Linear Algebra for MLDS - Homework 5

Coordinates, Rank, Linear Transformations, Matrices of L.T

Make sure to read and follow the "Homework Submission Instructions" file

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Exercise 1: Given $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

1. Prove that B is a basis for $\mathbb{R}^3$
2. Find the coordinate vector of $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ with repsect to the basis B.
3. Find a vector $v \in \mathbb{R}^3$ such that $[v]_B = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$
4. Find the coordinate vector $[w]_B$ for a general vector $w = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Exercise 2: For each of the following matices: $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 2 & 4 & -1 & 3 & 1 \end{pmatrix}$

Find bases for $\mathrm{Row}(A), \mathrm{Col}(A), \mathrm{Nul}(A)$, and $\mathrm{rank}(A), \mathrm{nullity}(A)$

Exercise 3: For each of the following, determine whether it is a linear transformation.

1. $F : \mathbb{R}^2 \to \mathbb{R}^3, F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ x \\ y \end{pmatrix}$
2. $F : \mathbb{R}^{2\times 2} \to \mathbb{R}, F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$
3. $F : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Exercise 4: For each the following linear tranformations: find $\dim(\mathrm{Range}(T)), \dim(\mathrm{Ker}(T))$ and determine whether T is invertible.

1. $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y + z \end{pmatrix}$
2. $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 2z \\ y + z \\ x + 2y \\ x + y - z \end{pmatrix}$

Exercise 5: Given $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ -x \end{pmatrix}$

1. Find the matrix of F with respect to B
2. Use that matrix to calculate $F \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$
3. Is the matrix singular? What does that say nabout F?

Exercise 6: Prove or disprove the following claims:

1. If $F : \mathbb{R}^3 \to \mathbb{R}^4$ is a one-to-one linear transformation, then $\dim(\text{Range}(F)) = 4$
2. If $F : \mathbb{R}^5 \to \mathbb{R}^3$ is a linear transformation and F is onto $\mathbb{R}^3$ then $\dim(\text{Ker}(F)) = 2$
3. There does not exist a linear transformation $F : \mathbb{R}^4 \to \mathbb{R}^3$ that is one-to-one.
4. If $F : \mathbb{R}^n \to \mathbb{R}^m$ is an invertible linear transformation then $n = m$
5. There does not exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\text{Ker}(T) = \text{Range}(T)$
6. Let T, S be two linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ such that $\text{Ker}(T) = \text{Ker}(S)$, then $\text{Range}(T) = \text{Range}(S)$