HW 2

Ex 1:

1)

$$\begin{aligned} x_1 + 3x_3 &= 7 \\ 2x_2 - 4x_3 &= -8 \\ 3x_1 + 2x_2 + 10x_3 &= 23 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 2 & -4 & -8 \\ 3 & 2 & 10 & 23 \end{array}\right) \xrightarrow{R_3 \to R_3 - 3R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 2 & -4 & -8 \\ 0 & 2 & 1 & 2 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 2 & -4 & -8 \\ 0 & 0 & 5 & 10 \end{array}\right) \xrightarrow[R_2 \to \frac{R_2}{2}]{R_3 \to \frac{R_3}{5}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow[R_2 \to R_2 + 2R_2]{R_1 \to R_1 - 3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 2 \end{array}\right)$$

<u>back to LS</u>

$x_1 = 1$

$x_2 = 0$

$x_3 = 2$

Ex 1: 2)

$$\begin{aligned} x_2 + 3x_3 &= 0 \\ 6x_1 - 3x_3 &= 4 \\ -2x_1 + 2x_2 &= 1 \end{aligned} \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 6 & 0 & -3 & 4 \\ -2 & 2 & 0 & 1 \end{array}\right) \xrightarrow[R_1 \leftrightarrow R_3]{R_3 \to -\frac{R_3}{2}}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -\frac{1}{2} \\ 6 & 0 & -3 & 4 \\ 0 & 1 & 3 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 6R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -\frac{1}{2} \\ 0 & 6 & -3 & 7 \\ 0 & 1 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to 6R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -\frac{1}{2} \\ 0 & 6 & -3 & 7 \\ 0 & 0 & 21 & -7 \end{array}\right) \xrightarrow{R_3 \to \frac{R_3}{7}}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -\frac{1}{2} \\ 0 & 6 & -3 & 7 \\ 0 & 0 & 3 & -1 \end{array}\right) \xrightarrow[R_3 \to \frac{R_3}{3}]{R_2 \to R_2 + R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -\frac{1}{2} \\ 0 & 6 & 0 & 6 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right) \xrightarrow[2.\ R_1 \to R_1 + R_2]{1.\ R_2 \to \frac{R_2}{6}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right) \xrightarrow{\text{back to LS}} \begin{aligned} x_1 &= \frac{1}{2} \\ x_2 &= 1 \\ x_3 &= -\frac{1}{3} \end{aligned}$$

Ex 1 3)

$$\begin{aligned} -4x_1+6x_2&=-12\\ x_1-x_2&=1\\ 2x_1+x_2&=17\\ 6x_1-8x_2&=0 \end{aligned} \longrightarrow \left(\begin{array}{cc|c} -4 & 6 & -12\\ 1 & -1 & 1\\ 2 & 1 & 17\\ 6 & -8 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{R_4 \to \frac{R_4}{2}\\ R_2 \leftrightarrow R_1}]{R_1 \to \frac{R_1}{2}} \left(\begin{array}{cc|c} 1 & -1 & 1\\ -2 & 3 & 6\\ 2 & 1 & 17\\ 3 & -4 & 0 \end{array}\right) \xrightarrow[R_4 \to R_4 - 3R]{R_2 \to R_2 + R_3}$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & -1 & 1\\ 0 & 4 & 23\\ 2 & 1 & 17\\ 0 & -1 & -3 \end{array}\right) \xrightarrow{R_4 \to 4\cdot R_4 + R_2} \left(\begin{array}{cc|c} 1 & -1 & 1\\ 0 & 4 & 23\\ 2 & 1 & 17\\ 0 & 0 & 11 \end{array}\right)$$

we got a line with zeros and a non-zero coefficient $\rightarrow$ no solution to the equations

Ex 2:

As there are 2 free variables, there are $5-2=3$ leading entries, ~~thats~~ thus the r.e.f will not have any zero rows and as a result there won't be any inconsistent rows. Using matrix A with any $\vec{b}$ will thus also not contain any inconsistent rows which means that there is a solution

Ex 3:

1. <u>false</u>

counter example: $m=3 > n=2$

$$\begin{aligned} x+y&=1 \\ 2x+2y&=2 \\ 4x+4y&=4 \end{aligned} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 4 & 4 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 - 2R_2}$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \text{infinite solutions}$$

2. <u>false</u>

counter example: $m=2 < n=3$

$$\begin{aligned} z+x+y&=3 \\ z+x+y&=6 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{array}\right] \rightarrow \text{we have inconsistent row} \rightarrow \underline{\text{no}} \text{ solution}$$

3. <u>true</u>

as $m<n$, after number of element row operations we will get at most $n-m$ free variables and because $b=\bar{0}$ we would not get any inconsistent rows, thus we will end up with infinite solutions

$m=2$ $n=3$: example

$$\begin{array}{c} \text{free variable} \\ \left[\begin{array}{ccc|c} 1 & 0 & x & 0 \\ 0 & 1 & x & 0 \end{array}\right] \end{array}$$

Ex 2 :

4. false

counter example : $n = 3 > n = 2$

$$\begin{aligned} x + y &= 0 \\ 2x + 2y &= 0 \\ 4x + 4y &= 0 \end{aligned} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \text{infinite solutions}$$

Ex_4:

1. $\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_3}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$

the matrix is invertible, the inverse matrix

2) $\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1}$

$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to 2R_3 - R_2}$

$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 2 \end{array}\right)$

there is a zero row $\Rightarrow$ the matrix is singular

Exs:
1) probe by contradiction

Let's say A is invertible with B as inverse matrix

so $A \cdot B = I$

$A^2 \cdot B = (A \cdot A) B = A \cdot (A \cdot B)$

$(A \cdot A) \cdot B = 0 \cdot B = 0$

$A \cdot (A \cdot B) = A \cdot I = A$ ⚡

$\Rightarrow$ A is singular

2) $A^2 - 2A + I = 0 \Rightarrow A^2 - 2A = -I$

multiply by -1:

$\Rightarrow -A^2 + 2A = I \Rightarrow A\underbrace{(-A + 2I)}_{B} = I$

$\Rightarrow$ A is invertible and the inverse matrix is $-A + 2I$

Ex6:

1. $(BA^2)^{-1} \Rightarrow$

$(BA^2)\cdot(BA^2)^{-1} = I \quad |\cdot B^{-1}$

$\Rightarrow A^2\cdot(BA^2)^{-1} = B^{-1} \quad |\cdot(A^{-1})^2$

$\Rightarrow (BA^2)^{-1} = (A^{-1})^2\cdot B^{-1}$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 3 & 1 \\ 3 & 1 & 3 \\ 1 & 3 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 13 & 5 & 17 \\ 17 & 12 & 21 \\ 15 & 14 & 18 \end{pmatrix} = (BA^2)^{-1}$$

2) $A\bar{x} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \quad |\cdot A^{-1}$

$$\Rightarrow \bar{x} = A^{-1}\cdot\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} 1\times3 + 2\times1 + (-1)\cdot1 \\ 1\times3 + 2\times1 + 0\times1 \\ 1\times3 + 1\times1 + 1\times1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 5 \end{pmatrix}$$

3. If A and B are non singular it means that both are row equivalent to I. from I it possible to get to A/B by element row operations

so:

$$A \xrightarrow[\text{element row operations}]{} I \xrightarrow[\text{element row operations}]{} B$$

4. $A^2 \bar{x} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \quad \backslash \cdot (A^{-1})^2$

$$\bar{x} = (A^{-1})^2 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \bar{x} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 14 \\ 14 \end{pmatrix}$$ ← one solution

also as A is non singular A^2 is also non singular thus $A^2 \bar{x} = \vec{b}$ will have only one solution.

Ex 7: 1) $P = (-1, 2, 3)$ $\bar{v} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

parametric representation:

$\left\{ (-1, 2, 3) + t \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$

2) $(x, y, z) = (-1, 2, 3) + t \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ $x = -1 + 3t$

$y = 2$

$z = 3 + t$

let's check for the first point

$(2, 2, 1)$

for x: $2 = -1 + 3t \Rightarrow t = 1$

for z: $1 = 3 + t \Rightarrow t = -2$ $\neq$ $\Rightarrow$ the point is not on the line

let's check the second point $\rightarrow (-10, 2, 0)$

for x: $-10 = -1 + 3t \Rightarrow t = -3$

for z: $0 = 3 + t \Rightarrow t = -3$ $=$ $\Rightarrow$ the point is on the line

$y = 2$ Given

3. $x = -1 + 3t$

$y = 2$

$z = 3 + t$

xy plane => every x and y while, $z = 0$

let's set z to be zero

$0 = 3 + t \Rightarrow t = -3$

$\Rightarrow x = -1 + 3 \cdot (-3) = -10$

thus the intersection of the line with xy plane is on the point $(-10, 2, 0)$