HW 6

1. $V = \left\{ \begin{matrix} a & b \\ b & -a \end{matrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

$\text{span}(V) = \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

$\text{basis}(V) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

$\dim(V) = 2$

2. $\text{Range } F = \left\{ \begin{pmatrix} b & 3a+b \\ 3a+b & -b \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

$= \text{span} \left\{ \begin{matrix} 0 & 3 \\ 3 & 0 \end{matrix}, \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right\}$

$\dim \text{range}(F) = 2 = \dim(V)$

thus $\text{range}(F) \subseteq V$

3.

$$F(v_1) = F\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

$$F(v_2) = F\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$[F v_1]_B = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$[F v_2]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$[F]_B = \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}$$

$\det([F]_B) \neq 0$ thus F is invertible

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right) \xrightarrow[R_1 \to R_1 - R_2]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|cc} 3 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to \frac{R_1}{3}} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$[F^{-1}]_B = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$$

4. $$\left[F\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right]_B = [F]_B \cdot \left[\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right]_B$$

$$\left[\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right]_B = \begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\Rightarrow \left[F\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right]_B = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$F\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = 0\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + 1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Ex 2:

1) $A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -2 & -3-\lambda \end{vmatrix}$

$(1-\lambda)(-3-\lambda) + 4 = 0$

$-3 + 3\lambda - \lambda + \lambda^2 + 4 = 0$

$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0$

$Am(\lambda = -1) = 2$

solution for HLS

$(A + 1I) = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow x + y = 0 \Rightarrow x = -y$
$y = t$ $\Rightarrow \bar{V} = \left\{ \begin{pmatrix} -t \\ t \end{pmatrix}, t \in \mathbb{R} \right\}$

eigenVector

$= \text{span}\left\{ (-1, 1) \right\}$

$\dim(V_{\lambda=-1}) = 1 = Gm(\lambda = -1)$

the matrix is not diagonalizable as

$Gm(\lambda = -1) = 1 \neq Am(\lambda = -1) = 2$

2. $A = \begin{pmatrix} 2 & -4 & 3 \\ 1 & -2 & 1 \\ -4 & 0 & -6 \end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -4 & 3 \\ 1 & -2-\lambda & 1 \\ -4 & 0 & -6-\lambda \end{vmatrix}$$

$$-4\begin{vmatrix} -4 & 3 \\ -2-\lambda & 1 \end{vmatrix} + (-6-\lambda)\begin{vmatrix} 2-\lambda & -4 \\ 1 & -2-\lambda \end{vmatrix}$$

$$-4(-4+6+3\lambda) + (-6-\lambda)\cdot(\lambda^2 - 2^2 + 4)$$

$$-4(2+3\lambda) + \lambda^2(-6-\lambda)$$

$$= -8 - 12\lambda - 6\lambda^2 - \lambda^3 = -(\lambda+2)^3$$

$AM(\lambda = -2) = 3$

$$(A+2I) = \begin{pmatrix} 4 & -4 & 3 \\ 1 & 0 & 1 \\ -4 & 0 & -4 \end{pmatrix} \xrightarrow{R_3 \to 4R_2 + R_3} \xrightarrow{R_2 \to R_1 - 4R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 4 & -4 & 3 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$-4x_2 - x_3 = 0$

$-4x_2 = x_3$

$x_2 = -\frac{x_3}{4}$

$4x_1 - 4\left(\frac{-x_3}{4}\right) + 3x_3 = 0$

$4x_1 + 4x_3 = 0$

$x_1 = -x_3$

$x_3 = 1 \Rightarrow x_2 = \frac{-1}{4} \quad x_1 = -1$

$\text{span}\left\{\begin{pmatrix} -1 \\ -\frac{1}{4} \\ 1 \end{pmatrix}\right\} \quad Gm(\lambda = -2) = 1$

thus the matrix is not diagonalizable ($Gm \neq Am$)

3. $A = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \det|A| = 0$ thus $\lambda_1 = 0$

If $\lambda = 2$ the second column will be zero in $(A - 2I)$ thus $\det|A - 2I| = 0 \Rightarrow \lambda_2 = 2$

sum of $\lambda = 2 + 0 + \lambda_3 = 5 \Rightarrow \lambda_3 = 3$

thus A is diagonalizable as there are 3 real distinct eigenvalues

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$(A - 0I) = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to 3R_2 + R_1} \begin{pmatrix} 3 & 0 & 2 \\ 0 & 6 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\lambda_1 = 0$

$x_3 = t$

$x_2 = \frac{-8t}{6} = \frac{-4t}{3}$

$x_1 = \frac{-2t}{3}$

$$\Rightarrow V_{\lambda_1} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \end{pmatrix}$$

$\lambda_2 = 2$

$$(A - 2I) = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + \frac{R_2}{2}}$$

$$R_2 \to \frac{R_2}{4}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$x_2 = t$

$x_3 = 0$

$x + 2x_3 = 0 \Rightarrow x_1 = 0$

$$V = \left\{ \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\} \quad \text{basis} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\lambda_3 = 3$

$$(A - 3I) = \begin{pmatrix} 0 & 0 & 2 \\ -1 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{pmatrix}$$

$x_2 = t$

$x_3 = 0$

$-x_1 - t = 0 \Rightarrow x_1 = -t$

$$\Rightarrow V = \left\{ \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} \middle| \, t \in R \right\}$$

basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$\text{so } P = \begin{pmatrix} -\frac{2}{3} & 0 & -1 \\ -\frac{4}{3} & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Ex$_3$: $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}^6$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} =$$

$$(2-\lambda)(3-\lambda)-2 = 6-2\lambda-3\lambda+\lambda^2-2$$

$$= \lambda^2-5\lambda+4 = \lambda^2-\lambda-4\lambda+4$$

$$\lambda(\lambda-1)-4(\lambda-1) = (\lambda-4)(\lambda-1)$$

$\lambda_1 = 1 \quad \lambda_2 = 4$

$$(A-1\cdot I) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$x_2 = t \Rightarrow x_1 = -2t$

$= \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

$$(A-4I) = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_1 + 2R} \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$$

$\text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad P = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to 2R_2 + R_1}$$

$$\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 0 & 3 & 1 & 2 \end{array} \xrightarrow{R_1 \to 3R_1 - R_2} \left(\begin{array}{cc|cc} -6 & 0 & 2 & -2 \\ 0 & 3 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[R_2 \to \frac{R_2}{3}]{R_1 \to \frac{R_1}{-6}} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \end{array}\right)$$

$$P^{-1} = \frac{1}{3} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A^6 = P D^6 P^{-1} = \frac{1}{3} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4^6 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\frac{1}{3} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 4^6 & 2 \cdot 4^6 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 + 4^6 & 2 \cdot 4^6 - 2 \\ 4^6 - 1 & 2 \cdot 4^6 + 1 \end{pmatrix}$$

Ex 4:

1. $Av = \lambda v$

$(2A)v = 2(Av) = 2\lambda v$

2. $A^3 \cdot v = A \cdot A (A \cdot v) = A \cdot A \cdot \lambda v = \lambda \cdot A \cdot (A \cdot v)$

$= \lambda \cdot A \cdot \lambda v = \lambda^2 \cdot (A \cdot v) = \lambda^2 \cdot \lambda \cdot v = \lambda^3 \cdot v$

3. $Av = \lambda v \quad | \cdot A^{-1} \Rightarrow \quad Iv = A^{-1} \lambda v \quad | : \lambda$

$= \frac{v}{\lambda} = A^{-1} v$

Ex 5: As A is diagonalizable $\Leftrightarrow$

$D = P^{-1} A P \Rightarrow D^k = P^{-1} A^k P = 0$

$\Rightarrow$ all λ are zero as D is diagonal

so if D is zero $\Rightarrow$

$A = P D P^{-1} = 0$ (as P cannot be zero because it's invertible)

Ex 6:

1. $T(A) = A^t$

$T(\alpha A) = (\alpha A)^T = \alpha A^T = \alpha T(A)$

$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$

so it is L.T

2. $E = \left\{ \underset{v_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{v_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$

$T(v_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $T(v_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $T(v_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$T(v_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$[T(v_1)]_E = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$[T(v_2)]_E = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$[T(v_3)]_E = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$[T(v_4)]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

3.

$$A = \left(\quad \right) \xrightarrow{c_2 \leftrightarrow c_3} -\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -1$$

$\det(A) = -1$ thus A is nonsingular and invertible and so is T

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = A^{-1}$$

$$T^{-1}\left(T(A)\right) = A$$

$$\Rightarrow T^{-1}\left(A^T\right) = A \Rightarrow T^{-1}(A) = A^T$$

4. $$[T]_E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\cdot(1-\lambda)\left(\lambda^2-1\right) = -(1-\lambda)^3\cdot(1+\lambda)$$

$\lambda_1 = 1 \quad AM = 3$

$\lambda_2 = -1 \quad Am = 1 \Rightarrow Gm = 1$

$\lambda_1 = 1$

$$\left(A - 1\cdot I\right) = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \xrightarrow[2.\ R_2 \leftrightarrow R_3]{1.\ R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} x_1 = t \\ x_2 = s \\ x_3 = s \\ x_4 = w \end{matrix} = \left\{ \begin{pmatrix} t \\ s \\ s \\ w \end{pmatrix} \middle| \; t, s, w \in \mathbb{R} \right\}$$

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow Gm = 3$$

thus $G_m = A_m$ for all λ

$\Rightarrow [T]_E$ is diagonalizable

5. $[T]_B = P[T]_E P^{-1}$

let's continue to find P

$\lambda_2 = -1$

$$(A + I\lambda) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow[R_4 \to \frac{R_4}{2}]{R_1 \to \frac{R_1}{2}} R_3 \to R_3 - R_2$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} x_3 = t \\ x_1 = 0 \\ x_4 = 0 \\ x_2 = -t \end{matrix} \Rightarrow \operatorname{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

so P comprised from the eigenvectors

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$[T]_B = P[T]_E P^{-1}$$

we know that $[T]_B$ is diagonal matrix

with the eigenvalues of $[T]_E$ on the diagonal

and P comprise of the eigen vectors as its columns

but p is also the change of matrix whose columns

are coordinate vectors of B relative to basis e

and as $[v_B]_e = v_B$ it means that the eigen vectors

themselfs form the basis B:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

7.1. $A = A^t$ we use sylvester's criterion:

$k=1 \Rightarrow |1| = 1 > 0$

$k=2 \Rightarrow \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 2-1 = 1 > 0$

$k=3 \Rightarrow \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 0 + 0 + 1 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$

$= 0$ thus A is PSD (not PD as det for $k=3$ is 0)