Combinatorics 2

David Tang

July 2020

1 Problems

1. Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \ldots, 2n\}$, for integers i, j with $0 \leq i, j \leq n$, such that: for all $0 \leq i, j \leq n$, the set $S_{i,j}$ has $i + j$ elements; and $S_{i,j} \subseteq S_{k,l}$ whenever $0 \leq i \leq k \leq n$ and $0 \leq j \leq l \leq n$.

2. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m .

3. In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The distance between two cities is defined to be the least possible numbers of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

4. Players A and B play a "paintful" game on the real line. Player A has a pot of paint with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In every round, player A picks some positive integer m and provides $1/2^m$ units of ink from the pot. Player B then picks an integer k and blackens the interval from $k/2^m$ to $(k+1)/2^m$ (some parts of this interval may have been blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0, 1]$ is not completely blackened. Decide whether there exists a strategy for player A to win in a finite number of moves.