LOGIC FOR PHILOSOPHY

1 Chapter one

2 Chapter two

2.1 Valuation functions for disjunction and equivalence

(a) We need to show that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\varphi \vee \chi)$=1 iff either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. We must first notice that $(\varphi \vee \chi)$ is just shorthand for $(\sim \varphi \rightarrow \chi)$, so that what we need to show is that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1 iff either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. This can be done by showing, first, that, if $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1 then $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1, and subsequently that, if $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1 then $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1.

Let us start by proving that if $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1, then either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. We first assume that $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1. Recall that we have defined the valuation function for $\rightarrow$ as follows: $V_{\mathscr{I}}(\phi \rightarrow \psi)$=1 iff either $V_{\mathscr{I}}(\phi)$=0 or $V_{\mathscr{I}}(\psi)$=1. Given our assumption, we can therefore say that either $V_{\mathscr{I}}(\sim \varphi)$=0 or $V_{\mathscr{I}}(\chi)$=1. But then, given the definition for the valuation function for $\sim$, which is $V_{\mathscr{I}}(\sim \phi)$=1 iff $V_{\mathscr{I}}(\phi)$=0, we can say that either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. This is what we wanted to show.

The second stage of the proof requires us to prove that if $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1 then $V_{\mathscr{I}}(\sim \varphi \rightarrow \chi)$=1. So, let us as-

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sume that either $V_{\mathscr{I}}(\varphi)$=1 or $V_{\mathscr{I}}(\chi)$=1. By the valuation function for ~, we see how our assumption implies that either $V_{\mathscr{I}}(\sim\varphi)$=0 or $V_{\mathscr{I}}(\chi)$=1. And so, subsequently, by the valuation function for →, we can demonstrate that our assumption implies that $V_{\mathscr{I}}(\sim\varphi\rightarrow\chi)$=1. □

(b) We need to show that for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\varphi\leftrightarrow\chi)=1$ iff $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$. We begin by noting that $\phi\leftrightarrow\psi$ is shorthand for $(\phi\rightarrow\psi)\wedge(\psi\rightarrow\phi)$, and that the latter in its turn is shorthand for $(\sim(\phi\rightarrow\psi)\rightarrow\sim(\psi\rightarrow\phi))$. This means that what we need to show is that, for any wff φ and χ, and any PL-interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ iff $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$. This can be done by showing, first that, if $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ then $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$, and, subsequently, that if $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$, then $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$.

Let us start by proving that if $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$ then $V_{\mathscr{I}}(\phi)=V_{\mathscr{I}}(\chi)$. We assume the antecedent of the conditional we are trying to prove, $V_{\mathscr{I}}(\sim((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi)))=1$. By the definition of the valuation function of ~, we see that $V_{\mathscr{I}}((\phi\rightarrow\chi)\rightarrow\sim(\chi\rightarrow\phi))=0$. If we rely on our definition of the valuation function of →, we can demonstrate that $V_{\mathscr{I}}(\phi\rightarrow\chi)=1$ and that $V_{\mathscr{I}}(\sim(\chi\rightarrow\phi))=0$. We now assume, for reductio, that $V_{\mathscr{I}}(\phi)\neq V_{\mathscr{I}}(\chi)$. This implies either that $\mathscr{I}(\phi)=1$ and $\mathscr{I}(\chi)=0$, or that $\mathscr{I}(\phi)=0$ and $\mathscr{I}(\chi)=1$ (but not both, of course). We will now demonstrate that each of these possibilities entails a contradiction.

So, first, let $\mathscr{I}(\phi)=1$ and $\mathscr{I}(\chi)=0$. The converse of the valuation function for → is that $V_{\mathscr{I}}(\phi\rightarrow\psi)=0$ iff $V_{\mathscr{I}}(\phi)=1$ and $V_{\mathscr{I}}(\psi)=0$. We see that our current assumption about the particular valuation interpretation for ψ and χ implies that $V_{\mathscr{I}}(\phi\rightarrow\chi)=0$. But this contradicts with a premise we derived from our antecedent.

This means that, at best, $\mathscr{I}(\phi)=0$ and $\mathscr{I}(\chi)=1$. But then, by the valuation functions for $\rightarrow$ and $\sim$, we see that $V_{\mathscr{I}}(\chi \rightarrow \psi) = 0$, and so that $V_{\mathscr{I}}(\sim(\chi \rightarrow \psi)) = 1$. This again contradicts with a premise we derived from our antecedent, and so we have shown that our antecedent together with $V_{\mathscr{I}}(\phi) \neq V_{\mathscr{I}}(\chi)$ leads to contradiction.

Now we still need to prove that if $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\chi)$, then $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow\sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\chi)$, and so can assume either that $\mathscr{I}(\phi) = 1$ and $\mathscr{I}(\chi) = 1$, or that $\mathscr{I}(\phi) = 0$ and $V\mathscr{I}(\chi) = 0$ (but, again, not both). We proceed again in two straightforward stages.

First, we demonstrate that if we assume $\mathscr{I}(\phi) = 1$ and that $\mathscr{I}(\chi) = 1$, it follows that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow\sim(\chi \rightarrow \phi))) = 1$. We assume that $V_{\mathscr{I}}(\phi) = 1$ and that $V_{\mathscr{I}}(\chi) = 1$. Using our definitions for the valuation function of $\rightarrow$ and $\sim$, we can say that $V_{\mathscr{I}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathscr{I}}(\sim(\chi \rightarrow \phi)) = 0$. But this implies that $V_{\mathscr{I}}((\phi \rightarrow \chi) \rightarrow\sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow\sim(\chi \rightarrow \phi))) = 1$. This is the desired result.

Second, we demonstrate that if we assume $\mathscr{I}(\phi) = 0$ and that $\mathscr{I}(\chi) = 0$, it equally follows that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow\sim(\chi \rightarrow \phi))) = 1$. Again, the valuation function of $\rightarrow$ and $\sim$ allow us to say that $V_{\mathscr{I}}(\phi \rightarrow \chi) = 1$ and that $V_{\mathscr{I}}(\sim(\chi \rightarrow \phi)) = 0$. As shown, this implies that $V_{\mathscr{I}}((\phi \rightarrow \chi) \rightarrow\sim(\chi \rightarrow \phi)) = 0$, which in turn implies that $V_{\mathscr{I}}(\sim((\phi \rightarrow \chi) \rightarrow\sim(\chi \rightarrow \phi))) = 1$. □

2.2 Logical consequence

(a) We need to establish that $\vDash [P \wedge (Q \vee R)] \rightarrow [(P \wedge Q) \vee (P \wedge R)]$. We can prove this by reductio. So let $\mathscr{I}$ be any PL-interpretation, and let us assume that $V_{\mathscr{I}}((P \wedge (Q \vee R)) \rightarrow ((P \wedge Q) \vee (P \wedge R))) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that (*i*) $V_{\mathscr{I}}(P \wedge (Q \vee R)) = 1$, and that (*ii*) $V_{\mathscr{I}}((P \wedge Q) \vee (P \wedge R)) = 0$. Given our definitions of the valuation function $\wedge$ and $\vee$, we can say that (*iii*) $\mathscr{I}(P) =$

1 (from i), and that (iv) $V_{\mathscr{I}}(P \wedge Q) = 0$ (from ii). By the valuation function of $\wedge$, this means that (v) $\mathscr{I}(Q) = 0$ (from iii & iv).

Now, given both these interpretations of P and Q, we can demonstrate that i implies that (vi) $V_{\mathscr{I}}(Q \vee R) = 1$ (by $\wedge$ and $\mathscr{I}(P)$), and so that $\mathscr{I}(R) = 1$ (by $\vee$, vi, & $\mathscr{I}(Q)$). But given our valuation function for $\wedge$, we see that our interpretations of P and Q also imply (vii) $V_{\mathscr{I}}(P \wedge Q) = 0$. And so we can show that ii implies that $V_{\mathscr{I}}(P \wedge R) = 0$ (by $\vee$ & vii), and so that $\mathscr{I}(R) = 0$ (by $\wedge$ & $\mathscr{I}(P)$). So, we have derived both $\mathscr{I}(R) = 1$ and $\mathscr{I}(R) = 0$, which is absurd. □

(b) We need to establish that $(P \leftrightarrow Q) \vee (R \leftrightarrow S) \not\models P \vee R$. We can do this by finding an interpretation of P, Q, R, and S such that $V_{\mathscr{I}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$ and $V_{\mathscr{I}}(P \vee R) = 0$. So, let $\mathscr{I}$ be an interpretation such that $\mathscr{I}(P) = 0$, $\mathscr{I}(Q) = 0$, $\mathscr{I}(R) = 0$, and $\mathscr{I}(P) = 0$. Given our definition of the valuation function $\leftrightarrow$, we can say that $V_{\mathscr{I}}(P \leftrightarrow Q) = 1$, and that $V_{\mathscr{I}}(R \leftrightarrow S) = 1$. And given our definition of the valuation function $\vee$ this implies that $V_{\mathscr{I}}(P \leftrightarrow Q) \vee (R \leftrightarrow S) = 1$. But then, given our definition of the valuation function $\vee$, we can say that $V_{\mathscr{I}}(P \vee R) = 0$. □

(c) We need to establish that $\sim (P \wedge Q)$ and $\sim P \vee \sim Q$ are semantically equivalent. This, in effect, means that proof is required for $\models (\sim (P \wedge Q)) \leftrightarrow (\sim P \vee \sim Q)$. We can proceed in two stages. First, we demonstrate that $\models (\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$, and, subsequently, that $\models (\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q))$.

We prove that $\models (\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q)$ by reductio. So let $\mathscr{I}$ be any PL-interpretation, and let us assume that $V_{\mathscr{I}}(\sim (P \wedge Q)) \rightarrow (\sim P \vee \sim Q) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that $V_{\mathscr{I}}(\sim (P \wedge Q)) = 1$, and that $V_{\mathscr{I}}(\sim P \vee \sim Q) = 0$. By our definition of the valuation functions $\sim$ and $\wedge$, we see that $V_{\mathscr{I}}(P \wedge Q) = 0$, and so that not both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. But by our definition of the valuation function $\vee$, we also see that $V_{\mathscr{I}}(\sim P) = 0$ and $V_{\mathscr{I}}(\sim Q) = 0$. Using again our definition for the valuation function $\sim$, this implies that both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. From our assumption, we have derived a contradiction.

We now prove that $\vDash (\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q))$, again by reductio. So let $\mathscr{I}$ be any PL-interpretation, and let us assume that $V_{\mathscr{I}}(\sim P \vee \sim Q) \rightarrow (\sim (P \wedge Q)) = 0$. Given our definition of the valuation function $\rightarrow$, we can say that $V_{\mathscr{I}}(\sim P \vee \sim Q) = 1$, and that $V_{\mathscr{I}}(\sim (P \wedge Q)) = 0$. By our definition of the valuation function $\vee$ and $\sim$, we see that not both $V_{\mathscr{I}}(\sim P) = 0$ and $V_{\mathscr{I}}(\sim Q) = 0$, and so that not both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. Now, by our definition of the valuation function $\sim$, we see that $V_{\mathscr{I}}(P \wedge Q) = 1$. Notice, however, that given the valuation function $\wedge$, it follows that both $\mathscr{I}(P) = 1$ and $\mathscr{I}(Q) = 1$. And again we have derived a contradiction from our assumption. □

2.3 Sequent proofs in PL

(a) We need to prove that $P \rightarrow (Q \rightarrow R) \Rightarrow (Q \wedge \sim R) \rightarrow \sim P$.

1	$\mathrm{P} \rightarrow (Q \rightarrow R) \Rightarrow P \rightarrow (Q \rightarrow R)$	(RA)
2	$\mathrm{Q} \wedge \sim R \Rightarrow Q \wedge \sim R$	(RA)
3	$\mathrm{P} \Rightarrow P$	(RA, for reductio)
4	$\mathrm{P}, \mathrm{P} \rightarrow (Q \rightarrow R) \Rightarrow Q \rightarrow R$	(1,3, $\rightarrow$E)
5	$\mathrm{Q} \wedge \sim R \Rightarrow Q$	(2, $\wedge$E)
6	$\mathrm{P}, \mathrm{P} \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow R$	(4, 5, $\rightarrow$E)
7	$\mathrm{P}, \mathrm{P} \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow \sim R$	(6, $\wedge$E)
8	$\mathrm{P}, \mathrm{P} \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow R \wedge \sim R$	(6, 7, $\wedge$I)
9	$\mathrm{P} \rightarrow (Q \rightarrow R), Q \wedge \sim R \Rightarrow \sim P$	(8, RAA)
10	$\mathrm{P} \rightarrow (Q \rightarrow R) \Rightarrow (Q \wedge \sim R) \rightarrow \sim P$	(9, $\rightarrow$I)

(b) We need to prove that $P, Q, R \Rightarrow P$.

1	$\mathrm{P} \Rightarrow P$	(RA)
2	$\mathrm{Q} \Rightarrow Q$	(RA)
3	$\mathrm{R} \Rightarrow R$	(RA)
4	$\mathrm{Q}, \mathrm{R} \Rightarrow Q \wedge R$	(2, 3, $\wedge$I)
5	$\mathrm{Q}, \mathrm{R} \Rightarrow Q$	(4 $\wedge$E)
6	$\mathrm{P}, \mathrm{Q}, \mathrm{R} \Rightarrow P \wedge Q$	(1, 5, $\wedge$I)
7	$\mathrm{P}, \mathrm{Q}, \mathrm{R} \Rightarrow P$	(6 $\wedge$E)

(c) We need to prove that $P \to Q, R \to Q \Rightarrow (P \vee R) \to Q$.

1	$P \Rightarrow P$	(RA)
2	$R \Rightarrow R$	(RA)
3	$P \vee R \Rightarrow P \vee R$	(RA)
4	$P \to Q \Rightarrow P \to Q$	(RA)
5	$R \to Q \Rightarrow R \to Q$	(RA)
6	$P \to Q, P \Rightarrow Q$	(1, 4, $\to$E)
7	$R \to Q, R \Rightarrow Q$	(2, 5, $\to$E)
8	$P \to Q, R \to Q, P \vee R \Rightarrow Q$	(3, 6, 7, $\vee$E)
9	$P \to Q, R \to Q \Rightarrow (P \vee R) \to Q$	(8, $\to$I)

2.4 Axiomatic proofs in PL

(a) We need to prove that $\vdash P \to P$

1	$(P \to ((P \to P) \to P)) \to ((P \to (P \to P)) \to (P \to P))$	PL2
2	$P \to ((P \to P) \to P)$	PL1
3	$(P \to (P \to P)) \to (P \to P)$	1,2,MP
4	$P \to (P \to P)$	PL1
5	$P \to P$	3,4 MP

(b) We need to prove that $\vdash (\sim P \to P) \to P$

1	$\sim P \to ((\sim P \to \sim P) \to \sim P) \to$ etc.	PL2
2	$\sim P \to ((\sim P \to \sim P) \to \sim P)$	PL1
3	$(\sim P \to (\sim P \to \sim P)) \to (\sim P \to \sim P)$	1,2,MP
4	$\sim P \to (\sim P \to \sim P)$	PL1
5	$\sim P \to \sim P$	3,4 MP
6	$(\sim P \to \sim P) \to (\sim P \to P) \to P$	PL3
7	$(\sim P \to P) \to P$	5,6,MP

(c) We need to prove that $\sim\sim P \vdash P$

1	$\sim\sim P$	premise
2	$\sim\sim P \to (\sim P \to \sim\sim P)$	PL1
3	$\sim P \to \sim\sim P$	1,2,MP
4	$(\sim P \to \sim\sim P) \to ((\sim P \to \sim P) \to P)$	PL3
5	$(\sim P \to \sim P) \to P$	3,4,MP
6	$\sim P \to \sim P$	premise, 2.4(b)
7	P	5,6,MP