Introduction to Tensor Networks

with applications to topological phases

Brayden Ware

September 6th 2014

Why tensor networks?

- Hilbert space is big
  - $\mathrm{Dim}\left(\bigotimes_{i=1}^{N}\mathcal{H}_i\right) = 2^N$
  - Generic states are maximally entangled
- Ground states of many-body systems are special
  - Tensor networks target area law states

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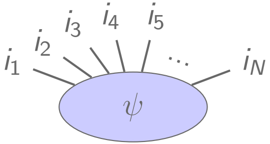

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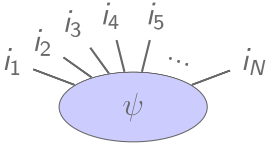

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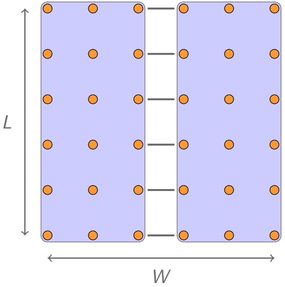

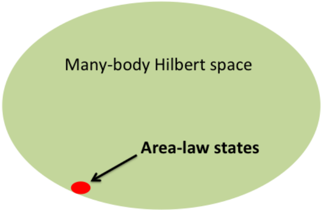

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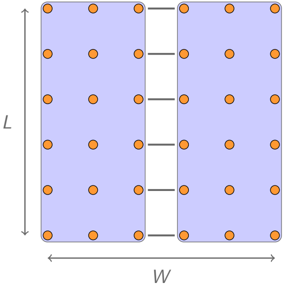

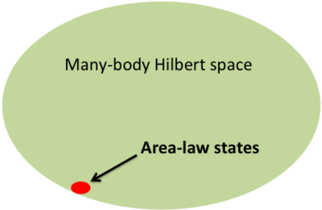

Why tensor networks?

- Tensor networks algorithms generalize renormalization group methods
  - Numerical RG (Wilson)

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  - Including non-chiral topological or SPT order

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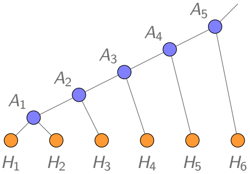

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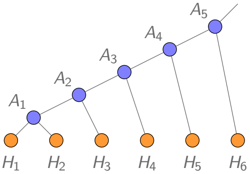

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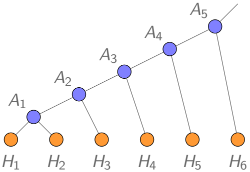

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  - Including non-chiral topological or SPT order

Outline

1. What is a tensor network?
2. AKLT: the canonical MPS
3. Constructing the toric code state

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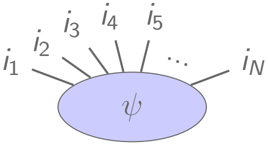
What is a tensor?

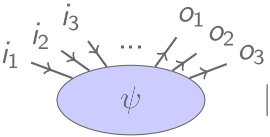
- ... a state of many qubits $|\psi\rangle \in \bigotimes_{i=1}^{N} \mathcal{H}_i$



$$|\psi\rangle = \sum \psi_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle$$

What is a tensor?

- ... a map between Hilbert spaces



$$|\psi\rangle = \sum \psi_{i_1 i_2 .. o_1 o_2 ..} |o_1 o_2 ..\rangle\langle i_1 i_2 ..|$$

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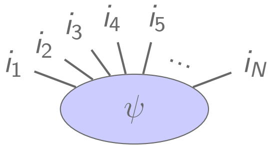
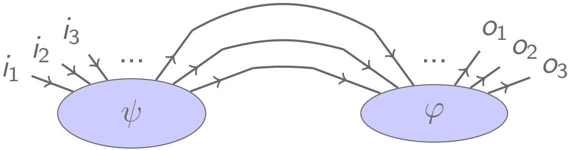


$$|\psi\rangle = \sum \psi_{i_1 i_2 \dots i_N} |i_1 i_2 \dots i_N\rangle$$

- with which we can form contractions



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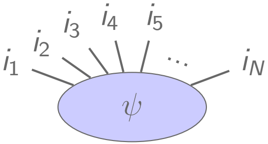
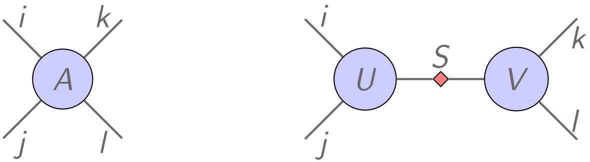


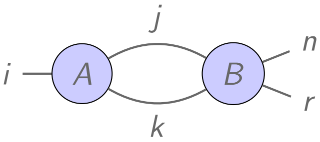
$$|\psi\rangle = \sum \psi_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle$$

- or decompose using SVD.



What is a tensor network?

- ... a contraction scheme for building tensors



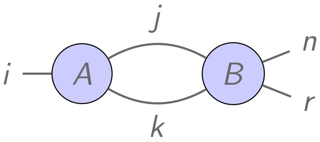
$$\sum_{jk} A_{ijk} B_{jknr}$$

- ... an ansatz for wavefunctions using a tensor network

Matrix Product State ansatz

What is a tensor network *state*?

- ... a contraction scheme for building tensors



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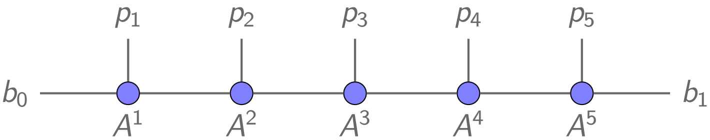
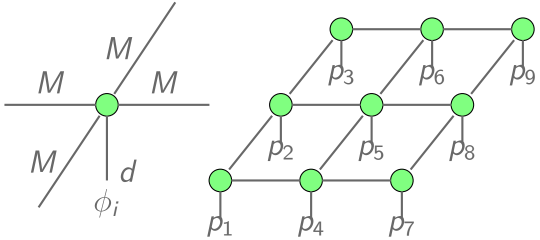


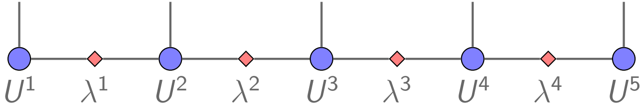
Matrix Product State ansatz

What is a tensor network *state*?

- ... with internal bond dimension M fixed.

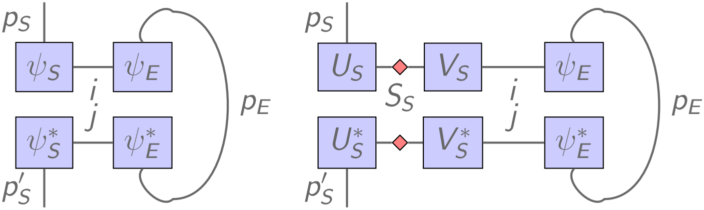


- With no bounds on M, anything is possible:



Working with tensor network states

- Rank of reduced density matrix (and entanglement entropy) bounded by total bond dimension



- Spectrum can be computed without using U_S
- U_S columns are orthogonal Schmidt states

$$|\psi\rangle = \sum_k U_S^{kp_s} |p_s\rangle \Lambda_k |p_E\rangle U_E^{kp_e}$$

Working with tensor network states

- Computing correlation functions in infinite MPS

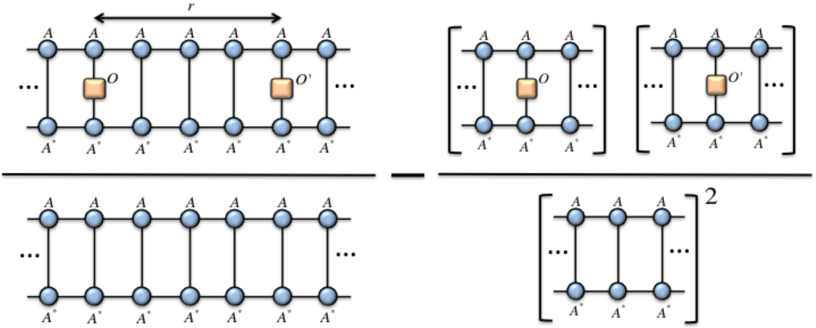


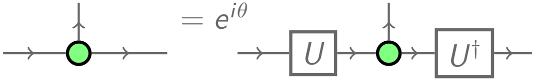
Diagram for $C(r) = \langle O_i O'_{i+r} \rangle - \langle O_i \rangle \langle O'_{i+r} \rangle$

- $\langle O_i O'_{i+r} \rangle = (v_L | T_O T_I^r T_{O'} | v_R)$

Outline

Symmetry and MPS

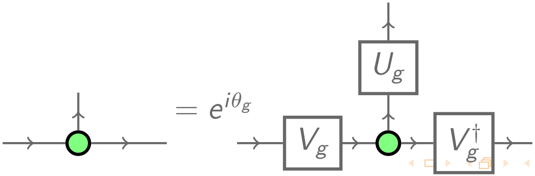
MPS representations are *gauge equivalent* if



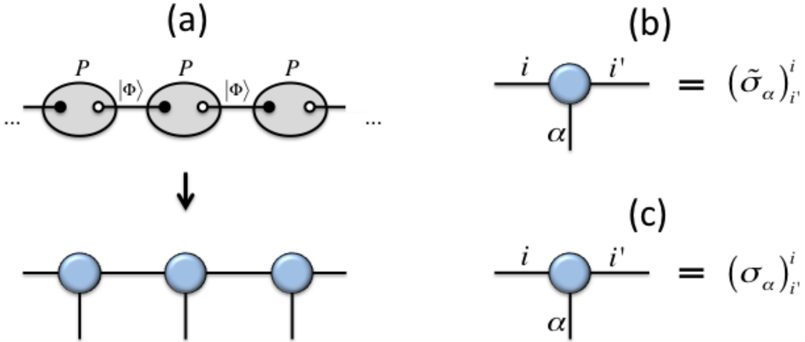
Theorem

Two (simple) MPS specify the same state if and only if they have a gauge equivalence between them.

Therefore symmetric MPS can be represented using a symmetric site tensor.



AKLT: the canonical MPS



Derivation of the site tensors for the AKLT state

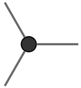
AKLT: Results

- Transfer matrix is simple (one eigenvalue of magnitude 1).
- Correlations decay with power $-\frac{1}{3}$
- Reduced density matrix has eigenvalues $\frac{1}{2}, \frac{1}{2}$
- Can't continuously tune to a product state without breaking SU(2) symmetry
- Two Schmidt states look like infinite system ground state far from boundary
- Schmidt states differ by a spin 1/2 degree of freedom living near boundary
- Degeneracy 2 on half-infinite chain, 1 on circle
- Advanced result: Can't continuously tune to a product state without breaking D_2 AND time-reversal AND spatial reflection symmetry

Outline

Nuts and bolts

- GHZ-state



$$|GHZ\rangle = |000\rangle + |111\rangle$$

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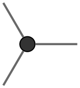

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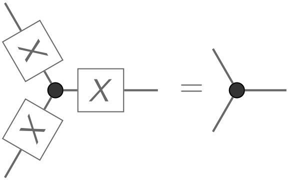
  - $\mathbf{Z}_2$ symmetric



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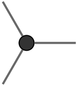

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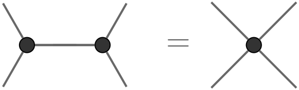
  - Contracting gives larger GHZ state



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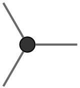

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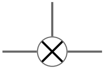
- GHZ-state in basis of X



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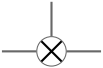

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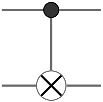
- C-NOT Gate



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- Using GHZ as the MPS site tensor creates a CAT state

Nuts and bolts

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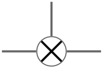

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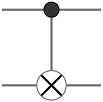

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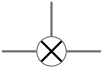

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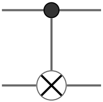

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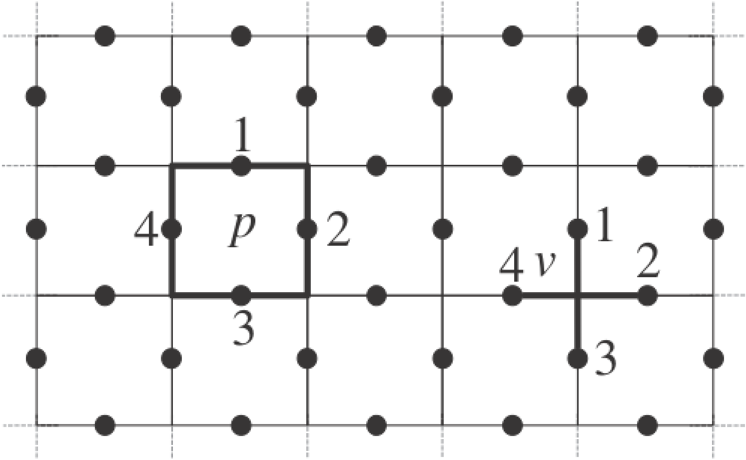
Toric Code State



Toric code

Toric Code

$$H = -J_a \sum_s A_s - J_b \sum_p B_p \ ,$$

where A_v and B_p are vertex and plaquette operators such that

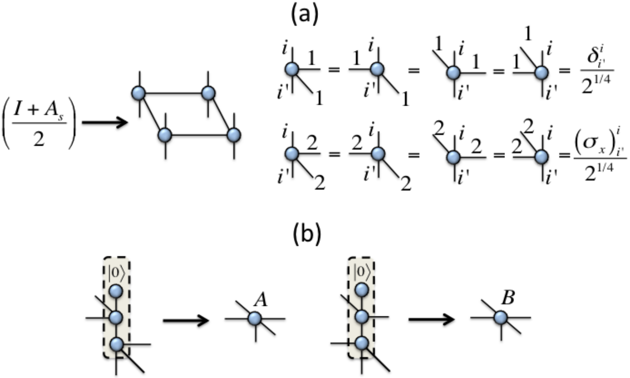
$$A_v = \prod_{i \in v} \sigma_z^i \ , \qquad B_p = \prod_{i \in p} \sigma_x^i \ .$$

$$|\Psi_{TC}\rangle = \prod_v \frac{(\mathbb{I} + A_v)}{2} \prod_p \frac{(\mathbb{I} + B_p)}{2} |00..\rangle = \prod_p \frac{(\mathbb{I} + B_p)}{2} |00..\rangle \ ,$$

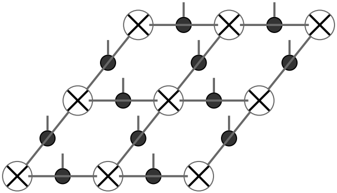
$|\Psi_{TC}\rangle$ is a equal weighted superpositions of all 'loops', where loops indicate positions of $|1\rangle$

Toric Code Site Tensors

- $\mathbb{I} + B_p$ is just a C-NOT-NOT-NOT-NOT
- To get a toric code PEPS, just apply this operator on all plaquettes to product state $|00..\rangle$



Toric Code Results



- Topological order signified by virtual level symmetry
- Around a trivial cycle, virtual symmetry leads to correction to entanglement area law.
- Around a nontrivial cycle, virtual symmetry maps to degenerate ground states
- This is generic for non-chiral topological order

Resources

A Practical Introduction to Tensor Networks

Orus, R. A Practical Introduction to Tensor Networks: Matrix Product States and Projected Entangled Pair States. arXiv [cond-mat.str-el] (2013). at <http://arxiv.org/abs/1306.2164>

Questions ?

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Bonus slides

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RVB States