AM-GM Inequality

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1 Introduction

Inequalities are a type of problem usually seen in full-answer format contests. It involves proving that a certain expression is greater than another expression given certain conditions. An example would be to prove that $x^2 + 1 \geq 2x$ for all non-negative real values x. AM-GM is one of the most basic but powerful ways to solve these inequalities. It asserts that:

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$$

for all non-negative real numbers x_1 to x_n. Equality holds when all values are equal.

Weighted AM-GM is AM-GM with weights, or basically having a number appear more than once. It states that if $w_1 + \cdots w_n = 1$ for positive reals w_1 to w_n, then

$$w_1 x_1 + \cdots w_n x_n \geq x_1^{w_1} \cdots x_n^{w_n}$$

Normal AM-GM is with weights $w_i = \frac{1}{n}$.

2 Proof

I will show a proof originally found by Cauchy. It involves a two step induction. First, we must prove that if AM-GM is true for n variables, it's true for $n-1$ variables. Secondly, we must also prove that it's true for $2n$ variables. The base case can be any positive integer. This induction works as any number can be reached by these two steps.

Base Case: $n = 2$

$$a + b - 2\sqrt{ab} = (\sqrt{a} + \sqrt{b})^2 \geq 0$$

Induction Case 1: n to $2n$

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} &= \frac{\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + \cdots + a_{2n}}{n}}{2n} \\
&\geq \frac{\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{a_{n+1} \cdots a_{2n}}}{2} \\
&\geq \sqrt[2n]{a_1 a_2 \cdots a_{2n}}
\end{aligned}$$

This is done using AM-GM twice, first in the n variable case, then in the 2 variable case.
Induction Case 2: n to $n-1$ Using the n variable case, with $a_n = \frac{a_1 + \cdots a_{n-1}}{n}$:

$$\frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + \cdots a_{n-1}}{n}}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} \frac{a_1 + \cdots a_{n-1}}{n-1}}$$
$$\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} \frac{a_1 + \cdots a_{n-1}}{n-1}}$$
$$(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1})^n \geq a_1 a_2 \cdots a_{n-1} (\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1})$$
$$(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1})^{n-1} \geq a_1 a_2 \cdots a_{n-1}$$
$$\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}$$

3 Examples

3.1 First Example

Let us try to prove the following:

$$a^4c + b^4a + c^4b \geq a^3bc + b^3ac + c^3ab$$

We seek to prove by weighted AM-GM. The strategy for proving something of the above form, a cyclic or symmetric sum on both sides, is to try to find weights w_1, w_2, w_3 such that $w_1 + w_2 + w_3 = 1$ and $(a^4c)^{w_1}(b^4a)^{w_2}(c^4b)^{w_3} = a^3bc$. Solving the system of equations for the exponents of a, b, c gives you that $w_1 = \frac{9}{13}, w_2 = \frac{3}{13}, w_3 = \frac{1}{13}$.
Now, we can seperate terms $a^4c + b^4a + c^4b$ as $\frac{1}{13}(9a^4c + 3b^4a + c^4b) + \frac{1}{13}(a^4c + 9b^4a + 3c^4b) + \frac{1}{13}(3a^4c + b^4a + 9c^4b)$. Using weighted AM-GM on each term, we have $\frac{1}{13}(9a^4c + 3b^4a + c^4b) \geq a^3bc$ and symmetrically for the other ones. Thus, we find that $a^4c + b^4a + c^4b = \frac{1}{13}(9a^4c + 3b^4a + c^4b) + \frac{1}{13}(a^4c + 9b^4a + 3c^4b) + \frac{1}{13}(3a^4c + b^4a + 9c^4b) \geq a^3bc + b^3ac + c^3ab$. This finishes the proof.

3.2 Second Example

Now let's try to prove a problem that doesn't even look like it has inequalities. Find all solutions to $p^3 + q^3 + 8 = 6pq$ for primes p, q.

First, we notice that $p^3 + q^3 + 8 = 6pq$ is a direct result of AM-GM. Thus, there is equality if and only if $p^3 = q^3 = 8$ or $p = q = 2$. This is indeed a solution, and must be the only solution due to the inequality.

3.3 Third Example

Now let's try to solve a really hard problem with only AM-GM. This is from the 2018 IMO Shortlist as question A7, the hardest algebra problem on the shortlist. It was one of my favourite proofs using AM-GM. Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a+b+c+d=100$.

First, we notice the cubic roots, and we need to maximize them, so we should try to use AM-GM the other way. We want three terms x, y, z such that $xyz = K\frac{a}{b+7}$, where K is an constant, and the sum $x+y+z$ cyclically will yield a constant. With some trial and error, one might guess that maximum occurs when $(a,b,c,d) = (49,1,49,1)$. Thus, at equality case, we have $\frac{a}{a+7} = \frac{7}{b+7}$. This is beautiful because we know that if we sum the terms $\frac{a}{a+7} + \frac{7}{b+7}$ cyclically, it equals a constant, 4. Now, let $x = \frac{a}{a+7}, y = \frac{7}{b+7}$. We find that if $xyz = K\frac{a}{b+7}$, then $z = \frac{K(a+7)}{7}$. But at equality, $x=y=z$, so we can solve for K to be $\frac{7}{64}$.

Thus, after a lot of scratch work, we find that $\sqrt[3]{\frac{7a}{64(b+7)}} \leq \frac{1}{3}\left(\frac{a+7}{64} + \frac{a}{a+7} + \frac{7}{b+7}\right)$ seems to maintain the equality case. Summing it cyclically, we have $\sqrt[3]{\frac{7}{64}}(\sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}}) \leq \frac{1}{3}(\frac{a+b+c+d+28}{64} + 4) = 2$ so $S \leq 2\sqrt[3]{\frac{64}{7}} = \frac{8}{\sqrt[3]{7}}$. This will hold when $(a,b,c,d) = (49,1,49,1)$, so we have found the maximum to be $\frac{8}{\sqrt[3]{7}}$. This example truly shows the power of AM-GM when solving even the hardest problems, but it requires a lot of work to find the exact terms to use AM-GM on. Working backwards from the equality case will help you find this.

4 Quick Tips

You should use AM-GM when:

1. The variables are positive reals or the negative case is easy to prove.
2. You see n-th roots being less than something.
3. Cyclic sum is greater than another cyclic sum.
4. Solving an equation that can't be directly done, such has having cubic terms.
5. An inequality where the sum is the greater side.

5 Problems

These are ordered roughly in increasing difficulty.

1. Find the minimum value of $a^2 + \frac{1}{a^2} + b^2 + \frac{1}{b^2}$ for real numbers a, b.
2. Prove that $\frac{n-1}{n} + \frac{a}{n} \geq \sqrt[n]{a}$ for all positive integers n and positive reals a.
3. (2018 CSMC) Suppose that $0^\circ < A < 90^\circ$ and $0^\circ < B < 90^\circ$ and
$$\left(4+\tan^2 A\right)\left(5+\tan^2 B\right) = \sqrt{320}\tan A \tan B$$
Determine all possible values of $\cos A \sin B$.
4. Prove that $(a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \geq 9$ for all positive reals a, b, c.

5. (1992 AIME) Triangle ABC has $AB = 9$ and $BC : AC = 40 : 41$. What's the largest area that this triangle can have?

6. Prove that $a^5 + b^5 + c^5 \geq a^2b^2c + a^2bc^2 + ab^2c^2$ for all positive reals a, b, c.

7. Prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$ for all positive reals a, b, c.

8. (2012 CMO) Show that $x^2 + xy^2 + xyz^2 \geq 4xyz - 4$ for all positive reals x, y, z.

9. (2017 CMO) Let a, b, and c be non-negative real numbers, no two of which are equal. Prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(a-c)^2} + \frac{c^2}{(a-b)^2} > 2$$

10. (2014 CMO) Let $a_1, a_2, \ldots, a_n$ be positive real numbers whose product is 1. Show that the sum $\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \frac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$ is greater than or equal to $\frac{2^n-1}{2^n}$.

These are super-challenging problems:

11. (2018 USAMO) Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

12. (2015 CMO) Let ABC be an acute-angled triangle with altitudes AD, BE, and CF. Let H be the orthocentre, that is, the point where the altitudes meet. Prove that

$$\frac{AB \cdot AC + BC \cdot CA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2.$$

13. (2004 USAMO) Let a, b, and c be positive real numbers Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$$

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