Combo 5

David Tang

August 2020

1   Problems

1. A 2^n by n matrix of 1s and -1s is such that its 2^n rows are pairwise distinct. An arbitrary subset of the entries of the matrix are changed to 0. Prove that there is a nonempty subset of the rows of the altered matrix that sum to the zero vector.

2. On an infinite square grid we place finitely many cars, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a move, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.

3. Let M be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.