Olin College of Engineering
Linearity 1, 2017

Studio 9 Problems
Due Wed. April 5, at the end of class (10:40 am)

Main Lessons from Studio 8

- The roots of the characteristic equation determine the exponents in the solution.
- If $\lambda = \alpha \pm i\omega$ are a pair of conjugate roots of the characteristic equation, the growth (or decay) rate is determined by the real part α while the frequency of oscillation is determined by the imaginary part ω.

(1) The following true statements are logically related. Explain *how* and *why*.

- Every second degree polynomial has two roots
- Every (linear, constant coefficient) second order scalar differential equation has a second degree characteristic polynomial.
- The homogeneous solution to a second order differential equation has two free parameters
- An initial position and velocity must be known to specify uniquely the solution to a second order differential equation.
- The nullspace of a second order differential operator is two dimensional

(2) Consider the inhomogeneous equation

$$\ddot{x} + 3\dot{x} + 4x = e^{it}$$

(a) Check that $x(t) = \frac{1}{3+3i}e^{it}$ is a particular solution. The coefficient $\frac{1}{3+3i}$ is called the **complex gain**.

(b) Check that

$$x(t) = ae^{-\frac{3}{2}t}\cos(\frac{\sqrt{7}}{2}t) + be^{-\frac{3}{2}t}\sin(\frac{\sqrt{7}}{2}t) + \frac{1}{3+3i}e^{it}$$

is the general solution. (*Hint:* Its easier if you make use of superposition and also if you rewrite the products of exponentials and sinusoids in terms of complex exponentials).

(c) The general solution above can be written as a **transient response** plus a **sustained oscillation**. Which part of the solution do each of these names refer to?

(3) You might be disturbed by the complex-valued force $e^{i\omega t}$. Consider now the related equation

$$\ddot{x} + 3\dot{x} + 4x = \sin(t)$$

with a real forcing function.

(a) Use a linear combination of the particular solution you checked above and its conjugate to find a particular solution to the inhomogeneous equation $\ddot{x} + 3\dot{x} + 4x = \sin(t)$.

(b) Your solution will be of the form $a\cos(t) + b\sin(t)$. Rewrite it as $A\sin(t+\phi)$.

(c) Relate the complex gain you found above to the parameters A and ϕ.

(4) Now vary the forcing frequency:

$$\ddot{x} + 3\dot{x} + 4x = e^{i\omega t}$$

(a) Seek a sustained oscillation of the form $x(t) = Ae^{i\phi}e^{i\omega t}$. Find A and ϕ as functions of ω.

(b) Produce a **Bode plot**, i.e. plot $\log A$ against $\log\omega$ and also ϕ against $\log\omega$

(5) Produce a Bode plot for the differential equation $x'''' + 3x'' + 2x' + 12x = e^{i\omega t}$. Which frequencies does this equation pass? which frequencies does it attenuate?

(6) Design problem: Find a differential equation which passes frequencies close to $\omega = 2$ and $\omega = 20$ while attenuating other frequencies.

(7) Analysis problem: Among second order damped and forced oscillators $\ddot{x} + 2Q\dot{x} + x = e^{i\omega t}$, for what values of Q is the maximum gain at $\omega = 0$?