

LOGIC FOR PHILOSOPHY

1 Chapter one

2 Chapter two

2.1 Valuation functions for disjunction and equivalence

(a) We need to show that for any wff φ and χ , and any PL-interpretation \mathcal{J} , $V_{\mathcal{J}}(\varphi \vee \chi)=1$ iff either $V_{\mathcal{J}}(\varphi)=1$ or $V_{\mathcal{J}}(\chi)=1$. We must first notice that $(\varphi \vee \chi)$ is just shorthand for $(\sim \varphi \rightarrow \chi)$, so that what we need to show is that for any wff φ and χ , and any PL-interpretation \mathcal{J} , $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$ iff either $V_{\mathcal{J}}(\varphi)=1$ or $V_{\mathcal{J}}(\chi)=1$. This can be done by showing, first, that, if $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$ then $V_{\mathcal{J}}(\varphi)=1$ or $V_{\mathcal{J}}(\chi)=1$, and subsequently that, if $V_{\mathcal{J}}(\varphi)=1$ or $V_{\mathcal{J}}(\chi)=1$ then $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$.

Let us start by proving that if $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$, then either $V_{\mathcal{J}}(\varphi)=1$ or $V_{\mathcal{J}}(\chi)=1$. We first assume that $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$. Recall that we have defined the valuation function for \rightarrow as follows: $V_{\mathcal{J}}(\phi \rightarrow \psi)=1$ iff either $V_{\mathcal{J}}(\phi)=0$ or $V_{\mathcal{J}}(\psi)=1$. Given our assumption, we can therefore say that either $V_{\mathcal{J}}(\sim \varphi)=0$ or $V_{\mathcal{J}}(\chi)=1$. But then, given the definition for the valuation function for \sim , which is $V_{\mathcal{J}}(\sim \phi)=1$ iff $V_{\mathcal{J}}(\phi)=0$, we can say that either $V_{\mathcal{J}}(\varphi)=1$ or $V_{\mathcal{J}}(\chi)=1$. This is what we wanted to show.

The second stage of the proof requires us to prove that if $V_{\mathcal{J}}(\varphi)=1$ or $V_{\mathcal{J}}(\chi)=1$ then $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$. So, let us as-

sume that either $V_{\mathcal{J}}(\varphi)=1$ or $V_{\mathcal{J}}(\chi)=1$. By the valuation function for \sim , we see how our assumption implies that either $V_{\mathcal{J}}(\sim \varphi)=0$ or $V_{\mathcal{J}}(\chi)=1$. And so, subsequently, by the valuation function for \rightarrow , we can demonstrate that our assumption implies that $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$. \square

(b) We need to show that for any wff φ and χ , and any PL-interpretation \mathcal{J} , $V_{\mathcal{J}}(\varphi \leftrightarrow \chi) = 1$ iff $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$. We begin by noting that $\phi \leftrightarrow \psi$ is shorthand for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, and that the latter in its turn is shorthand for $(\sim (\phi \rightarrow \psi) \rightarrow \sim (\psi \rightarrow \phi))$. This means that what we need to show is that, for any wff φ and χ , and any PL-interpretation \mathcal{J} , $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$ iff $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$. This can be done by showing, first that, if $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$ then $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$, and, subsequently, that if $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$, then $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$.

Let us start by proving that if $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$ then $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$. We assume the antecedent of the conditional we are trying to prove, $V_{\mathcal{J}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$. By the definition of the valuation function of \sim , we see that $V_{\mathcal{J}}((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi)) = 0$. If we rely on our definition of the valuation function of \rightarrow , we can demonstrate that $V_{\mathcal{J}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathcal{J}}(\sim (\chi \rightarrow \phi)) = 0$. We now assume, for reductio, that $V_{\mathcal{J}}(\varphi) \neq V_{\mathcal{J}}(\chi)$. This implies either that $\mathcal{J}(\varphi) = 1$ and $\mathcal{J}(\chi) = 0$, or that $\mathcal{J}(\varphi) = 0$ and $\mathcal{J}(\chi) = 1$ (but not both, of course). We will now demonstrate that each of these possibilities entails a contradiction.

So, first, let $\mathcal{J}(\varphi) = 1$ and $\mathcal{J}(\chi) = 0$. The converse of the valuation function for \rightarrow is that $V_{\mathcal{J}}(\phi \rightarrow \psi) = 0$ iff $V_{\mathcal{J}}(\phi) = 1$ and $V_{\mathcal{J}}(\psi) = 0$. We see that our current assumption about the particular valuation interpretation for ψ and χ implies that $V_{\mathcal{J}}(\phi \rightarrow \chi) = 0$. But this contradicts with a premise we derived from our antecedent.

This means that, at best, $\mathcal{J}(\varphi) = 0$ and $\mathcal{J}(\chi) = 1$. But

then, by the valuation functions for \rightarrow and \sim , we see that $V_{\mathcal{J}}(\chi \rightarrow \psi) = 0$, and so that $V_{\mathcal{J}}(\sim(\chi \rightarrow \psi)) = 1$. This again contradicts with a premise we derived from our antecedent, and so we have shown that our antecedent together with $V_{\mathcal{J}}(\phi) \neq V_{\mathcal{J}}(\chi)$ leads to contradiction.

Now we still need to prove that if $V_{\mathcal{J}}(\phi) = V_{\mathcal{J}}(\chi)$, then $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathcal{J}}(\phi) = V_{\mathcal{J}}(\chi)$, and so can assume either that $\mathcal{J}(\phi) = 1$ and $\mathcal{J}(\chi) = 1$, or that $\mathcal{J}(\phi) = 0$ and $V_{\mathcal{J}}(\chi) = 0$ (but, again, not both). We proceed again in two straightforward stages.

First, we demonstrate that if we assume $\mathcal{J}(\phi) = 1$ and that $\mathcal{J}(\chi) = 1$, it follows that $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathcal{J}}(\phi) = 1$ and that $V_{\mathcal{J}}(\chi) = 1$. Using our definitions for the valuation function of \rightarrow and \sim , we can say that $V_{\mathcal{J}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathcal{J}}(\sim(\chi \rightarrow \phi)) = 0$. But this implies that $V_{\mathcal{J}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. This is the desired result.

Second, we demonstrate that if we assume $\mathcal{J}(\phi) = 0$ and that $\mathcal{J}(\chi) = 0$, it equally follows that $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. Again, the valuation function of \rightarrow and \sim allow us to say that $V_{\mathcal{J}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathcal{J}}(\sim(\chi \rightarrow \phi)) = 0$. As shown, this implies that $V_{\mathcal{J}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. \square

2.2 Logical consequence

(a) We need to establish that $\models [P \wedge (Q \vee R)] \rightarrow [(P \wedge Q) \vee (P \wedge R)]$. We can prove this by reductio. So let \mathcal{J} be any PL-interpretation, and let us assume that $V_{\mathcal{J}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 0$. Given our definition of the valuation function \rightarrow , we can say that (i) $V_{\mathcal{J}}(P \wedge (Q \vee R)) = 1$, and that (ii) $V_{\mathcal{J}}((P \wedge Q) \vee (P \wedge R)) = 0$. Given our definitions of the valuation function \wedge and \vee , we can say that (iii) $\mathcal{J}(P) =$

1 (from *i*), and that (iv) $V_{\mathcal{J}}(P \wedge Q) = 0$ (from *ii*). By the valuation function for \wedge , this means that (v) $\mathcal{J}(Q) = 0$ (from *iii* & *iv*).

Now, given both these interpretations of P and Q , we can demonstrate that *i* implies that (vi) $V_{\mathcal{J}}(Q \vee R) = 1$ (by \wedge and $\mathcal{J}(P)$), and so that $\mathcal{J}(R) = 1$ (by \vee , *vi*, & $\mathcal{J}(Q)$). But given our valuation function for \wedge , we see that our interpretations of P and Q also imply (vii) $V_{\mathcal{J}}(P \wedge Q) = 0$. And so we can show that *ii* implies that $V_{\mathcal{J}}(P \wedge R) = 0$ (by \vee & *vii*), and so that $\mathcal{J}(R) = 0$ (by \wedge & $\mathcal{J}(P)$). So, we have derived both $\mathcal{J}(R) = 1$ and $\mathcal{J}(R) = 0$, which is absurd. \square

(b) We need to establish that $(P \leftrightarrow Q) \vee (R \leftrightarrow S) \not\models P \vee R$. We can do this by finding an interpretation of P , Q , R , and S such that $V_{\mathcal{J}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$ and $V_{\mathcal{J}}(P \vee R) = 0$. So, let \mathcal{J} be an interpretation such that $\mathcal{J}(P) = 0$, $\mathcal{J}(Q) = 0$, $\mathcal{J}(R) = 0$, and $\mathcal{J}(S) = 0$. Given our definition of the valuation function \leftrightarrow , we can say that $V_{\mathcal{J}}(P \leftrightarrow Q) = 1$, and that $V_{\mathcal{J}}(R \leftrightarrow S) = 1$. And given our definition of the valuation function \vee this implies that $V_{\mathcal{J}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$. But then, given our definition of the valuation function \vee , we can say that $V_{\mathcal{J}}(P \vee R) = 0$. \square

(c) We need to establish that $\sim(P \wedge Q)$ and $\sim P \vee \sim Q$ are semantically equivalent. This, in effect, means that proof is required for $\models (\sim(P \wedge Q)) \leftrightarrow (\sim P \vee \sim Q)$. We can proceed in two stages. First, we demonstrate that $\models (\sim(P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$, and, subsequently, that $\models (\sim P \vee \sim Q) \rightarrow (\sim(P \wedge Q))$.

We prove that $\models (\sim(P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$ by reductio. So let \mathcal{J} be any PL-interpretation, and let us assume that $V_{\mathcal{J}}(\sim(P \wedge Q)) \rightarrow (\sim P \vee \sim Q) = 0$. Given our definition of the valuation function \rightarrow , we can say that $V_{\mathcal{J}}(\sim(P \wedge Q)) = 1$, and that $V_{\mathcal{J}}(\sim P \vee \sim Q) = 0$. By our definition of the valuation functions \sim and \wedge , we see that $V_{\mathcal{J}}(P \wedge Q) = 0$, and so

that not both $\mathcal{J}(P) = 1$ and $\mathcal{J}(Q) = 1$. But by our definition of the valuation function \vee , we also see that $V_{\mathcal{J}}(\sim P) = 0$ and $V_{\mathcal{J}}(\sim Q) = 0$. Using again our definition for the valuation function \sim , this implies that both $\mathcal{J}(P) = 1$ and $\mathcal{J}(Q) = 1$. From our assumption, we have derived a contradiction.

We now prove that $\models (\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q))$, again by reductio. So let \mathcal{J} be any PL-interpretation, and let us assume that $V_{\mathcal{J}}(\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q)) = 0$. Given our definition of the valuation function \rightarrow , we can say that $V_{\mathcal{J}}(\sim P \vee \sim Q) = 1$, and that $V_{\mathcal{J}}(\sim (P \wedge Q)) = 0$. By our definition of the valuation function \vee and \sim , we see that not both $V_{\mathcal{J}}(\sim P) = 0$ and $V_{\mathcal{J}}(\sim Q) = 0$, and so that not both $\mathcal{J}(P) = 1$ and $\mathcal{J}(Q) = 1$. Now, by our definition of the valuation function \sim , we see that $V_{\mathcal{J}}(P \wedge Q) = 1$. Notice, however, that given the valuation function \wedge , it follows that both $\mathcal{J}(P) = 1$ and $\mathcal{J}(Q) = 1$. And again we have derived a contradiction from our assumption. \square

2.3 Sequent proofs in PL

(a) We need to prove that $P \rightarrow (Q \rightarrow R) \Rightarrow (Q \wedge \sim R) \rightarrow \sim P$.

1. $P \rightarrow (Q \rightarrow R) \Rightarrow P \rightarrow (Q \rightarrow R)$ (RA)
2. $Q \wedge \sim R \Rightarrow Q \wedge \sim R$ (RA)
3. $P \Rightarrow P$ (RA, for reductio)
4. $P, P \rightarrow (Q \rightarrow R) \Rightarrow Q \rightarrow R$ (1, 3, \rightarrow E)
5. $Q \wedge \sim R \Rightarrow Q$ (2, \wedge E)
6. $P, P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow R$ (4, 5, \rightarrow E)
7. $P, P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow \sim R$ (6, \wedge E)
8. $P, P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow R \wedge \sim R$ (6, 7, \wedge I)

9. $P \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow \sim P$ (8, RAA)
 10. $P \rightarrow (Q \rightarrow R) \Rightarrow (Q \wedge \sim R) \rightarrow \sim P$ (9, \rightarrow I)

(b) We need to prove that $P, Q, R \Rightarrow P$.

1. $P \Rightarrow P$ (RA)
 2. $Q \Rightarrow Q$ (RA)
 3. $R \Rightarrow R$ (RA)
 4. $Q, R \Rightarrow Q \wedge R$ (2, 3, \wedge I)
 5. $Q, R \Rightarrow Q$ (4 \wedge E)
 6. $P, Q, R \Rightarrow P \wedge Q$ (1, 5, \wedge I)
 7. $P, Q, R \Rightarrow P$ (6 \wedge E)

(c) We need to prove that $P \rightarrow Q, R \rightarrow Q \Rightarrow (P \vee R) \rightarrow Q$.

1. $P \Rightarrow P$ (RA)
 2. $R \Rightarrow R$ (RA)
 3. $P \vee R \Rightarrow P \vee R$ (RA)
 4. $P \rightarrow Q \Rightarrow P \rightarrow Q$ (RA)
 5. $R \rightarrow Q \Rightarrow R \rightarrow Q$ (RA)
 6. $P \rightarrow Q, P \Rightarrow Q$ (1, 4, \rightarrow E)
 7. $R \rightarrow Q, R \Rightarrow Q$ (2, 5, \rightarrow E)
 8. $P \rightarrow Q, R \rightarrow Q, P \vee R \Rightarrow Q$ (3, 6, 7, \vee E)
 9. $P \rightarrow Q, R \rightarrow Q \Rightarrow (P \vee R) \rightarrow Q$ (8, \rightarrow I)

2.4 Axiomatic proofs in PL

- (a) We need to prove that $\vdash P \rightarrow P$
- (b) We need to prove that $\vdash (P) \rightarrow P$
- (c) We need to prove that $\sim\sim P \vdash P$