



Introduction to Tensor Networks

with applications to topological phases

Brayden Ware

September 6th 2014

Why tensor networks?

- Hilbert space is **big**
 - $\text{Dim} \left(\bigotimes_{i=1}^N \mathcal{H}_i \right) = 2^N$
 - Generic states are maximally entangled
- Ground states of many-body systems are special
 - Tensor networks target area law states

Why tensor networks?

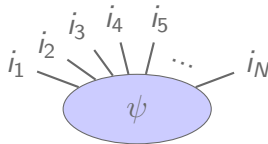
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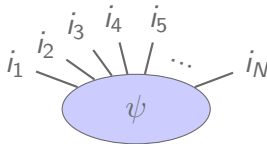
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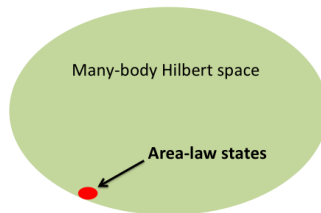
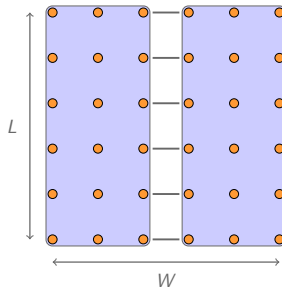
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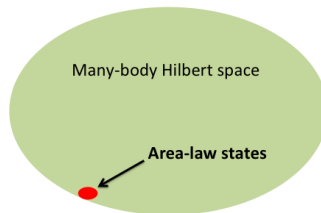
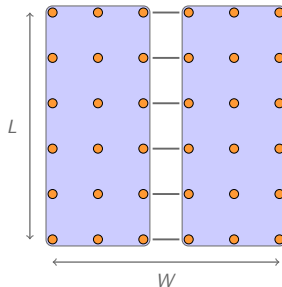
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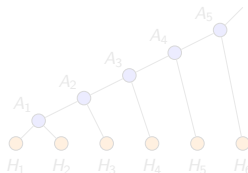
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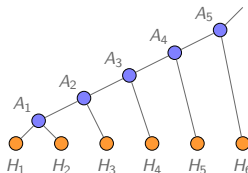
- Tensor networks algorithms generalize renormalization group methods
 - Numerical RG (Wilson)



- Interesting physical systems can be described by simple tensor networks
 - Including non-chiral topological or SPT order

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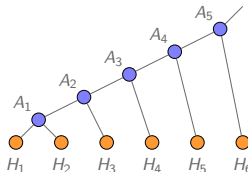
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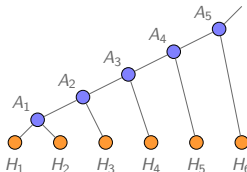
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Outline

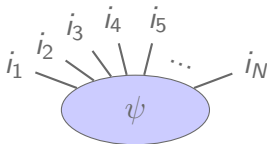
- 1 What is a tensor network?
- 2 AKLT: the canonical MPS
- 3 Constructing the toric code state

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What is a tensor?

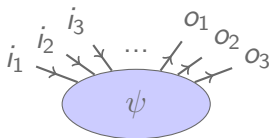
- ... a state of many qubits $|\psi\rangle \in \bigotimes_{i=1}^N \mathcal{H}_i$



$$|\psi\rangle = \sum \psi_{i_1 i_2 \dots i_N} |i_1 i_2 \dots i_N\rangle$$

What is a tensor?

- ... a map between Hilbert spaces

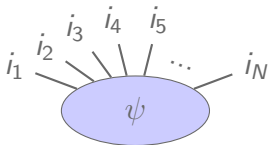


The diagram shows a central blue oval labeled ψ . On the left, three arrows labeled i_1, i_2, i_3 point towards the oval, with an ellipsis \dots between i_2 and i_3 . On the right, three arrows labeled o_1, o_2, o_3 point away from the oval, with an ellipsis \dots between o_1 and o_2 .

$$|\psi\rangle = \sum \psi_{i_1 i_2 \dots o_1 o_2 \dots} |o_1 o_2 \dots\rangle \langle i_1 i_2 \dots|$$

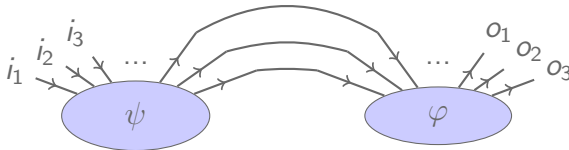
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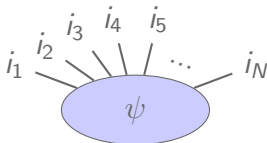
$$|\psi\rangle = \sum \psi_{i_1 i_2 \dots i_N} |i_1 i_2 \dots i_N\rangle$$

- with which we can form contractions



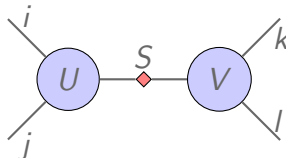
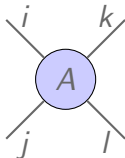
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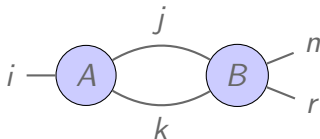
$$|\psi\rangle = \sum \psi_{i_1 i_2 \dots i_N} |i_1 i_2 \dots i_N\rangle$$

- or decompose using SVD.



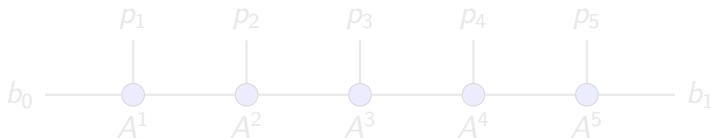
What is a tensor network?

- ... a contraction scheme for building tensors



$$\sum_{jk} A_{ijk} B_{jknr}$$

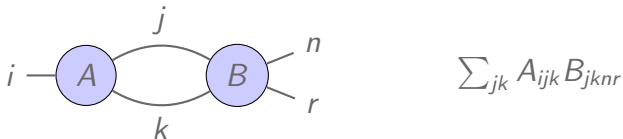
- ... an ansatz for wavefunctions using a tensor network



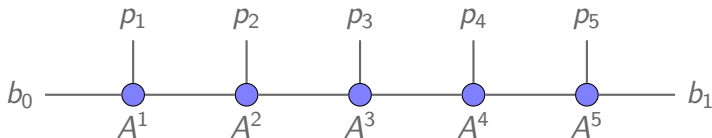
Matrix Product State ansatz

What is a tensor network *state*?

- ... a contraction scheme for building tensors



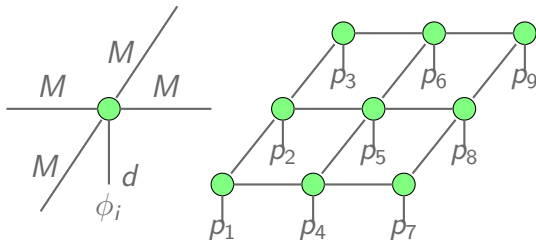
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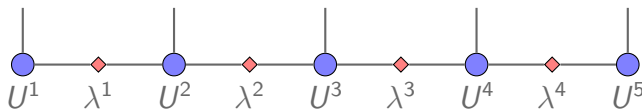
Matrix Product State ansatz

What is a tensor network state?

- ... with internal bond dimension M fixed.

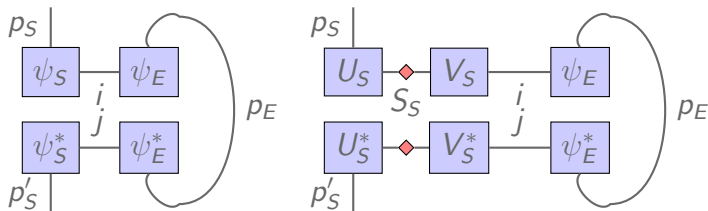


- With no bounds on M , anything is possible:



Working with tensor network states

- Rank of reduced density matrix (and entanglement entropy) bounded by total bond dimension



- Spectrum can be computed without using U_S
- U_S columns are orthogonal Schmidt states

$$|\psi\rangle = \sum_k U_S^{kp_S} |p_S\rangle \Lambda_k |p_E\rangle U_E^{kp_E}$$

Working with tensor network states

- Computing correlation functions in infinite MPS

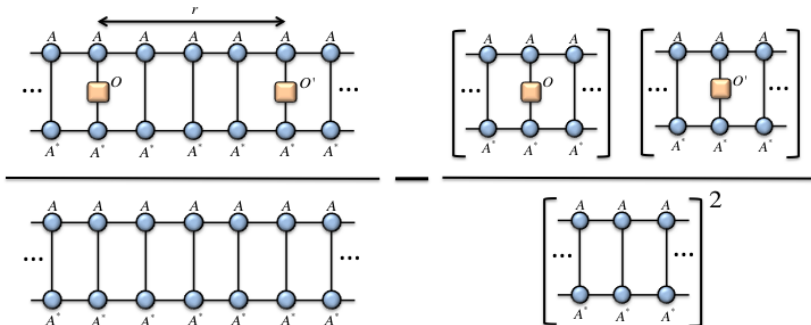


Diagram for $C(r) = \langle O_i O'_{i+r} \rangle - \langle O_i \rangle \langle O'_{i+r} \rangle$

- $\langle O_i O'_{i+r} \rangle = (v_L | T_O T_I^r T_{O'} | v_R)$

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Symmetry and MPS

MPS representations are *gauge equivalent* if

$$\begin{array}{c} \uparrow \\ \rightarrow \bullet \rightarrow \end{array} = e^{i\theta} \begin{array}{c} \rightarrow \boxed{U} \rightarrow \bullet \rightarrow \boxed{U^\dagger} \rightarrow \end{array}$$

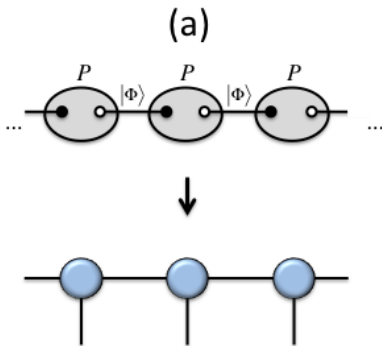
Theorem

Two (simple) MPS specify the same state if and only if they have a gauge equivalence between them.

Therefore symmetric MPS can be represented using a symmetric site tensor.

$$\begin{array}{c} \uparrow \\ \rightarrow \bullet \rightarrow \end{array} = e^{i\theta_g} \begin{array}{c} \uparrow \\ \boxed{U_g} \\ \rightarrow \boxed{V_g} \rightarrow \bullet \rightarrow \boxed{V_g^\dagger} \rightarrow \end{array}$$

AKLT: the canonical MPS



(b)

$$\text{Diagram (b)} = (\tilde{\sigma}_\alpha)^i_{i'}$$

(c)

$$\text{Diagram (c)} = (\sigma_\alpha)^i_{i'}$$

Derivation of the site tensors for the AKLT state

AKLT: Results


- Transfer matrix is simple (one eigenvalue of magnitude 1).
- Correlations decay with power $-\frac{1}{3}$
- Reduced density matrix has eigenvalues $\frac{1}{2}, \frac{1}{2}$
- Can't continuously tune to a product state without breaking $SU(2)$ symmetry
- Two Schmidt states look like infinite system ground state far from boundary
- Schmidt states differ by a spin $1/2$ degree of freedom living near boundary
- Degeneracy 2 on half-infinite chain, 1 on circle
- Advanced result: Can't continuously tune to a product state without breaking D_2 AND time-reversal AND spatial reflection symmetry

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
Nuts and bolts

■ GHZ-state

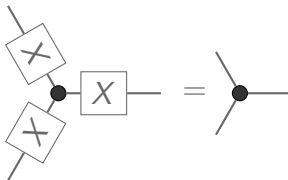

$$|GHZ\rangle = |000\rangle + |111\rangle$$

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

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■ \mathbb{Z}_2 symmetric



Nuts and bolts

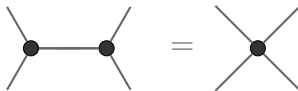
- GHZ-state



A diagram of a vertex representing a GHZ state. It consists of a central black dot with three lines extending from it: one to the right, one to the top-left, and one to the bottom-left.

$$|GHZ\rangle = |000\rangle + |111\rangle$$


- Contracting gives larger GHZ state



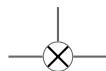
A diagram showing the contraction of two GHZ vertices. On the left, two vertices (each a black dot with three lines) are connected by a horizontal line between their rightmost lines. This is followed by an equals sign and a single vertex (a black dot with four lines extending from it: two to the top-left, two to the bottom-left).

Nuts and bolts

■ GHZ-state

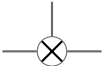

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■ GHZ-state in basis of X

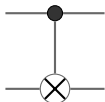

$$|X\rangle = |000\rangle + |011\rangle + |101\rangle + |110\rangle$$

Nuts and bolts

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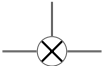
- C-NOT Gate



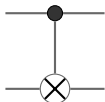
- GHZ can be used to synchronize many operators:
C-NOT-NOT-NOT-NOT
- Using GHZ as the MPS site tensor creates a CAT state

Nuts and bolts

- GHZ-state in basis of X


$$|X\rangle = |000\rangle + |011\rangle + |101\rangle + |110\rangle$$


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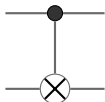
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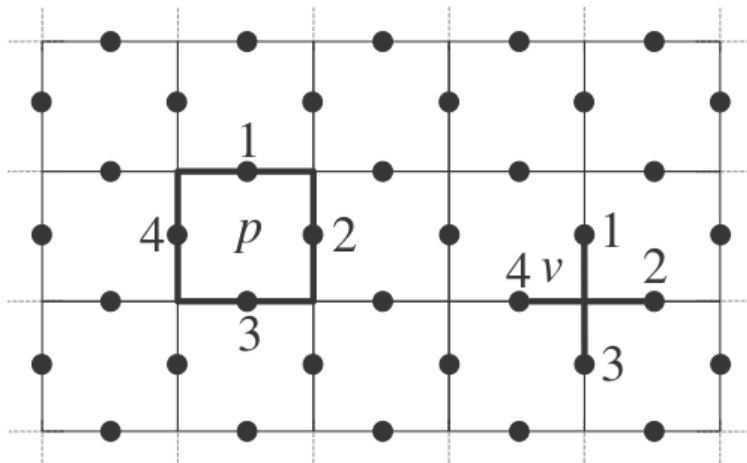

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- C-NOT Gate



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Toric Code State



Toric code

Toric Code

$$H = -J_a \sum_s A_s - J_b \sum_p B_p ,$$

where A_v and B_p are vertex and plaquette operators such that

$$A_v = \prod_{i \in v} \sigma_z^i , \quad B_p = \prod_{i \in p} \sigma_x^i .$$

$$|\Psi_{TC}\rangle = \prod_v \frac{(\mathbb{I} + A_v)}{2} \prod_p \frac{(\mathbb{I} + B_p)}{2} |00..\rangle = \prod_p \frac{(\mathbb{I} + B_p)}{2} |00..\rangle ,$$

$|\Psi_{TC}\rangle$ is a equal weighted superpositions of all 'loops', where loops indicate positions of $|1\rangle$

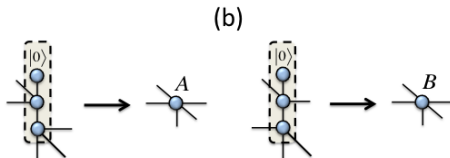
Toric Code Site Tensors

- $\mathbb{I} + B_p$ is just a C-NOT-NOT-NOT-NOT
- To get a toric code PEPS, just apply this operator on all plaquettes to product state $|00..\rangle$

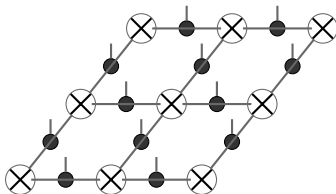
(a)

$$\left(\frac{I + A_x}{2}\right) \rightarrow \text{plaquette}$$

$$\begin{aligned} \begin{array}{c} i \\ \bullet \\ i' \end{array} \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} &= \frac{1}{2} \begin{array}{c} i \\ \bullet \\ i' \end{array} \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} = \frac{1}{2} \begin{array}{c} i \\ \bullet \\ i' \end{array} \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} = \frac{1}{2} \begin{array}{c} i \\ \bullet \\ i' \end{array} \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} = \frac{\delta_{i'}^i}{2^{1/4}} \\ \begin{array}{c} i \\ \bullet \\ i' \end{array} \begin{array}{c} 2 \\ \text{---} \\ 2 \end{array} &= \frac{2}{2} \begin{array}{c} i \\ \bullet \\ i' \end{array} \begin{array}{c} 2 \\ \text{---} \\ 2 \end{array} = \frac{2}{2} \begin{array}{c} i \\ \bullet \\ i' \end{array} \begin{array}{c} 2 \\ \text{---} \\ 2 \end{array} = \frac{2}{2} \begin{array}{c} i \\ \bullet \\ i' \end{array} \begin{array}{c} 2 \\ \text{---} \\ 2 \end{array} = \frac{(\sigma_x)_{i'}^i}{2^{1/4}} \end{aligned}$$



Toric Code Results



- Topological order signified by virtual level symmetry
- Around a trivial cycle, virtual symmetry leads to correction to entanglement area law.
- Around a nontrivial cycle, virtual symmetry maps to degenerate ground states
- This is generic for non-chiral topological order

Resources



A Practical Introduction to Tensor Networks

Orus, R. A Practical Introduction to Tensor Networks: Matrix Product States and Projected Entangled Pair States. arXiv [cond-mat.str-el] (2013). at <http://arxiv.org/abs/1306.2164>



Questions ?

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Bonus slides

Bonus slides

RVB States

(a)

$$\left| \begin{array}{cccc} \bullet & \circ & \bullet & \bullet \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \circ & \bullet & \circ \end{array} \right\rangle + \left| \begin{array}{cccc} \bullet & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \bullet \\ \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \bullet \end{array} \right\rangle + \dots$$

(b)

$$\frac{1}{1} \frac{3}{1} \frac{3}{3} = \frac{2}{2} \frac{3}{2} \frac{3}{3} = 1$$

(and rotations)