תרגיל בית 1

אלגברה לינארית

Ex 1:

a) $a = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

b) $\bar{a} + \bar{b} = \begin{pmatrix} 3+1 \\ 1+2 \\ -1-4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -5 \end{pmatrix}$

c) $\|\bar{a} + \bar{b}\| = \sqrt{4^2 + 3^2 + (-5)^2} = \sqrt{50}$

d) $(\bar{a} - 2\bar{b})(3\bar{a} + \bar{b}) = 3\bar{a}\cdot\bar{a} + \bar{a}\bar{b} - 6\bar{b}\bar{a} - 2\bar{b}\cdot\bar{b}$

$3 \cdot \|a\|^2 - 5\bar{b}\bar{a} - 2\|\bar{b}\|^2$

$\|a\|^2 = 3^2 + 1^2 + (-1)^2 = 11$

$\|b\|^2 = 1^2 + 2^2 + (-4)^2 = 21$

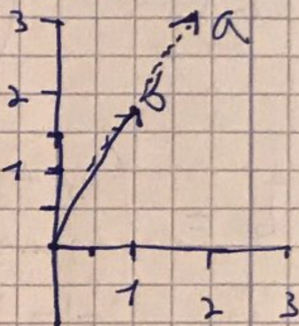
$\bar{b} \cdot \bar{a} = 3 \cdot 1 + 1 \cdot 2 + (-1) \cdot (-4) = 9$

$\Rightarrow 3 \cdot 11 - 5 \cdot 9 - 2 \cdot 21 = -54$

Ex 2: $\bar{a}$ is parallel to $\bar{b}$ if $\bar{a} = c\bar{b}$
where $c \in \mathbb{R}$

1.
$$\begin{pmatrix} t \\ 3 \end{pmatrix} = c \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} c \\ ct \end{pmatrix}$$

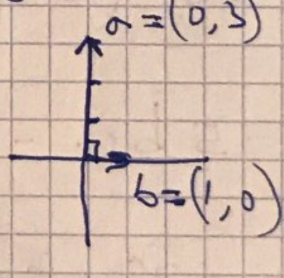
$$\Rightarrow \begin{matrix} t = c \\ 3 = ct \end{matrix} \Rightarrow 3 = t^2 \Rightarrow t = \pm\sqrt{3}$$



$$a = \begin{pmatrix} \sqrt{3} \\ 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

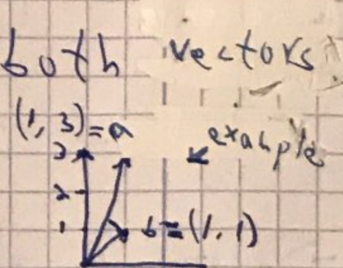
2. $\bar{a}$ and $\bar{b}$ are perpendicular
iff $\bar{a} \cdot \bar{b} = 0$

$$\Rightarrow t \cdot 1 + 3 \cdot t = 0 \Rightarrow t = 0$$



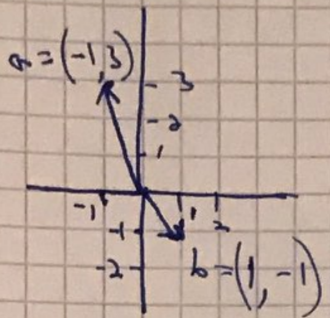
$$a = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

3. the angle between $\bar{a}$ and $\bar{b}$ is acute
when $t > 0$ and $t \neq \sqrt{3}$ as both vectors
will be located in the first quadrant



4. once $t < 0$ and $t \neq -\sqrt{3}$ vector b will be located in the IV quadrant and a will be located in the second quadrant thus the angle between them will be obtuse

example for Ex 2.4



Ex 3:

$$(\bar{a}+\bar{b})\cdot(\bar{a}-\bar{b})=0$$

$$\bar{a}\cdot\bar{a}+\bar{b}\bar{a}-\bar{a}\bar{b}-\bar{b}\cdot\bar{b}=$$

$$\|\bar{a}\|^2-\|b\|^2=0 \Rightarrow \|a\|^2=\|b\|^2$$

$\Rightarrow \|a\|=\|b\|$ as both norms are positive

Ex 4:

$$A: \begin{pmatrix}1 & 0 & 2\\ 2 & 2 & -1\\ 0 & 1 & 0\end{pmatrix}, \quad B=\begin{pmatrix}2 & 1 & -3\\ 2 & 5 & 4\\ 0 & 4 & 9\end{pmatrix}$$

a) $$A-2B=\begin{pmatrix}1 & 0 & 2\\ 3 & 2 & -1\\ 0 & 1 & 0\end{pmatrix}-\begin{pmatrix}4 & 2 & -6\\ 4 & 10 & 8\\ 0 & 8 & 18\end{pmatrix}=\begin{pmatrix}-3 & -2 & 8\\ -1 & -8 & -9\\ 0 & -7 & -18\end{pmatrix}$$

b) $$AB=\begin{pmatrix}1\cdot 2+0\cdot 2+2\cdot 0 & 1\times 1+0\times 5+2\times 4 & 1\times(-3)+0\times 4+2\times 9\\ 3\times 2+2\times 2+(-1)\times 0 & 3\times 1+2\times 5+(-1)\times 4 & 3\times(-3)+2\times 4+(-1)\times 9\\ 0\times 2+1\times 2+0\times 0 & 0\times 1+1\times 5+0\times 4 & 0\times(-3)+1\times 4+0\times 9\end{pmatrix}$$

$$AB=\begin{pmatrix}2 & 9 & 15\\ 10 & 9 & -10\\ 2 & 5 & 4\end{pmatrix}$$

c) $B^t \cdot A^t =$

$$B^t = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 5 & 4 \\ -3 & 4 & 9 \end{pmatrix}$$

$$A^t = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\Rightarrow B^t \cdot A^t = \begin{pmatrix} 2 & 10 & 2 \\ 9 & 9 & 5 \\ 15 & -10 & 4 \end{pmatrix}$$

D) $$A^2 = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 9 & 3 & 4 \\ 3 & 2 & -1 \end{pmatrix}$$

We can see that $B^t \cdot A^t = (A \cdot B)^t$

$$\Rightarrow \begin{pmatrix} 2 & 10 & 2 \\ 9 & 9 & 5 \\ 15 & -10 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 9 & 15 \\ 10 & 9 & -10 \\ 2 & 5 & 4 \end{pmatrix}^T$$

$$B^t \cdot A^t = (A \cdot B)^t$$

Ex 5 :

1. $(A + \alpha B)^T = A^T + (\alpha B)^T = A^T + \alpha B^T$

$= -A - \alpha B = -(A + \alpha B)$

2.1) $\left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T) =$

$\frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T)$ c.q.d

2) $\left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A + (-A^T))^T = \frac{1}{2}(A^T + (-A^T)^T)$

$= \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T)$

$\Rightarrow \left(\frac{1}{2}(A - A^T)\right)^T = -\frac{1}{2}(A - A^T)$ c.q.d

3) $\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \frac{A}{2} + \frac{A^T}{2} + \frac{A}{2} - \frac{A^T}{2}$

$= A$

4. $$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 4 & 3 \\ 0 & 2 & 2 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 2 \\ -1 & 3 & 2 \end{pmatrix}$$

$$B = A + A^T = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 4 & 3 \\ 0 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 2 \\ -1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & -1 \\ 4 & 8 & 5 \\ -1 & 5 & 4 \end{pmatrix}$$

$$C = A - A^T = \begin{pmatrix} -\cdot\cdot- \end{pmatrix} - \begin{pmatrix} -\cdot\cdot- \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\frac{1}{2}(B + C) = \frac{1}{2} \cdot \begin{pmatrix} 4 & 6 & -2 \\ 2 & 8 & 6 \\ 0 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 4 & 3 \\ 0 & 2 & 2 \end{pmatrix} = A$$

Ex 6:

1. False

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 0 & 3 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} \Rightarrow \quad A \cdot B \neq B \cdot A$$

2. True

$$A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax & bx \\ cx & dx \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} ax & bx \\ cx & dx \end{pmatrix}$$

$$\Rightarrow \quad A \cdot B = B \cdot A$$

3. false

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

4. false A can be Involutory matrix as:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^2 - I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

5. True

$$A(B + C^3) = AB + AC^3 = AB + A \cdot C \cdot C \cdot C$$

$$\Rightarrow A \cdot C \cdot C \cdot C = C \cdot A \cdot C \cdot C = C \cdot C \cdot A \cdot C = C \cdot C \cdot C \cdot A \Leftarrow$$

מתכונות הכפל מטריצות ונתון ש- $AC = CA$

$$\Rightarrow AB + C^3 \cdot A = BA + C^3 A = (B + C^3)A$$