Incomplete Information

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1 Introduction

Microeconomics is all just game theory, according to my economics professor last semester. Game theory in economics, at least at the first-year level, deals a lot with complete information, but games are so much more interesting when we have to extrapolate from incomplete information. In these games, you want to make sure that each move gives you as much information as possible, in order to guarantee that you land at the answer in the quickest manner. Let's take a look at some of these below.

2 Examples

2.1 First Example

We have an invisible space ship in n-dimensional space that starts at some rational point with some constant rational velocity (each component is rational). Prove that, if every second, we can make a guess as to a random point where the spaceship will be, we can guess in a way that guarantees we succeed in finite time.

2.2 Second Example

A deck of 52 cards is given. There are four suites each having cards numbered $1, 2, \ldots, 13$. The audience chooses some five cards with distinct numbers written on them. The assistant of the magician comes by, looks at the five cards and turns exactly one of them face down and arranges the other four cards in some order. Then the magician enters and with an agreement made beforehand with the assistant, he has to determine the face down card (both suite and number). Explain how the trick can be completed.

2.3 Third Example

Let's say you are trying to get from intersection X to intersection Y in a grid street layout with north-south streets being significantly wider than east-west streets. There are traffic lights at every intersection. What is the best way of going from X to Y?

2.4 Fourth Example

We have 13 coins such that 12 have equal weight and the last one has a different weight. We have a balance that can be used 3 times. Devise a strategy to find the coin with different weight.

3 Problems

1. Prior to the game John selects an integer greater than 100. Then Mary calls out an integer $d > 1$. If John's integer is divisible by d, then Mary wins. Otherwise, John subtracts d from his number and the game continues (with the new number). Mary is not allowed to call out any number twice. When John's number becomes negative, Mary loses. Does Mary have a winning strategy?

2. Given 365 cards, in which distinct numbers are written. We may ask for any three cards, the order of numbers written in them. Is it always possible to find out the order of all 365 cards by 2000 such questions?

3. A magician and his assistant present the following trick. Thirteen empty closed boxes are placed in a row. Then, the magician leaves the stage, and a random person from the audience is selected to put two coins into two boxes of their choice, one coin in each box, in front of the magician's assistant, i.e. the assistant knows which boxes contain coins. Then, the magician returns and his assistant is allowed to open one box that does not contain a coin. After that the magician must choose four boxes to be opened simultaneously. The goal of the magician is to open both boxes with coins. Construct a scheme by which the magician and his assistant can perform the trick successfully every time.

4. a) There are $2n + 1$ ($n > 2$) batteries. We don't know which batteries are good and which are bad but we know that the number of good batteries is greater by 1 than the number of bad batteries. A lamp uses two batteries, and it works only if both of them are good. What is the least number of attempts sufficient to make the lamp work?

   b) The same problem but the total number of batteries is $2n$ ($n > 2$) and the numbers of good and bad batteries are equal.

5. Let T be the set of ordered triples (x, y, z), where x, y, z are integers with $0 \leq x, y, z \leq 9$. Players A and B play the following guessing game. Player A chooses a triple (x, y, z) in T, and Player B has to discover A's triple in as few moves as possible. A move consists of the following: B gives A a triple (a, b, c) in T, and A replies by giving B the number $|x + y - a - b| + |y + z - b - c| + |z + x - c - a|$. Find the minimum number of moves that B needs to be sure of determining A's triple.

6. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order: The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1. A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1. The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1. Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

7. Alice has a map of Wonderland, a country consisting of $n \geq 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

8. The liar's guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

   At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

   After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

   1. If $n \geq 2^k$, then B can guarantee a win. 2. For all sufficiently large k, there exists an integer $n \geq (1.99)^k$ such that B cannot guarantee a win.