Combinatorics 1

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1 Problems

1. Consider a 2018×2019 board with integers in each unit square. Two unit squares are said to be neighbours if they share a common edge. In each turn, you choose some unit squares. Then for each chosen unit square the average of all its neighbours is calculated. Finally, after these calculations are done, the number in each chosen unit square is replaced by the corresponding average. Is it always possible to make the numbers in all squares become the same after finitely many turns?

2. An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A beam is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions: The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.) No two beams have intersecting interiors. The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam. What is the smallest positive number of beams that can be placed to satisfy these conditions?

3. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n. Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

4. Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular *edges* that meet at *vertices*. Notice that there are an even number of vertices on each circle. Given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice- once for each of the two circle that cross at that point. If the two colours agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.