

LOGIC FOR PHILOSOPHY

1 Chapter one

2 Chapter two

2.1 Disjunction and equivalence

(a) We need to show that for any wff φ and χ , and any PL-interpretation \mathcal{I} , $V_{\mathcal{I}}(\varphi \vee \chi)=1$ iff either $V_{\mathcal{I}}(\varphi)=1$ or $V_{\mathcal{I}}(\chi)=1$. We must first notice that $(\varphi \vee \chi)$ is just shorthand for $(\sim \varphi \rightarrow \chi)$, so that what we need to show is that for any wff φ and χ , and any PL-interpretation \mathcal{I} , $V_{\mathcal{I}}(\sim \varphi \rightarrow \chi)=1$ iff either $V_{\mathcal{I}}(\varphi)=1$ or $V_{\mathcal{I}}(\chi)=1$. This can be done by showing, first, that, if $V_{\mathcal{I}}(\sim \varphi \rightarrow \chi)=1$ then $V_{\mathcal{I}}(\varphi)=1$ or $V_{\mathcal{I}}(\chi)=1$, and subsequently that, if $V_{\mathcal{I}}(\varphi)=1$ or $V_{\mathcal{I}}(\chi)=1$ then $V_{\mathcal{I}}(\sim \varphi \rightarrow \chi)=1$.

Let us start by proving that if $V_{\mathcal{I}}(\sim \varphi \rightarrow \chi)=1$, then either $V_{\mathcal{I}}(\varphi)=1$ or $V_{\mathcal{I}}(\chi)=1$. We first assume that $V_{\mathcal{I}}(\sim \varphi \rightarrow \chi)=1$. Recall that we have defined the valuation function for \rightarrow as follows: $V_{\mathcal{I}}(\phi \rightarrow \psi)=1$ iff either $V_{\mathcal{I}}(\phi)=0$ or $V_{\mathcal{I}}(\psi)=1$. Given our assumption, we can therefore say that either $V_{\mathcal{I}}(\sim \varphi)=0$ or $V_{\mathcal{I}}(\chi)=1$. But then, given the definition for the valuation function for \sim , which is $V_{\mathcal{I}}(\sim \phi)=1$ iff $V_{\mathcal{I}}(\phi)=0$, we can say that either $V_{\mathcal{I}}(\varphi)=1$ or $V_{\mathcal{I}}(\chi)=1$. This is what we wanted to show.

The second stage of the proof requires us to prove that if $V_{\mathcal{I}}(\varphi)=1$ or $V_{\mathcal{I}}(\chi)=1$ then $V_{\mathcal{I}}(\sim \varphi \rightarrow \chi)=1$. So, let us assume that either $V_{\mathcal{I}}(\varphi)=1$ or $V_{\mathcal{I}}(\chi)=1$. By the valuation function for \sim , we see how our assumption implies that ei-

ther $V_{\mathcal{J}}(\sim \varphi)=0$ or $V_{\mathcal{J}}(\chi)=1$. And so, subsequently, by the valuation function for \rightarrow , we can demonstrate that our assumption implies that $V_{\mathcal{J}}(\sim \varphi \rightarrow \chi)=1$. \square

(b) We need to show that for any wff φ and χ , and any PL-interpretation \mathcal{J} , $V_{\mathcal{J}}(\varphi \leftrightarrow \chi) = 1$ iff $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$. We begin by noting that $\varphi \leftrightarrow \psi$ is shorthand for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, and that the latter in its turn is shorthand for $(\sim (\varphi \rightarrow \psi) \rightarrow \sim (\psi \rightarrow \varphi))$. This means that what we need to show is that, for any wff φ and χ , and any PL-interpretation \mathcal{J} , $V_{\mathcal{J}}(\sim ((\varphi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \varphi))) = 1$ iff $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$. This can be done by showing, first that, if $V_{\mathcal{J}}(\sim ((\varphi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \varphi))) = 1$ then $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$, and, subsequently, that if $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$, then $V_{\mathcal{J}}(\sim ((\varphi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \varphi))) = 1$.

Let us start by proving that if $V_{\mathcal{J}}(\sim ((\varphi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \varphi))) = 1$ then $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$. We proceed by reductio, and so start by assuming that $V_{\mathcal{J}}(\varphi) \neq V_{\mathcal{J}}(\chi)$. This allows us to assume that, say, $V_{\mathcal{J}}(\varphi) = 1$ and that $V_{\mathcal{J}}(\chi) = 0$. Recall that we have defined the valuation function for \rightarrow as follows: $V_{\mathcal{J}}(\varphi \rightarrow \psi) = 1$ iff either $V_{\mathcal{J}}(\varphi) = 0$ or $V_{\mathcal{J}}(\psi) = 1$. The converse of this is that $V_{\mathcal{J}}(\varphi \rightarrow \psi) = 0$ iff $V_{\mathcal{J}}(\varphi) = 1$ and $V_{\mathcal{J}}(\psi) = 0$. We see that our assumptions about the valuation interpretation for ψ and χ imply that $V_{\mathcal{J}}(\varphi \rightarrow \chi) = 0$. We now introduce the further assumption that $V_{\mathcal{J}}(\sim ((\varphi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \varphi))) = 1$, which is just the antecedent of the conditional we are trying to prove. By the definition of the valuation function of \sim , we see that $V_{\mathcal{J}}((\varphi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \varphi)) = 0$. But if we again rely on our definition of the valuation function of \rightarrow , we can demonstrate that $V_{\mathcal{J}}(\varphi \rightarrow \chi) = 1$ (and that $V_{\mathcal{J}}(\sim (\chi \rightarrow \varphi)) = 0$). But we have already demonstrated that $V_{\mathcal{J}}(\varphi \rightarrow \chi) = 0$, and so we have a contradiction. This allows us to reject the assumption that $V_{\mathcal{J}}(\varphi) \neq V_{\mathcal{J}}(\chi)$, and so we have shown that $V_{\mathcal{J}}(\sim ((\varphi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \varphi))) = 1$ then $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$.

Now we still need to prove that if $V_{\mathcal{J}}(\varphi) = V_{\mathcal{J}}(\chi)$, then

$V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathcal{J}}(\phi) = V_{\mathcal{J}}(\chi)$, and so can assume that either $V_{\mathcal{J}}(\phi) = 1$ and that $V_{\mathcal{J}}(\chi) = 1$, or that $V_{\mathcal{J}}(\phi) = 0$ and that $V_{\mathcal{J}}(\chi) = 0$. We proceed in two straightforward stages.

First, we demonstrate that if we assume $V_{\mathcal{J}}(\phi) = 1$ and that $V_{\mathcal{J}}(\chi) = 1$, it follows that $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathcal{J}}(\phi) = 1$ and that $V_{\mathcal{J}}(\chi) = 1$. Using our definitions for the valuation function of \rightarrow and \sim , we can say that $V_{\mathcal{J}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathcal{J}}(\sim(\chi \rightarrow \phi)) = 0$. But this implies that $V_{\mathcal{J}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. This is the desired result.

Second, we demonstrate that if we assume $V_{\mathcal{J}}(\phi) = 0$ and that $V_{\mathcal{J}}(\chi) = 0$, it equally follows that $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. Again, the valuation function of \rightarrow and \sim allow us to say that $V_{\mathcal{J}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathcal{J}}(\sim(\chi \rightarrow \phi)) = 0$. As shown, this implies that $V_{\mathcal{J}}((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathcal{J}}(\sim((\phi \rightarrow \chi) \rightarrow \sim(\chi \rightarrow \phi))) = 1$. \square

2.2 Logical consequence

(a) We need to establish that $\models [P \wedge (Q \vee R)] \rightarrow [(P \wedge Q) \vee (P \wedge R)]$. We can prove this by reductio. So let \mathcal{J} be any PL-interpretation, and let us assume that $V_{\mathcal{J}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 0$. Given our definition of the valuation function \rightarrow , we can say that (i) $V_{\mathcal{J}}(P \wedge (Q \vee R)) = 1$, and that (ii) $V_{\mathcal{J}}((P \wedge Q) \vee (P \wedge R)) = 0$. Given our definitions of the valuation function \wedge and \vee , we can say that (iii) $\mathcal{J}(P) = 1$ (from (i)), and that (iv) $V_{\mathcal{J}}(P \wedge Q) = 0$ (from (ii)), and so that (v) $\mathcal{J}(P) = 0$ (from (ii)). We have now derived a contradiction from our assumption (iii & v), and so $V_{\mathcal{J}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 1$. \square

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(b) We need to establish that $(P \leftrightarrow Q) \vee (R \leftrightarrow S) \not\models P \vee R$. We can do this by finding an interpretation of P , Q , R , and S such that $V_{\mathcal{J}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$ and $V_{\mathcal{J}}(P \vee R) = 0$. So, let \mathcal{J} be an interpretation such that $\mathcal{J}(P) = 0$, $\mathcal{J}(Q) = 0$, $\mathcal{J}(R) = 0$, and $\mathcal{J}(S) = 0$. Given our definition of the valuation function \leftrightarrow , we can say that $V_{\mathcal{J}}(P \leftrightarrow Q) = 1$, and that $V_{\mathcal{J}}(R \leftrightarrow S) = 1$. And given our definition of the valuation function \vee this implies that $V_{\mathcal{J}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$. But then, given our definition of the valuation function \vee , we can say that $V_{\mathcal{J}}(P \vee R) = 0$. \square

(c)