1 Snapshot Model

$$x(t) = v(\theta)f(t) + n(t)$$

$$X = VF + N$$

$$X = X_s + N$$

- x - (Nx1) Received signal vector at time t
- v - (NxD) Matrix of Steering Vectors
- f(t) - (Dx1) Zero Mean Random Vector that contains desired and possibly undesired signals (D signals in total)
- n(t) - (Nx1) Complex Additive White Gaussian Noise

- X - (NxT) Full Signal as received by the Array
- V - (NxD) Matrix of Steering Vectors
- F - (DxT) Zero Mean Random Vector that contains desired and possibly undesired signals (D signals in total)
- N - (NxT) Complex Additive White Gaussian Noise

2 Bartlett Beamforming

$$y = w^{\mathsf{H}} X$$

$$\begin{aligned}
y &= v^{\mathsf{H}}(\theta_s) X \\
f = y &= v^{\mathsf{H}}(\theta_s) X \\
\mathcal{P} &= \frac{1}{T} \sum_{t=0}^{T-1} |v^{\mathsf{H}}(\theta_s) x(t)|^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{P} &= Var(f) \\
&= E[(f - \mu_f)^2] \\
&= E[f^2] && \text{zero mean signal} \\
&= E[|v^{\mathsf{H}}(\theta_s) X|^2] && \text{ideal Beamformer} \\
&= \frac{1}{T} \sum_{t=0}^{T-1} |v^{\mathsf{H}}(\theta_s) x(t)|^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{P} &= \frac{1}{T} \sum_{t=0}^{T-1} |v^{\mathsf{H}}(\theta) x(t)|^2 \\
&= \frac{1}{T} \sum_{t=0}^{T-1} (v^{\mathsf{H}} x(t))(v^{\mathsf{H}} x(t))^{\mathsf{H}} \\
&= \frac{1}{T} \sum_{t=0}^{T-1} v^{\mathsf{H}} x(t) x^{\mathsf{H}}(t) v_s \\
&= v^{\mathsf{H}} \left(\frac{1}{T} \sum_{t=0}^{T-1} x(t) x^{\mathsf{H}}(t) \right) v \\
&= v^{\mathsf{H}} E[X X^{\mathsf{H}}] v \\
\mathcal{P}_B &= v^{\mathsf{H}} R_{XX} v
\end{aligned}$$

3 Capon Beamformer

$$\begin{aligned}
\hat{y} &= w^{\mathsf{H}} X \\
\hat{y} &= f \\
v^{\mathsf{H}}(\theta_s) w &= 1 \\
f = y &= v^{\mathsf{H}}(\theta_s) X \\
\mathcal{P} &= \frac{1}{T} \sum_{t=0}^{T-1} |v^{\mathsf{H}}(\theta_s) x(t)|^2
\end{aligned}$$

$$\begin{aligned}
\hat{y} &= w^{\mathsf{H}} X \\
&= w^{\mathsf{H}} (X_s + N) \\
&= w^{\mathsf{H}} X_s + w^{\mathsf{H}} N \\
&= f + w^{\mathsf{H}} N && \text{Distortionaless Response} \\
\hat{y} &= f + Y_n
\end{aligned}$$

$$\begin{aligned}
Var(\hat{y}) &= E[|\hat{y}|^2] + E[\hat{y}]^2 \\
&= E[|\hat{y}|^2] + E[f + Y_n]^2 \\
&= E[|\hat{y}|^2] + (E[f] + E[Y_n])^2 && \text{both } f \text{ and } Y_n \text{ are zero mean} \\
&= E[|\hat{y}|^2] + 0 \\
&= E[|w^{\mathsf{H}} X|^2] \\
&= E[(w^{\mathsf{H}} X)(w^{\mathsf{H}} X)^{\mathsf{H}}] \\
&= E[w^{\mathsf{H}} X X^{\mathsf{H}} w] \\
&= w^{\mathsf{H}} E[X X^{\mathsf{H}}] w \\
&= w^{\mathsf{H}} \left(\frac{1}{T} \sum_{t=0}^{T-1} x(t) x(t)^{\mathsf{H}} \right) w \\
&= w^{\mathsf{H}} R_{XX} w
\end{aligned}$$

$$\begin{aligned}
\text{Minimize } & w^{\mathsf{H}} R_{XX} w \\
\text{subject to : } & w^{\mathsf{H}} v_s = 1
\end{aligned}$$

$$\begin{aligned}
&\text{first construct the objective function} \\
&\mathcal{L} = w^{\mathsf{H}} R_{XX} w + \text{Re}[\lambda (w^{\mathsf{H}} v_s - 1)]
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L} &= w^{\mathsf{H}} R_{XX} w + \mathrm{Re}[\lambda(w^{\mathsf{H}} v_s - 1)] \\
&= w^{\mathsf{H}} R_{XX} w + \frac{\lambda}{2}(w^{\mathsf{H}} v_s - 1) + \left[\frac{\lambda}{2}(w^{\mathsf{H}} v_s - 1)\right]^* \\
&= w^{\mathsf{H}} R_{XX} w + \frac{\lambda}{2}(w^{\mathsf{H}} v_s - 1) + \frac{\lambda}{2}^*(w^{\mathsf{T}} v_s^* - 1) \\
&= w^{\mathsf{H}} R_{XX} w + \frac{\lambda}{2}(w^{\mathsf{H}} v_s - 1) + \frac{\lambda}{2}^*(v_s^{\mathsf{H}} w - 1)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L} = w^{\mathsf{H}} R_{XX} w + \frac{\lambda}{2}(w^{\mathsf{H}} v_s - 1) &+ \frac{\lambda}{2}^*(v_s^{\mathsf{H}} w - 1) \\
\frac{\partial \mathcal{L}}{\partial w^{\mathsf{H}}} = R_{XX} w + \frac{\lambda}{2} v_s &= 0 \\
R_{XX} w &= -\frac{\lambda}{2} v_s \\
w &= -\frac{\lambda}{2} R_{XX}^{-1} v_s
\end{aligned}
$$

$$
\begin{aligned}
w^{\mathsf{H}} v_s = v_s^{\mathsf{H}} w &= 1 \\
v_s^{\mathsf{H}} w = -\frac{\lambda}{2} v_s^{\mathsf{H}} R_{XX}^{-1} v_s &= 1 \\
-\frac{\lambda}{2} &= (v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1}
\end{aligned}
$$

$$
w_{mvdr} = (v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1} R_{XX}^{-1} v_s
$$

OLD

$$
\begin{aligned}
w_{mvdr} &= (v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1} R_{XX}^{-1} v_s \\
\mathcal{P} &= \frac{1}{T} \sum_{t=0}^{T-1} |w^{\mathsf{H}} x(t)|^2 \\
&= \frac{1}{T} \sum_{t=0}^{T-1} |(v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1} v_s^{\mathsf{H}} R_{XX}^{-1} x(t)|^2 \\
&= \left|(v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1}\right|^2 (v_s^{\mathsf{H}} R_{XX}^{-1}) \frac{1}{T} \sum_{t=0}^{T-1} |x(t)|^2 (R_{XX}^{-1} v_s) \\
&= \left|(v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1}\right|^2 (v_s^{\mathsf{H}} R_{XX}^{-1}) R_{XX} (R_{XX}^{-1} v_s) \\
&= \left|(v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1}\right|^2 v_s^{\mathsf{H}} R_{XX}^{-1} v_s \\
\mathcal{P}_{mvdr} &= (v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1}
\end{aligned}
$$

NEW

$$
\begin{aligned}
w_{mvdr} &= (v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1} R_{XX}^{-1} v_s \\
\mathcal{P} &= \frac{1}{T} \sum_{t=0}^{T-1} |w^{\mathsf{H}} x(t)|^2 \\
&= \frac{1}{T} \sum_{t=0}^{T-1} |(v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1} v_s^{\mathsf{H}} R_{XX}^{-1} x(t)|^2 \\
&= \left|(v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1}\right|^2 \frac{1}{T} \sum_{t=0}^{T-1} |(v_s^{\mathsf{H}} R_{XX}^{-1} x(t)|^2 \\
&= \left|(v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1}\right|^2 (v_s^{\mathsf{H}} R_{XX}^{-1} R_{XX}|^2
\end{aligned}
$$

NEW

$$
\mathcal{P}_{mvdr} = (v_s^{\mathsf{H}} R_{XX}^{-1} v_s)^{-1}
$$

Forward Backward Averaging

$$
R_{XX,fb} = \frac{1}{2} \left(R_{XX} + J R_{XX}^{*} J \right)
$$

Diagonal Loading

$$
R_{XX,dl} = R_{XX} + \sigma_L I
$$