Complexity Theory 236313 - Homework Assignment #2

Due December 27, 2022. Submit a single PDF file to the course site

December 19, 2022

Question 1. A language L is *self-reducible* if there exists a polytime oracle Turing machine M^L that decides L and satisfies the following property: for every input x, M^L asks the oracle only questions y of length $|y| < |x|$. Let SR be the set of all self-reducible languages. A language L is SR-complete if $L \in \mathsf{SR}$ and for every $L' \in \mathsf{SR}$ there is a polytime reduction $L' \leq_p L$.

Prove that TQBF is SR-complete.

Question 2. A language S is *sparse* if there exists a polynomial $s(n)$ s.t. $|\{x \in S : |x| \leq n\}| \leq s(n)$ for all $n \in \mathbb{N}$.

1. Prove that if $\mathsf{NP}^L \subseteq \mathsf{P}^{\mathsf{NP}}$ for all $L \in \mathsf{NP}$ then the polynimial hierarchy collapses.
2. Prove that $\mathsf{NP}^S \subseteq \mathsf{P}^{\mathsf{NP}}$ for any sparse language S.
   Hint: Begin by proving that for any polynomial $p(n)$ there exists a polytime deterministic TM with an oracle in NP that, given input 1^n, computes the number of words in S whose length is at most $p(n)$. Then, use the counting algorithm to simulate a computation from NP^S within NP.
   Remark: This implies that unless the polynomial hierarchy collapses, there exists some non-sparse language in NP.

Question 3. The Polynomial Hierarchy with an oracle A is defined as follows:

$$\Sigma_0^{p^A} = \mathsf{P}^A \quad , \quad \Sigma_{n+1}^{p^A} = \mathsf{NP}^{\Sigma_n^{p^A}} \quad , \quad \mathsf{PH}^A = \bigcup_{n \geq 0} \Sigma_n^{p^A} .$$

A language $L \in \mathsf{NP}$ is *n-low* if $\Sigma_n^{p^L} = \Sigma_n^p$. Let C_n be the class of languages in NP that are n-low, and let $\mathsf{LH} = \bigcup_{n \geq 0} \mathsf{C}_n$ denote the "low hierarchy".

1. Prove or disprove: $\mathsf{PH}^A = \mathsf{PH}$ for every oracle A.
2. Prove that $\mathsf{C}_n \subseteq \mathsf{C}_{n+1}$ for all n.
3. Prove that if $\mathsf{LH} = \mathsf{NP}$ then the polynomial hierarchy collapses.
4. To what known complexity class does C_0 equal? Prove your answer.
5. Prove that $\mathsf{C}_1 = \mathsf{NP} \cap \mathsf{coNP}$.

Question 4. Given $n \in \mathbb{N}$, let $\alpha(n)$ denote the minimal s s.t. every function $f\colon \{0,1\}^n \to \{0,1\}$ is computable with a circuit of size at most s. (where size is measured as the number of nodes in the circuit.)

1. (Hierarchy theorem for circuits) Prove that there exists a constant $c > 0$ s.t. for all n and for all $0 \leq r < \alpha(n)$ there exists a function $f\colon \{0,1\}^n \to \{0,1\}$ that cannot be computed by any circuit of size r but can be computed by a circuit of size $r + cn$.
   Hint: Consider pairs of functions that differ only on one assignment.

2. Prove that for all $k > 0$ there exists a language $L \in \mathsf{PH}$ that has circuit complexity $\geq n^k$.
   Hint: use the previous result.

Question 5. Given a function $h(n)$, let $\mathsf{P}/h(n)$ denote the class of all languages that are decidable with a polytime TM with "*advice*" $h(n)$, where the advice depends only on the input length (namely, inputs of the same length get the same advice). Formally, $L \in \mathsf{P}/h(n)$ if there exists a deterministic polytime Turing machine M and a series of advices $\{a_n\}_{n\in\mathbb{N}}$ such that: (1) $|a_n| \leq h(n)$, and (2) M accepts $x\$a_{|x|}$ iff $x \in L$.

1. Let $\mathsf{P/poly} = \bigcup_{c\geq 1} \mathsf{P}/n^c$. Prove that $L \in \mathsf{P/poly}$ iff there exists a polynomial family of circuits that recognizes L.

2. Prove or disprove: $\mathsf{P} = \mathsf{P}/1$.

3. Let $\mathsf{P}/\log = \bigcup_{c\geq 1} \mathsf{P}/c\log n$. Prove that if $\mathsf{NP} \subseteq \mathsf{P}/\log$ then $\mathsf{P} = \mathsf{NP}$.