Common Inequalities You Must Know

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0 Introduction

Inequalities are a type of problem usually seen in full-answer format contests. It involves proving that a certain expression is greater than another expression given certain conditions. An example would be to prove that $x^2 + 1 \geq 2x$ for all real values x.

1 The Trivial Inequality

This is the most basic way to prove an inequality and uses the fact that a square of a real number is never negative. It states that a sum of squares is greater or equal to zero, or $x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$ for all real values x_1 to x_n. This can be used to prove one case of the AM-GM inequality below, but also can be used to prove more complex inequalities. Equality case occurs when each term $x_i = 0$.

2 AM-GM

This is the most common type of inequality and can be very powerful if used correctly. It states that:

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$$

for all real numbers x_1 to x_n. I will provide a proof of the $n = 2$ case and leave the general case for you as an exercise. We wish to show that $\frac{a+b}{2} \geq \sqrt{ab}$, a, b are positive reals. This is equivalent to $a + b - 2\sqrt{ab} \geq 0$ or $(\sqrt{a} - \sqrt{b})^2 \geq 0$, which is true. Equality holds when all values are equal.

3 Rearrangement

This states that

$$x_n y_1 + \cdots + x_1 y_n \leq x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n \leq x_1 y_1 + \cdots + x_n y_n$$

for every choice of real numbers

$$x_1 \geq \cdots \geq x_n \text{ and } y_1 \geq \cdots \geq y_n$$

and every permutation $x_{\sigma(1)}, \cdots, x_{\sigma(n)}$ of $x_1, \cdots, x_n$.

In other words, out of two ordered sequences of reals, the maximum pairing to maximize the sum of the products is the largest with the largest, second largest with the second largest, and so on. And to minimize, we must pair the largest of one sequence with the smallest of the other, the second largest with the second smallest, and so on.

We may prove this using contradiction. Assume that another permutation of pairings is the maximal. Notice that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $(x_1 - x_2)(y_1 - y_2) \geq 0 \implies x_1y_1 + x_2y_2 > x_2y_1 + x_1y_2$. Let m be the maximal integer such that $\sigma(m) \neq m$, so we can let $\sigma(m) = k$. There would exist $j < m$ such that $\sigma(j) = m$. But $x_k > x_m$ and $y_j > y_m$ so $x_my_m + x_ky_j \geq x_ky_m + x_my_j$. This means there exists a rearrangement which is greater, which is a contradiction. Thus, we have $\sigma(m) = m$ $\forall m \in \{1, 2, ..., n\}$. To prove the minimal case, replace the values $x_1 \geq \cdots \geq x_n$ with $-x_n \geq \cdots \geq -x_1$ to flip the inequality while matching x_{n-j} with y_j.

4 Cauchy-Schwarz

It states that for any real numbers $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$,

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

A nice proof involves considering the polynomial

$$\sum_{i=1}^{n}(a_ix - b_i)^2 = Ax^2 - 2Bx + C$$

where A is $\sum_{i=1}^{n} a_i^2$, B is $\sum_{i=1}^{n} a_ib_i$, and C is $\sum_{i=1}^{n} b_i^2$. Thus, since this quadratic is a sum of squares, it is always greater or equal to zero, so the discriminant is always positive. This means that $4B^2-4AC \geq 0$ or $AC \leq B^2$. Equality holds when $\frac{a_i}{b_i}$ is a constant value.

5 Problems

A-Level problems are around AIME/COMC level while B-Level problems are olympiad level. The problems are roughly in order of difficulty.

5.1 A-Level

1. (AM-HM) Prove that
$$\frac{x_1 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$$
for all positive reals x_1 to x_n.

2. (RMS-AM) Prove that
$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n}$$
for all positive reals x_1 to x_n.

3. (Cosine Law Inequality) Prove that for any positive real angles A, B, C such that $A + B + C = 180°$, and any reals x, y, z, we have $x^2 + y^2 + z^2 \geq 2xycos(A) + 2yzcos(B) + 2xzcos(C)$.

4. (1996 AIME) Triangle ABC has $AB = 9$ and $BC : AC = 40 : 41$. What's the largest area that this triangle can have?

5. (2013 AMC 12B) Let a, b, and c be real numbers such that

$$a + b + c = 2, \text{ and}$$

$$a^2 + b^2 + c^2 = 12$$

. What is the difference between the maximum and minimum possible values of c?

6. Prove AM-GM in general, for all positive integers n.

5.2 B-Level

1. (1969 CMO) Let c be the length of the hypotenuse of a right triangle whose two other sides have lengths a and b. Prove that $a + b \leq c\sqrt{2}$. When does the equality hold?

2. (Mildorf Handout) Show that for all positive reals a, b, c, d,

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}$$

3. (2018 USAMO) Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

4. (2019 UTSMO) Prove that the following inequality is true for all positive real numbers a, b, c:

$$\left(\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a}\right)\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) \geq \frac{9}{4}$$

5. (IMO 1974) If a, b, c, d are positive reals, then determine the possible values of:

$$\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

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6 Hints

6.1 A-Level

1. Use Cauchy-Schwartz.

2. Square both sides then use Cauchy-Schwartz

3. The equation is a quadratic, which means the discriminant is non-negative.

4. Use Heron's formula for the area of a triangle to change it into the form Area $= \sqrt{AB}$ where $A + B$ is a constant. Then AM-GM.

5. Use Cauchy Schwarz to show that equality is rare.

6. Use induction. First let x_n be the smallest and x_{n+1} be the largest. If we let X be the average of all the numbers, use the fact that $(x_n + x_{n+1} - X)X - x_n x_{n+1} = (X - x_n)(x_{n+1} - X) \geq 0$.

6.2 B-Level

1. Square both sides and write c^2 in terms of a and b.

2. Move the sum to the LHS and then use Cauchy-Schwartz.

3. We may assume that a is the minimum. Then, add $2(ab + bc + ac)$ to both sides, factor out $a + b + c$ whenever possible to replace it, and then use AM-GM.

4. First use rearrangement on the left bracket to make every term a perfect square. Then, use Cauchy-Schwartz followed by AM-HM to finish.

5. The range is $(1, 2)$. We approach the lower bound as a approaches infinity and $b = d = \sqrt{a}$ and $c = 1$. The upper bound is approached by a, c approach infinity and $b = d = 1$. We can prove this by dropping terms or adding terms to the denominators.