2 REVIEW

Concepts and Vocabulary

1. Explain the meaning of each limit expression and illustrate with a sketch.
   (a) $\lim_{x\to a} f(x) = L$ (b) $\lim_{x\to a^+} f(x) = L$ (c) $\lim_{x\to a^-} f(x) = L$
   (d) $\lim_{x\to a} f(x) = \infty$ (e) $\lim_{x\to\infty} f(x) = L$
2. Describe several ways in which a limit can fail to exist. Illustrate each case with a sketch.
3. Give an example of a function such that $\lim_{x\to a} f(x) \neq f(a)$.
4. Explain the limit statement
$$\lim_{x\to a} f(x) = L$$
if and only if
$$\lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$
in your own words.
5. State the following Limit Laws.
   (a) Sum Law (b) Difference Law
   (c) Constant Multiple Law (d) Product Law
   (e) Quotient Law (f) Power Law
   (g) Root Law
6. Explain the Squeeze Theorem in your own words.
7. (a) What does it mean to say that the line $x = a$ is a vertical asymptote on the graph of $y = f(x)$? Carefully sketch an example of each possibility.
   (b) What does it mean to say that the line $y = L$ is a horizontal asymptote on the graph of $y = f(x)$? Carefully sketch an example of each possibility.
8. Consider the graph of each of the following functions. Which graphs have vertical asymptotes? Which have horizontal asymptotes?
   (a) $f(x) = x^4$ (b) $f(x) = \sin x$ (c) $f(x) = \tan x$
   (d) $f(x) = \tan^{-1} x$ (e) $f(x) = e^x$ (f) $f(x) = \ln x$
   (g) $f(x) = \dfrac{1}{x}$ (h) $f(x) = \sqrt{x}$
9. (a) What does it mean for f to be continuous at a?
   (b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?
10. (a) Give several examples of functions that are continuous on the interval $[-1, 1]$.
    (b) Give an example of a function that is not continuous on the interval $[0, 1]$.
11. Explain the Intermediate Value Theorem in your own words.

True–False Quiz

Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\lim_{x\to4}\left(\dfrac{2}{x-4} - \dfrac{8}{x-4}\right) = \lim_{x\to4}\dfrac{2x}{x-4} - \lim_{x\to4}\dfrac{8}{x-4}$
2. $\lim_{x\to1}\dfrac{x^2+6x-7}{x^2+5x-6} = \dfrac{\lim_{x\to1}(x^2+6x-7)}{\lim_{x\to1}(x^2+5x-6)}$
3. $\lim_{x\to1}\dfrac{x-3}{x^2+2x-4} = \dfrac{\lim_{x\to1}(x-3)}{\lim_{x\to1}(x^2+2x-4)}$
4. $\dfrac{x^2-9}{x-3} = x+3$
5. $\lim_{x\to3}\dfrac{x^2-9}{x-3} = \lim_{x\to3}(x+3)$
6. $\lim_{x\to4} 4|x-3| = \pm4$
7. $\lim_{x\to0}\dfrac{\sin x}{6x} = 0$
8. $\lim_{h\to0}\dfrac{4(x+h)-4x}{h} = -4$
9. $\lim_{x\to0}\dfrac{\sin 3x}{\sin 2x} = \dfrac{3}{2}$
10. If $\lim_{x\to5} f(x) = 2$ and $\lim_{x\to5} g(x) = 0$, then $\lim_{x\to5}\dfrac{f(x)}{g(x)}$ does not exist.
11. If $\lim_{x\to5} f(x) = 0$ and $\lim_{x\to5} g(x) = 0$, then $\lim_{x\to5}\dfrac{f(x)}{g(x)}$ does not exist.
12. If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}[f(x)+g(x)]$ does not exist.
13. If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}[f(x)+g(x)]$ does not exist.
14. If $\lim_{x\to6}[f(x)\,g(x)]$ exists, then the limit must be $f(6)\cdot g(6)$.
15. If p is a polynomial, then $\lim_{x\to b} p(x) = p(b)$.
16. If $\lim_{x\to0} f(x) = \infty$ and $\lim_{x\to0} g(x) = \infty$, then $\lim_{x\to0}[f(x)-g(x)] = 0$.
17. There can be two different horizontal asymptotes on the graph of a function.
18. If f has domain $[0, \infty)$ and the graph of f has no horizontal asymptote, then $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$.
19. If the line $x = 1$ is a vertical asymptote on the graph of a function f, then f is not defined at 1.

20. If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.

21. If f is continuous at 5 and $f(5) = 2$ and $f(4) = 3$, then $\lim_{x\to 2} f(4x^2 - 11) = 2$.

22. If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.

23. Let f be a function such that $\lim_{x\to 0} f(x) = 6$. Then there exists a positive number δ such that

$$\text{if} \quad 0 < |x| < \delta \qquad \text{then} \qquad |f(x) - 6| < 1$$

24. If $f(x) > 1$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

25. The equation $x^{10} - 10x^2 + 5 = 0$ has a solution in the interval $(0, 2)$.

26. If $f(x) = \dfrac{x^2 + 3x - 10}{x + 5}$, then for any real number r there exists a value c such that $f(c) = r$.

27. If f is continuous at a, then so is $|f|$.

28. If $|f|$ is continuous at a, then so is f.

29. Let $f(x) = \cos\left(\dfrac{x^2 + x}{x^3}\right)$.

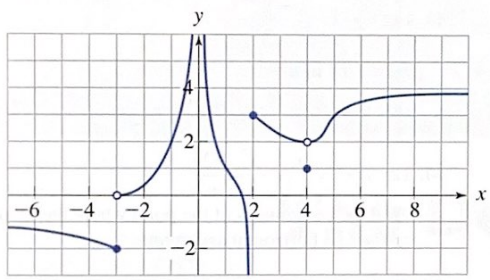
(a) $\lim_{x\to 0} f(x)$ exists.

(b) The line $y = 1$ is a horizontal asymptote on the graph of f.

(c) The line $x = 0$ is a vertical asymptote on the graph of f.

Exercises

1. The graph of f is shown in the figure.



(a) Find each limit, or explain why it does not exist.

(i) $\lim_{x\to 2^+} f(x)$ (ii) $\lim_{x\to -3^+} f(x)$ (iii) $\lim_{x\to -3} f(x)$

(iv) $\lim_{x\to 4} f(x)$ (v) $\lim_{x\to 0} f(x)$ (vi) $\lim_{x\to 2^-} f(x)$

(vii) $\lim_{x\to \infty} f(x)$ (viii) $\lim_{x\to -\infty} f(x)$

(b) Find the equation of each horizontal asymptote.

(c) Find the equation of each vertical asymptote.

(d) For what values is f discontinuous? Explain your reasoning.

2. Sketch the graph of a function that satisfies all of the following conditions.

$$\lim_{x\to -\infty} f(x) = -2, \quad \lim_{x\to \infty} f(x) = 0, \quad \lim_{x\to -3} f(x) = \infty,$$

$$\lim_{x\to 3^-} f(x) = -\infty, \quad \lim_{x\to 3^+} f(x) = 2,$$

f is continuous from the right at 3.

3. Find, if possible, a rational function $f(x)$ that satisfies all of the following conditions.

$$\lim_{x\to \infty} f(x) = 3, \quad \lim_{x\to -4} f(x) = \infty, \quad \lim_{x\to 2} f(x) = 3,$$

f is discontinuous at $x = 2$.

4. Find, if possible, a rational function $f(x)$ that satisfies all of the following conditions.

$$\lim_{x\to -3} |f(x)| = \infty, \quad \lim_{x\to \infty} f(x) = 2, \quad \lim_{x\to 1} f(x) = 2,$$

f is discontinuous at $x = 1$.

5. Find, if possible, a rational function $f(x)$ that satisfies all of the following conditions.

$$\lim_{x\to 3} f(x) = 10, \quad \lim_{x\to \infty} f(x) = 2, \quad \lim_{x\to -\infty} f(x) = 2,$$

$$f(0) = 2$$

6. Find a function $f(x)$ such that $\lim_{x\to 1} f(x) = \infty$ even though the limit corresponds to the indeterminate form $\dfrac{0}{0}$, or explain why this is not possible.

7–27 Find the limit.

7. $\lim_{x\to 1} e^{x^3 - x}$

8. $\lim_{x\to 3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

9. $\lim_{x\to -3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

10. $\lim_{x\to 1^+} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

11. $\lim_{h\to 0} \dfrac{(h - 1)^3 + 1}{h}$

12. $\lim_{t\to 2} \dfrac{t^2 - 4}{t^3 - 8}$

13. $\lim_{r\to 0} \dfrac{\sqrt{r}}{(r - 9)^2}$

14. $\lim_{v\to 4^+} \dfrac{4 - v}{|4 - v|}$

15. $\lim_{u\to 1} \dfrac{u^4 - 1}{u^3 + 5u^2 - 6u}$

16. $\lim_{x\to 3} \dfrac{\sqrt{x + 6} - x}{x^3 - 3x^2}$

17. $\lim_{x\to 0} \dfrac{\tan x}{x}$

18. $\lim_{x\to 1} \dfrac{\ln x}{x^2}$

19. $\lim_{x\to 0} \dfrac{|x|}{2|x|}$

20. $\lim_{x\to \infty} \dfrac{\sqrt{x^2 - 9}}{2x - 6}$

21. $\lim_{x\to -\infty} \dfrac{\sqrt{x^2 - 9}}{2x - 6}$

22. $\lim_{x\to \pi^-} \ln(\sin x)$

23. $\lim_{x\to -\infty} \dfrac{1 - 2x^2 - x^4}{5 + x - 3x^4}$

24. $\lim_{x\to \infty} \left(\sqrt{x^2 + 4x + 1} - x\right)$

25. $\lim_{x\to \infty} e^{x - x^2}$

26. $\lim_{x\to 0^+} \tan^{-1}\left(\dfrac{1}{x}\right)$

27. $\lim_{x\to 1} \left(\dfrac{1}{x - 1} + \dfrac{1}{x^2 - 3x + 2}\right)$

T **28–29** Use technology to conjecture the asymptotes on the graph of the function. Find each asymptote analytically.

28. $f(x) = \dfrac{\cos^2 x}{x^2}$

29. $g(x) = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$

30. If $2x - 1 \le f(x) \le x^2$ for $0 < x < 3$, find $\lim_{x \to 1} f(x)$.

31. Show that $\lim_{x \to 0} x^2 \cos\left(\dfrac{1}{x^2}\right) = 0$.

32. Find $\lim_{x \to \infty} \dfrac{10 \sin\left(\dfrac{4}{x}\right) - 6x^2 \cos\left(\dfrac{1}{x}\right)}{2x^2 + 3x + 2}$.

33–36 Prove the statement by using the precise definition of a limit.

33. $\lim_{x \to 2} (14 - 5x) = 4$

34. $\lim_{x \to 0} \sqrt[3]{x} = 0$

35. $\lim_{x \to 2} (x^2 - 3x) = -2$

36. $\lim_{x \to 4^+} \dfrac{2}{\sqrt{x - 4}} = \infty$

37. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \le x < 3 \\ (x - 3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit.

(i) $\lim_{x \to 0^+} f(x)$ **(ii)** $\lim_{x \to 0^-} f(x)$ **(iii)** $\lim_{x \to 0} f(x)$

(iv) $\lim_{x \to 3^-} f(x)$ **(v)** $\lim_{x \to 3^+} f(x)$ **(vi)** $\lim_{x \to 3} f(x)$

(b) Where is f discontinuous?

(c) Carefully sketch the graph of f.

38. Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \le x \le 2 \\ 2 - x & \text{if } 2 < x \le 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \ge 4 \end{cases}$$

(a) Determine whether g is continuous from the left, continuous from the right, or continuous at $x = 2$, 3, and 4.

(b) Carefully sketch the graph of g.

39–40 Find the domain and show that the function is continuous on its domain.

39. $h(x) = xe^{\sin x}$

40. $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$

41. The function

$$f(x) = \frac{\dfrac{6}{x} - 1}{x - 6}$$

is discontinuous at two different x-values. Find these two values. Explain why one of these discontinuities is removable and the other is not.

42. Find two different values of k such that the line $y = 10$ is a horizontal asymptote on the graph of

$$f(x) = \frac{kx + k \cdot 5^{-x}}{2x + 5^{-x}}$$

43. Let $f(x) = \dfrac{2 + 3^{1/x}}{7 + 3^{1/x}}$.

Find each limit.

(a) $\lim_{x \to \infty} f(x)$ **(b)** $\lim_{x \to 0^+} f(x)$ **(c)** $\lim_{x \to 0^-} f(x)$

44. Let $f(x) = \dfrac{x^3 - x^2 - 2x}{x^3 - 2x^2 + x}$.

Find the coordinates of the point where the graph of f crosses its horizontal asymptote.

45–46 Use the Intermediate Value Theorem to show that there is a solution to the equation in the given interval.

45. $x^5 - x^3 + 3x - 5 = 0$, $(1, 2)$

46. $\cos\sqrt{x} = e^x - 2$, $(0, 1)$

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True–False Quiz

1. False 3. True 5. True 7. False 9. True
11. False 13. True 15. True 17. True
19. False 21. True 23. True 25. True
27. True 29. (a) False (b) True (c) False

Exercises

1. (a) (i) 3 (ii) 0 (iii) DNE (iv) 2 (v) ∞ (vi) $-\infty$
(vii) 4 (viii) -1
(b) $y = -1, y = 4$ (c) $x = 0, x = 2$ (d) $x = -3, 0, 3, 4$

3. $f(x) = \dfrac{3(x^2+8)(x+2)}{x(x+4)(x+2)}$; answers may vary.

5. $f(x) = \dfrac{(2x^2-8)(x-3)}{(x-3)(x-2)^2}$; answers may vary.

7. 1 9. $\frac{3}{2}$ 11. 3 13. 0 15. $\frac{4}{7}$ 17. 1

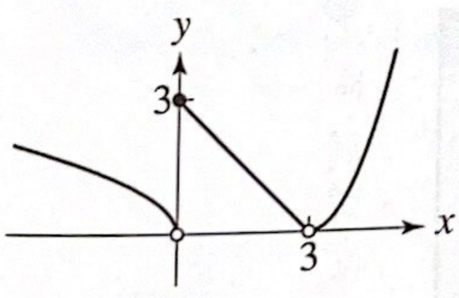
19. $\frac{1}{2}$ 21. $-\frac{1}{2}$ 23. $\frac{1}{3}$ 25. 0 27. -1

29. $y = -1, y = 1$ 33. Let $\delta = \dfrac{\varepsilon}{5}$.

35. Let $\delta = \min\left\{\dfrac{\varepsilon}{2}, 1\right\}$.

37. (a) (i) 3 (ii) 0 (iii) DNE (iv) 0 (v) 0 (vi) 0
(b) $x = 0, 3$

(c)



39. $\mathbb{R}$ 41. $x = 0$ (not removable) and $x = 6$ (removable)

43. (a) $\frac{3}{8}$ (b) 1 (c) $\frac{2}{7}$