Game Theory

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1 Introduction

Games can take many different forms on contests. For example, it could appear as the probability of a player winning given optimal play. In other cases, it asks for a winning strategy, or for the best strategy to maximize one's score. The trick to solve these problems is usually to play the game once or twice, or play a simpler version of the game. Usual solutions involve induction or strategy stealing, which are showcased in the examples below.

2 Examples

2.1 First Example

(HMMT February 2019 C3) A and B play a game using 4 fair coins. Initially both sides of each coin are white. Starting with A, they take turns to color one of the white sides either red or green. After all sides are colored, the 4 coins are tossed. If there are more red sides showing up, then A wins, and if there are more green sides showing up, then B wins. However, if there is an equal number of red sides and green sides, then neither of them wins. Given that both of them play optimally to maximize the probability of winning, what is the probability that A wins?

2.2 Second Example

(HMMT February 2018 C6) A stands at $(0, 0)$ and B stands at $(6, 8)$ in the Euclidean plane. A can only move 1 unit in the positive x or y direction, and B can only move 1 unit in the negative x or y direction. Each second, A and B see each other, independently pick a direction to move at the same time, and move to their new position. A catches B if A and B are ever at the same point. B wins if he is able to get to $(0, 0)$ without being caught; otherwise, A wins. Given that both of them play optimally to maximize their probability of winning, what is the probability that B wins?

2.3 Third Example

A and B are playing Chomp. In an m by n arragement of cookies (more than one), the bottom left one is poisoned. Each player makes a move by selecting a cookie and eating it and all cookies to the top and right of it. The person who gets poisoned loses. Prove that if A plays first, she will win.

3 Problems

Point weights correspond to approximate difficulty.

1. (2) Mr. Gregson and Mr. Wilson play a game with a circular table of radius 2019 and coins of radius 1. With Mr. Gregson playing first, they each take turns placing a coin on the table such that they do not overlap. The person who cannot place a coin loses. Who wins?

2. (3)Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

3. (3)A and B play a game on a 100 x 100 board. First, A plays 50 kings on the board and then B places a rook. They then move in turns beginning with A as follows. On his turn, A moves each of the kings one square in any direction. On his turn, B can move the rook horizontally or vertically by any number of squares without passing over a king. A's goal is to capture the rook and B's is to avoid capture. Is there a strategy allowing B to avoid capture indefinitely?

4. (3)Alice and Bob play a game. Initially, the nonnegative integers m and n are written on the blackboard. Alice and Bob take turns erasing a nonnegative integer from the board, replacing it with a lesser nonnegative integer that is different from all previously written nonnegative integers. Alice goes first. For which pairs (m, n) does she win?

5. (3)Alphonse and Beryl are playing a two person game with the following rules:

   - Initially there is a pile of N stones, with $N \geq 2$.
   - The players alternate turns, with Alphonse going first. On his first turn, he must remove at least 1 and at most $N-1$ stones from the pile.
   - If a player removes k stones on their turn, then the other player must remove at least 1 and at most $2k-1$ stones on their next turn.
   - The player who removes the last stone wins.

   Who wins the game?

6. (3)A game is played on a 23 x 23 board. The first player controls two white chips which start in the bottom-left and the top-right corners. The second player controls two black ones which start in the bottom-right and the top-left corners. The players move alternately. In each move, a player can move one of the chips under control to a vacant square which shares a common side with its current location. The first player wins if the two white chips are located on two squares sharing a common side. Can the second player prevents the first player from winning?

7. (4)Consider a 4 x 4 grid of squares. Aziraphale and Crowley play a game on this grid, alternating turns, with Aziraphale going first. On Aziraphale's turn, he may color any uncolored square red, and on Crowley's turn, he may color any uncolored square blue. The game ends when all the squares are colored, and Aziraphale's score is the area of the largest closed region that is entirely red. If Aziraphale wishes to maximize his score, Crowley wishes to minimize it, and both players play optimally, what will Aziraphale's score be?

8. (4)2002 cards with the numbers $1, 2, 3, \cdots, 2002$ written on them are put on a table face up. Two players in turns pick up a card from the table until all cards are gone. The player who gets the last digit of the sum of all numbers on his cards larger than his opponent, wins. Who has a winning strategy and how should one play to win?

9. (6)A and B wish to divide 25 coins, of denominations $1, 2, 3, \cdots, 25$ kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. A makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kopeks wins. Which player has a winning strategy?

10. (6) Prior to the game John selects an integer greater than 100. Then Mary calls out an integer $d > 1$. If John's integer is divisible by d, then Mary wins. Otherwise, John subtracts d from his number and the game continues (with the new number). Mary is not allowed to call out any number twice. When John's number becomes negative, Mary loses. Does Mary have a winning strategy?

11. (6)Two players A and B play alternatively in a convex polygon with $n \geq 5$ sides. In each turn, the corresponding player has to draw a diagonal that does not cut inside the polygon previously drawn diagonals. A player loses if after his turn, one quadrilateral is formed such that its two diagonals are not drawn. A starts the game. For each positive integer n, find a winning strategy for one of the players.

12. (6)A deck of 52 cards is given. There are four suites each having cards numbered $1, 2, \ldots, 13$. The audience chooses some five cards with distinct numbers written on them. The assistant of the magician comes by, looks at the five cards and turns exactly one of them face down and arranges all five cards in some order. Then the magician enters and with an agreement made beforehand with the assistant, he has to determine the face down card (both suite and number). Explain how the trick can be completed.

13. (9)Queenie and Horst play a game on a 20×20 chessboard. In the beginning the board is empty. In every turn, Horst places a black knight on an empty square in such a way that his new knight does not attack any previous knights. Then Queenie places a white queen on an empty square. The game gets finished when somebody cannot move.

    Find the maximal positive K such that, regardless of the strategy of Queenie, Horst can put at least K knights on the board.

14. (9)A and B are playing a game on a 2019 by 2019 chessboard. A places a knight on a square and then moves it. Then they take turns moving the knight to a square that was never used before. The person who cannot move loses. Who wins?