AIME Combinatorics

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1 Examples

1. Mrs. Toad has a class of 2017 students, with unhappiness levels $1, 2, \ldots, 2017$ respectively. Today in class, there is a group project and Mrs. Toad wants to split the class in exactly 15 groups. The unhappiness level of a group is the average unhappiness of its members, and the unhappiness of the class is the sum of the unhappiness of all 15 groups. What's the minimum unhappiness of the class Mrs. Toad can achieve by splitting the class into 15 groups?

2. Each of the integers $1, 2, ..., 729$ is written in its base-3 representation without leading zeroes. The numbers are then joined together in that order to form a continuous string of digits: 12101112202122... How many times in this string does the substring 012 appear?

3. There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical.

4. Let $SP_1P_2P_3EP_4P_5$ be a heptagon. A frog starts jumping at vertex S. From any vertex of the heptagon except E, the frog may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Find the number of distinct sequences of jumps of no more than 12 jumps that end at E.

2 Problems

1. Two ordered pairs (a, b) and (c, d), where a, b, c, d are real numbers, form a basis of the coordinate plane if $ad \neq bc$. Determine the number of ordered quadruples (a, b, c) of integers between 1 and 3 inclusive for which (a, b) and (c, d) form a basis for the coordinate plane.

2. An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

3. A soccer team has 22 available players. A fixed set of 11 players starts the game, while the other 11 are available as substitutes. During the game, the coach may make as many as 3 substitutions, where any one of the 11 players in the game is replaced by one of the substitutes. No player removed from the game may reenter the game, although a substitute entering the game may be replaced later. No two substitutions can happen at the same time. The players involved and the order of the substitutions matter. Let n be the number of ways the coach can make substitutions during the game (including the possibility of making no substitutions). Find the remainder when n is divided by 1000.

4. Marisa has a collection of $2^8 - 1 = 255$ distinct nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. For each step she takes two subsets chosen uniformly at random from the collection, and replaces them with either their union or their intersection, chosen randomly with equal probability. (The collection is allowed to contain repeated sets.) She repeats this process $2^8 - 2 = 254$ times until there is only one set left in the collection. What is the expected size of this set?

5. Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each k with $1 \leq k \leq 5$, at least one of the first k terms of the permutation is greater than k.

6. Sean is a biologist, and is looking at a strng of length 66 composed of the letters A, T, C, G. A substring of a string is a contiguous sequence of letters in the string. For example, the string $AGTC$ has 10 substrings: A, G, T, C, AG, GT, TC, AGT, GTC, $AGTC$. What is the maximum number of distinct substrings of the string Sean is looking at?

7. A point P lies at the center of square $ABCD$. A sequence of points $\{P_n\}$ is determined by $P_0 = P$, and given point P_i, point P_{i+1} is obtained by reflecting P_i over one of the four lines AB, BC, CD, DA, chosen uniformly at random and independently for each i. What is the probability that $P_8 = P$?

8. Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.