Linear Algebra for MLDS - Homework 6

Coordinates, Rank, Linear Transformations, Matrices of L.T

Make sure to read and follow the ”Homework Submission Instructions” file

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Exercise 1: Let $U = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ be a subspace of $\mathbb{R}^{2\times 2}$.

1. Find a basis for U and determine $\dim(U)$ (it is worthwhile to take a simple basis).
2. Let $F : U \to U$ by: $F \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} b & 3a+b \\ 3a+b & -b \end{pmatrix}$. explain why Range$(F) \subseteq U$
3. Find $[F]_B$ where B is the basis you found in 1
4. Explain why F is invertible and find $[F^{-1}]_B$
5. Calculate $F^{-1} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Exercise 2: For each of the following matrices, find all eigenvalues, their algebraic and geometric multiplicities, bases for the corresponding eigenspaces, and check whether it is diagonalizable.

If so, find the diagonal matrix D and invertible matrix P such that $D = P^{-1}AP$

1. $A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$
2. $A = \begin{pmatrix} 2 & -4 & 3 \\ 1 & -2 & 1 \\ -4 & 0 & -6 \end{pmatrix}$
3. $A = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Exercise 3: Use diagonalization to calculate $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}^6$

Exercise 4: Prove the following claims regarding an $n \times n$ matrix A with an eigenvalue λ and a corresponding eigenvector λ

1. 2λ is an eigenvalue of $2A$ with the same eigenvector v
2. λ^3 is an eigenvalue of A^3 with the same eigenvector v
3. If A is non-singular, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with the same eigenvector v

Exercise 5: If A is a diagonalizable $n \times n$ matrix, and there is a natural number k such that $A^k = 0$ then A is the zero matrix. (hint: think what must be the eigenvalues of A)

Exercise 6: In the space $V = \mathbb{R}^{2\times 2}$, define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(A) = A^t$

1. Verify that T is a linear transformation.
2. Find the matrix of T with respect to the standard basis E.
3. Show that that T is invertible and find T^{-1}
4. Show that $[T]_E$ is diagonalizable.
5. Find a basis B for $\mathbb{R}^{2\times 2}$ such that $[T]_B$ is diagonal, what can you say about the matrices in the basis?

Exercise 7: Regarding the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

1. Show that A is positive semi-definite
2. Find a square root for A