Linear Algebra for MLDS - Homework 4

Vectors Spaces, Subspaces, Basis and Dimension

Make sure to read and follow the "Homework Submission Instructions" file

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Exercise 1: For each of the following, determine whether it is a subspace of the given vector space, if it is, find a its dimension

1. $\left\{ \begin{pmatrix} a \\ 2 \\ c \end{pmatrix} \middle| \; a + c = 0 \right\} \subset \mathbb{R}^3$
2. $\left\{ A \in \mathbb{R}^{2\times 2} \;\middle|\; A \text{ is upper triangular } \right\} \subset \mathbb{R}^{2\times 2}$
3. $\left\{ A \in \mathbb{R}^{3\times 3} \;\middle|\; A \text{ is invertible } \right\} \subset \mathbb{R}^{3\times 3}$
4. $\left\{ A \in \mathbb{R}^{2\times 2} \;\middle|\; A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^{2\times 2}$

Exercise 2: Given: $v_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, v_4 = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, v_5 = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$ Look at the following sets: $S_1 = \{v_1, v_2\}, S_2 = \{v_3, v_4\}, S_3 = \{v_1, v_2, v_3\}, S_4 = \{v_2, v_4, v_5\}, S_5 = \{v_1, v_2, v_3, v_5\}, S_6 = \{v_2, v_3, v_4, v_5\}$

1. Which of $S_1, \ldots, S_6$ form bases to $\mathbb{R}^3$?
2. For the rest, find the dimension of $\text{span}(S_i)$

Exercise 3: In the vector space $V = \mathbb{R}^{2\times 2}$, define $U = \{A \in V \mid A^t = A\}, W = \{A \in V \mid A^t = -A\}$

1. Prove that U, W are subspaces of V
2. Find $\dim(U), \dim(W)$
3. Prove that $U + W = V$

Exercise 4: For $V = \mathbb{R}^3$, define $U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \; a + b + c = 0 \right\} W = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \; a - b + 2c = 0 \right\}$

1. Find bases for U, W
2. Find a basis for $U \cap W$
3. Find a basis for $U + W$

Exercise 5: Prove the following claims:

1. If S is a linearly independent set and $T \subseteq S$ then T is linearly independent
2. Let v_1, v_2, v_3, v_4 be vectors such that $v_4 \in \text{span}\{v_1, v_2, v_3\}$ then $\text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1, v_2, v_3\}$
3. Any two scalar 5×5 matrices are linearly dependent.
4. For any two vectors u, v in a vector space V, $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.
5. Let U, W be subspaces of a vector space V such that $\{v_1, v_2\}$ is a basis for U and $\{v_2, v_3, v_4\}$ is a basis for W, then $\dim(U + W) \le 4$