אלגברה לינארית, תרגיל 4

Ex1:

1) $\left\{ \begin{pmatrix} a \\ 2 \\ c \end{pmatrix} \middle| \, a+c=0 \right\} < \mathbb{R}^3$

let's assign $a=0$, $c=0$ $\Rightarrow \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

we cannot arrive to the zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

thus it's not a subspace.

2) $\left\{ A \in \mathbb{R}^{2\times 2} \,\middle|\, A \text{ is upper triangular} \right\} < \mathbb{R}^{2\times 2}$

first, check if zero vector contained

$U = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Rightarrow a,b,c=0 \Rightarrow U = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

now let's check addition closure:

$U = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \quad V = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$

$U+V = \begin{pmatrix} a+x & b+y \\ 0 & c+z \end{pmatrix} \Rightarrow$ still upper triangular matrix

and contained in the subspace

.2 Ex$_1$ (המשך)

let's check scalar multipilication closure

$$V = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \quad \alpha \in R$$

$$\alpha V = \begin{pmatrix} \alpha a & \alpha b \\ 0 & \alpha c \end{pmatrix}$$ still upper triangular matrix

$\Rightarrow$ the set satisfied all 3 conditions thus it is a supspace. it's dim=3 $\left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| a, b, c \in R \right\}$

3) $\{A \in R^{3\times3} \mid A \text{ is invertible}\} \subset R^{3\times3}$

let's check if zero vector contained in this set. No, as zero matrix is not invertible

so this is not a sub space

4) $\left\{ A \in R^{2\times2} \mid A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \subset R^{2\times2}$

let's a be $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} a+b=0 \\ c+d=0 \end{matrix} \Rightarrow \begin{matrix} b=-a \\ d=-c \end{matrix}$ so we can write A

like this $A = \begin{pmatrix} a & -a \\ c & -c \end{pmatrix}$

4 Ex. המשך

let's check if zero vector contained

$$A = \begin{pmatrix} a & -a \\ c & -c \end{pmatrix} \quad a = c = 0 \Rightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

let's check addition closure

$$U = \begin{pmatrix} a & -a \\ c & -c \end{pmatrix} \quad V = \begin{pmatrix} x & -x \\ y & -y \end{pmatrix} \quad U+V = \begin{pmatrix} a+x & -a-x \\ c+y & -c-y \end{pmatrix}$$

$$(U+V)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+x-a-x \\ c+y-c-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

let's check scalar multiplication closure:

$$U = \begin{pmatrix} a & -a \\ c & -c \end{pmatrix} \quad \alpha \in \mathbb{R} \quad \alpha U = \begin{pmatrix} \alpha a & -\alpha a \\ \alpha c & -\alpha c \end{pmatrix}$$

$$(\alpha U)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha a - \alpha a \\ \alpha c - \alpha c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ all 3 conditions are satisfied $\Rightarrow$ it is a subspace

$$\left\{ a\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, c\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \middle| a, c \in \mathbb{R} \right\}$$

dim = 2

Ex$_n$:

$\dim \mathbb{R}^3 = 3$

If the number of vector in S is less or more than 3, then S cannot be a basis for $\mathbb{R}^3$

In our case: S_1, S_2, S_5, S_6 cannot be a basis for $\mathbb{R}^3$. let's check S_4, S_3

S_3: let's check if S_3 span $\mathbb{R}^3$

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & \\ 3 & 0 & 2 & \\ 1 & 3 & 4 & \end{array}\right) \xrightarrow{R_3 \to R_3 + R_1} \left(\begin{array}{ccc|c} -1 & 2 & 1 & \\ 3 & 0 & 2 & \\ 0 & 5 & 5 & \end{array}\right) \xrightarrow{R_2 \to 3R_1 + R_2}$$

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & \\ 0 & 6 & 5 & \\ 0 & 5 & 5 & \end{array}\right) \xrightarrow[R_3 \to \frac{R_3}{5}]{R_2 \to \frac{R_2}{6}} \left(\begin{array}{ccc|c} -1 & 2 & 1 & \\ 0 & 1 & \frac{5}{6} & \\ 0 & 1 & 1 & \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} -1 & 2 & 1 & \\ 0 & 1 & \frac{5}{6} & \\ 0 & 0 & \frac{1}{6} & \end{array}\right)$$

We reached ref there is a solution so S_3 span $\mathbb{R}^3$. We can also see that the vectors in S_3 are linear independent (setting the "free" coefficients to zero). thus S_3 is a basis for $\mathbb{R}^3$

: Ex 2 1ª p.

let's check if s_4 is a basis for R^3

but we can observe that

$V_2 + V_5 = V_4$ that the vectors are linear

dependent so s_4 cannot be basis for R^3

2) let's find the dimension of the others:

s_1: span of s_1 is $\{c_1\bar{V}_1 + c_2\bar{V}_2 \mid c_1, c_2 \in R\}$

as the two vectors are linear independent the dimension

is 2

s_2: $\vec{V}_4 = 2\bar{V}_3$ thus the span of s_2

is $\{c_1\bar{V}_2 \mid c_1 \in R\}$ with dimension of 1

s_4: we saw that $V_2 + V_5 = V_4$ thus the

span of $s_4 = \{c_1\bar{V}_2 + c_2\bar{V}_5 \mid c_1, c_2 \in R\}$

so the dimension is 2

s_5: let's check linear dependency but

we know that $s_5 = s_3 + \bar{V}_5$ and s_3

span R^3 so $\bar{V}_5$ must be linear combination

of $\{V_1, V_2, V_3\}$ so the span of s_5 is:

: Ex₂ המשך

span S_5 : $\{c_1 v_1 + c_2 v_2 + c_3 v_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$

withe a dim = 3

S_6: We saw that $v_2 + v_5 = v_4$ so we can remove v_4. let's check L.i for the remaining three. $\{v_2, v_3, v_5\}$ but we can see that $2\bar{v}_3 - v_2 = \bar{v}_5$ so we can remove $\bar{v}_5$ as well so:

span S_6 = $\{c_1\bar{v}_2 + c_2\bar{v}_3 \mid c_1, c_2 \in \mathbb{R}\}$ with dimension of 2

Ex 3:

1, 2 (together)

$U = \{A \in V \mid A^t = A\}$ let A be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow \begin{matrix} b = c \\ a = a \\ d = d \end{matrix} \Rightarrow$ so U

represent matrix as $\left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \middle| a, b, d \in \mathbb{R} \right\}$

let's find its basis: $\left\{ a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| a, b, d \in \mathbb{R} \right\}$

$\Rightarrow$ span $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ are L.i so this is basis for $\bar{U}$

Conclusion: 1 $\bar{U}$ contains vector from V and is a span $\Rightarrow$ $\bar{U}$ is subspace of V

2 $\dim(U) = 3$

Let's do the same for W

$W = \{A \in V \mid A^t = -A\}$ let A be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \Rightarrow \begin{matrix} a = -a \Rightarrow a = 0 \\ c = -b \\ d = -d \Rightarrow d = 0 \end{matrix}$ thus:

סיום תרגיל Ex_3:

$$W = \left\{ \begin{matrix} 0 & b \\ -b & 0 \end{matrix} \;\middle|\; b \in R \right\}$$

$$= \text{span} \left\{ b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;\middle|\; b \in R \right\}$$

conclusion:

① W is a span of vectors in V $\Rightarrow$
W is subspace of V

② $\dim(W) = 1$

③ let's take $M \in V \cap W$

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ and should also be } \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} a = 0 \\ c = 0 \\ b = -b = b = 0 \end{matrix} \quad \Rightarrow V \cap W = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$\dim(V \cap W) = 0$ basis $= \emptyset$

$\dim(V + W) = \dim(V) + \dim(V) - \dim(V \cap W)$

$\Rightarrow \dim(V + W) = 4$
$\dim(V) = 4$

thus $V + W = V$

Ex 4:

1. $V = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a+b+c=0 \right\} \Rightarrow V = \left\{ \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix} \middle| a, b \in R \right\}$

$V = \left\{ a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \middle| a, b \in R \right\}$

basis of $V = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

$W = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a-b+2c=0 \right\} \Rightarrow W = \left\{ \begin{matrix} a \\ b \\ \frac{b-a}{2} \end{matrix} \middle| a, b \in R \right\}$

$W = \left\{ a \cdot \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, b \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \middle| a, b \in R \right\}$

basis of $W = \left\{ \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \right\}$

2. let take $M \in V \cap W$

$m = \begin{pmatrix} a \\ b \\ -b-a \end{pmatrix}$ and also $\begin{pmatrix} a \\ b \\ \frac{b-a}{2} \end{pmatrix} \Rightarrow -b-a = \frac{b-a}{2} \Rightarrow a = -3b$

$\Rightarrow M = \begin{pmatrix} -3b \\ b \\ 2b \end{pmatrix} = \operatorname{span} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \left\{ b \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \middle| b \in R \right\}$

basis of $V \cap W = \left\{ \begin{matrix} -3 \\ 1 \\ 2 \end{matrix} \right\}$

3. let's union the bases of V & W.

is spanning set fo V+W, let's remove vectors which are linear combination of the others

$$\left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \underset{v_2}{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}} \underset{v_3}{\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix}} \underset{v_4}{\begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}} \right\}$$ let's check l.i

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & -1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0.5 & 0.5 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0.5 & 1.5 \end{pmatrix}$$ we can see that $V_4 = 3V_3 - V_1 + V_2$

so the basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix} \right\}$

Exs:

1. $S = \{v_1 \dots v_k\}$ where $c_1\bar{v}_1 + \dots c_k\bar{v}_k = 0$

and $c_1 \dots c_k = 0$

So let t be $\{\bar{v}_1 \dots \bar{v}_n\}$ where $n \leq k$

$c_1\bar{v}_1 + \dots + c_n\bar{v}_n = 0$ will also imply $c_1 \dots c_n = 0$

otherwise there will be a solution to

$c_1 v_1 + \dots c_n\bar{v}_n + 0c_{n+1} \dots 0\bar{v}_k = 0$ where some c in $\{c_1, \dots, c_n\}$

that will not equal to zero contradicting the fact

that s is L.i

2. if $v_4 \in \{v_1, v_2, v_3\}$ it mean that it is linear

combination of the three vectors we showed

in class a theorem that: span$\{v_1, v_2, v_3, v_4\}$

= span$\{v_1, v_2, v_3\}$ Iff v_4 is linear combination

of v_1, v_2, v_3, which is our case.

3. scalar matrix is of the form:

$$A = \begin{pmatrix} a & 0 & \cdots & 0 \\ \vdots & a & & \\ \vdots & & a & \\ & & & a \\ 0 & & & a \end{pmatrix}$$ if we have another matrix $$B = \begin{pmatrix} b & \cdots & \cdots & 0 \\ \vdots & b & & \vdots \\ \vdots & & b & \vdots \\ \vdots & & b & \vdots \\ 0 & & & b \end{pmatrix}$$

so A can be express as $A = \frac{a}{b} B$ where $a, b \in \mathbb{R}$ and $b \neq 0$

4. let's prove by contradiction,

let's say that u, v are linear dependant, then there exists a and b, one of them not equal to zero s.t $au + bv = 0$

we can write $\bar{u}$ as: $\frac{u+v+u-v}{2}$

and $\bar{v}$ as: $\frac{u+v-(u-v)}{2}$

so $a\left(\frac{u+v+u-v}{2}\right) + b\left(\frac{u+v-(u-v)}{2}\right) = 0$

$\Rightarrow \frac{a+b}{2}(u+v) + \frac{a-b}{2}(u-v) = 0$

we know that $(u+v)$ and $(u-v)$ are linear intependent

so $\frac{a+b}{2} = \frac{a-b}{2} = 0 \Rightarrow \begin{matrix} a+b=0 \\ a-b=0 \end{matrix} \Rightarrow \begin{matrix} a=0 \\ b=0 \end{matrix}$

contradicting our initial assumtion. thus a and b must be zero $\Rightarrow$ $\bar{u}, \bar{v}$ are linear independent

.

$\leftarrow$ (end)

let's prove the other way:

If $\bar{u}$ and $\bar{v}$ are linear independent

let a and b be scalar s.t:

$$a(u-v) + b(u+v) = 0$$

we rewrite it a follows:

$$(a+b)u + (a-b)v = 0$$

so $a+b=0$ as $\bar{u}, \bar{v}$ L.i
$a-b=0$

$\Rightarrow a=b=0$ thus a and b must be zero

meaning $(u-v)$ and $(u+v)$ are linear independent

5. the basis of $V+W$ is the spaning set $V \cup W$ after removing vectors which are linear combination of the other

$V \cup W = \{v_1, v_2, v_3, v_4\}$ we know that v_2, v_3, v_4 are L.i and v_1, v_2 are L.i but ther is a chance that v_1 will be linear combination of $\{v_3, v_4\}$ and will need to be removed from the basis so the basis might have 4 or less vectors init thus $\dim(U+V) \leq 4$