,\underline{\xi})}{\varepsilon(t) - \varepsilon_o(t)} \qquad t \equiv (\vec{r},\tau) \; \varepsilon \; \Sigma x[0,T] \tag{6.68}$$

In terms of β, the intensity functions λ_j, $\hat{\lambda}$ can be expressed as

$$\lambda_j(t,\underline{\xi}) = \begin{cases} |\beta(t,\underline{\xi})-1|^2 \tilde{\lambda}(t) \quad , & j \in J_o(\tau) \\ |\beta(t,\underline{\xi})|^2 \tilde{\lambda}(t) \quad , & j \in J(\tau) \end{cases} \tag{6.69}$$

where

$$\tilde{\lambda}(t) \equiv |\varepsilon(t) - \varepsilon_o(t)|^2, \qquad t \in \Sigma x[0,T]. \tag{6.70}$$

Also,

$$\hat{\lambda}(t,\underline{\xi}) = \{\xi_o(\tau)|\beta(t,\underline{\xi})-1|^2 + \xi(\tau)|\beta(t,\underline{\xi})|^2\}\tilde{\lambda}(t) \tag{6.71}$$

where

$$\xi_o(\tau) \equiv \sum_{j\varepsilon J_o(\tau)} \xi_j \tag{6.72}$$

$$\xi(\tau) = \sum_{j\varepsilon J(\tau)} \xi_j$$

We also define the intensity ratios

$$r(t,\underline{\xi}) = \frac{\tilde{\lambda}(t)|\beta(t,\underline{\xi})|^2}{\hat{\lambda}(t,\underline{\xi})} \left(= \frac{\lambda_j(t,\underline{\xi})}{\hat{\lambda}(t,\underline{\xi})} \quad \text{if } j \in J(\tau) \right)$$

$$r_o(t,\underline{\xi}) = \frac{\tilde{\lambda}(t)|\beta(t,\underline{\xi})-1|^2}{\hat{\lambda}(t,\underline{\xi})} \left(= \frac{\lambda_j(t,\underline{\xi})}{\hat{\lambda}(t,\underline{\xi})} \quad \text{if } j \in J_o(t) \right) \tag{6.73}$$

Note that

$$\xi_o(\tau)\, r_o(t,\underline{\xi}) + \xi(\tau)\, r(t,\underline{\xi}) = 1 \tag{6.74}$$

so $\kappa(t,\underline{\xi})$ and $\kappa_o(t,\underline{\xi})$ are not independent parameters.
The potential a posteriori probabilities ρ_j are given in terms of the intensity ratios by

$$\rho_j(t,\underline{\xi}) = \begin{cases} \xi_j \kappa(t,\underline{\xi}) & , \ j \ \varepsilon \ J(\tau) \\ \xi_j \kappa_o(t,\underline{\xi}) = \xi_j \dfrac{1-\xi(\tau)\kappa(t,\underline{\xi})}{\xi_o(\tau)} & , \ j \ \varepsilon \ J_o(\tau) \end{cases} \tag{6.75}$$

The minimization problem in (3.37) is simplified from the general case because $\overline{C}''[\underline{\rho}(t,\underline{\xi}), \underline{\xi},\tau]$ depends on the feedback ℓ just through the single real parameter $\kappa(t,\underline{\xi})$. Since we are optimizing over a complex quantity we might expect to be able to solve half the problem, in some sense, without knowing the explicit analytic form of $\overline{C}''$. This is indeed the case; it can be shown that for generalized PPM signals sets the optimum $\beta(t,\underline{\xi})$ is real. This is consistent with our earlier intuitive observation in Section 6.2.4 that the optimal shift of a binary signal set should produce common phase.

One way to accomplish the desired minimization is to temporarily fix $\kappa(t,\underline{\xi})= \kappa$, then (for every $\kappa \geq 0$) optimize $\hat{\lambda}\overline{C}''$ subject to the constraint $\kappa(t,\underline{\xi})=\kappa$, and finally optimize the resulting expression over κ. For fixed κ, Re(β) must be given in terms of κ and Im(β) as a solution of the quadratic equation implied by (6.73) and (6.71),

$$[\mathrm{Re}(\beta)]^2(1-\kappa) + 2\xi_o\kappa\mathrm{Re}(\beta) - \xi_o\kappa + [\mathrm{Im}(\beta)]^2(1-\kappa) = 0 \tag{6.76}$$

which yields solutions

$$\mathrm{Re}(\beta) = \frac{r}{r - r_o} \pm \sqrt{\frac{r^2}{(r-r_o)^2} + \frac{r}{r-r_o} - [\mathrm{Im}(\beta)]^2} \qquad (6.77)$$

because $1-r = \xi_o(r_o-r)$.

In (6.76) and (6.77) and the following we drop all space, time, and probability vector arguments because the optimization in (3.37) is pointwise over t, $\underline{\xi}$.

Given r, $\bar{C}''$ is fixed (and negative), so the minimization problem reduces to maximizing $\hat{\lambda}$. Substituting (6.77) into (6.71) we obtain

$$\hat{\lambda} = \tilde{\lambda}\left[\frac{r+r_o}{(r-r_o)^2} \pm \frac{2}{r-r_o}\sqrt{\frac{rr_o}{(r-r_o)^2} - [\mathrm{Im}(\beta)]^2}\right], \qquad (6.78)$$

which is obviously maximized by the choice of sign in (6.77) to be the same as the sign of $r-r_o$, and $\mathrm{Im}(\beta)=0$.

Inserting these optimal choices into (6.77) and (6.78) we obtain

$$\beta = \frac{r + \sqrt{rr_o}}{r-r_o} \qquad (6.79)$$

and

$$\hat{\lambda} = \tilde{\lambda}\left[\frac{1}{\sqrt{r} - \sqrt{r_o}}\right]^2 \qquad (6.80)$$

At this point we may re-state the minimization problem in terms of varying the parameter r. Denoting by $\hat{\lambda}'$ the derivative of (6.80) with respect to r,

$$\hat{\lambda}' = \frac{\hat{\lambda}}{1-r}\left[1 + \frac{1}{\sqrt{rr_o}}\right] \tag{6.81}$$

we see that a necessary condition for optimality, obtaining by setting the derivative of $\hat{\lambda}\overline{C}''$ with respect to r equal to 0, is

$$0 = \hat{\lambda}'\,\overline{C}'' + \hat{\lambda}\left[\sum_{i\varepsilon J} \xi_i\left(\frac{\partial C(\underline{\rho})}{\partial \rho_i} - \frac{\partial C(\underline{\xi})}{\partial \xi_i}\right) + \sum_{i\varepsilon J_o} \xi_i\left(-\frac{\xi}{\xi_o}\right)\left(\frac{\partial C(\underline{\rho})}{\partial \rho_i} - \frac{\partial C(\underline{\xi})}{\partial \xi_i}\right)\right] \tag{6.82}$$

The expression in brackets may be simplified by the following observation. Defining $\widetilde{C}''(\underline{\rho},\underline{\xi},\tau) \equiv \overline{C}''(\underline{\xi},\underline{\rho},\tau)$, we have from (3.28) and (6.75),

$$\begin{aligned}\overline{C}'' + \widetilde{C}'' &= \sum_{i=1}^{M}(\rho_i-\xi_i)\left[\frac{\partial C(\underline{\rho})}{\partial \rho_i} - \frac{\partial C(\underline{\xi})}{\partial \xi_i}\right] \\ &= (r-1)\left[\sum_{i\varepsilon J} \xi_i\left(\frac{\partial C(\underline{\rho})}{\partial \rho_i} - \frac{\partial C(\underline{\xi})}{\partial \xi_i}\right) + \sum_{i\varepsilon J_o}\left(-\frac{\xi}{\xi_o}\right)\xi_i\left(\frac{\partial C(\underline{\rho})}{\partial \rho_i} - \frac{\partial C(\underline{\xi})}{\partial \xi_i}\right)\right]\end{aligned} \tag{6.83}$$

The expression in brackets is the same as the one in (6.82).

Therefore, using (6.81) along with (6.83) and dividing by $\frac{\hat{\lambda}}{r-1}$ we can reduce the necessary condition for optimality to

$$\sqrt{r r_0} = \frac{\overline{C}''}{\tilde{C}''} \tag{6.84}$$

The optimum feedback level ℓ for space-time point t and probability vector $\underline{\xi}$ is specified in terms of the optimum r satisfying (6.84) through (6.68) and (6.79). [We do not have to worry about possible division by zero in (6.77)-(6.81) because the case $r=r_0=1$ implying $\underline{\rho}=\underline{\xi}$ is non-optimum unless all feedback functions perform equally poorly. The convexity of $\overline{C}$ and (6.82), (6.83) imply that $\frac{d}{dr}(\hat{\lambda}\overline{C}'')$ is nonpositive at $r=0$ and nonnegative at $r=\frac{1}{\xi}$ ($r_0=0$), so an endpoint minimum not satisfying (6.84) is ruled out. We have not, however, excluded the possibility of multiple local extrema satisfying (6.84)].

Whereas the optimality condition in Chapter III only reduced the problem of finding an optimum feedback function to one of minimizing pointwise a function of a single complex variable, the necessary condition (6.84), and our previous result that the optimum β is real, together specify the solution to that pointwise minimization (for the case of generalized PPM signals). The right side of (6.84) is still non-explicit, depending on the cost function for the remaining time interval problem in the same way that (3.37) does. But (6.84) represents and equation to be solved for r, rather than an expression to be minimized over r or over the complex variable ℓ.

We remark that it is straightforward to extend the technique developed in this section to problems in which the receiver is subject to dark current μ_o or μ at times when the signal is ε_o or ε respectively. Results are briefly outlined for the realistic case of signal-independent dark current, $\mu_o=\mu$. We define the parameter β by (6.68), the signal-difference intensity $\tilde{\lambda}$ by (6.70), the noise-to-signal ratio ν by

$$\nu = \frac{\mu}{\tilde{\lambda}} \tag{6.85}$$

the average intensity $\hat{\lambda}^{\nu}$ by

$$\hat{\lambda}^{\nu} = [\xi_o|\beta-1|^2 + \xi|\beta|^2 + \nu]\ \tilde{\lambda} \tag{6.86}$$

the intensity ratios $\hbar^{\nu}$, $\hbar_o^{\nu}$ by

$$\begin{aligned} \hbar^{\nu} &= [|\beta|^2 + \nu]\ \frac{\tilde{\lambda}}{\hat{\lambda}} \\ \hbar_o^{\nu} &= [|\beta-1|^2 + \nu]\ \frac{\tilde{\lambda}}{\hat{\lambda}} \end{aligned} \tag{6.87}$$

and the potential a posteriori probabilities ρ_j^{ν} by

$$\rho_j^{\nu} = \begin{cases} \xi_j \hbar^{\nu}, & j\varepsilon J \\ \xi_j \hbar_o^{\nu}, & j\varepsilon J_o \end{cases} \tag{6.88}$$

Then the optimum $\beta=\beta^{\nu}$ is real and is given by

$$\beta^{\nu} = \frac{r^{\nu} + \sqrt{r^{\nu} r_o^{\nu} - \nu(r^{\nu}-r_o^{\nu})^2}}{r^{\nu}-r_o^{\nu}} \tag{6.89}$$

This value of β^{ν} produces an average intensity

$$\hat{\lambda}^{\nu} = \tilde{\lambda}\left[\frac{r^{\nu}+r_o^{\nu} + 2\sqrt{r^{\nu} r_o^{\nu} - \nu(r^{\nu}-r_o^{\nu})^2}}{(r^{\nu}-r_o^{\nu})^2}\right] \tag{6.90}$$

and the necessary condition obtained by setting the derivative of $\hat{\lambda}\overline{C}''$ equal to zero is

$$\sqrt{r^{\nu} r_o^{\nu} - \nu(r^{\nu}-r_o^{\nu})^2} = \frac{\overline{C}''}{\tilde{C}''} \tag{6.91}$$

where $\tilde{C}''$ is defined as before. [The quantity under the square root sign in (6.89)-(6.91) is constrained to be nonnegative by the definition (6.87).]

6.5.2 Application to Two-State Problems with Symmetric A Priori Cost

We conclude this section, as well as the chapter, by demonstrating that for M=2 the binary minimum error probability receiver satisfies the necessary condition (6.84) for an arbitrary cost matrix with symmetric a priori guess performance, as defined in (6.28). Applying the result in (6.29) to the $[\tau,T]$ problem we calculate the cost-to-go from time τ as

$$\overline{C}(\underline{\xi},\tau) = C\left[\xi_1\xi_2\ e^{-[E(T)-E(\tau)]}\right] \qquad (6.92)$$

Since for the binary minimum error probability receiver $\rho_1=\xi_2$ and $\rho_2=\xi_1$, we calculate $\overline{C}''[\underline{\rho},\underline{\xi},\tau]$, in terms of the derivative $C'(\cdot)$ of $C(\cdot)$, as

$$\overline{C}''(\underline{\rho},\underline{\xi},\tau) = C[\rho_1\rho_2\ e^{-E(T)+E(\tau)}] - C[\xi_1\xi_2\ e^{-E(T)+E(\tau)}]$$

$$- [(\rho_1-\xi_1)\xi_2+(\rho_2-\xi_2)\xi_1]e^{-E(T)+E(\tau)}\ C'[\xi_1\xi_2 e^{-E(T)+E(\tau)}]$$

$$= -\ (\xi_2-\xi_1)^2\ e^{-E(T)+E(\tau)}C'[\xi_1\xi_2\ e^{-E(T)+E(\tau)}] \qquad (6.93)$$

when $\rho_1= \xi_2$ and $\rho_2 = \xi_1$. Similarly,

$$\tilde{C}''(\underline{\rho},\underline{\xi},\tau) = \overline{C}''(\underline{\xi},\underline{\rho},\tau) = -(\xi_2-\xi_1)^2\ e^{-E(T)+E(\tau)}C'[\xi_1\xi_2\ e^{-E(T)+E(\tau)}] \qquad (6.94)$$

when $\xi_1=\rho_2$ and $\xi_2=\rho_1$. Therefore, the right side of (6.84) is unity and the optimum r should satisfy

$$1 = r r_o = r\ \frac{1-\xi_1 r}{\xi_2}\ , \qquad (6.95)$$

where we have made the identifications $\xi_1 \leftrightarrow \xi$, $\xi_2 \leftrightarrow \xi_o$ between the notation used in the two problems.

Solving the quadratic equation (6.95) and discarding the non-optimal root $r=r_o=1$, we find that

$$\kappa = \frac{\xi_2}{\xi_1}\ , \ \kappa_o = \frac{\xi_1}{\xi_2} \tag{6.96}$$

Noting that $\lambda_1/\lambda_2 = \kappa/\kappa_o = (\xi_2/\xi_1)^2$ we recall from (6.9) that this is precisely the intensity function ratio created by the minimum error probability receiver.

We have not quite proved that the minimum error probability receiver is the optimum feedback receiver for arbitrary symmetric costs C, because of the possibility of multiple local extrema satisfying (6.84). One example for which it is easy to prove sufficiency of (6.84) is

$$C(\xi_1\xi_2) = a\xi_1\xi_2 \quad , \quad a \geq 0 \tag{6.97}$$

For this case,

$$\overline{C}(\underline{\xi},\tau) = a\xi_1\xi_2\ e^{-[E(T)-E(\tau)]} \tag{6.98}$$

and it is easily shown from (6.98) that

$$\overline{C}''(\underline{\rho},\underline{\xi},\tau) = a(\rho_1-\xi_1)(\rho_2-\xi_2)\ e^{-E(T)+E(\tau)}$$

$$= -\ a(\xi_1\xi_2)^2(\kappa-\kappa_o)^2\ e^{-E(T)+E(\tau)} \tag{6.99}$$

Thus

$$\hat{\lambda}\overline{C}'' = -\ (\sqrt{\kappa}+\sqrt{\kappa_o})^2\ \ a\tilde{\lambda}(\xi_1\xi_2)^2\ e^{-E(T)+E(\tau)}\ , \tag{6.100}$$

which possesses a unique minimum at $\kappa = \frac{\xi_2}{\xi_1}$, $\kappa_o = \frac{\xi_1}{\xi_2}$ because

$$\xi_2\ \frac{d}{dr}\ (\sqrt{\kappa}+\sqrt{\kappa_o})^2 = (\xi_2-\xi_1) + (\kappa\kappa_o)^{-\frac{1}{2}}\ (\xi_2\kappa_o-\xi_1\kappa),\ \kappa\neq 0, \kappa_o\neq 0 \tag{6.101}$$

is zero at this point and

$$\xi_2^{\,2} \quad \frac{d^2}{dr^2} (\sqrt{r}+\sqrt{r_o})^2 = -\frac{1}{2} (r r_o)^{-\frac{3}{2}} \quad < \quad 0, \quad r\neq 0, r_o\neq 0, \qquad (6.102)$$

and the potential solutions $r=0$, $r_o=0$ are inferior since $\frac{1}{\xi_1\xi_2} > \max\left(\frac{1}{\xi_1}, \frac{1}{\xi_2}\right)$.

In particular, this verifies that the minimum error probability feedback is an interval optimum feedback function for the binary coherent state MMSE estimation problem in Section 6.3.1b.

CHAPTER VII

CONCLUSIONS

In this thesis we have defined an (ideally) implementable class of optical receivers which realize a certain subclass of the possible optical field measurements consistent with quantum mechanics. We characterized the measurement statistics of these receivers as a regular point process with a partially controllable intensity function, and we found conditions for determining the optimum receiver within this class. We also described a useful sub-optimum criterion. Then we characterized the realized subclass of quantum measurements in quantum communication terminology. Finally we presented a few simple communication problems for which we were able to demonstrate improved performance.

All of our results leave related questions unresolved. The optimality condition of Chapter III represents a complete answer to the feedback optimization problem posed there, but, as is typical with the dynamic programming method, the extra dimensionality resulting from having to solve a whole family of problems simultaneously is a prohibitive obstacle to explicit solution. This remark is even more applicable to realistic problems for which the finite number of states M should in fact be replaced by a continuum. It may be possible to obtain a simpler optimality condition by more fully exploiting the point process structure to discover a sufficient statistic less complex than the entire vector of a posteriori probabilities $\underline{\xi}(\tau;\underline{t})$.

In Chapter IV the optimum feedback receiver was shown, under certain conditions, to perform as well as the optimum quantum-consistent measurement realizable as a contingent sequence of arbitrary measurements performed on (ordered) infinitesimal time-samples of the received field. This correspondence is significant because it provides a bridge between the two analytical approaches to receiver design discussed in Section 1.1.2. Our thesis did not determine the least restrictive conditions under which the correspondence is valid, nor did it consider the quantum communication question of what class of measurements can be realized as contingent sequences of small time-sample measurements.

The main advantage of the (sub-optimum) incremental optimality criterion of Chapter V is that it allows the measurement (feedback level) at each time to be determined by minimizing an explicit function of the prior data. This is in contrast with the interval optimality criterion of Chapter III which requires that an optimum measurement <u>strategy</u> be pre-determined to cover all potential observations, before any data is recorded. We have found in Chapter VI that the incrementally optimum receiver can be interval optimum for some problems, but we have not determined <u>general</u> conditions under which it is optimum or near-optimum. It seems to be a reasonable optimality criterion for the causal estimation problem, and it may be worthwhile to continue our analysis of Example 5.3 and to investigate its connection with the work of Baras and Harger.[21]

Some of the specialized results of Chapter VI are significant. The optimum feedback receiver for the binary coherent state problem provides the first precise realization of the optimum quantum measurement by a structured receiver using conventional components. However, this result is of little practical benefit, because a simpler exponentially optimum structured receiver is already known[22] for this problem. Our near-optimum receiver for the M-ary PPM problem represents the first realization of an exponentially optimum quantum measurement and offers a 3 dB energy improvement over other known conventional receivers. Since we were able to obtain these results without directly applying the Chapter III interval optimality condition, it is likely that an interested researcher might discover other examples for which good feedback receivers can be determined by special techniques. Finally, we point out that a computer-aided solution of the simplified dynamic programming optimality condition of Section 6.5 may be feasible for the ternary (M=3) PPM problem or the binary problem with dark current. The latter would be useful in evaluating the sensitivity of our optimum binary receiver, and the former would furnish another opportunity to check whether optimum quantum performance can be precisely achieved by a feedback receiver.

We have formed no conclusions about the ultimate practicality of feedback receivers. Presently it appears that they rival abstract quantum measurements in analytical complexity, and we have not considered the engineering question of whether actual implementation of a feedback receiver is any less unrealistic. As we stated in the opening chapter, our approach to optical receiver design is a compromise between the two normally taken, representing intermediate levels of analytical difficulty, implementability, and attainable performance. So far, insufficient evidence has been accumulated to determine whether feedback receivers accomplish a reasonable trade-off among these factors.

APPENDIX TO CHAPTER III

3A.1 PROOF OF LEMMA 3.1

Decomposing $\underline{t}''$ into $(\underline{t},\underline{t}')$ we have

$$E_{j,\underline{t}''}\, C[j,\hat{j}(\underline{t}'')] = E_{\underline{t}}\, E_{j,\underline{t}'|\underline{t}} \left\{ C[j,\hat{j}(\underline{t},\underline{t}')] \,|\, 0<\underline{t}<\tau \right\}. \quad (3A.1)$$

For any fixed past feedback $\ell:[0,\tau]^+ \rightarrow \mathbb{C}$, we shall establish that the inner expectation for each $\underline{t}$ is a functional just of $\ell(\cdot:\ \underline{t},\ \cdot)$ and $\hat{j}(\underline{t},\cdot)$ for that same $\underline{t}$. Then the lemma follows directly from the remarks immediately preceding it because it is necessary and sufficient to minimize the inner expectation pointwise for each $\underline{t}\varepsilon[0,\tau]^*$, except on a set of measure zero.

(i) The dependence of the inner expectation on $\hat{j}(\cdot,\cdot)$ is through the cost function $C[j,\hat{j}(\underline{t},\cdot)]$ and thus for each $\underline{t}$ $E_{j,\underline{t}'|\underline{t}} \left\{ C[j,\hat{j}(\underline{t},\underline{t}')] \,|\, 0<\underline{t}<\tau \right\}$ is a functional of $\hat{j}(\underline{t},\cdot)$ only.

(ii) The dependence of the inner expectation on $\ell(\cdot:\cdot,\cdot)$ is through the conditional statistics of j and $\underline{t}'$ given $\underline{t}$. From (3.5), the conditional probabilities $\xi_j(\tau:\underline{t})$ depend only on the past feedback $\ell(\cdot:\cdot) : [0,\tau]^+ \rightarrow \mathbb{C}$. The conditional statistics of $\underline{t}'$ given $\underline{t}$ and j can be determined from the conditional probability density which is the ratio of two expressions of the form (3.4),

$$\frac{p_j(T:\underline{t},\underline{t}')}{p_j(\tau:\underline{t})} = \prod_i \lambda_j(t_i':\underline{t},\underline{t}_{i-1}') \prod_i \exp\left[-\int_{\Sigma} d\vec{r} \int_{\tau_{i-1}'}^{\tau_i'} d\tau'\ \lambda_j(\vec{r},\tau':\underline{t},\underline{t}_{i-1}') \right]$$

(3A.2)

This expression depends on the future feedback $\ell(\cdot:\cdot,\cdot)$ just through $\ell(\cdot:\underline{t},\cdot)$, and not at all on the past feedback.

QED

3A.2 PROOF OF LEMMA 3.2

From (ii) in the proof of Lemma 3.1 it is evident that the only dependence of the conditional statistics of j and $\underline{t}'$ given $\underline{t}$ on $\underline{\xi}$, $\underline{t}$ and $\ell(\cdot:\cdot):[0,\tau]^{+} \to \not{C}$ is through $\underline{\xi}(\tau:\underline{t})$. From Lemma 3.1, $j^{*}(\underline{t},\cdot)$ and $\ell^{*}(\cdot:\underline{t},\cdot)$ are chosen to jointly minimize $E_{j,\underline{t}'|t}\left\{C[j,\hat{j}(\underline{t},\underline{t}')]\,|\,0<\underline{t}<\tau\right\}$, a functional of $\hat{j}(\underline{t},\cdot)$, $\ell(\cdot:\underline{t},\cdot)$, and $\underline{\xi}(\tau:\underline{t})$ only. Therefore the joint solution of the minimization problem, $j^{*}(\underline{t},\cdot)$, $\ell^{*}(\cdot:\underline{t},\cdot)$, depends only on $\underline{\xi}(\tau:\underline{t})$.

QED

3A.3 PROOF OF LEMMA 3.3

Write $\overline{C}(\underline{\xi},\tau) = E_{\underline{t}}\, E_{j,\underline{t}'|\underline{t}}\, C\Big[j,\hat{j}[\underline{\xi}(T:\underline{t},\underline{t}';\underline{\xi},\tau)]\Big]$,

$$\tau<\underline{t}<\tau+\Delta<\underline{t}'<T \qquad (3A.3)$$

and

$$\overline{C}\Big[\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\tau+\Delta\Big] = E_{j,\underline{t}'}\, C\Big[j,\hat{j}[\underline{\xi}(T:\underline{t}';\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\tau+\Delta)]\Big]$$

$$\tau<\underline{t}<\tau+\Delta<\underline{t}'<T \qquad (3A.4)$$

It is obvious from the state propagation equations (3.6)-(3.8) that the particular initial condition used in (3A.4) guarantees that the decision function $\hat{j}$ is evaluated at the same probability vector on the right hand sides of both (3A.3) and (3A.4); i.e.,

$$\underline{\xi}(T:\underline{t}';\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\tau+\Delta) = \underline{\xi}(T:\underline{t},\underline{t}';\underline{\xi},\tau) \qquad (3A.5)$$

Thus, (3A.4) may be replaced by

$$\overline{C}\Big[\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\ \tau+\Delta\Big] = E_{j,\underline{t}'}\, C\Big[j,\hat{j}[\underline{\xi}(T:\underline{t},\underline{t}';\underline{\xi},\tau)]\Big]$$

$$\tau<\underline{t}<\tau+\Delta<\underline{t}'<T \qquad (3A.6)$$

Conditioned on j, the statistics of the event location vectors $\underline{t}'$ that appear in (3A.3) and (3A.6) are identical; the conditional probability density for both is given by the expression (3A.2) with τ replaced by $\tau+\Delta$. Furthermore, the unconditional statistics of j in (3A.6) and the conditional (on $\underline{t}$) statistics of j in (3A.3) are both given by

$\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau)$, regarded as the a priori probability vector in (3A.6) and as the a posteriori probability vector (given $\underline{t}$) in (3A.3). Thus the expectation in (3A.6) and the inner one in (3A.3) are with respect to the same probability measure and hence they are equal.

QED

3A.4 PROOF OF THEOREM 3.1

We first prove a trivial lemma.

Lemma 3A.1 $\overline{C}(\cdot,\cdot)$ is uniformly bounded, i.e., $\overline{C}(\underline{\xi},\tau) \leq C_{max} < \infty$ for all $\underline{\xi}\varepsilon P$, $\tau\varepsilon[0,T]$.

Proof:

$$\text{Let } C_{max} \equiv \max_{j,\hat{j}\varepsilon\hat{R}} |C(j,\hat{j})| \qquad \hat{R} \text{ finite} \tag{3A.7}$$

then

$$\left|\overline{C}(\underline{\xi},T)\right| = \left|\sum_{j=1}^{M} \xi_j \; C(j,\hat{j}(\underline{\xi})\right| \leq C_{max} \tag{3A.8}$$

and from (3.13), (3.14)

$$\left|\overline{C}(\underline{\xi},\tau)\right| = \left|E_{\underline{\xi}'} \; \overline{C}(\underline{\xi}',T)\right| \leq C_{max} \tag{3A.9}$$

where $\underline{\xi}' = \underline{\xi}(T:\underline{t}';\underline{\xi},\tau)$, $\tau<\underline{t}'<T$

QED

Then Theorem 3.1 is proved as follows:

(a) Using (3.15), (3.19), (3.20), (3.21) and Lemma 3A.1 to evaluate (3.12), we have

$$\begin{aligned} E_{\underline{t}} \; \overline{C}\left[\underline{\xi}(\tau+\Delta:\underline{t};\underline{\xi},\tau),\tau+\Delta\right] &= \int_{\Sigma} d\vec{r}_1 \int_{\tau}^{\tau+\Delta} d\tau_1 \; \hat{\lambda}(\vec{r}_1,\tau_1,\underline{\xi}) \; \overline{C}\left[\underline{\rho}(\vec{r}_1,\tau_1,\underline{\xi}),\tau+\Delta\right] \\ &+ \left[1-\Delta \int_{\Sigma} d\vec{r} \; \hat{\lambda}(\vec{r},\tau,\underline{\xi})\right] \overline{C}\left[\underline{\xi}-\Delta\int_{\Sigma} d\vec{r} \; \hat{\lambda}(\vec{r},\tau,\underline{\xi})(\underline{\rho}(\vec{r},\tau,\underline{\xi})- \underline{\xi}) + \underline{o}(\Delta), \; \tau+\Delta\right] \\ &+ o(\Delta) \end{aligned} \tag{3A.10}$$

For small enough Δ, $\hat{\lambda}(\vec{r}_1,\tau_1,\underline{\xi})$ and $\underline{\rho}(\vec{r}_1,\tau_1,\underline{\xi})$ are continuous for $\tau_1 \in [\tau,\tau+\Delta]$ (this fact follows from the left-continuity condition on the corresponding event-dependent intensities $\lambda_j(\cdot)$). Therefore the inner integral in the first term in (3A.10) is

$$\int_{\tau}^{\tau+\Delta} d\tau_1 \; \hat{\lambda}(\vec{r}_1,\tau_1,\underline{\xi}) \; \overline{C}[\underline{\rho}(\vec{r}_1,\tau_1,\underline{\xi}),\tau+\Delta] = \Delta \; \hat{\lambda}(\vec{r}_1,\tau,\underline{\xi}) \; \overline{C}[\underline{\rho}(\vec{r}_1,\tau,\underline{\xi}),\tau+\Delta] + o(\Delta) \qquad (3A.11)$$

Using definition (3.23) we may write $\overline{C}(\cdot)$ in the second term of (3A.10) as

$$\overline{C}[\underline{\xi} - \Delta\int_{\Sigma} d\vec{r} \; \hat{\lambda}(\vec{r},\tau,\underline{\xi})(\underline{\rho}(\vec{r},\tau,\underline{\xi})-\underline{\xi}) + \underline{o}(\Delta), \; \tau+\Delta]$$

$$= \overline{C}(\underline{\xi},\tau+\Delta) + \Delta\overline{C}'(\underline{\xi},\underline{x},\tau+\Delta) + o(\Delta) \qquad (3A.12)$$

where $\underline{x} = - \int_{\Sigma} d\vec{r} \; \hat{\lambda}(\vec{r},\tau,\underline{\xi})[\underline{\rho}(\vec{r},\tau,\underline{\xi})-\underline{\xi}]$. Finally, two of the integrations over dummy variables $\vec{r}_1,\vec{r}$ may be combined to give the result (3.25).

(b) Replace τ and $\tau+\Delta$ in (a) with $\tau-\Delta$ and τ respectively.

$$\overline{C}(\underline{\xi},\tau-\Delta)-\overline{C}(\underline{\xi},\tau) = \Delta \int_{\Sigma} d\vec{r} \; \hat{\lambda}(\vec{r},\tau-\Delta,\underline{\xi}) \left\{\overline{C}[\underline{\rho}(\vec{r},\tau-\Delta,\underline{\xi}),\tau] - \overline{C}(\underline{\xi},\tau)\right\}$$

$$+ \Delta \; \overline{C}'[\underline{\xi},-\int_{\Sigma} d\vec{r} \; \hat{\lambda}(\vec{r},\tau-\Delta,\underline{\xi})(\underline{\rho}(\vec{r},\tau-\Delta,\underline{\xi})-\underline{\xi}), \; \tau] + o(\Delta)$$

(3A.13)

Since $\dfrac{\partial \overline{C}(\underline{\xi},\tau)}{\partial \underline{\xi}}$ exists, $\overline{C}'(\cdot)$ may be evaluated from (3.24) as

$$\overline{C}'\left[\underline{\xi}, -\int_{\Sigma} d\vec{r}\ \hat{\lambda}(\vec{r},\tau-\Delta,\underline{\xi})(\underline{\rho}(\vec{r},\tau-\Delta,\underline{\xi})-\underline{\xi}),\ \tau\right]$$

$$= \int_{\Sigma} d\vec{r}\ \hat{\lambda}(\vec{r},\tau-\Delta,\underline{\xi})(\underline{\rho}(\vec{r},\tau-\Delta,\underline{\xi})-\underline{\xi})^T \frac{\partial \overline{C}(\underline{\xi},\tau)}{\partial \underline{\xi}} + \frac{o(\Delta)}{\Delta} \tag{3A.14}$$

Dividing (3A.13) by Δ as taking the limit as $\Delta \downarrow 0$ yields (3.27).

(c) Replace $\tau-\Delta$ and τ in (3A.13) and (3A.14) with τ and $\tau+\Delta$ respectively. Then (3.31) follows by taking $\Delta \downarrow 0$ after division by Δ.

(d) From (c) and (b) the left- and right-hand derivatives both exist and are equal, so $\overline{C}(\underline{\xi},\tau)$ is differentiable.

QED

3A.5 PROOF OF LEMMA 3.4

(if) Take $\tau=\tau_o$. Then $\underline{\xi}_o \varepsilon P_{\tau_o}$ because $\Pr\left[\underline{\xi}(\tau_o:\underline{t};\underline{\xi}_o,\tau_o)=\underline{\xi}_o\right] = 1$.
(only if) Suppose, for some τ, that the set P_τ of probability vectors $\underline{\xi}$ for which ℓ^*_τ minimizes $\overline{C}(\underline{\xi},\tau)$ is not large enough; i.e., $\Pr\left[\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o)\ \varepsilon\ P_\tau\right] < 1$. Then construct a new <u>event-dependent</u> feedback function for times after τ_o as follows: For all times $\tau'<\tau$, define $\ell(\vec{r},\tau':\underline{t}) \equiv \ell^*(\vec{r},\tau':\underline{t})$. For any time $\tau' \geq \tau$ decompose the event vector into components $(\underline{t},\underline{t}')$ referring to locations of events which occur before τ and after τ respectively. The definition of $\ell(\vec{r},\tau':\underline{t},\underline{t}')$ will depend on whether or not $\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o)\ \varepsilon\ P_\tau$. If $\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o)\ \varepsilon\ P_\tau$, define $\ell(\vec{r},\tau':\underline{t},\underline{t}') \equiv \ell^*(r,\tau':\underline{t},\underline{t}')$. If on the other hand the partial event vector $\underline{t}$ is such that $\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o) \notin P_\tau$, it is possible to find a state-dependent feedback function $\ell^{\underline{t}} : \Sigma x[\tau,T]\ xP \rightarrow ¢$ which results in a lower average cost $\overline{C}\left[\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o),\ \tau\right]$ than ℓ^*_τ; in this case define the event-dependent feedback by $\ell(\vec{r},\tau':\underline{t},\underline{t}') = \ell^{\underline{t}}\left[\vec{r},\tau',\underline{\xi}(\tau':\underline{t}';\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o),\ \tau)\right]$. Denote by $\overline{C}$ the average cost function for the event-dependent feedback so constructed, and by $\overline{C}^*$ the average cost for ℓ^*. Then

$$\overline{C}^*(\underline{\xi}_o,\tau_o) - \overline{C}(\underline{\xi}_o,\tau_o) = \underset{\underline{t}\ \varepsilon\ [\tau_o,\tau]^*}{E_{\underline{t}}} \left\{\overline{C}^*\left[\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o),\ \tau\right] - \overline{C}\left[\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o),\tau\right]\right\} \tag{3A.15}$$

This follows from (3.12) and the fact that ℓ and ℓ^* are identical for times $\tau' < \tau$, which implies that the statistics of $\underline{t}$ are the same for both ℓ and ℓ^*. Since ℓ was chosen to make $\overline{C}(\cdot,\tau) \leq \overline{C}^*(\cdot,\tau)$, (3A.15) implies that $\overline{C}^*(\underline{\xi}_o,\tau_o)$ is non-optimal unless $\overline{C}(\cdot,\tau) = \overline{C}^*(\cdot,\tau)$ almost everywhere. But by construction of ℓ, $\overline{C}(\underline{\xi},\tau) < \overline{C}^*(\underline{\xi},\tau)$ whenever $\underline{\xi} \notin P_\tau$. Thus $\overline{C}^*(\underline{\xi}_o,\tau_o)$ is optimal only if $\Pr\left[\underline{\xi}(\tau:\underline{t};\underline{\xi}_o,\tau_o) \notin P_\tau\right] = 0$.

<u>QED</u>

3A.6 PROOF OF LEMMA 3.5

a.(only if) $\overline{C}(\underline{\xi},\tau)$ must be minimized for $\tau=T$ and all $\underline{\xi}$ ε P.
(if) Let $j^*(\underline{\xi})$ satisfy (3.33) for all $\underline{\xi}$ ε P and let $\hat{j}(\cdot)$ be an arbitrary decision function. Denote the average cost functions associated with j^* and ℓ and with $\hat{j}$ and ℓ by $\overline{C}^*(\underline{\xi},\tau)$ and $\overline{C}(\underline{\xi},\tau)$ respectively. Then, from (3.12),

$$\overline{C}^*(\underline{\xi},\tau) = E_{\underline{t}} \; \overline{C}^*\left[\underline{\xi}(T:\underline{t};\underline{\xi},\tau),\ T\right], \qquad \underline{t}\ \varepsilon[\tau,T]^*$$

$$\leq E_{\underline{t}} \; \overline{C}\left[\underline{\xi}(T:\underline{t};\underline{\xi},\tau),\ T\right] = \overline{C}(\underline{\xi},\tau), \tag{3A.16}$$

since from (3.33) $\overline{C}^*(\cdot,T) \leq \overline{C}(\cdot,T)$.

b.(if) Let $j^*(\underline{\xi})$ satisfy (3.33) for $\underline{\xi}$ ε $\hat{P} \subseteq P$ and let $\hat{j}(\cdot)$ be an arbitrary decision function. As in (a), let $\overline{C}^*$, $\overline{C}$ denote the corresponding average cost functions. Then if (3.34) is satisfied,

$$\overline{C}^*(\underline{\xi}_o,\tau_o) = E_{\underline{t}} \; \overline{C}^*\left[\underline{\xi}(T:\underline{t};\underline{\xi}_o,\tau_o),\ T\right], \qquad \underline{t}\ \varepsilon[0,T]^*$$

$$\leq E_{\underline{t}} \; \overline{C}\left[\underline{\xi}(T:\underline{t};\underline{\xi}_o,\tau_o),\ T\right] \equiv \overline{C}(\underline{\xi}_o,\tau_o), \tag{3A.17}$$

because $\overline{C}^*(\cdot,T) \leq \overline{C}(\cdot,T)$ almost everywhere.
(only if) Conversely if $\overline{C}^*(\underline{\xi}_o,\tau_o)$ is minimum, construct a particular $\hat{j}$ by defining $\hat{j}(\underline{\xi}) \equiv j^*(\underline{\xi})$ for $\underline{\xi}$ ε $\hat{P}$ and choosing $\hat{j}(\underline{\xi})$ to not satisfy (3.34) for all $\underline{\xi} \notin \hat{P}$. Again denote the corresponding average costs by $\overline{C}^*$, $\overline{C}$.

Then

$$\overline{C}^{*}(\underline{\xi}_o,\tau_o) - \overline{C}(\underline{\xi}_o,\tau_o) = E_{\underline{t}}\Big\{\overline{C}^{*}\Big[\underline{\xi}(T:\underline{t};\underline{\xi}_o,\tau_o),T\Big]$$

$$- \overline{C}\Big[\underline{\xi}(T:\underline{t};\underline{\xi}_o,\tau_o), T\Big]\Big\} \qquad (3A.18)$$

is nonnegative because $\hat{j}$ was chosen to make $\overline{C}(\cdot,T) \leq \overline{C}^{*}(\cdot,T)$, and is strictly positive unless $\overline{C}(\cdot,T) = \overline{C}^{*}(\cdot,T)$ almost everywhere. Therefore this condition is necessary if $\overline{C}^{*}(\underline{\xi}_o,\tau_o)$ is minimal.

QED

3A.7 PROOF OF LEMMA 3.6

(I) First consider the case τ=T. Since j^* is uniformly optimum, Lemma 3.5a implies that for every $\underline{\xi} \in P$,

$$\bar{C}^*(\underline{\xi},T) = \sum_{j=1}^{M} \xi_j \; C\left[j,j^*(\underline{\xi})\right] \leq \sum_{j=1}^{M} \xi_j \; C\left[j,k\right] \text{ for all } k\varepsilon \hat{R} \tag{3A.19}$$

A solution to (3A.19) always exists because $\hat{R}$ is finite. For some $\underline{\xi} \in P$ there may be several solutions. For any $\underline{x}$, $\underline{o}(\Delta)$ with $\underline{\xi}+\Delta\underline{x} + \underline{o}(\Delta) \in P$ and $\lim_{\Delta\to 0} \frac{|\underline{o}(\Delta)|}{\Delta} = 0$, there is always a simultaneously optimum decision function $j^*(\underline{\xi},\underline{x})$ which essentially solves (3A.19) for all probability vectors $\underline{\xi}+\Delta\underline{x} + \underline{o}(\Delta)$ if Δ is small enough and positive, $0\leq\Delta\leq\delta$, $\delta>0$. This can be shown in the following manner:

(i) If $\hat{j}(\underline{\xi}) = k$ is non-optimum for $\underline{\xi}$, then (3A.19) is a strict inequality for any optimum $j^*(\underline{\xi})$. Let $\gamma>0$ equal the difference,

$$\gamma \equiv \sum_{j=1}^{M} \xi_j \; C(j,k) - \sum_{j=1}^{M} \xi_j \; C(j,j^*(\underline{\xi})) \tag{3A.20}$$

Then

$$\sum_{j=1}^{M} \left[\xi_j+\Delta x_j+o_j(\Delta)\right] \; C(j,k) - \sum_{j=1}^{M} \left[\xi_j+\Delta x_j+o_j(\Delta)\right] \; C(j,j^*(\underline{\xi}))$$

$$= \gamma + \sum_{j=1}^{M} \left[\Delta x_j+o_j(\Delta)\right] \; \left[C(j,k) - C(j,j^*(\underline{\xi}))\right] \tag{3A.21}$$

Since $\left|\Delta x_j+o_j(\Delta)\right| \leq 1$ and $\left|C(\cdot,\cdot)\right| \leq C_{max}$, the right hand side is positive for small enough Δ. This shows that, for small enough Δ (not necessarily positive), the optimum decision for a priori probabilities $\underline{\xi}+\Delta\underline{x} + \underline{o}(\Delta)$ must equal one of the optimum decisions for a priori probabilities $\underline{\xi}$.

(ii) Let j^* simultaneously optimize $\overline{C}(\underline{\xi},T)$ and $\overline{C}(\underline{\xi}',T)$ for $\underline{\xi},\underline{\xi}'\varepsilon$ P. Then j^* also optimizes $\overline{C}[(1-\lambda)\underline{\xi}+\lambda\underline{\xi}',T]$ for all $\lambda\varepsilon[0,1]$; i.e., for all a priori probability vectors on the straight line segment joining $\underline{\xi}$ and $\underline{\xi}'$, because for any $k \varepsilon \hat{R}$:

$$\sum_{j=1}^{M}\left[(1-\lambda)\xi_j + \lambda\xi_j'\right] C(j,k) = (1-\lambda)\sum_{j=1}^{M}\xi_j\ C(j,k) + \lambda\sum_{j=1}^{M}\xi_j'\ C(j,k)$$

$$\geq (1-\lambda)\sum_{j=1}^{M}\xi_j\ C(j,j^*) + \lambda\sum_{j=1}^{M}\xi_j'\ C(j,j^*)$$

$$= \sum_{j=1}^{M}\left[(1-\lambda)\xi_j + \lambda\xi_j'\right] C(j,j^*) \qquad (3A.22)$$

(iii) Let δ_1 be any positive number which causes the right side of (3A.21) to be positive for all $0\leq\Delta\leq\delta_1$ and all non-optimal k. Let δ_2 be any positive number which achieves the same result when $\underline{o}(\Delta)$ is replaced by 0 in (3A.21). Now let $\delta=\min(\delta_1,\delta_2)$. There is a decision $j^*(\underline{\xi},\underline{x})$ which is simultaneously optimum for $\underline{\xi}$ and for $\underline{\xi}+\delta\underline{x}$. By (ii), it is also optimum for all probability vectors on the straight line between them, $\underline{\xi}+\Delta\underline{x}$, $0\leq\Delta\leq\delta$.

(iv) By choice of δ in (iii), there are also decisions $j^*(\Delta)$ which are (pairwise) simultaneously optimum for $\underline{\xi}$ and $\underline{\xi}+\Delta\underline{x} + \underline{o}(\Delta)$, for any $0 \leq \Delta \leq \delta$. Although $j^*(\Delta)$ does not necessarily equal $j^*(\underline{\xi},\underline{x})$ for all Δ, the additional cost which results from using $j^*(\underline{\xi},\underline{x})$ in place of $j^*(\Delta)$ is small. Specifically,

$$\sum_{j=1}^{M}\left[\xi_j+\Delta x_j\right] C\left[j,j^*(\underline{\xi},\underline{x})\right] \leq \sum_{j=1}^{M}\left[\xi_j+\Delta x_j\right] C\left[j,j^*(\Delta)\right] \tag{3A.23}$$

and

$$\sum_{j=1}^{M}\left[\xi_j+\Delta x_j+o_j(\Delta)\right] C\left[j,j^*(\Delta)\right] \leq \sum_{j=1}^{M}\left[\xi_j+\Delta x_j+o_j(\Delta)\right] C\left[j,j^*(\underline{\xi},\underline{x})\right]$$

since $j^*(\underline{\xi},\underline{x})$ and $j^*(\Delta)$ are optimum for $\underline{\xi}+\Delta\underline{x}$ and $\underline{\xi}+\Delta\underline{x} + \underline{o}(\Delta)$ respectively. We note that $\sum_{j=1}^{M} \xi_j \; C\left[j,j^*(\Delta)\right] = \sum_{j=1}^{M} \xi_j \; C\left[j,j^*(\underline{\xi},\underline{x})\right]$ because $j^*(\Delta)$ and $j^*(\underline{\xi},\underline{x})$ were both chosen to be optimum for probability vector $\underline{\xi}$. After subtracting these common terms and dividing by Δ in (3A.23), we find that

$$\left|\sum_{j=1}^{M} x_j\left\{ C\left[j,j^*(\Delta)\right] - C\left[j,j^*(\underline{\xi},\underline{x})\right]\right\}\right| \leq \frac{o(\Delta)}{\Delta} \tag{3A.24}$$

since $\left|C(\cdot,\cdot)\right| \leq C_{max}$.

(v) Thus for $0<\Delta<\delta$,

$$\frac{\overline{C}(\underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta),\ T)-\overline{C}^*(\underline{\xi},T)}{\Delta}=\frac{1}{\Delta}\left\{\sum_{j=1}^{M}\left[\xi_j+\Delta x_j+o(\Delta)\right]C\left[j,j^*(\Delta)\right]-\sum_{j=1}^{M}\xi_j\ C\left[j,j^*(\underline{\xi},\underline{x})\right]\right\}$$

$$=\sum_{j=1}^{M}x_j\ C\left[j,j^*(\Delta)\right]+\frac{o(\Delta)}{\Delta},\quad \text{since } \sum_{j=1}^{M}\xi_j C\left[j,j^*(\Delta)\right]=\sum_{j=1}^{M}\xi_j C\left[j,j^*(\underline{\xi},\underline{x})\right]$$

$$\text{and } \left|C(\cdot,\cdot)\right|\leq C_{max}$$

$$=\sum_{j=1}^{M}x_j\ C\left[j,j^*(\underline{\xi},\underline{x})\right]+\frac{o(\Delta)}{\Delta}$$

from (3A.24), so

$\overline{C}^{*'}(\underline{\xi},\underline{x},T)\equiv\lim_{\Delta\downarrow 0}\dfrac{\overline{C}^*(\underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta),T)-\overline{C}^*(\underline{\xi},T)}{\Delta}$ exists and can be evaluated as

$$\overline{C}^{*'}(\underline{\xi},\underline{x},T)=\sum_{j=1}^{M}x_j\ C\left[j,j^*(\underline{\xi},\underline{x})\right]\qquad(3A.25)$$

(II) This proves the lemma for $\tau=T$. The result for smaller τ can be proved by using (3.12) with $\tau+\Delta$ replaced by T.

$$\overline{C}^*[\underline{\xi},\tau] = E_{\underline{t}}\, \overline{C}^*[\underline{\xi}(T:\underline{t}\ ;\ \underline{\xi},\tau),T] \quad , \qquad \underline{t}\varepsilon[\tau,T]^*$$
$$\overline{C}^*[\underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta),\tau] = E_{\underline{t}}\, \overline{C}^*[\underline{\xi}(T:\underline{t}\ ;\ \underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta),\tau),T] \ , \quad \underline{t}\varepsilon[\tau,T]^* \tag{3A.26}$$

We define

$$\underline{\xi}(T:\underline{t}\ ;\ \underline{\xi},\tau) \equiv \underline{\xi}'(\underline{t})$$

and (3A.27)

$$\underline{\xi}(T:\underline{t}\ ;\ \underline{\xi}+\Delta\underline{x}+\underline{o}(\Delta),\tau) \equiv \underline{\xi}'(\underline{t}) + \underline{\delta\xi}'(\underline{t})$$

In terms of $\underline{\xi}'(\underline{t})$ and $\underline{\delta\xi}'(\underline{t})$, the difference between the costs in (3A.26) may be written as

$$\frac{\overline{C}^*[\underline{\xi} + \Delta\underline{x} + \underline{o}(\Delta),\tau] - \overline{C}^*(\underline{\xi},\tau)}{\Delta} = E_{\underline{t}} \left\{ \frac{\overline{C}^*[\underline{\xi}'(\underline{t}) + \underline{\delta\xi}'(\underline{t}),T] - \overline{C}^*[\underline{\xi}'(\underline{t}),T]}{\Delta} \right\} \tag{3A.28}$$

The term inside the braces in (3A.28) is bounded by

$\dfrac{C_{max}}{\Delta} \sum\limits_{j=1}^{M} |\delta\xi_j'(\underline{t})|$, because

$$\overline{C}^*[\underline{\xi}'+ \underline{\delta\xi}',T] - \overline{C}^*[\underline{\xi}',T] \leq \sum_{j=1}^{M}(\xi_j' + \delta\xi_j')C[j,j^*(\underline{\xi}')] - \sum_{j=1}^{M} \xi_j'\, C[j,j^*(\underline{\xi}')]$$

$$= \sum_{j=1}^{M} \delta\xi_j'\, C[j,j^*(\underline{\xi}')] \tag{3A.29}$$

and

$$\overline{C}^*[\underline{\xi}',T] - \overline{C}^*[\underline{\xi}' + \underline{\delta\xi}',T] \leq \sum_{j=1}^{M} \xi_j'\, C[j,j^*(\underline{\xi}' + \underline{\delta\xi}')] - \sum_{j=1}^{M}(\xi_j' + \delta\xi_j')C[j,j^*(\underline{\xi}'+\underline{\delta\xi}')]$$

$$= - \sum_{j=1}^{M} \delta\xi_j'\, C[j,j^*(\underline{\xi}' + \underline{\delta\xi}')] \tag{3A.30}$$

Therefore,

$$|\,\overline{C}^*[\underline{\xi}' + \underline{\delta\xi}',T] - \overline{C}^*[\underline{\xi},T]| \leq \max\left\{\sum_{j=1}^{M}|\delta\xi_j'| \cdot |C[j,j^*(\underline{\xi}')]|\right.,$$

$$\left.\sum_{j=1}^{M} |\delta\xi_j'| \cdot |C[j,j^*(\underline{\xi}' + \underline{\delta\xi}')]|\right\} \leq C_{max} \sum_{j=1}^{M}|\delta\xi_j'| \quad (3A.31)$$

Differentiating $\underline{\xi}(T:\underline{t}\ ;\ \underline{\xi}_o,0)$ with respect to $\underline{\xi}_o$ for an arbitrary initial condition $\underline{\xi}_o \equiv [\xi_1^o, \ldots, \xi_M^o]^T$, we obtain from (3.5)

$$\frac{\partial}{\partial\xi_i^o}\ \xi_j(T:\underline{t};\ \underline{\xi}_o,0) = \begin{cases} \dfrac{1}{\xi_j}\ \xi_j(T:\underline{t};\ \underline{\xi}_o,0)[1 - \xi_j(T:\underline{t};\ \underline{\xi}_o,0)], & i=j \\ \\ -\dfrac{1}{\xi_i}\ \xi_i(T:\underline{t};\ \underline{\xi}_o,0)\xi_j(T:\underline{t};\ \underline{\xi}_o,0), & i\neq j \end{cases} \quad (3A.32)$$

By the mean value theorem,

$$\delta\xi_j'(\underline{t}) = \sum_{i=1}^{M} [\Delta x_i + o_i(\Delta)]\ \frac{\partial}{\partial\xi_i^o}\ \xi_j(T:\underline{t};\ \underline{\xi}_o,0) \quad (3A.33)$$

for some $\underline{\xi}_o$ between $\underline{\xi}$ and $\underline{\xi} + \Delta\underline{x} + \underline{o}(\Delta)$. Choosing Δ small enough that $\dfrac{|\underline{o}(\Delta)|}{\Delta} \leq 1$, we obtain

$$\frac{C_{max}}{\Delta} \sum_{j=1}^{M} |\delta\xi_j'(\underline{t})| \leq \sum_{j=1}^{M} \sum_{i=1}^{M} (|x_i| + 1)\ \frac{\xi_i(T:\underline{t};\ \underline{\xi}_o,0)}{\xi_i} \quad (3A.34)$$

which has finite expectation (over $\underline{t}$) because $E_{\underline{t}}\ \xi_i(T:\underline{t};\ \underline{\xi}_o,0) = \xi_i^o$.

Since, from part (I), the limit as $\Delta\downarrow 0$ of the term in braces in (3A.28) exists for all $\underline{t}$, the dominated convergence theorem may be applied to calculate the limit of the left hand side as the expected value of this limit; i.e., $\lim_{\Delta \downarrow 0} \dfrac{\overline{C}^*[\underline{\xi} + \Delta \underline{x} + \underline{o}(\Delta),\tau] - \overline{C}^*[\underline{\xi},\tau]}{\Delta}$ exists and is given by

$$\overline{C}^{*\prime}[\underline{\xi},\underline{x},\tau] = E_{\underline{t}}\, \overline{C}^{*\prime}[\underline{\xi}'(\underline{t}),\underline{x}'(\underline{t}),T] \quad , \tag{3A.35}$$

where

$$x_j'(\underline{t}) = \sum_{i=1}^{M} x_i \frac{\partial}{\partial \xi_i} \xi_j(T:\underline{t};\ \underline{\xi},0)$$

$$= x_j \frac{\xi_j'(\underline{t})}{\xi_j} - \sum_{i=1}^{M} x_i \frac{\xi_i'(\underline{t})}{\xi_i} \xi_j'(\underline{t}) \tag{3A.36}$$

<u>QED</u>

3A.8 CONVEXITY OF THE OPTIMUM COST

In this section we show that the optimum average cost is a convex $\cap$ function of the a priori probabilities. More precisely, we obtain the following lemma.

Lemma 3A.2. Let $\overline{C}(\underline{\xi},\tau)$ denote the average cost achieved by any feedback function ℓ in conjunction with a uniformly optimum state-dependent decision function j^*. For any $\underline{\xi}''\epsilon P$, $\underline{\xi}'\epsilon P$ and $0\leq\gamma\leq 1$, it follows that $\underline{\xi} \equiv \gamma\underline{\xi}'+(1-\gamma)\underline{\xi}''\epsilon P$ and

$$\overline{C}(\underline{\xi},\tau) \geq \gamma\overline{C}(\underline{\xi}',\tau) + (1-\gamma)\,\overline{C}(\underline{\xi}'',\tau) \tag{3A.37}$$

i.e., $\overline{C}(\underline{\xi},\tau)$ is a convex $\cap$ function of $\underline{\xi}$.

Proof: We first verify (3A.37) for $\tau = T$.

$$\begin{aligned}
\overline{C}(\underline{\xi},T) &= \sum_{j=1}^{M} [\gamma\xi_j' + (1-\gamma)\xi_j'']\, C[j,j^*(\underline{\xi})] \\
&= \gamma \sum_{j=1}^{M} \xi_j'\, C[j,j^*(\underline{\xi})] + (1-\gamma) \sum_{j=1}^{M} \xi_j''\, C[j,j^*(\underline{\xi})] \\
&\geq \gamma \sum_{j=1}^{M} \xi_j'\, C[j,j^*(\underline{\xi}')] + (1-\gamma) \sum_{j=1}^{M} \xi_j''\, C[j,j^*(\underline{\xi}'')] \\
&= \gamma\, \overline{C}(\underline{\xi}',T) + (1-\gamma)\, \overline{C}(\underline{\xi}'',T)
\end{aligned} \tag{3A.38}$$

where the inequality follows from the assumed optimality of $j^*(\underline{\xi}')$ and $j^*(\underline{\xi}'')$ for probability vectors $\underline{\xi}'$ and $\underline{\xi}''$ respectively.

For $\tau < T$ we can use Lemma 3.3 to write

$$\overline{C}(\underline{\xi},\tau) = E_{\underline{t}}\, \overline{C}[\underline{\xi}(T:\underline{t};\, \underline{\xi},\tau),T] \qquad \underline{t}\epsilon[\tau,T]^* \tag{3A.39}$$

Letting $p_j(\underline{t})$ denote the conditional probability density of $\underline{t}$ under H_j (given by an expression similar to (3.4)), we can express $\underline{\xi}(T:\underline{t};\ \underline{\xi},\tau)$ as

$$\xi_j(T:\underline{t};\ \underline{\xi},\tau) = \frac{\xi_j\, p_j(\underline{t})}{\sum_{i=1}^{M} \xi_i p_i(\underline{t})}$$

$$= \gamma'(\underline{t})\ \xi_j(T:\underline{t};\ \underline{\xi}',\tau) + [1-\gamma'(\underline{t})]\ \xi_j(T:\underline{t};\ \underline{\xi}'',\tau) \quad (3A.40)$$

where

$$\gamma'(\underline{t}) \equiv \gamma \frac{\sum_{i=1}^{M} \xi_i'\, p_i(\underline{t})}{\sum_{i=1}^{M} \xi_i\, p_i(\underline{t})}\ ,\ 1-\gamma'(\underline{t}) = (1-\gamma) \frac{\sum_{i=1}^{M} \xi_i''\, p_i(\underline{t})}{\sum_{i=1}^{M} \xi_i\, p_i(\underline{t})} \quad (3A.41)$$

Applying (3A.38) pointwise, we obtain

$$p(\underline{t})\ \overline{C}[\underline{\xi}(T:\underline{t};\ \underline{\xi},\tau),T] \geq \gamma\ p'(\underline{t})\ \overline{C}[\underline{\xi}(T:\underline{t};\ \underline{\xi}',\tau),T]$$

$$+\ (1-\gamma)\ p''(\underline{t})\ \overline{C}[\underline{\xi}(T:\underline{t};\ \underline{\xi}'',\tau),T] \quad (3A.42)$$

where

$$p(\underline{t}) \equiv \sum_{i=1}^{M} \xi_i\ p_i(\underline{t})$$

$$p'(\underline{t}) \equiv \sum_{i=1}^{M} \xi_i'\ p_i(\underline{t}) \quad (3A.43)$$

and

$$p''(\underline{t}) \equiv \sum_{i=1}^{M} \xi_i''\ p_i(\underline{t})$$

are the probability densities for $\underline{t}$ assuming a priori probabilities $\underline{\xi}$, $\underline{\xi}'$, and $\underline{\xi}''$, respectively. By integrating (3A.42) over $\underline{t}$ and applying (3A.39) we obtain (3A.37).

QED

We remark that this proof depends only on the assumed optimality of the state-dependent decision function $j^*(\underline{\xi})$ and not on the conditional probability densities $p_j(\underline{t})$ for the data $\underline{t}$. Thus the lemma is also applicable to average costs obtained by using optimum decisions in conjunction with arbitrary quantum measurements.

3A.9 PROOF OF THEOREM 3.2

We shall write

$$\overline{C}(\underline{\xi},\tau-\Delta) = \overline{C}(\underline{\xi},\tau) + \Delta\, h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})] + o(\Delta) \qquad (3A.44)$$

where $h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})]$ is a linear functional of $\overline{C}(\underline{\cdot},\tau)$ which is also dependent on $\ell(\cdot,\tau^-,\underline{\xi})$. From (3.25) (with τ replaced by $\tau-\Delta$ and $\tau+\Delta$ replaced by τ),

$$h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})] = \int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau^-,\underline{\xi})\ [\overline{C}(\underline{\rho}(\vec{r},\tau^-,\underline{\xi}),\tau) - \overline{C}(\underline{\xi},\tau)]$$
$$+ \overline{C}'[\underline{\xi},-\int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau^-,\underline{\xi})(\underline{\rho}(\vec{r},\tau^-,\underline{\xi})-\underline{\xi}),\tau]$$
$$(3A.45)$$

We must show that ℓ^* is optimum if and **only** if $\ell^*(\cdot,\tau^-,\underline{\xi})$ minimizes $h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})]$ at least for all reachable states $\underline{\xi}$ at each time τ.

Necessity of (3.35) is proved as follows, for the case considered in Theorem 3.2b. Suppose that ℓ^* is optimum for the $\underline{\xi}_0,\tau_0$ problem. Let $\overline{C}^*(\cdot,\cdot)$ denote the cost-to-go function achieved by ℓ^* and $h^*[\cdot,\cdot;\cdot]$ the corresponding cost increment functional in (3A.45). Let P_τ denote the set of states $\underline{\xi}$ for which $\ell^*(\cdot,\tau^-,\underline{\xi})$ minimizes $h^*[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})]$. Then for $\underline{\xi}\notin P_\tau$ we can find $\ell(\cdot,\tau^-,\underline{\xi})$ with $h^*[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})] < h^*[\underline{\xi},\tau;\ell^*(\cdot,\tau^-,\underline{\xi})]$. Letting $\overline{C}(\cdot,\cdot)$ denote the cost-to-go function achieved by the feedback function equal to ℓ^* on $\Sigma x[\tau,T]xP$ and equal to ℓ

on $\Sigma x[\tau-\Delta,\tau]xP$, we have for $\underline{\xi}\notin P_\tau$,

$$\overline{C}(\underline{\xi},\tau-\Delta) = \overline{C}^*(\underline{\xi},\tau) + \Delta\ h^*[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})] + o(\Delta)$$

$$\overline{C}^*(\underline{\xi},\tau-\Delta) = \overline{C}^*(\underline{\xi},\tau) + \Delta\ h^*[\underline{\xi},\tau;\ell^*(\cdot,\tau^-,\underline{\xi})] + o(\Delta) \tag{3A.46}$$

Thus

$$\frac{\overline{C}(\underline{\xi},\tau-\Delta) - \overline{C}^*(\underline{\xi},\tau-\Delta)}{\Delta} = h^*[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})] - h^*[\underline{\xi},\tau;\ell^*(\cdot,\tau^-,\underline{\xi})] + \frac{o(\Delta)}{\Delta} \tag{3A.47}$$

which may be made negative by choosing Δ small enough. Thus, $\underline{\xi}\notin P_\tau$ implies that $\overline{C}^*(\underline{\xi},\tau-\Delta)$ is non-optimum for small enough Δ. But according to Lemma 3.4, $\overline{C}^*(\underline{\xi},\tau-\Delta)$ is optimum for all states $\underline{\xi}$ which are reachable at time $\tau-\Delta$ from $\underline{\xi}_o$ at time τ_o. Therefore, P_τ must include all states reachable at $\tau-\Delta$ for arbitrarily small $\Delta>0$. The necessity proof for Part b is completed by letting $\Delta\downarrow 0$.

In Part a, in order for ℓ^* to be uniformly optimum, it must minimize $\overline{C}(\underline{\xi},\tau)$ for all $\underline{\xi},\tau$. Thus, all states are reachable from some $\underline{\xi},\tau$ and hence $h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})]$ must be minimized for all $\underline{\xi}$ at each time τ.

Sufficiency of (3.35) is proved by subdividing the interval $[\tau_o,T]$ into a large number, N, of subintervals of length $\Delta = (T-\tau_o)\frac{1}{N}$, and using induction. Let $\tau_i \equiv \tau_o + i\Delta$. Suppose we have already shown that $\overline{C}^*(\underline{\xi},\tau_i)$ approximates

$\inf\limits_{\ell} \overline{C}(\underline{\xi},\tau_i)$ uniformly (over reachable states $\underline{\xi}$) within $\sum\limits_{k=i}^{N} o_k(\Delta)$. [This induction hypothesis is obviously satisfied for i=N, because the a priori guess performance $\overline{C}(\underline{\xi},T)$ is independent of ℓ.] Then for an arbitrary ℓ, achieving cost $\overline{C}^{\ell}(\cdot,\cdot)$, we may use Lemma 3.3 to conclude that the cost $\overline{C}(\underline{\xi},\tau_{i-1})$ obtained by replacing ℓ_{τ_i} with $\ell^*_{\tau_i}$ is at worst $\sum\limits_{k=i}^{N} o_k(\Delta)$ greater than $\overline{C}^{\ell}(\underline{\xi},\tau_{i-1})$, if $\underline{\xi},\tau_{i-1}$ is reachable from $\underline{\xi}_o,\tau_o$, because

$$\begin{aligned}
&\overline{C}(\underline{\xi},\tau_{i-1}) - \overline{C}^{\ell}(\underline{\xi},\tau_{i-1}) \\
&\quad = E_{\underline{t}} \left[\overline{C}^*[\underline{\xi}(\tau_i:\underline{t};\underline{\xi},\tau_{i-1}),\tau_i] - \overline{C}^{\ell}[\underline{\xi}(\tau_i:\underline{t};\underline{\xi},\tau_{i-1}),\tau_i]\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{t} \in [\tau_{i-1},\tau_i]^* \\
&\quad \leq \sum_{k=i}^{N} o_k(\Delta) \qquad\qquad (3A.48)
\end{aligned}$$

Next we write from (3A.44)

$$\begin{aligned}
&\overline{C}^*(\underline{\xi},\tau_{i-1}) - \overline{C}(\underline{\xi},\tau_{i-1}) \\
&\quad = \Delta\left\{h^*[\underline{\xi},\tau_i;\ell^*(\cdot,\tau_i^-,\underline{\xi})] - h^*[\underline{\xi},\tau_i;\ell(\cdot,\tau_i^-,\underline{\xi})]\right\} + o_{i-1}(\Delta) \\
&\quad \leq o_{i-1}(\Delta) \qquad\qquad (3A.49)
\end{aligned}$$

because the first term is nonpositive by the assumption that $\ell^*(\cdot,\tau_i^-,\underline{\xi})$ minimizes $h^*[\underline{\xi},\tau_i;\ell(\cdot,\tau_i^-,\underline{\xi})]$. We may easily argue that this approximation is uniform over $\underline{\xi}$ by considering the origin of the $o(\Delta)$ term in (3A.44).

Combining (3A.48) and (3A.49) and taking the infimum over ℓ, we obtain

$$\overline{C}^*(\underline{\xi},\tau_{i-1}) - \inf_{\ell} \overline{C}^{\ell}(\underline{\xi},\tau_{i-1}) \leq \sum_{k=i-1}^{N} o_k(\Delta) \qquad (3A.50)$$

Thus we have verified the induction hypothesis for τ_{i-1}. Continuing, we find that for any $\Delta>0$,

$$\overline{C}^*(\underline{\xi}_o,\tau_o) - \inf_{\ell} \overline{C}^{\ell}(\underline{\xi}_o,\tau_o) \leq \sum_{k=1}^{N} o_k(\Delta) = \frac{o(\Delta)}{\Delta} \qquad (3A.51)$$

By letting $\Delta \downarrow 0$ we conclude that $\overline{C}^*(\underline{\xi}_o,\tau_o)$ is optimum.

<u>QED</u>

3A.10 PROOF OF THEOREM 3.3

When (3.32) is applicable for the cost propagation, the cost increment $h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})]$ defined in (3A.45) in the previous section may be written as

$$h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})] = \int_\Sigma d\vec{r}\ \hat{\lambda}(\vec{r},\tau^-,\underline{\xi})\ \overline{C}''[\underline{\rho}(\vec{r},\tau^-,\underline{\xi}),\underline{\xi},\tau] \tag{3A.52}$$

For each $\vec{r}$, the integrand in (3A.52) can be minimized over K-bounded complex numbers $\ell(\vec{r},\tau^-,\underline{\xi})$ because it is a continuous function of $\ell(\vec{r},\tau^-,\underline{\xi})$ over the compact set $\mathbb{C}_K$. [Continuity follows from the regularity assumptions of Chapter II and the expressions (3.17), (3.18), and (3A.35) for $\hat{\lambda}(\cdot)$, $\underline{\rho}(\cdot)$, and $\overline{C}'(\cdot)$.] The function $\ell^K(\cdot,\tau^-,\underline{\xi})$ determined by this pointwise minimization procedure minimizes $h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})]$.

In the more general case when only the one-sided directional derivatives exist, $h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})]$ can still be minimized. The regularity assumptions of Chapter II imply that the set of K-bounded complex functions $\ell(\cdot,\tau^-,\underline{\xi})$ defined on the closed area Σ is compact under the sup norm. [The sup norm is defined by $||\ell(\cdot,\tau^-,\underline{\xi})-\ell'(\cdot,\tau^-,\underline{\xi})|| = \sup_{\vec{r}\in\Sigma}|\ell(\vec{r},\tau^-,\underline{\xi})-\ell'(\vec{r},\tau^-,\underline{\xi})|$.]
With this norm on $\ell(\cdot,\tau^-,\underline{\xi})$ the functional $h[\underline{\xi},\tau;\ell(\cdot,\tau^-,\underline{\xi})]$ is continuous and therefore achieves its minimizing value over K-bounded complex functions $\ell(\cdot,\tau^-,\underline{\xi})$.

Thus we have shown that $(3.35)_K$ possesses a solution for every $\underline{\xi},\tau$, and K. By the same arguments used in the previous section to prove Theorem 3.2, we may conclude that the feedback function defined over $\vec{r},\tau,\underline{\xi}$ in terms of these optimal solutions for each $\underline{\xi},\tau$ is uniformly optimum _among K-bounded feedback functions_. Similarly, any feedback function minimizing $(3.35)_K$ for all states $\underline{\xi}$ reachable at each time τ from $\underline{\xi}_o,\tau_o$ is optimum _among K-bounded feedback functions_ for the $\underline{\xi}_o,\tau_o$ problem. Since any regular feedback function ℓ is K-bounded for some K, the infimum of $\overline{C}$ over ℓ may be calculated in each case as a limit of the K-bounded optima as $K \to \infty$.

QED

3A.11 PROOF OF THEOREM 3.4

We first prove (i) for both Part a and Part b, by induction, as in the proof of Theorem 3.2. We suppose that $\overline{C}^{\Delta}(\underline{\xi},\tau_i)$ approximates $\inf_{\ell} \overline{C}(\underline{\xi},\tau_i)$ uniformly (over reachable states $\underline{\xi}$) within $\sum_{k=i}^{N} o_k(\Delta)$, and we attempt to verify the same property for τ_{i-1}. Let ℓ be arbitrary and achieve cost $\overline{C}^{\ell}(\cdot,\cdot)$. The cost $\overline{C}(\underline{\xi},\tau_{i-1})$ obtained by replacing ℓ_{τ_i} with $\ell^{\Delta}_{\tau_i}$ is at worst $\sum_{k=i}^{N} o_k(\Delta)$ greater than $\overline{C}^{\ell}(\underline{\xi},\tau_i)$, if $\underline{\xi},\tau_{i-1}$ is reachable from $\underline{\xi}_o,\tau_o$, because from Lemma 3.3,

$$\overline{C}(\underline{\xi},\tau_{i-1}) - \overline{C}^{\ell}(\underline{\xi},\tau_{i-1}) \leq \sum_{k=i}^{N} o_k(\Delta) , \qquad (3A.52)$$

just as in (3A.48). Furthermore, by the same argument that justified (3A.49), we also conclude that

$$\overline{C}^{\Delta}(\underline{\xi},\tau_{i-1}) - \overline{C}(\underline{\xi},\tau_{i-1}) \leq o_{i-1}(\Delta) \qquad (3A.53)$$

Therefore,

$$\overline{C}^{\Delta}(\underline{\xi},\tau_{i-1}) - \inf_{\ell} \overline{C}^{\ell}(\underline{\xi},\tau_{i-1}) \leq \sum_{k=i-1}^{N} o_k(\Delta) \qquad (3A.54)$$

which verifies the induction hypothesis for τ_{i-1}. Continuing, we find that

$$\overline{C}^{\Delta}(\underline{\xi}_o,\tau_o) - \inf_{\ell} \overline{C}^{\ell}(\underline{\xi}_o,\tau_o) \leq \sum_{i=1}^{N} o_i(\Delta) = \frac{o(\Delta)}{\Delta} \qquad (3A.55)$$

and conclude by letting $\Delta\downarrow 0$ that

$$\lim_{\Delta\downarrow 0} \overline{C}^{\Delta}(\underline{\xi}_o,\tau_o) = \inf_{\ell} \overline{C}^{\ell}(\underline{\xi}_o,\tau_o) \tag{3A.56}$$

A similar limiting relationship $(3A.56)_K$ is true when both optimizations are performed over the class of K-bounded regular feedback functions instead of the class of all regular feedback functions. Since all regular feedback functions are K-bounded for some K, both sides of $(3A.56)_K$ may be evaluated as limits of the corresponding K-bounded optima as $K \to \infty$.

QED

APPENDIX TO CHAPTER IV

4A.1 PRELIMINARY LEMMA

In this section we state and prove a simple lemma which is useful in the proofs of the theorems in this chapter.

Lemma 4A.1 The performance of the optimum quantum measurement is unaffected by a common translation of the signal set; i.e., the optimum costs for signal sets $\{S_j(\tau)\varepsilon_o(\vec{r})\}_{j=1}^{M}$ and $\{[S_j(\tau) + S_o(\tau)]\ \varepsilon_o(\vec{r})\}_{j=1}^{M}$, corresponding to the sets of quantum coherent states $\{|\alpha_j>\}_{j=1}^{M}$ and $\{|\alpha_j^o>\}_{j=1}^{M}$ respectively, are identical for any (deterministic) displacement field $S_o(\tau)\varepsilon_o(\vec{r})$.

Proof: The inner products $<\alpha_j^o|\alpha_j^o>$ are calculated in a straightforward manner from (4.4), (4.5) as

$$<\alpha_j^o|\alpha_j^o>$$
$$= e^{-\frac{1}{2}[E_j+E_k-2\gamma_{jk}\sqrt{E_j\,E_k}]}\ e^{i\ \mathrm{Im}[\gamma_{ok}\sqrt{E_oE_k}\ -\ \gamma_{oj}\ \sqrt{E_oE_j}]}$$

$$= (\ |\alpha_j>\ e^{i\ \mathrm{Im}[\gamma_{oj}\sqrt{E_oE_j}]}\ ,\ |\alpha_k>\ e^{i\ \mathrm{Im}[\gamma_{ok}\sqrt{E_oE_k}]}\ ) \tag{4A.1}$$

where

$$\gamma_{oj}\ \sqrt{E_oE_j}\ \equiv \int_0^T d\tau\ S_o^*(\tau)\ S_j(\tau)$$

$$\gamma_{ok}\ \sqrt{E_oE_k}\ \equiv \int_0^T d\tau\ S_o^*(\tau)\ S_k(\tau) \tag{4A.2}$$

$$E_o\ \equiv \int_0^T d\tau|S_o(\tau)|^2$$

Therefore the Hilbert space H' generated by the basis states $\{|\alpha_j^o>\}$ is isomorphic to the space H containing $\{|\alpha_j>\}$. Furthermore, the phase factors $e^{i\,\mathrm{Im}[\gamma_{oj}\sqrt{E_o E_j}]}$ by which the states in H corresponding to $|\alpha_j^o>$ differ from $|\alpha_j>$ are unimportant, because each appears in the performance expression (4.7) only in conjunction with its conjugate. Therefore the two problems possess the same optimal realization in H.

QED

We remark that the performance of the optimum feedback receiver obviously possesses the same invariance property expressed in Lemma 4A.1.

4A.2 PROOF OF THEOREM 4.1

We shall suppress the explicit dependence of the coherent states and the optimum measurement operators on Δ or N, replacing $\{Q^N_{\beta,\Delta}\}$ by $\{Q_\beta\}$ and $\{|\alpha^N_j>_N\}$ by $\{|\alpha_j>\}$. Then

$$\overline{C}^*_\Delta = \sum_{j=1}^{M} \sum_{\beta\varepsilon B} \xi_j <\alpha_j|Q_\beta|\alpha_j> C[j,\hat{j}^*(\beta)]$$

$$= \sum_{j=1}^{M} \sum_{\hat{j}\varepsilon\hat{R}} \xi_j <\alpha_j|Q_{\hat{j}}|\alpha_j> C(j,\hat{j})$$

$$= \sum_j \xi_j \Big[<\alpha_j|Q_{\hat{j}_o}|\alpha_j> C(j,\hat{j}_o) + \sum_{\hat{j}\neq\hat{j}_o} <\alpha_j|Q_{\hat{j}}|\alpha_j> C(j,\hat{j})\Big]$$

$$= \sum_j \xi_j C(j,\hat{j}_o) + \sum_j \sum_{\hat{j}\neq\hat{j}_o} \xi_j <\alpha_j|Q_{\hat{j}}|\alpha_j> [C(j,\hat{j})-C(j,\hat{j}_o)], \tag{4A.3}$$

where $Q_{\hat{j}} \equiv \sum_{\beta:\hat{j}^*(\beta)=\hat{j}} Q_\beta$ and the last equality follows from the the condition $Q_{\hat{j}_o} = I - \sum_{\hat{j}\neq\hat{j}_o} Q_{\hat{j}}$.

The first term in (4A.3) is $\overline{C}^*_0$, so the difference $\overline{C}^*_0 - \overline{C}^*_\Delta$ is given by

$$\overline{C}^*_0 - \overline{C}^*_\Delta = \sum_j \sum_{\hat{j}\neq\hat{j}_o} \xi_j <\alpha_j|Q_{\hat{j}}|\alpha_j> [C(j,\hat{j}_o)-C(j,\hat{j})] \tag{4A.4}$$

By the assumptions of Section 3.1.2, at least two of the signals, say S_1 and S_2 for concreteness, are unequal during

$[T-\Delta,T]$. From Lemma 4A.1 we may assume $S_1=0$, corresponding to the coherent state $|\alpha_1=0\rangle$. We may write

$$|\alpha_j\rangle = e^{-\frac{1}{2}E_j}|0\rangle + |\phi_j\rangle \qquad (4A.5)$$

where $\langle 0|\phi_j\rangle = 0$ and $\langle\phi_j|\phi_j\rangle = 1-e^{-E_j}$.

Thus

$$\bar{C}_0^* - \bar{C}_\Delta^* = \sum_{\hat{j}\neq\hat{j}_o}\sum_j \xi_j \left\{\langle 0|Q_{\hat{j}}|0\rangle\, e^{-E_j} + 2\,\mathrm{Re}[\langle 0|Q_{\hat{j}}|\phi_j\rangle]e^{-\frac{1}{2}E_j} + \langle\phi_j|Q_{\hat{j}}|\phi_j\rangle\right\}[C(j,\hat{j}_o)-C(j,\hat{j})] \qquad (4A.6)$$

Since $0 \le Q_{\hat{j}} \le I_H$,

$$\langle\phi_j|Q_{\hat{j}}|\phi_j\rangle \le \langle\phi_j|\phi_j\rangle \le E_j \qquad (4A.7)$$

and $$|\langle 0|Q_{\hat{j}}|\phi_j\rangle| \le \sqrt{\langle\phi_j|Q_{\hat{j}}|\phi_j\rangle\langle 0|Q_{\hat{j}}|0\rangle} \le \sqrt{E_j}\,\sqrt{\langle 0|Q_{\hat{j}}|0\rangle}\,.$$

These two relations imply that $\langle 0|Q_{\hat{j}}|0\rangle$ is small for any $\hat{j}$ which is non-optimum for $\Delta=0$, specifically

$$\limsup_{\Delta\downarrow 0} \frac{\langle 0|Q_{\hat{j}}|0\rangle}{\Delta} = K < \infty \qquad \hat{j}\notin J_o \qquad (4A.8)$$

because otherwise the first term in (4A.6) is dominant and negative for small enough Δ, which contradicts the assumed optimality of $\{Q_{\hat{j}}\}$. These statements are proved as follows. Suppose it is possible to find $\hat{j}$, and $\Delta_n \downarrow 0$ with $\langle 0|Q_{\hat{j}_1}|0\rangle \equiv K_n \Delta_n$, $K_n \uparrow \infty$. The $\hat{j}_1$th term in (4A.6) must be non-negative for all $\Delta \geq 0$ because otherwise a cost smaller than $\overline{C}^*_\Delta$ could be obtained by setting $Q_{\hat{j}_1} = 0$ (and incrementing $Q_{\hat{j}_o}$ by $Q_{\hat{j}_1}$). But, defining $-\delta \equiv \sum_j \xi_j [C(j,\hat{j}_o)-C(j,\hat{j}_1)]$ which is negative (and independent of Δ) from the assumed non-optimality of $\hat{j}_1$ for $\Delta=0$, we have

$$\sum_j \xi_j \langle\alpha_j|Q_{\hat{j}_1}|\alpha_j\rangle\, [C(j,\hat{j}_o)-C(j,\hat{j}_1)]$$

$$= -\delta K_n \Delta_n + \sum_j \xi_j \Big[\langle 0|Q_{\hat{j}_1}|0\rangle (e^{-E_j}-1) + \langle\phi_j|Q_{\hat{j}_1}|\phi_j\rangle + 2\,\mathrm{Re}[\langle 0|Q_{\hat{j}_1}|\phi_j\rangle] e^{-\frac{1}{2}E_j}\Big]\Big[C(j,\hat{j}_o)-C(j,\hat{j}_1)\Big]$$

$$\leq -\delta K_n \Delta_n + \sum_j \xi_j \Big[\langle 0|Q_{\hat{j}_1}|0\rangle (1-e^{-E_j}) + \langle\phi_j|Q_{\hat{j}_1}|\phi_j\rangle + 2\,|\langle 0|Q_{\hat{j}_1}|\phi_j\rangle|\Big] \cdot |C(j,\hat{j}_o)-C(j,\hat{j}_1)|$$

$$\leq -\delta K_n \Delta_n + \sum_j \xi_j \Big[E_j + E_j + 2\sqrt{E_j}\sqrt{K_n \Delta_n}\Big] \cdot 2\, C_{max}$$

$$= -\delta K_n \Delta_n + 4\, C_{max} \Big[\sum_j \xi_j E_j + \sqrt{K_n \Delta_n} \sum \xi_j \sqrt{E_j}\Big] \tag{4A.9}$$

The ratio of the magnitude of the two terms in (4A.9) becomes arbitrarily large as $K_n \uparrow \infty$ because $\lim_{\Delta_n \downarrow 0} \frac{E_j}{\Delta_n} = |S_j(T^-)|^2 < \infty$, so eventually the negative first term dominates. This completes the proof of (4A.8) for $\hat{j} \notin J_o$.

Now we are prepared to proceed with the proof of Theorem 4.1. We write $|\phi_j\rangle$ in (4A.5) as

$$|\phi_j\rangle = \langle\phi|\phi_j\rangle|\phi\rangle + |\phi_j'\rangle \tag{4A.10}$$

where

$$|\phi\rangle \equiv \frac{|\phi_2\rangle}{\sqrt{\langle\phi_2|\phi_2\rangle}} \tag{4A.11}$$

is a normalized vector in the direction $|\phi_2\rangle$. Calculating $\langle\phi_j'|\phi_j'\rangle$, we find that

$$\langle\phi_j'|\phi_j'\rangle = \langle\phi_j|\phi_j\rangle - |\langle\phi|\phi_j\rangle|^2 \tag{4A.12}$$

where

$$\langle\phi_j|\phi_j\rangle = 1-e^{-E_j}$$

and

$$\langle\phi|\phi_j\rangle = \frac{e^{-\frac{1}{2}(E_j+E_2)}}{\sqrt{1-e^{-E_2}}}\left(e^{\gamma_{j2}\sqrt{E_jE_2}}-1\right) \tag{4A.13}$$

Using the small Δ expressions $o(E_j) = o(E_2) = o(\sqrt{E_jE_2}) = o(\Delta)$, we obtain

$$\langle\phi_j|\phi_j\rangle = E_j + o(\Delta) \tag{4A.14}$$

$$|\langle\phi|\phi_j\rangle|^2 = |\gamma_{j2}|^2E_j + o(\Delta)$$

Therefore

$$<\phi_j^{!}|\phi_j^{!}> = o(\Delta) \tag{4A.15}$$

because from (4.12), $|\gamma_{j2}|^2 = 1 + \frac{o(\Delta)}{\Delta}$ (4A.16)

We re-write the cost increment in (4A.6), using the orthogonal expansion (4A.10). The terms involving $<\phi|Q_{\hat{j}}|\phi_j^{!}>$, $<\phi_j^{!}|Q_{\hat{j}}|\phi>$, and $<\phi_j^{!}|Q_{\hat{j}}|\phi_j^{!}>$ are $o(\Delta)$, so

$$\begin{aligned}\overline{C}_0^* - \overline{C}_\Delta^* = \sum_{\hat{j} \neq \hat{j}_o} \sum_{j} \xi_j \Big\{ &<0|Q_{\hat{j}}|0> e^{-E_j} + 2\ \mathrm{Re}\left[<0|Q_{\hat{j}}|\phi><\phi|\phi_j> e^{-\frac{1}{2}E_j}\right] \\ &+ <\phi|Q_{\hat{j}}|\phi>|<\phi|\phi_j>|^2 + 2\ \mathrm{Re}\left[<0|Q_{\hat{j}}|\phi_j'> e^{-\frac{1}{2}E_j}\right]\Big\} \left[C(j,\hat{j}_o) - C(j,\hat{j})\right] \\ &+ o(\Delta)\end{aligned} \tag{4A.17}$$

For any $\hat{j}$ which is non-optimum for $\Delta=0$, (4A.8) implies that the last term in braces in (4A.17) is also $o(\Delta)$, because the positive definiteness of $Q_{\hat{j}}$ and the orthogonality of $|0>$ and $|\phi_j^{!}>$ require

$$|<0|Q_{\hat{j}}|\phi_j^{!}>| \leq \sqrt{<0|Q_{\hat{j}}|0><\phi_j^{!}|Q_{\hat{j}}|\phi_j^{!}>} \leq \sqrt{K\Delta}\ \sqrt{o(\Delta)} \tag{4A.18}$$

for small enough Δ. Thus, under the conditions of Part a of Theorem 4.1 only the first three terms inside the braces in (4A.17) need be retained for an $o(\Delta)$ approximation to $\overline{C}_\Delta^*$.

For an approximation valid within $o(\sqrt{\Delta})$, only the first two terms in braces in (4A.17) are essential. This approximation will be used to prove Part b of Theorem 4.1.

4A.2.1 <u>Proof of Part a</u>

We shall optimize the $o(\Delta)$ approximation to $\overline{C}_0^*-\overline{C}_\Delta^*$ in (4A.17) by choosing a set of real and complex numbers $<0|Q_{\hat{j}}|0>$, $<\phi|Q_{\hat{j}}|\phi>$, $<0|Q_{\hat{j}}|\phi>$, subject only to the constraints $0 \leq <0|Q_{\hat{j}}|0>$, $0 \leq <\phi|Q_{\hat{j}}|\phi>$, $|<0|Q_{\hat{j}}|\phi>|^2 \leq <0|Q_{\hat{j}}|0><\phi|Q_{\hat{j}}|\phi>$, $\hat{j}\neq\hat{j}_o$, and $\sum_{\hat{j}\neq\hat{j}_o} <\phi|Q_{\hat{j}}|\phi> \leq 1$, which are required properties of the set of measurement operators $\{Q_{\hat{j}}\}$. The result will be an $o(\Delta)$ approximation to the optimum cost increment if, in addition, $<0|Q_{\hat{j}_o}|0> \geq 0$ and $|<0|Q_{\hat{j}_o}|\phi>|^2 \leq <0|Q_{\hat{j}_o}|0><\phi|Q_{\hat{j}}|\phi>$, because it is straightforward to construct a set of measurment operators $\{Q_{\hat{j}}\}$ having any $<0|Q_{\hat{j}}|0>$, $<0|Q_{\hat{j}}|\phi>$, $<\phi|Q_{\hat{j}}|\phi>$ matrix elements which satisfy these constraints.

For any fixed $\{<0|Q_{\hat{j}}|0>\}$ and $\{<\phi|Q_{\hat{j}}|\phi>\}$, the off-diagonal elements $\{<0|Q_{\hat{j}}|\phi>\}$ are optimally chosen to make the second term in braces in (4A.17) as large as possible for each $\hat{j}$. This is achieved by letting the magnitude of $<0|Q_{\hat{j}}|\phi>$ be as large as possible and selecting its phase to cancel the phase of $\sum_j \xi_j <\phi|\phi_j> [C(j,\hat{j}_o)-C(j,\hat{j})]$. Thus we may substitute

$$<0|Q_{\hat{j}}|\phi> = \sqrt{<0|Q_{\hat{j}}|0><\phi|Q_{\hat{j}}|\phi>} \; \frac{\sum_j \xi_j <\phi|\phi_j> e^{-\frac{1}{2}E_j}[C(j,\hat{j}_o)-C(j,\hat{j})]}{\left|\sum_j \xi_j <\phi|\phi_j> e^{-\frac{1}{2}E_j}[C(j,\hat{j}_o)-C(j,\hat{j})]\right|}$$

$$\hat{j}\neq\hat{j}_o \qquad (4A.19)$$

into (4A.17) and maximize the resulting expression over $\{\langle 0|Q_{\hat{j}}|0\rangle\}$, $\{\langle\phi|Q_{\hat{j}}|\phi\rangle\}$. That expression involves, for each $\hat{j}$ and fixed $\langle\phi|Q_{\hat{j}}|\phi\rangle$, a quadratic function of $\sqrt{\langle 0|Q_{\hat{j}}|0\rangle}$ of the form ax^2+bx+c, $a<0$, which has a unique maximum at $x=-\frac{b}{a}$. Thus, the optimum choice of $\{\langle 0|Q_{\hat{j}}|0\rangle\}$ is given by

$$\sqrt{\langle 0|Q_{\hat{j}}|0\rangle} = -\frac{\left|\sum_j \xi_j \langle\phi|\phi_j\rangle\, e^{-\frac{1}{2}E_j}[C(j,\hat{j}_0)-C(j,\hat{j})]\right|\sqrt{\langle\phi|Q_{\hat{j}}|\phi\rangle}}{\sum_j \xi_j\, e^{-E_j}\,[C(j,\hat{j}_0)-C(j,\hat{j})]}, \qquad \hat{j}\neq\hat{j}_0 \tag{4A.20}$$

We see that $\langle 0|Q_{\hat{j}}|0\rangle \sim \Delta$, $\hat{j}\neq\hat{j}_0$ because $|\langle\phi|\phi_j\rangle| \sim \sqrt{\Delta}$. It is always possible to satisfy the constraint

$\langle 0|Q_{\hat{j}_0}|0\rangle \equiv 1-\sum_{\hat{j}\neq\hat{j}_0}\langle 0|Q_{\hat{j}}|0\rangle \geq 0$ by restricting our attention to small enough Δ.

With (4A.19) and (4A.20) substituted into (4A.17), the expression for $\overline{C}_0^*-\overline{C}_\Delta^*$ as a function just of $\{\langle\phi|Q_{\hat{j}}|\phi\rangle\}$ reduces to

$$\overline{C}_0^*-\overline{C}_\Delta^* = \sum_{\hat{j}\neq\hat{j}_0}\langle\phi|Q_{\hat{j}}|\phi\rangle\left\{\sum_j \xi_j|\langle\phi|\phi_j\rangle|^2\,[C(j,\hat{j}_0)-C(j,\hat{j})] - \frac{\left|\sum_j \xi_j\langle\phi|\phi_j\rangle\, e^{-\frac{1}{2}E_j}\,[C(j,\hat{j}_0)-C(j,\hat{j})]\right|^2}{\sum_j \xi_j\, e^{-E_j}\,[C(j,\hat{j}_0)-C(j,\hat{j})]}\right\} + o(\Delta) \tag{4A.21}$$

This expression is maximized within $o(\Delta)$, subject to

$0 \leq \langle\phi|Q_{\hat{j}}|\phi\rangle$, $\sum_{\hat{j}\neq\hat{j}_o} \langle\phi|Q_{\hat{j}}|\phi\rangle \leq 1$, by

$$\begin{aligned} &\langle\phi|Q_{\hat{j}_1}|\phi\rangle = 1 \\ &\langle\phi|Q_{\hat{j}}|\phi\rangle = 0 \text{ for all } \hat{j}\neq\hat{j}_o,\hat{j}_1 \end{aligned} \qquad (4A.22)$$

where $\hat{j}=\hat{j}_1$ maximizes the expression in braces in (4A.21) over all $\hat{j}\varepsilon\hat{R}$. [If this expression is negative for all $\hat{j}\neq\hat{j}_o$, then $\hat{j}_1=\hat{j}_o$ and the optimal solution is $\langle\phi|Q_{\hat{j}}|\phi\rangle = 0$ for all $\hat{j}\neq\hat{j}_o$.]

The constraint $|\langle 0|Q_{\hat{j}_o}|\phi\rangle|^2 \leq \langle 0|Q_{\hat{j}_o}|0\rangle \langle\phi|Q_{\hat{j}_o}|\phi\rangle$ can be satisfied by modifying (4A.22) slightly to read $\langle\phi|Q_{\hat{j}_1}|\phi\rangle = 1 - \langle 0|Q_{\hat{j}_1}|0\rangle$, which from (4A.20) and (4A.21) does not affect $\bar{C}_0^*-\bar{C}_\Delta^*$ within $o(\Delta)$.

Finally, we evaluate the two terms in braces in (4A.21) for small Δ and $\hat{j}=\hat{j}_1$. Using (4A.13), (4A.14), (4A.16), and (4.12), we obtain

$$\sum_j \xi_j \; |\langle\phi|\phi_j\rangle|^2[C(j,\hat{j}_o)-C(j,\hat{j}_1)] = \Delta \sum \xi_j \; |S_j(T^-)|^2[C(j,\hat{j}_o)-C(j,\hat{j}_1)] + o(\Delta)$$

and

$$\frac{\left|\sum_j \xi_j \; \langle\phi|\phi_j\rangle \; e^{-\frac{1}{2}E_j}[C(j,\hat{j}_o)-C(j,\hat{j}_1)]\right|^2}{\sum_j \xi_j \; e^{-E_j} \; [C(j,\hat{j}_o)-C(j,\hat{j}_1)]} = \Delta \; \frac{\left|\sum_j \xi_j S_j(T^-)[C(j,\hat{j}_o)-C(j,\hat{j}_1)]\right|^2}{\sum_j \xi_j[C(j,\hat{j}_o)-C(j,\hat{j}_1)]} + o(\Delta) \qquad (4A.23)$$

The difference of the two expressions on the right side of (4A.23) is $\eta_{\hat{j}_1}\Delta + o(\Delta)$, where $\eta_{\hat{j}_1}$ is defined in (4.19). Therefore

$$\lim_{\Delta \downarrow 0} \frac{\overline{C}_0^* - \overline{C}_\Delta^*}{\Delta} = \eta_{\hat{j}_1} \tag{4A.24}$$

Now we verify that identical performance can be achieved within $o(\Delta)$ by a feedback receiver with feedback $\ell(\vec{r},\tau)$ given by (4.21). From (4.17), we have for $\hat{j}(0) = \hat{j}_o$ and $\hat{j}(1) = \hat{j}_1$,

$$\overline{C}_0^* - \overline{C}_\Delta^\ell = \Delta \sum_j \xi_j \int_\Sigma d\vec{r} \, |S_j(T^-)\varepsilon_o(\vec{r}) + \ell(\vec{r},T^-)|^2 [C(j,\hat{j}_o) - C(j,\hat{j}_1)] + o(\Delta) \tag{4A.25}$$

and for $\ell(\vec{r},\tau)$ given by (4.21),

$$\begin{aligned}
\overline{C}_0^* - \overline{C}_\Delta^* &= \Delta \sum_j \xi_j \left| S_j(T^-) - \frac{\sum_i \xi_i S_i(T^-)[C(i,\hat{j}_o) - C(i,\hat{j}_1)]}{\sum_i \xi_i [C(i,\hat{j}_o) - C(i,\hat{j}_1)]} \right|^2 [C(j,\hat{j}_o) - C(j,\hat{j}_1)] + o(\Delta) \\
&= \Delta \left\{ \sum_j \xi_j |S_j(T^-)|^2 [C(j,\hat{j}_o) - C(j,\hat{j}_1)] - \frac{\left|\sum_j \xi_j S_j(T^-)[C(j,\hat{j}_o) - C(j,\hat{j}_1)]\right|^2}{\sum_j \xi_j [C(j,\hat{j}_o) - C(j,\hat{j}_1)]} \right\} + o(\Delta)
\end{aligned} \tag{4A.26}$$

Therefore

$$\lim_{\Delta \downarrow 0} \frac{\overline{C}_0^* - \overline{C}_\Delta^\ell}{\Delta} = \eta_{\hat{j}_1}$$

<u>QED</u>

4A.2.2 Proof of Part b

We examine the $o(\sqrt{\Delta})$ approximation to $\overline{C}_0^*-\overline{C}_\Delta^*$ obtained from (4A.17). We know from (4A.8) that $\langle 0|Q_{\hat{j}}|0\rangle \sim \Delta$ (and $\langle 0|Q_{\hat{j}}|\phi\rangle \sim \sqrt{\Delta}$) for any $j\notin J_o$. Therefore the only terms that contribute to the $o(\sqrt{\Delta})$ approximation to (4A.17) are those with $\hat{j}\varepsilon J_o$. Thus we obtain, for an arbitrary $\hat{j}'\varepsilon J_o$,

$$\overline{C}_0^*-\overline{C}_\Delta^* = \sum_{\hat{j}\varepsilon J_o}\left\{\langle 0|Q_{\hat{j}}|0\rangle \sum_j \xi_j[C(j,\hat{j}')-C(j,\hat{j})] + \sqrt{\Delta}\ 2\mathrm{Re}\left[\langle 0|Q_{\hat{j}}|\phi\rangle \sum_j \xi_j \ \langle\phi|\phi_j\rangle \ [C(j,\hat{j}')-C(j,\hat{j})]\right]\right\} + o(\sqrt{\Delta}) \tag{4A.27}$$

The first term in braces is zero by the assumed co-optimality of $\hat{j}$ and $\hat{j}'$ for $\Delta=0$. For the second term we can use (4.12) and (4A.13) to write

$$\langle\phi|\phi_j\rangle = S_j^*(T^-)\ \frac{S_2(T^-)}{|S_2(T^-)|}\ \sqrt{\Delta} + o(\sqrt{\Delta}) \tag{4A.28}$$

Then, with $\eta_{\hat{j}_o\hat{j}_1}$ defined as in (4.24), we can upper bound $\overline{C}_0^*-\overline{C}_\Delta^*$ by

$$\overline{C}_0^*-\overline{C}_\Delta^* \leq 2\ \sqrt{\Delta}\ |\eta_{\hat{j}_o\hat{j}_1}| + o(\sqrt{\Delta}) \tag{4A.29}$$

because the Schwarz inequality and the positive definiteness of $\{Q_{\hat{j}}\}$ imply that

$$\sum_{\hat{j}} |\langle 0|Q_{\hat{j}}|\phi\rangle| \leq \sum_{\hat{j}} \langle 0|Q_{\hat{j}}|0\rangle\langle\phi|Q_{\hat{j}}|\phi\rangle$$

$$\leq \left|\sum_{\hat{j}}\langle 0|Q_{\hat{j}}|0\rangle\right|^{\frac{1}{2}} \left|\sum_{\hat{j}}\langle\phi|Q_{\hat{j}}|\phi\rangle\right|^{\frac{1}{2}} \tag{4A.30}$$

$$= 1$$

The lower bound in (4.25) is obtained by taking $\langle 0|Q_{\hat{j}_o}|\phi\rangle = \frac{1}{2}$, $\langle 0|Q_{\hat{j}_1}|\phi\rangle = -\frac{1}{2}$ (with $\langle 0|Q_{\hat{j}_o}|0\rangle = \langle 0|Q_{\hat{j}_1}|0\rangle = \langle\phi|Q_{\hat{j}_o}|\phi\rangle = \langle\phi|Q_{\hat{j}_1}|\phi\rangle = \frac{1}{2}$) and $Q_{\hat{j}} = 0$, for $\hat{j}\neq\hat{j}_o,\hat{j}_1$.

We remark that the optimum performance is never as good as that specified by the upper bound and for many signals sets (e.g., all $S_j(T^-)$ real) it is exactly equal to the lower bound.

To show that $\overline{C}_0^* - \inf_{\ell} \overline{C}_\Delta^\ell$ also has the $\sqrt{\Delta}$-dependence, we consider a particular feedback function sequence defined by

$$\ell^\Delta(\vec{r},\tau) = \Delta^{-\frac{1}{2}} \varepsilon_o(\vec{r})\, z$$

where

$$z \equiv \frac{\eta_{\hat{j}_o\hat{j}_1}}{|\eta_{\hat{j}_o\hat{j}_1}|} , \tag{4A.31}$$

and a decision function

$$\hat{j}(0) = \hat{j}_o , \quad \hat{j}(n) = 1 \quad , n \geq 1 \tag{4A.32}$$

Then from (4.17),

$$\overline{C}_0^* - \overline{C}_\Delta^{\ell^\Delta} = \sum_{n=1}^{\infty} \sum_j \xi_j |S_j(T^-)\sqrt{\Delta} + z|^{2n} \frac{1}{n!} e^{-|S_j(T^-)\sqrt{\Delta}+z|^2} [C(j,\hat{j}_o) - C(j,\hat{j}_1)] + o(\sqrt{\Delta})$$

$$= e^{-|z|^2} \sqrt{\Delta} \sum_{n=1}^{\infty} \sum_j \xi_j [n|z|^{2(n-1)} - 1] \frac{1}{n!} 2\mathrm{Re}[z^* S_j(T^-)][C(j,\hat{j}_o) - C(j,\hat{j}_1)] + o(\sqrt{\Delta})$$

$$= e^{-|z|^2} \sum_{n=1}^{\infty} [n|z|^{2(n-1)} - 1] \frac{1}{n!} \, 2|\eta_{\hat{j}_o \hat{j}_1}| \sqrt{\Delta} + o(\sqrt{\Delta})$$

$$= e^{-|z|^2} \, 2|\eta_{\hat{j}_o \hat{j}_1}| \sqrt{\Delta} + o(\sqrt{\Delta})$$

$$= \frac{2}{e} |\eta_{\hat{j}_o \hat{j}_1}| \sqrt{\Delta} + o(\sqrt{\Delta}) \tag{4A.33}$$

because, from (4A.31), $|z|=1$.

<u>QED</u>

4A.3 PROOF OF LEMMA 4.1

State sufficiency and the expression (4.32a) are proved by arguments exactly analogous to those used in proving Lemmas 3.1, 3.2, 3.3, because those arguments did not rely on the particular form of the point data statistics. The expression (4.32b) is derived from (4.32a) with the aid of the identity

$$\sum_{\beta} \sum_{j} \xi_j \; {}_i\langle\alpha_j^i \mid Q_\beta^i(\underline{\xi}) \mid \alpha_j^i\rangle_i \; (\underline{\xi}^i(\beta;\underline{\xi}) - \underline{\xi})^T \frac{\partial \bar{C}(\underline{\xi},\tau_i)}{\partial \underline{\xi}} = 0 \tag{4A.34}$$

which follows from (4.34) and the relation
$\sum_{\beta} {}_i\langle\alpha_j^i \mid Q_\beta^i(\underline{\xi}) \mid \alpha_j^i\rangle_i = 1.$

The optimality condition of Part b is proved by application of the dynamic programming optimality principle, which states that if
$\{Q_\beta^i(\underline{\xi})\}, \{Q_\beta^{i+1}(\underline{\xi})\},\ldots,\{Q_\beta^N(\underline{\xi})\}$ is an optimum measurement sequence for the $[\tau_{i-1},T]$ problem starting in state $\underline{\xi}_o$, then $\{Q_\beta^{i+1}(\underline{\xi})\},\ldots,\{Q_\beta^N(\underline{\xi})\}$ is optimum for the $[\tau_i,T]$ problem starting in state $\underline{\xi}^i(\beta;\underline{\xi}_o)$, at least for reachable states $\underline{\xi}^i(\beta;\underline{\xi}_o)$. [This principle is an exact analog of the one obtained in Lemma 3.4 for feedback receivers.] The necessity of minimizing the cost increment in (4.32b) follows directly from this principle, and sufficiency is proved by induction, as in Theorem 3.2.

QED

4A.4 PROOF OF THEOREM 4.2

The proof proceeds by induction. We suppose that a feedback function ℓ_i^Δ has been found which approximates, within $\sum_{k=i+1}^{N} o_k(\Delta)$, the performance of the optimum contingent measurement sequence for the $[\tau_i, T]$ problem. We then show that the cost increment in (4.32b) can be achieved within $o_i(\Delta)$ by this ℓ_i^Δ and an appropriately defined extension of it backward in time to the interval $[\tau_{i-1}, \tau_i]$. Since the a priori costs $\bar{C}_\Delta^*(\underline{\xi}, T)$, $\bar{C}_\Delta^\ell(\underline{\xi}, T)$ are identical, this will prove that the feedback function ℓ_0^Δ constructed in this manner achieves performance $\bar{C}^{\ell_0^\Delta}(\underline{\xi}, 0)$ which approximates $\bar{C}_\Delta^*(\underline{\xi}, 0)$ within $\sum_{k=1}^{N} o_k(\Delta) = \frac{o(\Delta)}{\Delta}$.

Then, if an optimum feedback function ℓ exists, we have the inequalities

$$\bar{C}_\Delta^*(\underline{\xi},0) \le \bar{C}^\ell(\underline{\xi},0) \le \bar{C}^{\ell_0^\Delta}(\underline{\xi},0) \le \bar{C}_\Delta^*(\underline{\xi},0) + \frac{o(\Delta)}{\Delta} \tag{4A.35}$$

This implies that $\lim_{\Delta \downarrow 0} \bar{C}_\Delta^*(\underline{\xi},0)$ exists and equals $\bar{C}^\ell(\underline{\xi},0)$.

Thus it only remains for us to verify that a feedback function minimizes the cost increment in (4.32b) within $o(\Delta)$. We shall suppress the index i, the variable length Δ, and the probability vector $\underline{\xi}$ for clarity, denoting $Q^i_{\beta,\Delta}(\underline{\xi})$ by Q_β and $|\alpha_j^i\rangle_i$ by $|\alpha_j\rangle$. We use the same orthogonal expression for $|\alpha_j\rangle$ as the one presented in the proof of Theorem 4.1. [It may be necessary to re-number the hypotheses at each stage of the induction procedure in order to guarantee that $|\alpha_2^i\rangle_i \neq |\alpha_1^i\rangle_i \equiv |0\rangle_i$. This will always be possible because of our assumption that at least two signals are unequal in every nonzero length interval.] We write

$$|\alpha_j\rangle = e^{-\frac{1}{2}E_j}\,|0\rangle \;=\; \langle\phi|\phi_j\rangle|\phi\rangle \;+\; |\phi_j'\rangle \tag{4A.36}$$

where

$$E_j = |S_j|^2\Delta + o(\Delta), \qquad S_j \equiv S_j(\tau_{i-1})$$

$$\langle\phi|\phi_j\rangle = S_j^*\frac{S_2}{|S_2|}\sqrt{\Delta} + o(\Delta) \tag{4A.37}$$

$$\langle\phi_j'|\phi_j'\rangle = o(\Delta)$$

and we obtain the following expression for the a posteriori probabilities,

$$\xi_j^i(\beta;\underline{\xi}) = \xi_j \frac{\langle\alpha_j|Q_\beta|\alpha_j\rangle}{\sum\limits_k \xi_k \langle\alpha_k|Q_\beta|\alpha_k\rangle} = \xi_j \frac{\chi_j^\beta}{\hat{\chi}^\beta} \tag{4A.38}$$

where

$$\chi_j^\beta \equiv [1-|S_j|^2\Delta]\langle 0|Q_\beta|0\rangle + 2\,\mathrm{Re}[S_j^*\frac{S_2}{|S_2|}\langle 0|Q_\beta|\phi\rangle]\sqrt{\Delta}$$

$$+ |S_j|^2\Delta\,\langle\phi|Q_\beta|\phi\rangle + 2\,\mathrm{Re}[\langle 0|Q_\beta|\phi_j'\rangle] + o(\Delta)$$

and (4A.39)

$$\hat{\chi}^\beta \equiv \sum_j \xi_j\,\chi_j^\beta$$

We consider three separate cases for measurement outcomes β.

(I) $\underline{B_I = \{\beta: \lim_{\Delta\downarrow 0} <0|Q_\beta|0> = 0\}}$. The differentiability condition (iii) implies $<0|Q_\beta|0> = a_\beta \Delta + o(\Delta)$, $a_\beta \geq 0$, for $\beta \varepsilon B_I$. Positive definiteness of Q_β implies that $<0|Q_\beta|\phi_j'> = o(\Delta)$ and $|<0|Q_\beta|\phi>|^2 \leq a_\beta \Delta + o(\Delta)$, so for $\beta\varepsilon B_I$ we may evaluate χ_j^β in (4A.39) as

$$\chi_j^\beta = (\lambda_j^\beta + \mu^\beta)<\phi|Q_\beta|\phi> \Delta \quad + \quad o(\Delta) \qquad (4A.40)$$

where

$$\lambda_j^\beta \;\; = |S_j + \ell^\beta|^2 \; \Delta$$

$$\ell^\beta \sqrt{\Delta} = \frac{<0|Q_\beta|\phi>}{<\phi|Q_\beta|\phi>} \frac{S_2}{|S_2|} \qquad (4A.41)$$

$$\mu^\beta \Delta = \frac{<0|Q_\beta|0><\phi|Q_\beta|\phi> - |<0|Q_\beta|\phi>|^2}{<\phi|Q_\beta|\phi>^2} \geq 0$$

Therefore the contribution to the cost increment in (4.32b) of measurement outcomes $\beta \varepsilon B_I$ is

$$\sum_{\beta \varepsilon B_I} \sum_j \xi_j <\alpha_j|Q_\beta|\alpha_j> \overline{C}_\Delta^{*''}[\underline{\xi}^1(\beta;\underline{\xi}),\underline{\xi},\tau_i]$$

$$= \Delta \sum_{\beta\varepsilon B_I} <\phi|Q_\beta|\phi> (\hat{\lambda}^\beta + \mu^\beta) \; \overline{C}_\Delta^{*''}[\underline{\rho}^\beta,\underline{\xi},\tau_i] \; + \; o(\Delta) \qquad (4A.42)$$

where

$$\hat{\lambda}^{\beta} = \sum_j \xi_j \lambda_j^{\beta}$$

$$\rho_j^{\beta} = \xi_j \frac{\lambda_j^{\beta} + \mu^{\beta}}{\hat{\lambda}^{\beta} + \mu^{\beta}} \tag{4A.43}$$

Since each term in the sum is negative ($\overline{C}_{\Delta}^{*}$ is convex $\cap$ by an argument identical to the one given in Lemma 3A.2) and $\{\langle\phi|Q_\beta|\phi\rangle\}$ are nonnegative probabilities, the sum in (4A.42) is minimized over $\{Q_\beta\}_{\beta\varepsilon B_I}$, subject to $\sum_{\beta\varepsilon B_I} Q_\beta \equiv Q_{B_I}$, by locating all the probability at the most negative term $\beta = \beta_I$.

We note that for any fixed ℓ^{β_I}, the right side of (4A.42) is minimized by $\mu^{\beta_I} = 0$, because

$$\frac{\partial}{\partial\mu^{\beta}} [(\hat{\lambda}^{\beta}+\mu^{\beta})\, \overline{C}_{\Delta}^{*''}(\underline{\rho}^{\beta},\underline{\xi},\tau_i)] = - \overline{C}_{\Delta}^{*''}(\underline{\xi},\underline{\rho}^{\beta},\tau_i) \geq 0 \tag{4A.44}$$

[This observation is equivalent to the analogous feedback receiver principle that performance is always degraded by a signal-independent dark current.]

Next we note that $\{Q_\beta\}$ can be optimum only if ℓ^{β_I} maximizes $\hat{\lambda}^{\beta_I}|\overline{C}_{\Delta}^{*''}(\underline{\rho}^{\beta_I}, \underline{\xi}, \tau_i)|$. We define the optimum (within $o(\Delta)$) feedback level $\ell^{\Delta}(\vec{r},\tau)$ in terms of this optimum ℓ^{β_I},

$$\ell^{\Delta}(\vec{r},\tau) = \ell^{\beta_I} \varepsilon_o(\vec{r}) \tag{4A.45}$$

(II) $B_{II} = \{\beta: \lim_{\Delta\downarrow 0} \langle 0|Q_\beta|0\rangle = b_\beta \neq 0, \text{ and } \lim_{\Delta\downarrow 0} \langle\phi|Q_\beta|\phi\rangle = 0\}$.

For $\beta\epsilon B_{II}$, positive definiteness of Q_β implies $\lim_{\Delta\downarrow 0} \langle 0|Q_\beta|\phi\rangle = 0$ also, so the a posteriori probability evaluated in (4A.38) differs from ξ_j by $o(\sqrt{\Delta})$. Since $\overline{C}^*_\Delta(\underline{\xi},\tau_i)$ is twice differentiable at $\underline{\xi}$, it follows from Taylor's theorem that $\overline{C}^{*\prime\prime}_\Delta[\underline{\xi}^i(\beta;\underline{\xi}),\underline{\xi},\tau_i] = o(\Delta)$, $\beta\epsilon B_{II}$.

(III) $B_{III} = \{\beta: \lim_{\Delta\downarrow 0} \langle 0|Q_\beta|0\rangle = b_\beta \neq 0, \text{ and } \lim_{\Delta\downarrow 0} \langle\phi|Q_\beta|\phi\rangle \neq 0\}$.

We compute an $o(\sqrt{\Delta})$ approximation to $\xi^i_j(\beta;\underline{\xi}) - \xi_j$, because, as in (II), $o(\sqrt{\Delta})$ terms contribute $o(\Delta)$ to the second-order difference $\overline{C}^{*\prime\prime}_\Delta$. From (4A.38) and (4A.39),

$$\xi^i_j(\beta;\underline{\xi}) - \xi_j = \xi_j \; 2 \operatorname{Re}[(S_j-\hat{S})^* \frac{S_2}{|S_2|} \frac{\langle 0|Q_\beta|\phi\rangle}{\langle 0|Q_\beta|0\rangle}] \sqrt{\Delta} + o(\sqrt{\Delta}) \tag{4A.45}$$

where

$$\hat{S} \equiv \sum_j \xi_j \; S_j \tag{4A.46}$$

By Taylor's theorem, we have

$$\overline{C}^{*\prime\prime}_\Delta[\underline{\xi}^i(\beta;\underline{\xi}),\underline{\xi},\tau_i]$$
$$= \Delta \frac{1}{2} \sum_j \sum_k \xi_j \; \xi_k \; 2 \operatorname{Re}[(S_j-\hat{S})^* \frac{S_2}{|S_2|} \frac{\langle 0|Q_\beta|\phi\rangle}{\langle 0|Q_\beta|0\rangle}] \frac{\partial^2 \overline{C}^*_\Delta(\underline{\xi},\tau_i)}{\partial\xi_j \partial\xi_k}$$
$$\cdot \; 2 \operatorname{Re}[(S_j-\hat{S})^* \frac{S_2}{|S_2|} \frac{\langle 0|Q_\beta|\phi\rangle}{\langle 0|Q_\beta|0\rangle}] + o(\Delta) \tag{4A.47}$$

We can upper bound this expression by

$$|\overline{C}_{\Delta}^{*''}[\underline{\xi}^{i}(\beta;\underline{\xi}),\underline{\xi},\tau_{i}]|$$

$$\leq 2\Delta \frac{<\phi|Q_{\beta}|\phi>}{<0|Q_{\beta}|0>} \sum_{j}\sum_{k} \xi_{j}\ \xi_{k} \left|S_{j}-\hat{S}\right|\cdot\left|\frac{\partial^{2}\overline{C}_{\Delta}^{*}(\underline{\xi},\tau_{i})}{\partial\xi_{j}\partial\xi_{k}}\right|\cdot\left|S_{k}-\hat{S}\right| + o(\Delta)$$

$$\leq \Delta \frac{<\phi|Q_{\beta}|\phi>}{<0|Q_{\beta}|0>} \hat{\lambda}^{\beta_{I}} |\overline{C}_{\Delta}^{*''}(\underline{\rho}^{\beta_{I}},\underline{\xi},\tau_{i})| + o(\Delta) \qquad (4A.48)$$

where the second inequality follows from assumption (v) and the assumed optimality of $\ell^{\beta_{I}}$. Therefore, the contribution to the cost increment in (4.32b) of measurement outcomes $\beta\varepsilon B_{III}$ is bounded by

$$\sum_{\beta\varepsilon B_{III}} \sum_{j} \xi_{j} <\alpha_{j}|Q_{\beta}|\alpha_{j}> |\overline{C}_{\Delta}^{*''}[\underline{\xi}^{i}(\beta;\underline{\xi}),\underline{\xi},\tau_{i}]|$$

$$\leq \Delta <\phi|Q_{B_{III}}|\phi> \hat{\lambda}^{\beta_{I}} |\overline{C}_{\Delta}^{*''}(\underline{\rho}^{\beta_{I}},\underline{\xi},\tau_{i})| + o(\Delta) \qquad (4A.49)$$

where

$$Q_{B_{III}} \equiv \sum_{\beta\varepsilon B_{III}} Q_{\beta} \qquad (4A.50)$$

Combining the results of (I), (II), and (III), we can bound the cost increment in (4.32b) by

$$|\overline{C}^{*}(\underline{\xi},\tau_{i-1}) - \overline{C}^{*}(\underline{\xi},\tau_{i})|$$

$$\leq \Delta\ (<\phi|Q_{B_{I}}|\phi> + <\phi|Q_{B_{III}}|\phi>)\ \hat{\lambda}^{\beta_I}\ |\overline{C}_{\Delta}^{*''}(\underline{\rho}^{\beta_I},\underline{\xi},\tau_i)| + o(\Delta) \qquad (4A.51)$$

This bound can obviously be achieved by the optimum choice

$$\begin{aligned} <\phi|Q_{B_I}|\phi> &= 1 \\ <\phi|Q_{B_{III}}|\phi> &= 0 \end{aligned} \qquad (4A.52)$$

When a feedback receiver with feedback level $\ell^{\Delta}(\vec{r},\tau)$ given by (4A.45) is used in place of the optimum measurement during $[\tau_{i-1},\tau_i]$, it achieves an identical (within $o(\Delta)$) cost increment

$$\overline{C}^{\ell^{\Delta}}(\underline{\xi},\tau_{i-1}) - \overline{C}_{\Delta}^{*}(\underline{\xi},\tau_i) =$$

$$\Delta\int d\vec{r}\,|S_j(\tau_{i-1})\varepsilon_o(\vec{r})+\ell(\vec{r},\tau)|^2\overline{C}_{\Delta}^{*''}\left[\xi_j\ \frac{|S_j(\tau_{i-1})\varepsilon_o(\vec{r})+\ell(\vec{r},\tau)|^2}{\sum_k \xi_k|S_k(\tau_{i-1})\varepsilon_o(\vec{r})+\ell(\vec{r},\tau)|^2},\underline{\xi},\tau_i\right] + o(\Delta)$$

$$= \Delta\ \hat{\lambda}^{\beta_I}\ \overline{C}_{\Delta}^{*''}(\underline{\rho}^{\beta_I},\underline{\xi},\tau_i) + o(\Delta) \qquad (4A.53)$$

The proof of the theorem is completed by induction, as outlined in the introductory paragraphs.

<u>QED</u>

5A.1 PROOF OF THEOREM 5.2

Case I: $J_o(\underline{\xi},\tau) = \{j_o\}$; i.e., there is an unambiguous best a priori guess. Then $a_{\hat{j}\hat{j}_o}(\tau,\underline{\xi}) \neq 0$ except for $\hat{j}=\hat{j}_o=j_o$. From (5.18) and the assumption that $C(j,\hat{j};\underline{\xi},\tau)$ is independent of $\underline{\xi}$,

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) = \sum_{j=1}^{M} \xi_j \ \lambda_j(\vec{r},\tau,\underline{\xi}) \ [C(j,\hat{j};\underline{\xi},\tau) - C(j,j_o;\underline{\xi},\tau)] \tag{5A.1}$$

$$\text{where} \quad \hat{j} \equiv j^{*}[\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau] \tag{5A.2}$$

For fixed $\hat{j}\neq j_o$, the right hand side is quadratic in $\ell(\vec{r},\tau,\underline{\xi})$ and achieves minimum value

$$\tilde{c}_{\hat{j}}(\underline{\xi},\tau,\vec{r}) = d_{\hat{j}j_o}(\vec{r},\tau,\underline{\xi}) - \frac{\left|b_{\hat{j}j_o}(\vec{r},\tau,\underline{\xi})\right|^2}{a_{\hat{j}j_o}(\tau,\underline{\xi})} \quad , \ \hat{j}\varepsilon\hat{R}-\{j_o\} \tag{5A.3}$$

for feedback

$$\ell_{\hat{j}}(\underline{\xi},\tau,\vec{r}) = - \frac{b_{\hat{j}j_o}(\vec{r},\tau,\underline{\xi})}{a_{\hat{j}j_o}(\tau,\underline{\xi})} \qquad \hat{j}\varepsilon\hat{R}-\{j_o\} \tag{5A.4}$$

Thus

$$\tilde{c}(\underline{\xi},\tau,\vec{r}) \geq \tilde{c}_{\hat{j}}(\underline{\xi},\tau,\vec{r}) \geq \tilde{c}_{j_1}(\underline{\xi},\tau,\vec{r}) \tag{5A.5}$$

where $\hat{j}=j_1$ minimizes $\tilde{c}_{\hat{j}}(\underline{\xi},\tau,\vec{r})$ and we have included the additional definition

$$\tilde{c}_{j_o}(\underline{\xi},\tau,\vec{r}) \equiv 0 \tag{5A.6}$$

The minimum differential cost on the right side of (5A.5) is achieved by ℓ_{j_1} determined from (5A.4) or else by arbitrary ℓ if $c_{j_1}(\underline{\xi},\tau,\vec{r}) = c_{j_o}(\underline{\xi},\tau,\vec{r}) = 0$.

Case II: For some $(j_1,j_o)\varepsilon\ J_o(\underline{\xi},\tau)\mathrm{x}\ J_o(\underline{\xi},\tau)$, $b_{j_1j_o}(\cdot,\tau,\underline{\xi}) \neq 0$ (almost everywhere in Σ). Let $\delta>0$ and

$$\ell(\vec{r},\tau,\underline{\xi}) = - K(\vec{r},\tau,\underline{\xi})\ b_{j_1j_o}(\vec{r},\tau,\underline{\xi}),\ \text{where } K(\vec{r},\tau,\underline{\xi}) = \frac{1}{\delta|b_{j_1j_o}(\vec{r},\tau,\underline{\xi})|}$$

at points where $b_{j_1j_o}(\vec{r},\tau,\underline{\xi}) \neq 0$. Then

$$\hat{\lambda}(\vec{r},\tau,\underline{\xi}) = 1/\delta^2 \left\{1 + 2\delta|b_{j_1j_o}(\vec{r},\tau,\underline{\xi})|\ \mathrm{Re}\left[\sum_{j=1}^{M}\xi_j\frac{\varepsilon_j(\vec{r},\tau)}{b_{j_1j_o}(\vec{r},\tau,\underline{\xi})}\right] + \delta^2\sum_{j=1}^{M}\xi_j|\varepsilon_j(\vec{r},\tau)|^2\right\} \quad (5A.7)$$

and

$$\rho_j(\vec{r},\tau,\underline{\xi})$$

$$= \xi_j \frac{1 + 2\delta|b_{j_1j_o}(\vec{r},\tau,\underline{\xi})|\mathrm{Re}\left[\frac{\varepsilon_j(\vec{r},\tau)}{b_{j_1j_o}(\vec{r},\tau,\underline{\xi})}\right] + \delta^2|\varepsilon_j(\vec{r},\tau)|^2}{1 + 2\delta|b_{j_1j_o}(\vec{r},\tau,\underline{\xi})|\mathrm{Re}\left[\sum_{i=1}^{M}\xi_i\frac{\varepsilon_i(\vec{r},\tau)}{b_{j_1j_o}(\vec{r},\tau,\underline{\xi})}\right] + \delta^2\sum_{i=1}^{M}\xi_i|\varepsilon_i(\vec{r},\tau)|}$$

(5A.8)

$$= \xi_j \left\{1 + 2\delta|b_{j_1j_o}(\vec{r},\tau,\underline{\xi})|\mathrm{Re}\left[\frac{\varepsilon_j(\vec{r},\tau) - \sum_{i=1}^{M}\xi_i\varepsilon_i(\vec{r},\tau)}{b_{j_1j_o}(\vec{r},\tau,\underline{\xi})}\right]\right\} + o(\delta,\vec{r}),$$

where $\int_{\Sigma}|o(\delta,\vec{r})|d\vec{r} = o(\delta)$. We obtain an upper bound on $\overline{c}(\underline{\xi},\tau)$ in (5.14) or (5.20) by replacing $j^*[\underline{\rho}(\vec{r},\tau,\underline{\xi}),\tau]$ with j_1 and $j^*(\underline{\xi},\tau)$ with j_o.

$$\overline{c}(\underline{\xi},\tau) \le \int d\vec{r}\ \hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi})[C(j,j_1;\underline{\xi},\tau)-C(j,j_o;\underline{\xi},\tau)]$$

$$= \frac{1}{\delta^2}\left\{\int d\vec{r} \sum_{j=1}^{M} 2\xi_j\delta|b_{j_1j_o}(\vec{r},\tau,\underline{\xi})|\,\mathrm{Re}\left[\frac{\varepsilon_j(\vec{r},\tau)-\sum_{i=1}^{M}\xi_i\varepsilon_i(\vec{r},\tau)}{b_{j_1j_o}(\vec{r},\tau,\underline{\xi})}\right]\right.$$

$$\left.\cdot\left[C(j,j_1;\underline{\xi},\tau)-C(j,j_o;\underline{\xi},\tau)\right]+o(\delta)\right\}$$

$$= \frac{1}{\delta^2}\left\{\int d\vec{r}\ 2\delta|b_{j_1j_o}(\vec{r},\tau,\underline{\xi})|\,\mathrm{Re}\left[\frac{-b_{j_1j_o}(\vec{r},\tau,\underline{\xi})}{b_{j_1j_o}(\vec{r},\tau,\underline{\xi})}\right] + o(\delta)\right\}$$

$$= \frac{1}{\delta}\left[-\int d\vec{r}\ 2|b_{j_1j_o}(\vec{r},\tau,\underline{\xi})| + \frac{o(\delta)}{\delta}\right] \qquad (5A.9)$$

By assumption the first term in (5A.9) is nonzero, so the right hand side may be made negative and arbitrarily large in magnitude by letting $\delta\downarrow 0$. [We used the assumption that j_o,j_1 are co-optimum for $\underline{\xi},\tau$ in deriving both the second and third lines of (5A.9).]

<u>Case III:</u> For all $(\hat{j},\hat{j}_o) \in J_o(\underline{\xi},\tau) \times J_o(\underline{\xi},\tau)$, $b_{\hat{j}\hat{j}_o}(\cdot,\tau,\underline{\xi}) = 0$ (almost everywhere in Σ). In this case, letting

$j_o \equiv j^*(\underline{\xi},\underline{x}(\underline{\xi},\tau),\tau) \varepsilon J_o(\underline{\xi},\tau)$ and $j_1 \equiv j^*[\underline{\rho}(\vec{r},\tau,\underline{\xi})]$, we evaluate the integrand in the first term in (5.20) for two cases, $j_1 \varepsilon J_o(\underline{\xi},\tau)$ and $j_1 \notin J_o(\underline{\xi},\tau)$:

$$\hat{\lambda}(\vec{r},\tau,\underline{\xi}) \sum_{j=1}^{M} \rho_j(\vec{r},\tau,\underline{\xi}) \, [C(j,j_1;\underline{\xi},\tau) - C(j,j_o;\underline{\xi},\tau)]$$
$$= \begin{cases} d_{j_1 j_o}(\vec{r},\tau,\underline{\xi}) & j_1 \varepsilon \, J_o(\underline{\xi},\tau) \\ \sum_{j=1}^{M} \xi_j \lambda_j(\vec{r},\tau,\underline{\xi}) \, [C(j,j_1;\underline{\xi},\tau) - C(j,j_o;\underline{\xi},\tau)] & , \; j_1 \notin J_o(\underline{\xi},\tau) \end{cases} \tag{5A.10}$$

In the second case the expression is a convex U quadratic function of ℓ achieving minimum value $d_{j_1 j_o}(\vec{r},\tau,\underline{\xi}) - \dfrac{|b_{j_1 j_o}(\vec{r},\tau,\underline{\xi})|^2}{a_{j_1 j_o}(\vec{r},\tau,\underline{\xi})}$ for $\ell(\vec{r},\tau,\underline{\xi}) = - \dfrac{b_{j_1 j_o}(\vec{r},\tau,\underline{\xi})}{a_{j_1 j_o}(\vec{r},\tau,\underline{\xi})}$. In the first case the evaluation is independent of $\ell(\vec{r},\tau,\underline{\xi})$. Thus

$$\begin{aligned} \overline{c}(\underline{\xi},\tau) &\geq \int d\vec{r} \; \eta_{j_1 j_o}(\underline{\xi},\tau,\vec{r}) \text{ for any } j_o \varepsilon J_o(\underline{\xi},\tau), \; j_1 \varepsilon \hat{R} \\ &\geq \int d\vec{r} \min_{\substack{j_1 \varepsilon \hat{R} \\ j_o \varepsilon J_o(\underline{\xi},\tau)}} \eta_{j_1 j_o}(\underline{\xi},\tau,\vec{r}) \end{aligned} \tag{5A.11}$$

The cost on the right side of (5A.11) is achieved by the feedback specified in (5.39).

QED

5A.2 PROOF OF THEOREM 5.4

As discussed in Section 5.9, we have already established by Theorem 4.1a the equality of the differential cost increments $-\overline{c}(\underline{\xi},\tau)$ achieved by the incrementally optimum feedback receiver and the incrementally optimum quantum measurement, for those points $\underline{\xi}$ which determine a uniquely optimum a priori guess. Referring to the proof of Theorem 5.2 we see that the only other conditions which allow the optimum feedback to exist are those of Case III. When the received field is not spatially modulated (i.e., $\varepsilon_j(\vec{r},\tau) = S_j(\tau)\varepsilon_0(\vec{r})$), the condition $b_{\hat{j}\hat{j}_0}(\vec{r},\tau,\underline{\xi}) = 0$ (a.e.) for all $\hat{j},\hat{j}_0 \varepsilon J_0$ reduces to

$$\sum_j \xi_j \, S_j(\tau) \, C(j,\hat{j}) = \sum_j \xi_j \, S_j(\tau) \, C(j,\hat{j}_0) \text{ for all } \hat{j},\hat{j}_0 \varepsilon J_0, \tag{5A.12}$$

where we have written $J_0 \equiv J_0(\underline{\xi},\tau)$ and $C(j,\hat{j}) \equiv C(j,\hat{j};\, \underline{\xi},\tau)$.

Referring to (4A.17) in the proof of Theorem 4.1, we evaluate the cost increment $\overline{C}_0^* - \overline{C}_\Delta^*$ achieved by the (incrementally) optimum quantum measurement $\{Q_{\hat{j}}\}_{\hat{j}\varepsilon\hat{R}}$, for an arbitrarily chosen $j_0 \varepsilon J_0$. The summation in (4A.17) may be divided into two parts, over $\hat{j}\varepsilon J_0$ and $\hat{j}\notin J_0$. We first maximize the sum over $\hat{j}\notin J_0$ subject to the constraint of nonnegative definite $Q_{\hat{j}}$'s and $\sum_{\hat{j}\notin J_0} \langle\phi|Q_{\hat{j}}|\phi\rangle \equiv 1-\langle\phi|Q_{J_0}|\phi\rangle \leq 1$. By the same reasoning that led to (4A.21)-(4A.24), we can conclude that it is optimum to concentrate all the available probability in the index $\hat{j}'$ which results in the largest differential cost increment. In the notation of

Theorem 5.2, this optimum cost increment due to terms $\hat{j} \notin J_o$ is

$$[1- \langle\phi|Q_{J_o}|\phi\rangle] \int d\vec{r} \left[d_{\hat{j}}^{\,\prime}{}_{j_o}(\vec{r},\tau,\underline{\xi}) - \frac{|b_{\hat{j}}^{\,\prime}{}_{j_o}(\vec{r},\tau,\underline{\xi})|^2}{a_{\hat{j}}^{\,\prime}{}_{j_o}(\vec{r},\tau,\underline{\xi})} \right] + o(\Delta) \qquad (5A.13)$$

The evaluation of the sum in (4A.17) over $\hat{j} \epsilon J_o$ is a little more involved. Using (4A.37) to evaluate $\langle\phi|\phi_j\rangle$, we see immediately that the second term in braces in (4A.17) contributes $o(\Delta)$ to the cost increment, because of the condition (5A.12). We show that the fourth term in braces in (4A.17) also contributes $o(\Delta)$ by expanding $|\phi_j'\rangle$ as

$$|\phi_j'\rangle = \langle\phi_3'|\phi_j'\rangle\, |\phi_3'\rangle + |\phi_j^2\rangle$$

$$|\phi_j^2\rangle = \langle\phi_4^2|\phi_j^2\rangle\, |\phi_4^2\rangle + |\phi_j^3\rangle$$

$$\vdots$$

$$|\phi_j^k\rangle = \langle\phi_{k+2}^k|\phi_j^k\rangle\, |\phi_{k+2}^k\rangle + |\phi_j^{k+1}\rangle$$

$$\vdots$$

$$|\phi_j^{M-2}\rangle = \langle\phi_M^{M-2}|\phi_j^{M-2}\rangle\, |\phi_M^{M-2}\rangle = \begin{cases} |\phi_M^{M-2}\rangle, & j=M \\ 0, & j\neq M \end{cases}$$

From (4A.10) and (4A.5) we calculate

$$\begin{aligned} \langle\phi_3'|\phi_j'\rangle &= \langle\alpha_3|\alpha_j\rangle - \langle\alpha_3|0\rangle\langle 0|\alpha_j\rangle - \langle\phi_3|\phi\rangle\langle\phi|\phi_j\rangle \\ &= o(\Delta) \end{aligned} \qquad (5A.15)$$

where the second equality follows from the small Δ expressions (4.12) and (4A.37). Similarly, by induction we show that

$$<\phi^k_{k+2}|\phi^k_j> \;=\; o(\Delta) \qquad \text{for all } k \geq 2 \tag{5A.16}$$

Therefore we have shown that the fourth term in braces in (4A.17) also contributes $o(\Delta)$ to the cost increment.

After performing the inner sum over j, we find that the terms in (4A.17) which are not $o(\Delta)$ for $\hat{j}\varepsilon J_o$ reduce to

$$\sum_{\hat{j}\varepsilon J_o} [<\phi|Q_{\hat{j}}|\phi> - <0|Q_{\hat{j}}|0>] \int d\vec{r}\; d_{\hat{j}j_o}(\vec{r},\tau,\underline{\xi}) \tag{5A.17}$$

The expression (5A.17) is maximized, subject to

$$\sum_{\hat{j}\varepsilon J_o} <\phi|Q_{\hat{j}}|\phi> \;=\; <\phi|Q_{J_o}|\phi> \leq 1, \text{ by}$$

$$<\phi|Q_{\hat{j}_1}|\phi> \;=\; <\phi|Q_{J_o}|\phi>$$

and

$$<0|Q_{\hat{j}_2}|0> \;=\; 1 \tag{5A.18}$$

where $\hat{j}_1$ and $\hat{j}_2$ respectively maximize and minimize $d_{\hat{j}j_o}(\vec{r},\tau,\underline{\xi})$ over $\hat{j}\varepsilon J_o$.

We insert these choices into (4A.17) and optimize the resulting expression over $<\phi|Q_{J_o}|\phi>$.

$$\overline{C}_0^* - \overline{C}_\Delta^* = \langle\phi|Q_{J_o}|\phi\rangle \int d\vec{r}[d_{\hat{j}_1 j_o}(\vec{r},\tau,\underline{\xi}) - d_{\hat{j}_2 j_o}(\vec{r},\tau,\underline{\xi})]$$

$$+ [1 - \langle\phi|Q_{J_o}|\phi\rangle] \int d\vec{r}\left[d_{\hat{j}' j_o}(\vec{r},\tau,\xi) - d_{\hat{j}_2 j_o}(\vec{r},\tau,\underline{\xi}) - \frac{|b_{\hat{j}' j_o}(\vec{r},\tau,\underline{\xi})|^2}{a_{\hat{j}' j_o}(\vec{r},\tau,\underline{\xi})}\right] + o(\Delta)$$

$$= \langle\phi|Q_{J_o}|\phi\rangle \int d\vec{r}\, d_{\hat{j}_1 \hat{j}_2}(\vec{r},\tau,\underline{\xi})$$

$$+ [1 - \langle\phi|Q_{J_o}|\phi\rangle] \int d\vec{r}\left[d_{\hat{j}' \hat{j}_2}(\vec{r},\tau,\underline{\xi}) - \frac{|b_{\hat{j}' \hat{j}_2}(\vec{r},\tau,\underline{\xi})|^2}{a_{\hat{j}' \hat{j}_2}(\vec{r},\tau,\underline{\xi})}\right] + o(\Delta)$$

(5A.19)

where the second equality is due to the relations

$$d_{jk}(\cdot) - d_{ik}(\cdot) = d_{ji}(\cdot)$$

and

$$\left.\begin{aligned} b_{jk}(\cdot) &= b_{ji}(\cdot) \\ a_{jk}(\cdot) &= a_{ji}(\cdot) \end{aligned}\right\} \quad \text{for all } i,k \in J_o \qquad (5A.20)$$

which follow from (5.37) and the assumed condition (5A.12).

Thus we have succeeded in eliminating the arbitrarily chosen index $j_o \in J_o$. Clearly, (5A.19) is maximized by

$\langle\phi|Q_{J_o}|\phi\rangle = 1$ or $1-\langle\phi|Q_{J_o}|\phi\rangle = 1$, depending on whether the first or second coefficient is larger. This leads to the same cost increment obtained in Theorem 5.2.

Finally, we must show that the interval costs $\overline{C}^{\ell}(\underline{\xi}_o,0)$ and $\overline{C}_{\Delta}^{*}(\underline{\xi}_o,0)$ achieved by the incrementally optimum feedback receiver and the incrementally optimum quantum measurement sequence are also identical in the limit as $\Delta\downarrow 0$. This fact follows immediately upon evaluation of $\overline{C}^{\ell}(\underline{\xi}_o,0)$ and $\overline{C}_{\Delta}^{*}(\underline{\xi}_o,0)$ from their respective backward-time cost propagation equations.

<u>QED</u>

6A.1 BINARY COHERENT STATE DETECTION OVER AN INFINITE TIME INTERVAL

By substituting T_o for T in (6.21) we see that the receiver that achieves minimum error probability for the binary coherent state problem over the interval [0,T] is also optimum for the corresponding $[0,T_o]$-interval problem, for any $T_o \varepsilon [0,T]$. Therefore it is reasonable to allow a potentially infinite signaling interval ($T=\infty$) and consider the behavior of the optimum feedback receiver specified by (6.1) or (6.22) over arbitrarily long subintervals $[0,T_o]$, $T_o \to \infty$.

We can evaluate the limiting error probability from (6.21) as

$$\lim_{T_o \to \infty} P_e(\underline{\xi}_o, T_o) = \frac{1}{2}\left[1-\sqrt{1-4\xi_1{}^o\xi_2{}^o\, e^{-E(\infty)}}\right] \tag{6A.1}$$

where

$$E(\infty) \equiv \int_0^\infty d\tau \int_\Sigma d\vec{r}\,|\varepsilon_1(\vec{r},\tau) - \varepsilon_2(\vec{r},\tau)|^2 \tag{6A.2}$$

is the energy in the signal difference over the infinite signaling interval. Arbitrarily good performance is attainable only if $E(\infty) = \infty$. If $E(\infty)<\infty$, there is a residual error probability that cannot be eliminated by infinitely long observation. In both cases the decision process is finite, in the sense that it is impossible for the receiver to record an infinite sequence of counts (and corresponding decision changes).

We determine the probability that the final count occurs before T_o, for any $0<T_o<\infty$.

$$\Pr\,[\text{final count occurs before } T_o] = \Pr\,[0 \text{ counts after } T_o] \tag{6A.3}$$

For the optimum receiver the counting intensity at time T_o has one of two values $\left[\frac{f(T_o)+1}{f(T_o)-1} \pm 1\right]^2 \left|\frac{\varepsilon_1(\vec{r},T_o)-\varepsilon_2(\vec{r},T_o)}{2}\right|^2$, with probabilities $P_e^+(\underline{\xi}_o,T_o) = \xi_+[1-P_{ev}^+(T_o)] + \xi_- P_{ev}^-(T_o)$ and $1-P_e^+(\underline{\xi}_o,T_o)$, respectively. Furthermore, <u>in the absence of counts occurring after T_o</u>, the intensity changes deterministically for $\tau>T_o$, according to $\left[\frac{f(\tau)+1}{f(\tau)-1} \pm 1\right]^2 \left|\frac{\varepsilon_1(\vec{r},\tau)-\varepsilon_2(\vec{r},\tau)}{2}\right|^2$.

Therefore the probability in (6A.3) is calculated as

$$\begin{aligned}\Pr\,&[\text{final count occurs before } T_o]\\ &= P_e^+(\underline{\xi}_o,T_o)\exp\left\{-\int_{T_o}^{\infty} d\tau\,\frac{\lambda(\tau)}{4}\left[\frac{f(\tau)+1}{f(\tau)-1}+1\right]^2\right\}\\ &+\left[1-P_e^+(\underline{\xi}_o,T_o)\right]\exp\left\{-\int_{T_o}^{\infty} d\tau\,\frac{\lambda(\tau)}{4}\left[\frac{f(\tau)+1}{f(\tau)-1}-1\right]^2\right\}\end{aligned} \tag{6A.4}$$

Using the relation (determined from (6.14))

$$\frac{f(\tau)+1}{f(\tau)-1} = \left[1-4\xi_1^{\,o}\xi_2^{\,o}\,e^{-E(\tau)}\right]^{-\frac{1}{2}} \tag{6A.5}$$

we can compute the integrals in (6A.4) by making the substitution $4\xi_1^{o}\xi_2^{o}\ e^{-E(\tau)} = \cos^2 x$. Then, with the help of (6.21), the expression in (6A.4) is evaluated as

$$\Pr\ [\text{final count occurs before } T_o] = \left[\frac{1-4\xi_1^{o}\xi_2^{o}\ e^{-E(T_o)}}{1-4\xi_1^{o}\xi_2^{o}\ e^{-E(\infty)}}\right]^{\frac{1}{4}} \tag{6A.6}$$

Since

$$\Pr\ [\text{final count}] \geq \Pr[\text{final count occurs before } T\ ] \tag{6A.7}$$

for all T_o, it follows that

$$\Pr[\text{final count}] \geq \lim_{T_o \uparrow \infty} \left[\frac{1-4\xi_1^{o}\xi_2^{o}\ e^{-E(T_o)}}{1-4\xi_1^{o}\xi_2^{o}\ e^{-E(\infty)}}\right]^{\frac{1}{4}} = 1 \tag{6A.8}$$

Thus we have shown that there is always a final count. Furthermore, letting the random variable τ_f denote the time of occurrence of the final count (with the understanding that $\tau_f \equiv 0$ if no counts are observed in $[0,\infty)$) we have obtained the probability distribution function for τ_f.

$$P_{\tau_f}(T_o) \equiv \Pr(\tau_f \leq T_o) = \begin{cases} 0, & T_o < 0 \\ \left[\dfrac{1-4\xi_1^{o}\xi_2^{o}\ e^{-E(T_o)}}{1-4\xi_1^{o}\xi_2^{o}\ e^{-E(\infty)}}\right]^{\frac{1}{4}}, & T_o \geq 0 \end{cases} \tag{6A.9}$$

6A.2 UNIFORM NEAR-OPTIMALITY OF THE CONDITIONAL SIGNAL NULLER FOR DETECTION OF PPM SIGNALS

We let $x=e^{-E_o}$ in the expressions (6.60), (6.61) for P_M, P_M^* and calculate the derivatives

$$\frac{d^2}{dx^2} P_M = (M-1)(1-x)^{M-2}$$

$$\frac{d^2}{dx^2} P_M^* = \frac{1}{2} (M-1) \left[(1-x)(1+(M-1)x)\right]^{\frac{3}{2}} \qquad (6A.10)$$

Now let

$$h(x,M) = \log \frac{d^2 P_M/dx^2}{d^2 P_M^*/dx^2} = \log 2(1-x)^{M-\frac{1}{2}} [1+(M-1)x]^{\frac{3}{2}} \qquad (6A.11)$$

We easily observe that $\frac{\partial^2 h(x,M)}{\partial x^2} \leq 0$ everywhere, so, for fixed M, h(x,M) is maximized by setting $\frac{\partial h(x,M)}{\partial x} = 0$. This results in

$$h(x,M) \leq h[x_o(M),M] = \log\left\{2\left[\frac{M(M-\frac{1}{2})}{M^2-1}\right]^{M-\frac{1}{2}}\left[\frac{\frac{3}{2}M}{M+1}\right]^{\frac{3}{2}}\right\} \qquad (6A.12)$$

where

$$x_o(M) = \frac{1}{2}\,\frac{M-2}{(M+1)(M-1)} \qquad (6A.13)$$

Next we show that $h[x_o(M),M]$ increases monotonically with M. We use expression (6A.12) to define $h[x_o(M),M]$ for all <u>real</u> $M \geq 2$ and calculate the derivative

$$\frac{d}{dM} h[x_o(M),M] = \log \left[1-\frac{1}{2}\frac{M-2}{M^2-1}\right] + \frac{1}{2}\frac{M-2}{M(M-1)}$$

$$= \log (1-\gamma) + \gamma(1+\frac{1}{M})$$

$$\geq \log (1-\gamma) + \gamma + 2\gamma^2 \qquad (6A.14)$$

where

$$\gamma \equiv \frac{1}{2}\frac{M-2}{M^2-1} \leq \frac{1}{2M} \leq \frac{1}{4} \qquad (6A.15)$$

The last expression in (6A.14) is nonnegative because it is zero at $\gamma=0$ and monotonically increasing for $0 \leq \gamma \leq \frac{3}{4}$. Therefore we can conclude that $h[x_o(M),M]$ is maximized at $M=\infty$.

$$h[x_o(M),M] \leq \lim_{M_o\to\infty} h[x_o(M_o), M_o]$$

$$= \log [3\sqrt{\frac{3}{2e}}\,] \qquad (6A.16)$$

Combining (6A.11), (6A.12), and (6A.16), we obtain the inequality

$$\frac{d^2}{dx^2} P_M \leq 3\sqrt{\frac{3}{2e}}\ \frac{d^2}{dx^2} P_M^* \qquad \text{for all } x,M, \qquad (6A.17)$$

which can be integrated twice to yield

$$P_M \leq 3\sqrt{\frac{3}{2e}}\ P_M^* \qquad \text{for all } x,M, \qquad (6A.18)$$

because P_M, P_M^*, $\frac{d}{dx} P_M$, $\frac{d}{dx} P_M^*$ are all zero at $x=0$.

REFERENCES

1. R. M. Gagliardi and S. Karp, Optical Communications, Wiley-Interscience, New York, 1976.

2. C. W. Helstrom, J. W. S. Liu, and J. P. Gordon, "Quantum-Mechanical Communication Theory," Proc. IEEE, vol. 58, pp. 1578-1598, October, 1970.

3. R. S. Kennedy and S. Karp, eds., Proc. M.I.T.-N.A.S.A. Workshop on Optical Space Communication, NASA SP-217, 1969.

4. E. V. Hoversten, D. L. Snyder, R. O. Harger, and K. Kurimoto, "Direct-Detection Optical Communication Receivers," IEEE Trans. Commun., vol. COM-22, pp. 17-27, January, 1974.

5. L. Mandel, E. C. G. Sudarshan, and E. Wolf, Proc. Phys. Soc., vol. 84, pp. 435-444, September, 1964.

6. R. S. Kennedy, Lecture notes, "Advanced Topics in Optical Communication Research" (6.454T), M.I.T., Cambridge, Mass., 1974.

7. D. L. Snyder, Random Point Processes, Wiley-Interscience, New York, 1975.

8. E. V. Hoversten, R. O. Harger, and S. J. Halme, "Communication Theory for the Turbulent Atmosphere," Proc. IEEE, vol. 58, pp. 1626-1650, October, 1970.

9. W. K. Pratt, Laser Communication Systems, John Wiley and Sons, New York, 1969.

10. M. Ross, Laser Receivers, John Wiley and Sons, New York, 1969.

11. I. Rubin, "Regular Point Processes and Their Detection," IEEE Trans. Inform. Theory, vol. IT-18, pp. 547-557, September, 1972.

12. J. R. Clark, "Estimation for Poisson Processes with Applications in Optical Communication," Ph.D. thesis, Dept. of Elec. Eng., M.I.T., Cambridge, Mass., September, 1971.

13. H. L. Van Trees, Detection, Estimation, and Modulation Theory (Part I), John Wiley and Sons, New York, 1968.

14. R. E. Bellman, _Dynamic Programming_, Princeton Univ. Press, Princeton, New Jersey, 1957.

15. S. E. Dreyfus, _Dynamic Programming and the Calculus of Variations_, Academic Press, New York, 1965.

16. H. J. Kushner, _Introduction to Stochastic Control_, Holt, Rinehart, and Winston, Inc., New York, 1971.

17. H. P. Yuen, "Communication Theory of Quantum Systems," Ph.D. thesis, Dept. of Elec. Eng., M.I.T., Cambridge, Mass., June, 1970.

18. P. A. Benioff, "Decision Procedures in Quantum Mechanics," _J. Math. Phys._, vol. 13, pp. 908-915, June, 1972.

19. V. W. Chan, "Characterization of Measurements in Quantum Communications," Ph.D. thesis, Dept. of Elec. Eng., M.I.T., Cambridge, Mass., September, 1974.

20. J. E. Kaufmann and V. W. Chan, "An Optimum Sequential Quantum Receiver for Binary Message Sequences," presented at the 1976 Conf. on Inform. Sciences and Systems, The Johns Hopkins Univ., Baltimore, Maryland, April, 1976.

21. J. S. Baras and R. O. Harger, "Multiparameter Filtering with Quantum Measurements," presented at the 1975 Conf. on Inform. Sciences and Systems, The Johns Hopkins Univ., Baltimore, Maryland, April, 1975.

22. R. S. Kennedy, "A Near-Optimum Receiver for the Binary Coherent State Quantum Channel," Quart. Prog. Rep. #108, pp. 219-225, Res. Lab. of Electronics, M.I.T., Cambridge, Mass., January, 1973.

23. C. W. Helstrom, "Detection Theory and Quantum Mechanics (II)," _Inform. Contr._, vol. 13, pp. 156-171, August, 1968.

24. S. D. Personick, "Efficient Analog Communication over Quantum Channels," Tech. Rep. #477, Res. Lab. of Electronics, M.I.T., Cambridge, Mass., May, 1970.

25. J. W. S. Liu, "Reliability of Quantum-Mechanical Communication Systems," Tech. Rep. #468, Res. Lab. of Electronics, M.I.T., Cambridge, Mass., December, 1968.

26. P. M. Fishman, "Statistical Inference for Space-Time Point Processes," D.Sc. thesis, Sever Institute of Technology, Washington Univ., St. Louis, Missouri, June, 1974.

27. A. E. Bryson, Jr., and Y. Ho, Applied Optimal Control, Ginn and Company, Waltham, Mass., 1969.

28. A.S. Holevo, "Statistical Decision Theory for Quantum Systems," J. Multivariate Analysis, vol. 3, pp. 337-394, December, 1973.

29. R. J. Glauber, "Coherent and Incoherent States of the Radiation Field," Phys. Rev., vol. 131, pp. 2766-2788, September, 1963.

BIOGRAPHICAL NOTE

Samuel Joseph Dolinar, Jr. was born on July 10, 1949, in Latrobe, Pennsylvania. After completing his primary and secondary education in parochial schools in southwestern Pennsylvania, he has been attending the Massachusetts Institute of Technology, where he received the degrees of Bachelor of Science in Physics, Master of Science in Electrical Engineering, and Electrical Engineer, simultaneously, in June, 1973.

Mr. Dolinar is a member of Sigma Xi.