LOGIC FOR PHILOSOPHY

1 Chapter one

2 Chapter two

2.1 Valuation functions for disjunction and equivalence

(a) We need to show that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\varphi \lor \chi)=1$ iff either $V_{\mathscr{I}}(\varphi)=1$ or $V_{\mathscr{I}}(\chi)=1$. We must first notice that $(\varphi \lor \chi)$ is just shorthand for $(\sim \varphi \to \chi)$, so that what we need to show is that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\sim \varphi \to \chi)=1$ iff either $V_{\mathscr{I}}(\varphi)=1$ or $V_{\mathscr{I}}(\chi)=1$. This can be done by showing, first, that, if $V_{\mathscr{I}}(\sim \varphi \to \chi)=1$ then $V_{\mathscr{I}}(\varphi)=1$ or $V_{\mathscr{I}}(\chi)=1$, and subsequently that, if $V_{\mathscr{I}}(\varphi)=1$ or $V_{\mathscr{I}}(\chi)=1$ then $V_{\mathscr{I}}(\sim \varphi \to \chi)=1$.

Let us start by proving that if $V_{\mathscr{I}}(\sim \varphi \to \chi)=1$, then either $V_{\mathscr{I}}(\varphi)=1$ or $V_{\mathscr{I}}(\chi)=1$. We first assume that $V_{\mathscr{I}}(\sim \varphi \to \chi)=1$. Recall that we have defined the valuation function for $\to$ as follows: $V_{\mathscr{I}}(\phi \to \psi)=1$ iff either $V_{\mathscr{I}}(\phi)=0$ or $V_{\mathscr{I}}(\psi)=1$. Given our assumption, we can therefore say that either $V_{\mathscr{I}}(\sim \varphi)=0$ or $V_{\mathscr{I}}(\chi)=1$. But then, given the definition for the valuation function for $\sim$, which is $V_{\mathscr{I}}(\sim \phi)=1$ iff $V_{\mathscr{I}}(\phi)=0$, we can say that either $V_{\mathscr{I}}(\varphi)=1$ or $V_{\mathscr{I}}(\chi)=1$. This is what we wanted to show.

The second stage of the proof requires us to prove that if $V_{\mathscr{I}}(\varphi)=1$ or $V_{\mathscr{I}}(\chi)=1$ then $V_{\mathscr{I}}(\sim \varphi \to \chi)=1$. So, let us as-

sume that either $V_{\mathscr{J}}(\varphi)=1$ or $V_{\mathscr{J}}(\chi)=1$. By the valuation function for $\sim$, we see how our assumption implies that either $V_{\mathscr{J}}(\sim\varphi)=0$ or $V_{\mathscr{J}}(\chi)=1$. And so, subsequently, by the valuation function for $\rightarrow$, we can demonstrate that our assumption implies that $V_{\mathscr{J}}(\sim\varphi\rightarrow\chi)=1$. $\square$

(b) We need to show that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{J}}(\varphi\leftrightarrow\chi)=1$ iff $V_{\mathscr{J}}(\phi)=V_{\mathscr{J}}(\chi)$. We begin by noting that $\phi\leftrightarrow\psi$ is shorthand for $(\phi\rightarrow\psi)\wedge(\psi\rightarrow\phi)$, and that the latter in its turn is shorthand for $(\sim(\phi\rightarrow\psi)\rightarrow\sim(\psi\rightarrow\phi))$. This means that what we need to show is that, for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{J}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ iff $V_{\mathscr{J}}(\phi)=V_{\mathscr{J}}(\chi)$. This can be done by showing, first that, if $V_{\mathscr{J}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ then $V_{\mathscr{J}}(\phi)=V_{\mathscr{J}}(\chi)$, and, subsequently, that if $V_{\mathscr{J}}(\phi)=V_{\mathscr{J}}(\chi)$, then $V_{\mathscr{J}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$.

Let us start by proving that if $V_{\mathscr{J}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ then $V_{\mathscr{J}}(\phi)=V_{\mathscr{J}}(\chi)$. We assume the antecedent of the conditional we are trying to prove, $V_{\mathscr{J}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$. By the definition of the valuation function of $\sim$, we see that $V_{\mathscr{J}}((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi))=0$. If we rely on our definition of the valuation function of $\rightarrow$, we can demonstrate that $V_{\mathscr{J}}(\phi\rightarrow\chi)=1$ and that $V_{\mathscr{J}}(\sim(\chi\rightarrow\phi))=0$. We now assume, for reductio, that $V_{\mathscr{J}}(\phi)\neq V_{\mathscr{J}}(\chi)$. This implies either that $\mathscr{J}(\phi)=1$ and $\mathscr{J}(\chi)=0$, or that $\mathscr{J}(\phi)=0$ and $\mathscr{J}(\chi)=1$ (but not both, of course). We will now demonstrate that each of these possibilities entails a contradiction.

So, first, let $\mathscr{J}(\phi)=1$ and $\mathscr{J}(\chi)=0$. The converse of the valuation function for $\rightarrow$ is that $V_{\mathscr{J}}(\phi\rightarrow\psi)=0$ iff $V_{\mathscr{J}}(\phi)=1$ and $V_{\mathscr{J}}(\psi)=0$. We see that our current assumption about the particular valuation interpretation for ψ and χ implies that $V_{\mathscr{J}}(\phi\rightarrow\chi)=0$. But this contradicts with a premise we derived from our antecedent.

This means that, at best, $\mathscr{J}(\phi)=0$ and $\mathscr{J}(\chi)=1$. But

then, by the valuation functions for $\rightarrow$ and $\sim$, we see that $V_{\mathscr{J}}(\chi \rightarrow \psi) = 0$, and so that $V_{\mathscr{J}}(\sim(\chi \rightarrow \psi)) = 1$. This again contradicts with a premise we derived from our antecedent, and so we have shown that our antecedent together with $V_{\mathscr{J}}(\phi) \neq V_{\mathscr{J}}(\chi)$ leads to contradiction.

Now we still need to prove that if $V_{\mathscr{J}}(\phi) = V_{\mathscr{J}}(\chi)$, then $V_{\mathscr{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathscr{J}}(\phi) = V_{\mathscr{J}}(\chi)$, and so can assume either that $\mathscr{J}(\phi) = 1$ and $\mathscr{J}(\chi) = 1$, or that $\mathscr{J}(\phi) = 0$ and $V_{\mathscr{J}}(\chi) = 0$ (but, again, not both). We proceed again in two straightforward stages.

First, we demonstrate that if we assume $\mathscr{J}(\phi) = 1$ and that $\mathscr{J}(\chi) = 1$, it follows that $V_{\mathscr{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathscr{J}}(\phi) = 1$ and that $V_{\mathscr{J}}(\chi) = 1$. Using our definitions for the valuation function of $\rightarrow$ and $\sim$, we can say that $V_{\mathscr{J}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathscr{J}}(\sim(\chi \rightarrow \phi)) = 0$. But this implies that $V_{\mathscr{J}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathscr{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. This is the desired result.

Second, we demonstrate that if we assume $\mathscr{J}(\phi) = 0$ and that $\mathscr{J}(\chi) = 0$, it equally follows that $V_{\mathscr{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$. Again, the valuation function of $\rightarrow$ and $\sim$ allow us to say that $V_{\mathscr{J}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathscr{J}}(\sim(\chi \rightarrow \phi)) = 0$. As shown, this implies that $V_{\mathscr{J}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathscr{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. $\square$

2.2   Logical consequence

(a)   We need to establish that $\vDash [P \wedge (Q \vee R)] \rightarrow [(P \wedge Q) \vee (P \wedge R)]$. We can prove this by reductio. So let $\mathscr{J}$ be any PL-interpretation, and let us assume that $V_{\mathscr{J}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that (*i*) $V_{\mathscr{J}}(P \wedge (Q \vee R)) = 1$, and that (*ii*) $V_{\mathscr{J}}((P \wedge Q) \vee (P \wedge R)) = 0$. Given our definitions of the valuation function $\wedge$ and $\vee$, we can say that (*iii*) $\mathscr{J}(P) =$

1 (from *i*), and that (*iv*) $V_{\mathscr{J}}(P \wedge Q) = 0$ (from *ii*). By the valuation function of $\wedge$, this means that (*v*) $\mathscr{J}(Q) = 0$ (from *iii* & *iv*).

Now, given both these interpretations of P and Q, we can demonstrate that *i* implies that (*vi*) $V_{\mathscr{J}}(Q \vee R) = 1$ (by $\wedge$ and $\mathscr{J}(P)$), and so that $\mathscr{J}(R) = 1$ (by $\vee$, *vi*, & $\mathscr{J}(Q)$). But given our valuation function for $\wedge$, we see that our interpretations of P and Q also imply (*vii*) $V_{\mathscr{J}}(P \wedge Q) = 0$. And so we can show that *ii* implies that $V_{\mathscr{J}}(P \wedge R) = 0$ (by $\vee$ & *vii*), and so that $\mathscr{J}(R) = 0$ (by $\wedge$ & $\mathscr{J}(P)$). So, we have derived both $\mathscr{J}(R) = 1$ and $\mathscr{J}(R) = 0$, which is absurd. $\square$

(b) We need to establish that $(P \leftrightarrow Q) \vee (R \leftrightarrow S) \nvDash P \vee R$. We can do this by finding an interpretation of P, Q, R, and S such that $V_{\mathscr{J}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$ and $V_{\mathscr{J}}(P \vee R) = 0$. So, let $\mathscr{J}$ be an interpretation such that $\mathscr{J}(P) = 0$, $\mathscr{J}(Q) = 0$, $\mathscr{J}(R) = 0$, and $\mathscr{J}(P) = 0$. Given our definition of the valuation function $\leftrightarrow$, we can say that $V_{\mathscr{J}}(P \leftrightarrow Q) = 1$, and that $V_{\mathscr{J}}(R \leftrightarrow S) = 1$. And given our definition of the valuation function $\vee$ this implies that $V_{\mathscr{J}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$. But then, given our definition of the valuation function $\vee$, we can say that $V_{\mathscr{J}}(P \vee R) = 0$. $\square$

(c) We need to establish that $\sim (P \wedge Q)$ and $\sim P \vee \sim Q$ are semantically equivalent. This, in effect, means that proof is required for $\vDash (\sim (P \wedge Q)) \leftrightarrow (\sim P \vee \sim Q)$. We can proceed in two stages. First, we demonstrate that $\vDash (\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$, and, subsequently, that $\vDash (\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q))$.

We prove that $\vDash (\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$ by reductio. So let $\mathscr{J}$ be any PL-interpretation, and let us assume that $V_{\mathscr{J}}(\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that $V_{\mathscr{J}}(\sim (P \wedge Q)) = 1$, and that $V_{\mathscr{J}}(\sim P \vee \sim Q) = 0$. By our definition of the valuation functions $\sim$ and $\wedge$, we see that $V_{\mathscr{J}}(P \wedge Q) = 0$, and so

that not both $\mathscr{J}(P) = 1$ and $\mathscr{J}(Q) = 1$. But by our definition of the valuation function $\lor$, we also see that $V_{\mathscr{J}}(\sim P) = 0$ and $V_{\mathscr{J}}(\sim Q) = 0$. Using again our definition for the valuation function $\sim$, this implies that both $\mathscr{J}(P) = 1$ and $\mathscr{J}(Q) = 1$. From our assumption, we have derived a contradiction.

We now prove that $\vDash (\sim P \lor \sim Q) \rightarrow (\sim (P \land Q))$, again by reductio. So let $\mathscr{J}$ be any PL-interpretation, and let us assume that $V_{\mathscr{J}}(\sim P \lor \sim Q) \rightarrow (\sim (P \land Q)) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that $V_{\mathscr{J}}(\sim P \lor \sim Q) = 1$, and that $V_{\mathscr{J}}(\sim (P \land Q)) = 0$. By our definition of the valuation function $\lor$ and $\sim$, we see that not both $V_{\mathscr{J}}(\sim P) = 0$ and $V_{\mathscr{J}}(\sim Q) = 0$, and so that not both $\mathscr{J}(P) = 1$ and $\mathscr{J}(Q) = 1$. Now, by our definition of the valuation function $\sim$, we see that $V_{\mathscr{J}}(P \land Q) = 1$. Notice, however, that given the valuation function $\land$, it follows that both $\mathscr{J}(P) = 1$ and $\mathscr{J}(Q) = 1$. And again we have derived a contradiction from our assumption. $\square$

2.3 Sequent proofs in PL

(a) We need to prove that $P \rightarrow (Q \rightarrow R) \Rightarrow (Q \land \sim R) \rightarrow \sim P$.

1. $P \rightarrow (Q \rightarrow R) \Rightarrow P \rightarrow (Q \rightarrow R)$ (RA)

2. $Q \land \sim R \Rightarrow Q \land \sim R$ (RA)

3. $P \Rightarrow P$ (RA, for reductio)

4. $P, P \rightarrow (Q \rightarrow R) \Rightarrow Q \rightarrow R$ (1,3, $\rightarrow$E)

5. $Q \land \sim R \Rightarrow Q$ (2, $\land$E)

6. $P, P \rightarrow (Q \rightarrow R), Q \land \sim R \Rightarrow R$ (4, 5, $\rightarrow$E)

7. $P, P \rightarrow (Q \rightarrow R), Q \land \sim R \Rightarrow \sim R$ (6, $\land$E)

8. $P, P \rightarrow (Q \rightarrow R), Q \land \sim R \Rightarrow R \land \sim R$ (6, 7, $\land$I)

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9. $P \to (Q \to R), Q \wedge {\sim} R \Rightarrow {\sim} P$ (8, RAA)

10. $P \to (Q \to R) \Rightarrow (Q \wedge {\sim} R) \to {\sim} P$ (9, $\to$I)

(b)   We need to prove that $P, Q, R \Rightarrow P$.

1. $P \Rightarrow P$ (RA)

2. $Q \Rightarrow Q$ (RA)

3. $R \Rightarrow R$ (RA)

4. $Q, R \Rightarrow Q \wedge R$ (2, 3, $\wedge$I)

5. $Q, R \Rightarrow Q$ (4 $\wedge$E)

6. $P, Q, R \Rightarrow P \wedge Q$ (1, 5, $\wedge$I)

7. $P, Q, R \Rightarrow P$ (6 $\wedge$E)

(c)   We need to prove that $P \to Q, R \to Q \Rightarrow (P \vee R) \to Q$.

1. $P \Rightarrow P$ (RA)

2. $R \Rightarrow R$ (RA)

3. $P \vee R \Rightarrow P \vee R$ (RA)

4. $P \to Q \Rightarrow P \to Q$ (RA)

5. $R \to Q \Rightarrow R \to Q$ (RA)

6. $P \to Q, P \Rightarrow Q$ (1, 4, $\to$E)

7. $R \to Q, R \Rightarrow Q$ (2, 5, $\to$E)

8. $P \to Q, R \to Q, P \vee R \Rightarrow Q$ (3, 6, 7, $\vee$E)

9. $P \to Q, R \to Q \Rightarrow (P \vee R) \to Q$ (8, $\to$I)

2.4   Axiomatic proofs in PL

(a)   We need to prove that $\vdash P \to P$

1. $(P \to ((P \to P) \to P)) \to ((P \to (P \to P)) \to (P \to P))$ PL2

2. $P \to ((P \to P) \to P)$ PL1

3. $(P \to (P \to P)) \to (P \to P)$ 1,2,MP

4. $P \to (P \to P)$ PL1

5. $P \to P$ 3,4 MP

(b)   We need to prove that $\vdash (\sim P \to P) \to P$

1. $(\sim P \to ((\sim P \to \sim P) \to \sim P)) \to$
   $((\sim P \to (\sim P \to \sim P)) \to (\sim P \to \sim P))$ PL2

2. $\sim P \to ((\sim P \to \sim P) \to \sim P)$ PL1

3. $(\sim P \to (\sim P \to \sim P)) \to (\sim P \to \sim P)$ 1,2,MP

4. $\sim P \to (\sim P \to \sim P)$ PL1

5. $\sim P \to \sim P$ 3,4 MP

6. $(\sim P \to \sim P) \to (\sim P \to P) \to P$ PL3

7. $(\sim P \to P) \to P$ 5,6,MP

(c)   We need to prove that $\sim\sim P \vdash P$