LOGIC FOR PHILOSOPHY

1 Chapter one

2 Chapter two

2.1 Valuation functions for disjunction and equivalence

(a) We need to show that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\varphi \vee \chi)$=1 iff either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. We must first notice that $(\varphi \vee \chi)$ is just shorthand for $(\sim \varphi \rightarrow \chi)$, so that what we need to show is that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1 iff either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. This can be done by showing, first, that, if $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1 then $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1, and subsequently that, if $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1 then $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1.

Let us start by proving that if $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1, then either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. We first assume that $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1. Recall that we have defined the valuation function for $\rightarrow$ as follows: $V_{\mathscr{I}}(\phi \rightarrow \psi)$=1 iff either $V_{\mathscr{I}}(\phi)$=0 or $V_{\mathscr{I}}(\psi)$=1. Given our assumption, we can therefore say that either $V_{\mathscr{I}}(\sim \varphi)$=0 or $V_{\mathscr{I}}(\chi)$=1. But then, given the definition for the valuation function for $\sim$, which is $V_{\mathscr{I}}(\sim \phi)$=1 iff $V_{\mathscr{I}}(\phi)$=0, we can say that either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. This is what we wanted to show.

The second stage of the proof requires us to prove that if $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1 then $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1. So, let us as-

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sume that either $V_{\mathscr{I}}(\varphi)=1$ or $V_{\mathscr{I}}(\chi)=1$. By the valuation function for $\sim$, we see how our assumption implies that either $V_{\mathscr{I}}(\sim\varphi)=0$ or $V_{\mathscr{I}}(\chi)=1$. And so, subsequently, by the valuation function for $\rightarrow$, we can demonstrate that our assumption implies that $V_{\mathscr{I}}(\sim\varphi\rightarrow\chi)=1$. □

(b) We need to show that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\varphi\leftrightarrow\chi)=1$ iff $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$. We begin by noting that $\phi\leftrightarrow\psi$ is shorthand for $(\phi\rightarrow\psi)\wedge(\psi\rightarrow\phi)$, and that the latter in its turn is shorthand for $(\sim(\phi\rightarrow\psi)\rightarrow\sim(\psi\rightarrow\phi))$. This means that what we need to show is that, for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ iff $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$. This can be done by showing, first that, if $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ then $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$, and, subsequently, that if $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$, then $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$.

Let us start by proving that if $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ then $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$. We proceed by reductio, and so start by assuming that $V_{\mathscr{I}}(\phi)\neq V_{\mathscr{I}}(\chi)$. This allows us to assume that, say, $V_{\mathscr{I}}(\phi)=1$ and that $V_{\mathscr{I}}(\chi)=0$. Recall that we have defined the valuation function for $\rightarrow$ as follows: $V_{\mathscr{I}}(\phi\rightarrow\psi)=1$ iff either $V_{\mathscr{I}}(\phi)=0$ or $V_{\mathscr{I}}(\psi)=1$. The converse of this is that $V_{\mathscr{I}}(\phi\rightarrow\psi)=0$ iff $V_{\mathscr{I}}(\phi)=1$ and $V_{\mathscr{I}}(\psi)=0$. We see that our assumptions about the valuation interpretation for ψ and χ imply that $V_{\mathscr{I}}(\phi\rightarrow\chi)=0$. We now introduce the further assumption that $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$, which is just the antecedent of the conditional we are trying to prove. By the definition of the valuation function of $\sim$, we see that $V_{\mathscr{I}}((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi))=0$. But if we again rely on our definition of the valuation function of $\rightarrow$, we can demonstrate that $V_{\mathscr{I}}(\phi\rightarrow\chi)=1$ (and that $V_{\mathscr{I}}(\sim(\chi\rightarrow\phi))=0$). But we have already demonstrated that $V_{\mathscr{I}}(\phi\rightarrow\chi)=0$, and so we have a contradiction. This allows us to reject the assumption that $V_{\mathscr{I}}(\phi)\neq V_{\mathscr{I}}(\chi)$, and so we have shown that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$ then $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\chi)$.

Now we still need to prove that if $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\chi)$, then $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\chi)$, and so can assume that either $V_{\mathscr{I}}(\phi) = 1$ and that $V_{\mathscr{I}}(\chi) = 1$, or that $V_{\mathscr{I}}(\phi) = 0$ and that $V_{\mathscr{I}}(\chi) = 0$. We proceed in two straightforward stages.

First, we demonstrate that if we assume $V_{\mathscr{I}}(\phi) = 1$ and that $V_{\mathscr{I}}(\chi) = 1$, it follows that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathscr{I}}(\phi) = 1$ and that $V_{\mathscr{I}}(\chi) = 1$. Using our definitions for the valuation function of $\rightarrow$ and $\sim$, we can say that $V_{\mathscr{I}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathscr{I}}(\sim(\chi \rightarrow \phi)) = 0$. But this implies that $V_{\mathscr{I}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. This is the desired result.

Second, we demonstrate that if we assume $V_{\mathscr{I}}(\phi) = 0$ and that $V_{\mathscr{I}}(\chi) = 0$, it equally follows that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. Again, the valuation function of $\rightarrow$ and $\sim$ allow us to say that $V_{\mathscr{I}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathscr{I}}(\sim(\chi \rightarrow \phi)) = 0$. As shown, this implies that $V_{\mathscr{I}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. □

2.2 Logical consequence

(a) We need to establish that $\models [P \wedge (Q \vee R)] \rightarrow [(P \wedge Q) \vee (P \wedge R)]$. We can prove this by reductio. So let $\mathscr{I}$ be any PL-interpretation, and let us assume that $V_{\mathscr{I}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that (*i*) $V_{\mathscr{I}}(P \wedge (Q \vee R)) = 1$, and that (*ii*) $V_{\mathscr{I}}((P \wedge Q) \vee (P \wedge R)) = 0$. Given our definitions of the valuation function $\wedge$ and $\vee$, we can say that (*iii*) $\mathscr{I}(P) = 1$ (from (*i*)), and that (*iv*) $V_{\mathscr{I}}(P \wedge Q) = 0$ (from (*ii*)), and so that (*v*) $\mathscr{I}(P) = 0$ (from (*ii*)). We have now derived a contradiction from our assumption (*iii* & *v*), and so $V_{\mathscr{I}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 1$. □

(b) We need to establish that $(P \leftrightarrow Q) \vee (R \leftrightarrow S) \nvDash P \vee R$. We can do this by finding an interpretation of P, Q, R, and S such that $V_{\mathscr{I}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$ and $V_{\mathscr{I}}(P \vee R) = 0$. So, let $\mathscr{I}$ be an interpretation such that $\mathscr{I}(P) = 0$, $\mathscr{I}(Q) = 0$, $\mathscr{I}(R) = 0$, and $\mathscr{S}(P) = 0$. Given our definition of the valuation function $\leftrightarrow$, we can say that $V_{\mathscr{I}}(P \leftrightarrow Q) = 1$, and that $V_{\mathscr{I}}(R \leftrightarrow S) = 1$. And given our definition of the valuation function $\vee$ this implies that $V_{\mathscr{I}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$. But then, given our definition of the valuation function $\vee$, we can say that $V_{\mathscr{I}}(P \vee R) = 0$. □

(c) We need to establish that $\sim (P \wedge Q)$ and $\sim P \vee \sim Q$ are semantically equivalent. This, in effect, means that proof is required for $\vDash (\sim (P \wedge Q)) \leftrightarrow (\sim P \vee \sim Q)$. We can proceed in two stages. First, we demonstrate that $\vDash (\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$, and, subsequently, that $\vDash (\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q))$.

We prove that $\vDash (\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$ by reductio. So let $\mathscr{I}$ be any PL-interpretation, and let us assume that $V_{\mathscr{I}}(\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that $V_{\mathscr{I}}(\sim (P \wedge Q)) = 1$, and that $V_{\mathscr{I}}(\sim P \vee \sim Q) = 0$. By our definition of the valuation functions $\sim$ and $\wedge$, we see that $V_{\mathscr{I}}(P \wedge Q) = 0$, and so that not both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. But by our definition of the valuation function $\vee$, we also see that $V_{\mathscr{I}}(\sim P) = 0$ and $V_{\mathscr{I}}(\sim Q) = 0$. Using again our definition for the valuation function $\sim$, this implies that both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. From our assumption, we have derived a contradiction.

We now prove that $\vDash (\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q))$, again by reductio. So let $\mathscr{I}$ be any PL-interpretation, and let us assume that $V_{\mathscr{I}}(\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q)) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that $V_{\mathscr{I}}(\sim P \vee \sim Q) = 1$, and that $V_{\mathscr{I}}(\sim (P \wedge Q)) = 0$. By our definition of the valuation function $\vee$ and $\sim$, we see that not both $V_{\mathscr{I}}(\sim P) = 0$ and $V_{\mathscr{I}}(\sim Q) = 0$, and so that not both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. Now, by our definition of the valuation function ~, we see that $V_{\mathscr{I}}(P \wedge Q) = 1$. Notice, however, that given the valuation function ∧, it follows that both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. And again we have derived a contradiction from our assumption. □

2.3 Sequent proofs in PL

(a) We need to prove that $P \to (Q \to R) \Rightarrow (Q \wedge \sim R) \to \sim P$

(b) We need to prove that $P, Q, R \Rightarrow P$

(c) We need to prove that $P \to Q, R \to Q \Rightarrow (P \vee R) \to Q$.

1. $P \Rightarrow P$

   (RA)

2. $R \Rightarrow R$ (RA)
3. $P \vee R \Rightarrow P \vee R$ (RA)
4. $P \to Q \Rightarrow P \to Q$

   (RA)

5. $R \to Q \Rightarrow R \to Q$ (RA)
6. $P \to Q, P \Rightarrow Q$ (1, 4, →E)
7. $R \to Q, R \Rightarrow Q$ (2, 5, →E)
8. $P \to Q, R \to Q, P \vee R \Rightarrow Q$ (3, 6, 7, ∨E)
9. $P \to Q, R \to Q \Rightarrow (P \vee R) \to Q$ (8, →I)