LOGIC FOR PHILOSOPHY

1 Chapter one

2 Chapter two

2.1 Valuation functions for disjunction and equivalence

(a) We need to show that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\varphi \vee \chi)$=1 iff either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. We must first notice that $(\varphi \vee \chi)$ is just shorthand for $(\sim \varphi \rightarrow \chi)$, so that what we need to show is that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1 iff either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. This can be done by showing, first, that, if $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1 then $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1, and subsequently that, if $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1 then $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1.

Let us start by proving that if $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1, then either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. We first assume that $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1. Recall that we have defined the valuation function for $\rightarrow$ as follows: $V_{\mathscr{I}}(\phi \rightarrow \psi)$=1 iff either $V_{\mathscr{I}}(\phi)$=0 or $V_{\mathscr{I}}(\psi)$=1. Given our assumption, we can therefore say that either $V_{\mathscr{I}}(\sim \varphi)$=0 or $V_{\mathscr{I}}(\chi)$=1. But then, given the definition for the valuation function for $\sim$, which is $V_{\mathscr{I}}(\sim \phi)$=1 iff $V_{\mathscr{I}}(\phi)$=0, we can say that either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. This is what we wanted to show.

The second stage of the proof requires us to prove that if $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1 then $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1. So, let us as-

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sume that either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. By the valuation function for ~, we see how our assumption implies that either $V_{\mathscr{I}}(\sim\varphi)$=0 or $V_{\mathscr{I}}(\chi)$=1. And so, subsequently, by the valuation function for →, we can demonstrate that our assumption implies that $V_{\mathscr{I}}(\sim\varphi\rightarrow\chi)$=1. □

(b) We need to show that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\varphi\leftrightarrow\chi)=1$ iff $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$. We begin by noting that $\phi\leftrightarrow\psi$ is shorthand for $(\phi\rightarrow\psi)\wedge(\psi\rightarrow\phi)$, and that the latter in its turn is shorthand for $(\sim(\phi\rightarrow\psi)\rightarrow\sim(\psi\rightarrow\phi))$. This means that what we need to show is that, for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ iff $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$. This can be done by showing, first that, if $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ then $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$, and, subsequently, that if $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$, then $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$.

Let us start by proving that if $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ then $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$. We assume the antecedent of the conditional we are trying to prove, $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$. By the definition of the valuation function of ~, we see that $V_{\mathscr{I}}((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi))=0$. If we rely on our definition of the valuation function of →, we can demonstrate that $V_{\mathscr{I}}(\phi\rightarrow\chi)=1$ and that $V_{\mathscr{I}}(\sim(\chi\rightarrow\phi))=0$. We now assume, for reductio, that $V_{\mathscr{I}}(\phi)\neq V_{\mathscr{I}}(\chi)$. This implies either that $\mathscr{I}(\phi)=1$ and $\mathscr{I}(\chi)=0$, or that $\mathscr{I}(\phi)=0$ and $\mathscr{I}(\chi)=1$ (but not both, of course). We will now demonstrate that each of these possibilities entails a contradiction.

So, first, let $\mathscr{I}(\phi)=1$ and $\mathscr{I}(\chi)=0$. The converse of the valuation function for → is that $V_{\mathscr{I}}(\phi\rightarrow\psi)=0$ iff $V_{\mathscr{I}}(\phi)=1$ and $V_{\mathscr{I}}(\psi)=0$. We see that our current assumption about the particular valuation interpretation for ψ and χ implies that $V_{\mathscr{I}}(\phi\rightarrow\chi)=0$. But this contradicts with a premise we derived from our antecedent.

This means that, at best, $\mathscr{I}(\phi)=0$ and $\mathscr{I}(\chi)=1$. But then, by the valuation functions for $\rightarrow$ and $\sim$, we see that $V_{\mathscr{I}}(\chi \rightarrow \psi) = 0$, and so that $V_{\mathscr{I}}(\sim(\chi \rightarrow \psi)) = 1$. This again contradicts with a premise we derived from our antecedent, and so we have shown that our antecedent together with $V_{\mathscr{I}}(\phi) \neq V_{\mathscr{I}}(\chi)$ leads to contradiction.

Now we still need to prove that if $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\chi)$, then $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\chi)$, and so can assume either that $\mathscr{I}(\phi) = 1$ and $\mathscr{I}(\chi) = 1$, or that $\mathscr{I}(\phi) = 0$ and $V\mathscr{I}(\chi) = 0$ (but, again, not both). We proceed again in two straightforward stages.

First, we demonstrate that if we assume $\mathscr{I}(\phi) = 1$ and that $\mathscr{I}(\chi) = 1$, it follows that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathscr{I}}(\phi) = 1$ and that $V_{\mathscr{I}}(\chi) = 1$. Using our definitions for the valuation function of $\rightarrow$ and $\sim$, we can say that $V_{\mathscr{I}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathscr{I}}(\sim(\chi \rightarrow \phi)) = 0$. But this implies that $V_{\mathscr{I}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. This is the desired result.

Second, we demonstrate that if we assume $\mathscr{I}(\phi) = 0$ and that $\mathscr{I}(\chi) = 0$, it equally follows that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. Again, the valuation function of $\rightarrow$ and $\sim$ allow us to say that $V_{\mathscr{I}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathscr{I}}(\sim(\chi \rightarrow \phi)) = 0$. As shown, this implies that $V_{\mathscr{I}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. □

2.2 Logical consequence

(a) We need to establish that $\vDash [P \wedge (Q \vee R)] \rightarrow [(P \wedge Q) \vee (P \wedge R)]$. We can prove this by reductio. So let $\mathscr{I}$ be any PL-interpretation, and let us assume that $V_{\mathscr{I}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that (*i*) $V_{\mathscr{I}}(P \wedge (Q \vee R)) = 1$, and that (*ii*) $V_{\mathscr{I}}((P \wedge Q) \vee (P \wedge R)) = 0$. Given our definitions of the valuation function $\wedge$ and $\vee$, we can say that (*iii*) $\mathscr{I}(P) =$

1 (from (i)), and that (iv) $V_{\mathscr{I}}(P \wedge Q) = 0$ (from (ii)), and so that (v) $\mathscr{I}(P) = 0$ (from (ii)). We have now derived a contradiction from our assumption (iii & v), and so $V_{\mathscr{I}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 1$. □

(b) We need to establish that $(P \leftrightarrow Q) \vee (R \leftrightarrow S) \nvDash P \vee R$. We can do this by finding an interpretation of P, Q, R, and S such that $V_{\mathscr{I}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$ and $V_{\mathscr{I}}(P \vee R) = 0$. So, let $\mathscr{I}$ be an interpretation such that $\mathscr{I}(P) = 0$, $\mathscr{I}(Q) = 0$, $\mathscr{I}(R) = 0$, and $\mathscr{I}(P) = 0$. Given our definition of the valuation function $\leftrightarrow$, we can say that $V_{\mathscr{I}}(P \leftrightarrow Q) = 1$, and that $V_{\mathscr{I}}(R \leftrightarrow S) = 1$. And given our definition of the valuation function $\vee$ this implies that $V_{\mathscr{I}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$. But then, given our definition of the valuation function $\vee$, we can say that $V_{\mathscr{I}}(P \vee R) = 0$. □

(c) We need to establish that $\sim (P \wedge Q)$ and $\sim P \vee \sim Q$ are semantically equivalent. This, in effect, means that proof is required for $\vDash (\sim (P \wedge Q)) \leftrightarrow (\sim P \vee \sim Q)$. We can proceed in two stages. First, we demonstrate that $\vDash (\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$, and, subsequently, that $\vDash (\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q))$.

We prove that $\vDash (\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$ by reductio. So let $\mathscr{I}$ be any PL-interpretation, and let us assume that $V_{\mathscr{I}}(\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that $V_{\mathscr{I}}(\sim (P \wedge Q)) = 1$, and that $V_{\mathscr{I}}(\sim P \vee \sim Q) = 0$. By our definition of the valuation functions $\sim$ and $\wedge$, we see that $V_{\mathscr{I}}(P \wedge Q) = 0$, and so that not both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. But by our definition of the valuation function $\vee$, we also see that $V_{\mathscr{I}}(\sim P) = 0$ and $V_{\mathscr{I}}(\sim Q) = 0$. Using again our definition for the valuation function $\sim$, this implies that both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. From our assumption, we have derived a contradiction.

We now prove that $\vDash (\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q))$, again by reductio. So let $\mathscr{I}$ be any PL-interpretation, and let us assume that $V_{\mathscr{I}}(\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q)) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that $V_{\mathscr{I}}(\sim P \vee \sim Q) = 1$, and that $V_{\mathscr{I}}(\sim (P \wedge Q)) = 0$. By our definition of the valuation function $\vee$ and $\sim$, we see that not both $V_{\mathscr{I}}(\sim P) = 0$ and $V_{\mathscr{I}}(\sim Q) = 0$, and so that not both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. Now, by our definition of the valuation function $\sim$, we see that $V_{\mathscr{I}}(P \wedge Q) = 1$. Notice, however, that given the valuation function $\wedge$, it follows that both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. And again we have derived a contradiction from our assumption. □

2.3 Sequent proofs in PL

(a) We need to prove that $P \rightarrow (Q \rightarrow R) \Rightarrow (Q \wedge \sim R) \rightarrow \sim P$.

1. $P \rightarrow (Q \rightarrow R) \Rightarrow P \rightarrow (Q \rightarrow R)$ (RA)
2. $Q \wedge \sim R \Rightarrow Q \wedge \sim R$ (RA)
3. $P \Rightarrow P$ (RA, for reductio)
4. $P, P \rightarrow (Q \rightarrow R) \Rightarrow Q \rightarrow R$ (1,3, $\rightarrow$E)
5. $Q \wedge \sim R \Rightarrow Q$ (2, $\wedge$E)
6. $P, P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow R$ (4, 5, $\rightarrow$E)
7. $P, P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow \sim R$ (6, $\wedge$E)
8. $P, P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow R \wedge \sim R$ (6, 7, $\wedge$I)
9. $P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow \sim P$ (8, RAA)
10. $P \rightarrow (Q \rightarrow R) \Rightarrow (Q \wedge \sim R) \rightarrow \sim P$ (9, $\rightarrow$I)

(b) We need to prove that $P, Q, R \Rightarrow P$.

1. $P \Rightarrow P$ (RA)
2. $Q \Rightarrow Q$ (RA)
3. $R \Rightarrow R$ (RA)
4. $Q, R \Rightarrow Q \wedge R$ (2, 3, $\wedge$I)
5. $Q, R \Rightarrow Q$ (4 $\wedge$E)
6. $P, Q, R \Rightarrow P \wedge Q$ (1, 5, $\wedge$I)
7. $P, Q, R \Rightarrow P$ (6 $\wedge$E)

(c) We need to prove that $P \rightarrow Q, R \rightarrow Q \Rightarrow (P \vee R) \rightarrow Q$.

1. $P \Rightarrow P$ (RA)
2. $R \Rightarrow R$ (RA)
3. $P \vee R \Rightarrow P \vee R$ (RA)
4. $P \rightarrow Q \Rightarrow P \rightarrow Q$ (RA)
5. $R \rightarrow Q \Rightarrow R \rightarrow Q$ (RA)
6. $P \rightarrow Q, P \Rightarrow Q$ (1, 4, $\rightarrow$E)
7. $R \rightarrow Q, R \Rightarrow Q$ (2, 5, $\rightarrow$E)
8. $P \rightarrow Q, R \rightarrow Q, P \vee R \Rightarrow Q$ (3, 6, 7, $\vee$E)
9. $P \rightarrow Q, R \rightarrow Q \Rightarrow (P \vee R) \rightarrow Q$ (8, $\rightarrow$I)

2.4 Axiomatic proofs in PL

(a) We need to prove that $\vdash P \rightarrow P$

(b) We need to prove that $\vdash (P) \rightarrow P$

(c) We need to prove that $\sim\sim P \vdash P$