Lab 7 Report
Ian Taulli
Computational Physics

Part 1

The Lax scheme does not reflect the waveform because the equation only includes terms that depend on u_{j+1}^n and u_{j-1}^n, but no terms with u_j^n. Consider the wave moving to the right toward a fixed end, the position just to the right of the zero line only has a value on the next iteration because the value to its right is not zero. But with a fixed end, the advancing line of zeros meets the final position on the right–which is also zero–so all the positions on the next iteration are zero and any iterations after that remain zero, preventing a reflection.
For discretization factor $\frac{c\Delta t}{\Delta x} > 1$, the Leapfrog method "blows up" within a few steps of the initial frame. But, for $\frac{c\Delta t}{\Delta x} < 1$ there is numerical dispersion. This implies that the Leapfrog method also follows the Courant criterion. So the only value of the discretization factor that satisfies the von Neumann stability condition, and maintains the shape of the waveform is to have $\frac{c\Delta t}{\Delta x} = 1$.

Part 2

Starting with the explicit Lax-Wendorff stepping equation

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left[\frac{1}{2}\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n + u_{j-1}^n - 2u_j^n\right)\right] \tag{1}$$

Insert the von Neumann solution

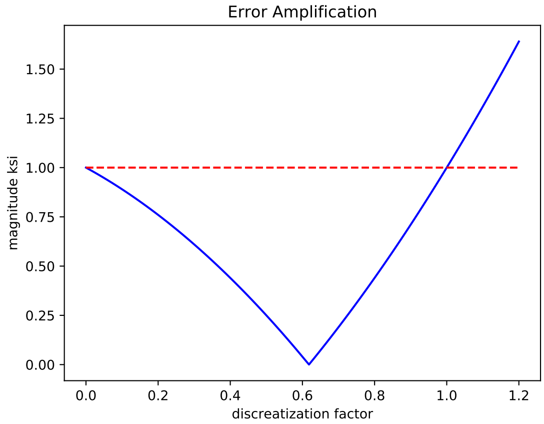
$$\xi A^n e^{ikj\Delta x} = A^n e^{ikj\Delta x}\left[1 - \frac{c\Delta t}{\Delta x}\left(\frac{1}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) - \frac{c\Delta t}{2\Delta x}\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)\right)\right] \tag{2}$$

The magnitude of ξ is then

$$|\xi| = \sqrt{\left(1 - \frac{c\Delta t}{\Delta x} + \left(\frac{c\Delta t}{\Delta x}\right)^2 \cos(k\Delta t)\right)^2 + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin(k\Delta t)^2} \tag{3}$$

I plotted this equation for ξ as a function of the discretization factor and the quantity $k\Delta t$. By inspection, the maximum values of ξ for a given value of $\frac{c\Delta t}{\Delta x}$ occurs when $k\Delta t = \pi$. So the relevant curve is

$$|\xi| = |1 - \frac{c\Delta t}{\Delta x} - \left(\frac{c\Delta t}{\Delta x}\right)^2 | \tag{4}$$



As you can see the Lax-Wendorff scheme still obeys the Courant criterion, and suffers from numerical dispersion just like the Lax scheme and the Leapfrog method. Implying that having $\frac{c\Delta t}{\Delta x} = 1$ is optimal for Lax-Wendorff as well.