AIME Combinatorics 2

David Tang

May 2020

1 Examples

1. Define an ordered triple (A, B, C) of sets to be minimally intersecting if $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \emptyset$. For example, $(\{1, 2\}, \{2, 3\}, \{1, 3, 4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Find the remainder when N is divided by 1000.

   Note: $|S|$ represents the number of elements in the set S.

2. Let $m \geq 3$ be an integer and let $S = \{3, 4, 5, \ldots, m\}$. Find the smallest value of m such that for every partition of S into two subsets, at least one of the subsets contains integers a, b, and c (not necessarily distinct) such that $ab = c$.

   Note: a partition of S is a pair of sets A, B such that $A \cap B = \emptyset$, $A \cup B = S$.

3. Kelvin the Frog likes numbers whose digits strictly decrease, but numbers that violate this condition in at most one place are good enough. In other words, if d_i denotes the ith digit, then $d_i \leq d_{(i+1)}$ for at most one value of i. For example, Kelvin likes the numbers 43210, 132, and 3, but not the numbers 1337 and 123. How many 5-digit numbers does Kelvin like?

4. Given an 8 x 8 checkerboard with alternating white and black squares, how many ways are there to choose four black squares and four white squares so that no two of the eight chosen squares are in the same row or column?

2 Problems

1. For positive integers n, let S_n be the set of integers x such that n distinct lines, no three concurrent, can divide a plane into x regions (for example, $S_2 = \{3, 4\}$, because the plane is divided into 3 regions if the two lines are parallel, and 4 regions otherwise). What is the minimum i such that S_i contains at least 4 elements?

2. Find the number of ordered triple (a, b, c) where a, b, and c are positive integers, a is a factor of b, a is a factor of c, and $a + b + c = 100$.

3. Kathy has 5 red cards and 5 green cards. She shuffles the 10 cards and lays out 5 of the cards in a row in a random order. She will be happy if and only if all the red cards laid out are adjacent and all the green cards laid out are adjacent. For example, card orders $RRGGG, GGGGR$, or $RRRRR$ will make Kathy happy, but $RRRGR$ will not. The probability that Kathy will be happy is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. In the 6×4 grid shown, 12 of the 24 squares are to be shaded so that there are two shaded squares in each row and three shaded squares in each column. Let N be the number of shadings with this property. Find the remainder when N is divided by 1000.

5. Let N be the number of ordered pairs of nonempty sets $\mathcal{A}$ and $\mathcal{B}$ that have the following properties:

   • $\mathcal{A} \cup \mathcal{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, • $\mathcal{A} \cap \mathcal{B} = \emptyset$, • The number of elements of $\mathcal{A}$ is not an element of $\mathcal{A}$, • The number of elements of $\mathcal{B}$ is not an element of $\mathcal{B}$.

   Find N.

6. Define a T-grid to be a 3×3 matrix which satisfies the following two properties:

   (1) Exactly five of the entries are 1's, and the remaining four entries are 0's. (2) Among the eight rows, columns, and long diagonals (the long diagonals are $\{a_{13}, a_{22}, a_{31}\}$ and $\{a_{11}, a_{22}, a_{33}\}$, no more than one of the eight has all three entries equal.

   Find the number of distinct T-grids.

7. Kelvin the Frog is hopping on a number line (extending to infinity in both directions). Kelvin starts at 0. Every minute, he has a $\frac{1}{3}$ chance of moving 1 unit left, a $\frac{1}{3}$ chance of moving 1 unit right, and $\frac{1}{3}$ chance of getting eaten. Find the expected number of times Kelvin returns to 0 (not including the start) before he gets eaten.

8. Eli, Joy, Paul, and Sam want to form a company; the company will have 16 shares to split among the 4 people. The following constraints are imposed:

   1) Every person must get a positive integer number of shares, and all 16 shares must be given out.
   2) No one person can have more shares than the other three people combined.

   Assuming that shares are indistinguishable, but people are distinguishable, in how many ways can the shares be given out?