AIME Combo 3

David Tang

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1 Examples

1. Yannick is playing a game with 100 rounds, starting with 1 coin. During each round, there is an $n\%$ chance that he gains an extra coin, where n is the number of coins he has at the beginning of the round. What is the expected number of coins he will have at the end of the game?

2. Petya and Vasya play a game with a pile of cards. For each subset of five different variables from the set $\{x_1, \cdots, x_{10}\}$ there is a single card with their product written on the card. With Petya starting, Petya and Vasya alternate choosing one card from the pile of cards. After all cards have been drawn from the pile, Vasya assigns numerical values to the variables as he wants, except that he must ensure $0 \leq x_1 \leq \cdots \leq x_{10}$. Can Vasya make his assignations in such a way that ensures the sum of the products on his cards is greater than the sum of the products on Petya's cards?

3. A magician and his assistant present the following trick. Thirteen empty closed boxes are placed in a row. Then, the magician leaves the stage, and a random person from the audience is selected to put two coins into two boxes of their choice, one coin in each box, in front of the magician's assistant, i.e. the assistant knows which boxes contain coins. Then, the magician returns and his assistant is allowed to open one box that does not contain a coin. After that the magician must choose four boxes to be opened simultaneously. The goal of the magician is to open both boxes with coins. Construct a scheme by which the magician and his assistant can perform the trick successfully every time.

4. There is a row of 100 squares each containing a counter. Any 2 neighbouring counters can be swapped for 1 dollar and any 2 counters that have exactly 4 counters between them can be swapped for free. What is the least amount of money that must be spent to rearrange the counters in reverse order?

2 Problems

1. In a shooting match, eight clay targets are arranged in two hanging columns of three targets each and one column of two targets. A marksman is to break all the targets according to the following rules:

   1) The marksman first chooses a column from which a target is to be broken. 2) The marksman must then break the lowest remaining target in the chosen column.

   If the rules are followed, in how many different orders can the eight targets be broken?

2. The probability that a set of three distinct vertices chosen at random from among the vertices of a regular n-gon determine an obtuse triangle is $\frac{93}{125}$. Find the sum of all possible values of n.

3. Find the number of functions $f(x)$ from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$ that satisfy $f(f(x)) = f(f(f(x)))$ for all x in $\{1, 2, 3, 4, 5\}$.

4. Consider arrangements of the 9 numbers $1, 2, 3, \ldots, 9$ in a 3×3 array. For each such arrangement, let a_1, a_2, and a_3 be the medians of the numbers in rows 1, 2, and 3 respectively, and let m be the median of $\{a_1, a_2, a_3\}$. Let Q be the number of arrangements for which $m = 5$. Find the remainder when Q is divided by 1000.

5. Call a set S product-free if there do not exist $a, b, c \in S$ (not necessarily distinct) such that $ab = c$. For example, the empty set and the set $\{16, 20\}$ are product-free, whereas the sets $\{4, 16\}$ and $\{2, 8, 16\}$ are not product-free. Find the number of product-free subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

6. Five towns are connected by a system of roads. There is exactly one road connecting each pair of towns. Find the number of ways there are to make all the roads one-way in such a way that it is still possible to get from any town to any other town using the roads (possibly passing through other towns on the way).

7. A $10 \times 10 \times 10$ grid of points consists of all points in space of the form (i, j, k), where i, j, and k are integers between 1 and 10, inclusive. Find the number of different lines that contain exactly 8 of these points.

8. For every subset T of $U = \{1, 2, 3, \ldots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.