Olin College of Engineering
Linearity 1, 2017

Studio Problems
Due Wed. 22, at the end of class (10:40 am)

(1) Models

Differential equations are fundamental in engineering science. They arise as models whenever there is a rule which allows us to predict a short time into the future given complete knowledge of the present. To know the field of differential equations is to know how to predict the future.

Here are some examples:

- The temperature T of a body cools to the ambient temperature T_{amb} according to Newton's law of cooling $\dot{T} = hA(T_{amb} - T)$. The parameters h and A are the heat transfer coefficient and surface area of heat transfer, respectively.
- Newton's second law of motion is written $F = ma$. An ideal spring with rest length L_{rest} satisfies Hooke's law wherein the restoring force F when the spring has length L is proportional to the extension or compression from rest length. Recalling that acceleration is the second derivative of displacement, Newton's second law becomes the differential equation $k(L - L_{rest}) = m\ddot{L}$.

Write down a differential equation for each of the following physical situations. Identify any parameters in your equation.

(a) Radioactive decay of Uranium-238

(b) The motion of a pendulum

(c) Voltage across a capacitor in an RC circuit

(2) Checking A Guess

A **differential equation** is an equation involving an unknown function as well as its derivatives. A **solution** of a differential equation is a function which makes the differential equation balance.

Checking whether or not a given function is a solution of a differential equation is much easier than finding the function to begin with. For example, the function $x(t) = 3e^{4t} - \frac{1}{4}$ is a solution to the differential equation $\frac{dx}{dt} = 4x + 1$. You have not yet learned how to find this function. Having found it, to check that it is a solution requires that the left and right side of the equation balance:

$$\frac{dx}{dt} = \frac{d}{dt}(3e^{4t} - \frac{1}{4}) = 12e^{4t} \text{ and } 4x+1 = 4(3e^{4t} - \frac{1}{4}) + 1 = 12e^{4t} - 1 + 1 = 12e^{4t}, \text{ thus } \frac{dx}{dt} = 4x.$$

(a) Check that $x(t) = e^{4t}$ is a solution of the equation $\dot{x} = 4x$.

(b) Check that $x(t) = -\frac{1}{4}$ is a solution of the equation $\dot{x} = 4x + 1$.

(c) Check that $x(t) = 3e^{4t} - \frac{1}{4}$ is also a solution of the equation $\dot{x} = 4x + 1$.

(d) Check that $x(t) = \sin(2t)$ is a solution of the equation $\ddot{x} + 4x = 0$

(e) Check that $x(t) = 3\sin(2t) + 2\cos(2t)$ is also a solution of the equation $\ddot{x} + 4x = 0$.

(3) Making a Guess For each of the following differential equations, find the values of λ for which $x(t) = e^{\lambda t}$ is a solution.

(a) $\dot{x} + 6x = 0$

(b) $\ddot{x} - 9x = 0$

(c) $x''' - x' = 0$

(4) Homogeneous and Particular

(a) You have already checked that e^{4t} solves the **homogeneous** equation $\dot{x} = 4x$ and that $x_p(t) = -\frac{1}{4}$ is a particular solution to the inhomogeneous equation $\dot{x} = 4x + 1$. Check that $x(t) = ax_h(t) + x_p(t)$ solves the inhomogeneous equation $\frac{dx}{dt} = 4x + 1$.

(b) A differential equation, together with enough information at $t = 0$ to specify a unique solution is called an **initial value problem** (IVP). Show that $x(t) = x_0 e^{4t} + \frac{1}{4}(e^{4t} - 1)$ solves the initial value problem

$$\begin{cases} \frac{dx}{dt} = 4x + 1 \\ x(0) = x_0 \end{cases}$$

* *Optional* Show that every solution to $\dot{x} = 4x + 1$ is of the form $ax_h(t) + x_p(t)$.

(c) In this next sequence of problems we will study the differential equation

$$y'' + 4y = \sin(3x)$$

i. Check that $y_1(x) = \sin(2x)$ solves the **homogeneous** equation $y'' + 4y = 0$.

ii. Check that $y_2(x) = \cos(2x)$ also solves $y'' + 4y = 0$.

iii. Check that $a\cos(2x) + b\sin(2x)$ also solves $y'' + 4y = 0$ for any choice of a and b.

iv. Check that $y_p(x) = \frac{-1}{5}\sin(3x)$ is a particular solution to the **inhomogeneous** equation $y'' + 4y = \sin(3x)$.

v. Check that $y = a_1 y_1 + a_2 y_2 + y_p$ solves $y'' + 4y = \sin(3x)$.