

# LOGIC FOR PHILOSOPHY

## 1 Chapter one

## 2 Chapter two

### 2.1 Valuation functions for disjunction and equivalence

(a) We need to show that for any wff  $\varphi$  and  $\chi$ , and any PL-interpretation  $\mathcal{J}$ ,  $V_{\mathcal{J}}(\varphi \vee \chi)=1$  iff either  $V_{\mathcal{J}}(\varphi)=1$  or  $V_{\mathcal{J}}(\chi)=1$ . We must first notice that  $(\varphi \vee \chi)$  is just shorthand for  $(\sim \varphi \rightarrow \chi)$ , so that what we need to show is that for any wff  $\varphi$  and  $\chi$ , and any PL-interpretation  $\mathcal{J}$ ,  $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$  iff either  $V_{\mathcal{J}}(\varphi)=1$  or  $V_{\mathcal{J}}(\chi)=1$ . This can be done by showing, first, that, if  $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$  then  $V_{\mathcal{J}}(\varphi)=1$  or  $V_{\mathcal{J}}(\chi)=1$ , and subsequently that, if  $V_{\mathcal{J}}(\varphi)=1$  or  $V_{\mathcal{J}}(\chi)=1$  then  $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$ .

Let us start by proving that if  $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$ , then either  $V_{\mathcal{J}}(\varphi)=1$  or  $V_{\mathcal{J}}(\chi)=1$ . We first assume that  $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$ . Recall that we have defined the valuation function for  $\rightarrow$  as follows:  $V_{\mathcal{J}}(\phi \rightarrow \psi)=1$  iff either  $V_{\mathcal{J}}(\phi)=0$  or  $V_{\mathcal{J}}(\psi)=1$ . Given our assumption, we can therefore say that either  $V_{\mathcal{J}}(\sim \varphi)=0$  or  $V_{\mathcal{J}}(\chi)=1$ . But then, given the definition for the valuation function for  $\sim$ , which is  $V_{\mathcal{J}}(\sim \phi)=1$  iff  $V_{\mathcal{J}}(\phi)=0$ , we can say that either  $V_{\mathcal{J}}(\varphi)=1$  or  $V_{\mathcal{J}}(\chi)=1$ . This is what we wanted to show.

The second stage of the proof requires us to prove that if  $V_{\mathcal{J}}(\varphi)=1$  or  $V_{\mathcal{J}}(\chi)=1$  then  $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$ . So, let us as-

sume that either  $V_{\mathcal{J}}(\varphi)=1$  or  $V_{\mathcal{J}}(\chi)=1$ . By the valuation function for  $\sim$ , we see how our assumption implies that either  $V_{\mathcal{J}}(\sim \varphi)=0$  or  $V_{\mathcal{J}}(\chi)=1$ . And so, subsequently, by the valuation function for  $\rightarrow$ , we can demonstrate that our assumption implies that  $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$ .  $\square$

**(b)** We need to show that for any wff  $\varphi$  and  $\chi$ , and any PL-interpretation  $\mathcal{J}$ ,  $V_{\mathcal{J}}(\varphi \leftrightarrow \chi) = 1$  iff  $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$ . We begin by noting that  $\phi \leftrightarrow \psi$  is shorthand for  $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ , and that the latter in its turn is shorthand for  $(\sim (\phi \rightarrow \psi) \rightarrow \sim (\psi \rightarrow \phi))$ . This means that what we need to show is that, for any wff  $\varphi$  and  $\chi$ , and any PL-interpretation  $\mathcal{J}$ ,  $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$  iff  $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$ . This can be done by showing, first that, if  $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$  then  $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$ , and, subsequently, that if  $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$ , then  $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$ .

Let us start by proving that if  $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$  then  $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$ . We assume the antecedent of the conditional we are trying to prove,  $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$ . By the definition of the valuation function of  $\sim$ , we see that  $V_{\mathcal{J}}((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi)) = 0$ . If we rely on our definition of the valuation function of  $\rightarrow$ , we can demonstrate that  $V_{\mathcal{J}}(\phi \rightarrow \chi) = 1$  and that  $V_{\mathcal{J}}(\sim (\chi \rightarrow \phi)) = 0$ . We now assume, for reductio, that  $V_{\mathcal{J}}(\varphi) \neq V_{\mathcal{J}}(\chi)$ . This implies either that  $\mathcal{J}(\varphi) = 1$  and  $\mathcal{J}(\chi) = 0$ , or that  $\mathcal{J}(\varphi) = 0$  and  $\mathcal{J}(\chi) = 1$  (but not both, of course). We will now demonstrate that each of these possibilities entails a contradiction.

So, first, let  $\mathcal{J}(\varphi) = 1$  and  $\mathcal{J}(\chi) = 0$ . The converse of the valuation function for  $\rightarrow$  is that  $V_{\mathcal{J}}(\phi \rightarrow \psi) = 0$  iff  $V_{\mathcal{J}}(\phi) = 1$  and  $V_{\mathcal{J}}(\psi) = 0$ . We see that our current assumption about the particular valuation interpretation for  $\psi$  and  $\chi$  implies that  $V_{\mathcal{J}}(\phi \rightarrow \chi) = 0$ . But this contradicts with a premise we derived from our antecedent.

This means that, at best,  $\mathcal{J}(\varphi) = 0$  and  $\mathcal{J}(\chi) = 1$ . But

then, by the valuation functions for  $\rightarrow$  and  $\sim$ , we see that  $V_{\mathcal{J}}(\chi \rightarrow \psi) = 0$ , and so that  $V_{\mathcal{J}}(\sim(\chi \rightarrow \psi)) = 1$ . This again contradicts with a premise we derived from our antecedent, and so we have shown that our antecedent together with  $V_{\mathcal{J}}(\phi) \neq V_{\mathcal{J}}(\chi)$  leads to contradiction.

Now we still need to prove that if  $V_{\mathcal{J}}(\phi) = V_{\mathcal{J}}(\chi)$ , then  $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$ . We assume that  $V_{\mathcal{J}}(\phi) = V_{\mathcal{J}}(\chi)$ , and so can assume either that  $\mathcal{J}(\phi) = 1$  and  $\mathcal{J}(\chi) = 1$ , or that  $\mathcal{J}(\phi) = 0$  and  $V_{\mathcal{J}}(\chi) = 0$  (but, again, not both). We proceed again in two straightforward stages.

First, we demonstrate that if we assume  $\mathcal{J}(\phi) = 1$  and that  $\mathcal{J}(\chi) = 1$ , it follows that  $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$ . We assume that  $V_{\mathcal{J}}(\phi) = 1$  and that  $V_{\mathcal{J}}(\chi) = 1$ . Using our definitions for the valuation function of  $\rightarrow$  and  $\sim$ , we can say that  $V_{\mathcal{J}}(\phi \rightarrow \chi) = 1$  and that  $V_{\mathcal{J}}(\sim(\chi \rightarrow \phi)) = 0$ . But this implies that  $V_{\mathcal{J}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$ , which in turn implies that  $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$ . This is the desired result.

Second, we demonstrate that if we assume  $\mathcal{J}(\phi) = 0$  and that  $\mathcal{J}(\chi) = 0$ , it equally follows that  $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$ . Again, the valuation function of  $\rightarrow$  and  $\sim$  allow us to say that  $V_{\mathcal{J}}(\phi \rightarrow \chi) = 1$  and that  $V_{\mathcal{J}}(\sim(\chi \rightarrow \phi)) = 0$ . As shown, this implies that  $V_{\mathcal{J}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$ , which in turn implies that  $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$ .  $\square$

## 2.2 Logical consequence

(a) We need to establish that  $\models [P \wedge (Q \vee R)] \rightarrow [(P \wedge Q) \vee (P \wedge R)]$ . We can prove this by reductio. So let  $\mathcal{J}$  be any PL-interpretation, and let us assume that  $V_{\mathcal{J}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 0$ . Given our definition of the valuation function  $\rightarrow$ , we can say that (i)  $V_{\mathcal{J}}(P \wedge (Q \vee R)) = 1$ , and that (ii)  $V_{\mathcal{J}}((P \wedge Q) \vee (P \wedge R)) = 0$ . Given our definitions of the valuation function  $\wedge$  and  $\vee$ , we can say that (iii)  $\mathcal{J}(P) =$

1 (from *i*), and that (iv)  $V_{\mathcal{J}}(P \wedge Q) = 0$  (from *ii*). By the valuation function for  $\wedge$ , this means that (v)  $\mathcal{J}(Q) = 0$  (from *iii* & *iv*).

Now, given both these interpretations of  $P$  and  $Q$ , we can demonstrate that *i* implies that (vi)  $V_{\mathcal{J}}(Q \vee R) = 1$  (by  $\wedge$  and  $\mathcal{J}(P)$ ), and so that  $\mathcal{J}(R) = 1$  (by  $\vee$ , *vi*, &  $\mathcal{J}(Q)$ ). But given our valuation function for  $\wedge$ , we see that our interpretations of  $P$  and  $Q$  also imply (vii)  $V_{\mathcal{J}}(P \wedge Q) = 0$ . And so we can show that *ii* implies that  $V_{\mathcal{J}}(P \wedge R) = 0$  (by  $\vee$  & *vii*), and so that  $\mathcal{J}(R) = 0$  (by  $\wedge$  &  $\mathcal{J}(P)$ ). So, we have derived both  $\mathcal{J}(R) = 1$  and  $\mathcal{J}(R) = 0$ , which is absurd.  $\square$

**(b)** We need to establish that  $(P \leftrightarrow Q) \vee (R \leftrightarrow S) \not\models P \vee R$ . We can do this by finding an interpretation of  $P$ ,  $Q$ ,  $R$ , and  $S$  such that  $V_{\mathcal{J}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$  and  $V_{\mathcal{J}}(P \vee R) = 0$ . So, let  $\mathcal{J}$  be an interpretation such that  $\mathcal{J}(P) = 0$ ,  $\mathcal{J}(Q) = 0$ ,  $\mathcal{J}(R) = 0$ , and  $\mathcal{J}(S) = 0$ . Given our definition of the valuation function  $\leftrightarrow$ , we can say that  $V_{\mathcal{J}}(P \leftrightarrow Q) = 1$ , and that  $V_{\mathcal{J}}(R \leftrightarrow S) = 1$ . And given our definition of the valuation function  $\vee$  this implies that  $V_{\mathcal{J}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$ . But then, given our definition of the valuation function  $\vee$ , we can say that  $V_{\mathcal{J}}(P \vee R) = 0$ .  $\square$

**(c)** We need to establish that  $\sim(P \wedge Q)$  and  $\sim P \vee \sim Q$  are semantically equivalent. This, in effect, means that proof is required for  $\models (\sim(P \wedge Q)) \leftrightarrow (\sim P \vee \sim Q)$ . We can proceed in two stages. First, we demonstrate that  $\models (\sim(P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$ , and, subsequently, that  $\models (\sim P \vee \sim Q) \rightarrow (\sim(P \wedge Q))$ .

We prove that  $\models (\sim(P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$  by reductio. So let  $\mathcal{J}$  be any PL-interpretation, and let us assume that  $V_{\mathcal{J}}(\sim(P \wedge Q)) \rightarrow (\sim P \vee \sim Q) = 0$ . Given our definition of the valuation function  $\rightarrow$ , we can say that  $V_{\mathcal{J}}(\sim(P \wedge Q)) = 1$ , and that  $V_{\mathcal{J}}(\sim P \vee \sim Q) = 0$ . By our definition of the valuation functions  $\sim$  and  $\wedge$ , we see that  $V_{\mathcal{J}}(P \wedge Q) = 0$ , and so

that not both  $\mathcal{J}(P) = 1$  and  $\mathcal{J}(Q) = 1$ . But by our definition of the valuation function  $\vee$ , we also see that  $V_{\mathcal{J}}(\sim P) = 0$  and  $V_{\mathcal{J}}(\sim Q) = 0$ . Using again our definition for the valuation function  $\sim$ , this implies that both  $\mathcal{J}(P) = 1$  and  $\mathcal{J}(Q) = 1$ . From our assumption, we have derived a contradiction.

We now prove that  $\models (\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q))$ , again by reductio. So let  $\mathcal{J}$  be any PL-interpretation, and let us assume that  $V_{\mathcal{J}}(\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q)) = 0$ . Given our definition of the valuation function  $\rightarrow$ , we can say that  $V_{\mathcal{J}}(\sim P \vee \sim Q) = 1$ , and that  $V_{\mathcal{J}}(\sim (P \wedge Q)) = 0$ . By our definition of the valuation function  $\vee$  and  $\sim$ , we see that not both  $V_{\mathcal{J}}(\sim P) = 0$  and  $V_{\mathcal{J}}(\sim Q) = 0$ , and so that not both  $\mathcal{J}(P) = 1$  and  $\mathcal{J}(Q) = 1$ . Now, by our definition of the valuation function  $\sim$ , we see that  $V_{\mathcal{J}}(P \wedge Q) = 1$ . Notice, however, that given the valuation function  $\wedge$ , it follows that both  $\mathcal{J}(P) = 1$  and  $\mathcal{J}(Q) = 1$ . And again we have derived a contradiction from our assumption.  $\square$

## 2.3 Sequent proofs in PL

(a) We need to prove that  $P \rightarrow (Q \rightarrow R) \Rightarrow (Q \wedge \sim R) \rightarrow \sim P$ .

1.  $P \rightarrow (Q \rightarrow R) \Rightarrow P \rightarrow (Q \rightarrow R)$  (RA)
2.  $Q \wedge \sim R \Rightarrow Q \wedge \sim R$  (RA)
3.  $P \Rightarrow P$  (RA, for reductio)
4.  $P, P \rightarrow (Q \rightarrow R) \Rightarrow Q \rightarrow R$  (1, 3,  $\rightarrow$ E)
5.  $Q \wedge \sim R \Rightarrow Q$  (2,  $\wedge$ E)
6.  $P, P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow R$  (4, 5,  $\rightarrow$ E)
7.  $P, P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow \sim R$  (6,  $\wedge$ E)
8.  $P, P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow R \wedge \sim R$  (6, 7,  $\wedge$ I)

9.  $P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow \sim P$  (8, RAA)  
 10.  $P \rightarrow (Q \rightarrow R) \Rightarrow (Q \wedge \sim R) \rightarrow \sim P$  (9,  $\rightarrow$ I)

**(b)** We need to prove that  $P, Q, R \Rightarrow P$ .

1.  $P \Rightarrow P$  (RA)  
 2.  $Q \Rightarrow Q$  (RA)  
 3.  $R \Rightarrow R$  (RA)  
 4.  $Q, R \Rightarrow Q \wedge R$  (2, 3,  $\wedge$ I)  
 5.  $Q, R \Rightarrow Q$  (4  $\wedge$ E)  
 6.  $P, Q, R \Rightarrow P \wedge Q$  (1, 5,  $\wedge$ I)  
 7.  $P, Q, R \Rightarrow P$  (6  $\wedge$ E)

**(c)** We need to prove that  $P \rightarrow Q, R \rightarrow Q \Rightarrow (P \vee R) \rightarrow Q$ .

1.  $P \Rightarrow P$  (RA)  
 2.  $R \Rightarrow R$  (RA)  
 3.  $P \vee R \Rightarrow P \vee R$  (RA)  
 4.  $P \rightarrow Q \Rightarrow P \rightarrow Q$  (RA)  
 5.  $R \rightarrow Q \Rightarrow R \rightarrow Q$  (RA)  
 6.  $P \rightarrow Q, P \Rightarrow Q$  (1, 4,  $\rightarrow$ E)  
 7.  $R \rightarrow Q, R \Rightarrow Q$  (2, 5,  $\rightarrow$ E)  
 8.  $P \rightarrow Q, R \rightarrow Q, P \vee R \Rightarrow Q$  (3, 6, 7,  $\vee$ E)  
 9.  $P \rightarrow Q, R \rightarrow Q \Rightarrow (P \vee R) \rightarrow Q$  (8,  $\rightarrow$ I)

## 2.4 Axiomatic proofs in PL

- (a) We need to prove that  $\vdash P \rightarrow P$
- (b) We need to prove that  $\vdash (P) \rightarrow P$
- (c) We need to prove that  $\sim\sim P \vdash P$