Olin College of Engineering
Linearity 1, 2016

Studio 4 Problems

Due Wednesday, February 15, at the end of class (10:40 am)

Lessons developed in or extrapolated from Studio 3

- Row reduction matrices can be used to rewrite linear systems of equations in reduced echelon form.
- Eigenvectors turn matrix powers into scalar powers.
- When the same matrix is applied over and over again, eigenvectors form a very convenient coordinate system.
- Eigenvalues determine rates of decay.
- The eigenvector associated with the dominant eigenvalue determines long term behavior.
- Given a transition matrix in a Markov Chain,
  - the eigenvector with eigenvalue one is interpreted as a steady-state probability distribution.
  - the eigenvector with eigenvalue one has positive entries.
  - the other eigenvectors are interpreted as transient deviations from the steady-state distribution.
  - the other eigenvectors have entries which sum to zero.

(1) Review

Review the first three studio assignments. Complete any problems that you left incomplete. Summarize what you have learned so far this semester. Below are some definitions and explanations that might help you consolidate your knowledge.

* Definitions

A **linear combination** of vectors $\mathbf{v_1}, \ldots, \mathbf{v}_n$ is a vector which can be written as $c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$ for some scalar coefficients $c_1, \ldots c_n$.

The **span** of a collection of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v}_n\}$ is the set of all possible linear combinations of those vectors.

A collection of vectors is called **linearly independent** if the only way to write the zero vector as a linear combination of those vectors is by choosing every coefficient to be zero.

A **vector space** is a set of vectors which contains every linear combination of the vectors in the set (e.g. a plane).

A **basis** (for a vector space) is a collection of linearly independent vectors that span the vector space. Equivalently $\mathbf{v_1}, \ldots \mathbf{v_n}$ is a basis for $\mathbb{R}^n$ if for every vector $\mathbf{x}$ in $\mathbb{R}^n$ there is one and only one choice of coefficients $c_1, \ldots c_n$ so that $\mathbf{x} = c_1\mathbf{v}_1 + \ldots c_n\mathbf{v_n}$.

A **coordinate system** is another term for a basis.

Vectors are **orthogonal** if their dot product is zero.

When $A\mathbf{v} = \lambda\mathbf{v}$ for some vector $\mathbf{v}$ and scalar λ we say that $\mathbf{v}$ is an eigenvector and λ is an eigenvalue.

Solutions to an equation of the form $A\mathbf{x} = \mathbf{0}$ are called **homogeneous** solutions.

Any particular solution to the equation $A\mathbf{x} = \mathbf{b}$ is called a **particular** solution.

The **Nullspace** of a matrix A is the vector space of solutions to the homogeneous problem $A\mathbf{x} = \mathbf{0}$.

The **general solution** to an inhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is given by $c_1\mathbf{x}_1 + \dots c_m\mathbf{x}_m + \mathbf{x}_p$ where the vectors $\mathbf{x}_1, \dots \mathbf{x}_m$ form a basis for the nullspace of A and the vector $\mathbf{x}_p$ is a particular solution of the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$.

(a) Define the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

i. Are any of the following bases for $\mathbb{R}^4$: $\{\mathbf{v}_1, \dots \mathbf{v}_5\}$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1, \mathbf{v}_2\}$? *Why?*

ii. Which of the following are bases for $\mathbb{R}^4$: $\{\mathbf{v}_1, \dots \mathbf{v}_4\}$, $\{\mathbf{v}_2, \dots \mathbf{v}_5\}$, $\{\mathbf{v}_3, \dots \mathbf{v}_6\}$?

iii. The vector $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ can be written as a linear combination of the vectors $\mathbf{v}_1, \dots \mathbf{v}_6$:

$$\mathbf{w} = \frac{1}{4}\mathbf{v_1} - \frac{3}{4}\mathbf{v_2} + \mathbf{v_3} - \frac{1}{2}\mathbf{v_4}$$

Find all of the ways that $\mathbf{w}$ can be written as a linear combination of $\mathbf{v}_1, \dots \mathbf{v}_6$.

The vector of coefficients $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_6 \end{bmatrix}$ will be in the following (or an equivalent) form: $\mathbf{c} = k\mathbf{c}_h + \mathbf{c}_p$. Here (for an appropriate matrix A) the **homogeneous** solution $\mathbf{c}_h$ solves $A\mathbf{c}_h = \mathbf{0}$ and the **particular** solution $\mathbf{c}_p$ solves $A\mathbf{c}_p = \mathbf{b}$ for appropriate vector $\mathbf{b}$.

iv. What is the intersection of the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ and the plane spanned by $\mathbf{v}_3$ and $\mathbf{v}_4$?

(2) Eigenvalues, Eigenvectors and Nullspaces

(a) Define the **diagonal** matrix $D = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.1 \end{bmatrix}$.

i. Describe what D does geometrically and use this information to find its eigenvalues and eigenvectors.

ii. Solve the eigenvalue equation $D\mathbf{v} = \lambda\mathbf{v}$ algebraically for λ and $\mathbf{v}$. For most values of λ you will only have the zero solution for $\mathbf{v}$. The eigenvalues are those λ for which the equation has a **non-trivial** (different from the zero vector) solution.

iii. How many eigenvalues are there? *Why?*

iv. How many (linearly independent) eigenvectors are there? *Why?*

v. Can you read the eigenvalues and eigenvectors directly from D? How?

vi. The eigenvalues of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are the solutions to the quadratic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Find the eigenvalues of D by using this equation as a check on your earlier computation.

(b) Let $A = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 0.5 \end{bmatrix}$.

i. Solve the eigenvalue equation $A\mathbf{v} = \lambda\mathbf{v}$ for λ and $\mathbf{v}$.

ii. Solve the difference equation $\mathbf{x}_{n+1} = A\mathbf{x}_n$ with initial condition $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Make use of the eigenvalues and eigenvectors like you learned last week.

iii. Plot/sketch the points $\mathbf{x}_n$ for $n = 1, 2, \ldots$ on a set of coordinate axes.

iv. Give a qualitative description of the behavior of the points $\mathbf{x}_n$.

v. The difference equation you have solved is called a **fast-slow** system. *Why?*

(c) Recall the rotation matrix R_θ from previous work and use the D defined in part (a) to consider the difference equation $\mathbf{y}_{n+1} = R_{\pi/4} D R_{\pi/4}^{-1} \mathbf{y}_n$.

i. Using only your geometric understanding of D and $R_{\pi/4}$ give a quantitative description of the behavior of $\mathbf{y}_n$ as n increases.

ii. Fact or Fiction: $\mathbf{y}_n = R_{\pi/4}\mathbf{x}_n$.

iii. Fact or Fiction: A and D have the same eigenvalues.

iv. Fact or Fiction: The columns of $R_{\pi/4}$ are the eigenvectors of A.

v. We say that A and D are **similar**. Why?

vi. Explain the following diagram:

$$\begin{tikzcd}
\mathbf{x_0} \arrow[r, "D"] \arrow[d, "R_\theta"'] & \mathbf{x_1} \arrow[r, "D"] & \mathbf{x_2} \arrow[d, "R_{-\theta}"] \\
\mathbf{y_0} \arrow[r, "A"] & \mathbf{y_1} \arrow[r, "A"] & \mathbf{y_2}
\end{tikzcd}$$

(d) Consider the matrix $A_\varepsilon = \begin{bmatrix} 0.5 & 0.5-\epsilon \\ 0.5-\epsilon & 0.5 \end{bmatrix}$ and the difference equation $\mathbf{x}_{n+1} = A_\varepsilon \mathbf{x}_n$. Play with this equation enough so that if you were given an initial condition $\mathbf{x_0}$ and an ϵ between 0 and 1 you would be able to sketch the trajectory ($\mathbf{x}_n$ for $n = 1, 2, \ldots$). Describe the dependence on ε and $\mathbf{x}_0$ in words and pictures.

(e) Compare and contrast the solutions of $\mathbf{x}_{n+1} = A_\varepsilon \mathbf{x}_n$ with the solutions of $\mathbf{y}_{n+1} = D_\varepsilon \mathbf{y}_n$ with $D_\varepsilon = \begin{bmatrix} 1-\varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}$.

(f) Fact or Fiction: If λ is an eigenvalue of an $n \times n$ matrix A then
   i. the columns of $A - \lambda I$ are linearly dependent.
   ii. there is some non-zero vector that is orthogonal to each of the rows of $A - \lambda I$.
   iii. The matrix $A - \lambda I$ is not invertible.

(3) Determinants

Let A be an $n \times n$ matrix. The following are equivalent:

- The rows of A are linearly independent.
- The columns of A are linearly independent.
- The matrix A is invertible.
- The **nullspace** of A is **trivial**.
- The equation $A\mathbf{x} = \mathbf{b}$ has one solution $\mathbf{x}$ for each right hand side $\mathbf{b}$.
- After row reducing the matrix A, one or more zeros are **pivots** (lie on the main diagonal).

The formulas are simple for 2×2 matrices, but quite complicated for large matrices:

$$\begin{aligned}
\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= ad - bc \\
\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) &= (aei + bfg + cdh) - (afh + bdi + ceg) \\
\det(A) &= ((a_{11}a_{22}\ldots a_{nn}) + \ldots) - ((a_{12}a_{21}a_{33}a_{44}\ldots a_{nn}) + \ldots) + \ldots \\
& \text{ } n! - 2 \text{ terms not shown}
\end{aligned}$$

(a) Revisit question 3h from studio 1. Can you rephrase your answer in terms of the determinant? (If you did the optional extension of this question, look to see how close you got to defining the determinant).

(b) Recall the matrices

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

from studio 2, section 1. Compute the determinant of each of these matrices and check that it is consistent with the number of solutions to the equation $A\mathbf{x} = \mathbf{b}$ that you found in studio 2.

(c) Fact or fiction:

i. $\det(R_\theta) = 1$.

ii. $\det(R_\theta D R_{-\theta}) = \det(D)$ for any 2×2 matrix D.

iii. $\det(PDP^{-1}) = \det(D)$ for any $n \times n$ matrix D and invertible $n \times n$ matrix

iv. The determinant of the matrix $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$ is $e(ad - bc)$

(d) det is a degree n polynomial in the entries of an $n \times n$ matrix. Check that the nice properties below hold in each of the following cases

i. For illustrative examples that you pick.

ii. For general 2×2 matrices

iii. *optional* for general $n \times n$ matrices.

- $\det(AB) = \det(A)\det(B)$
- $\det(A) \neq 0$ if and only if A is invertible.
- if D is diagonal, $\det(D)$ is the product of the entries of D.
- $|\det(A)|$ is the volume (or area, depending on the dimension) of the parallelopiped (parallelogram in n dimensions) whose sides are the columns of A.

(4) Computing Eigenvalues

λ is an eigenvalue of A if and only if the matrix $A - \lambda I$ is not invertibe. The characteristic polynomial of a matrix A is defined as $\det(A - \lambda I)$. The roots of the characteristic polynomial are the eigenvalues of a matrix.

(a) Verify the claim from earlier in the studio that the characteristic polynomial of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\lambda^2 - (a+d)\lambda + (ad - bc)$.

(b) Find the characteristic polynomial of the matrix $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$.

(c) Find the characteristic polynomial of the matrix $\begin{bmatrix} a & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(5) Impulse Response

Consider the scalar difference equation

$$\begin{cases} x_{n+1} = \frac{1}{2}x_n & n > 0 \\ x_{n+1} = \frac{1}{2}x_n + 3 & n = 0 \\ x_0 = 0 \end{cases}$$

Solving this differential equation directly is not too hard. You first find $x_1 = 3$ and then $x_n = 3\left(\frac{1}{2}\right)^{n-1}$.

Similarly, consider

$$\begin{cases} x_{n+1} = \frac{1}{2}x_n & n \neq 4 \\ x_{n+1} = \frac{1}{2}x_n + 3 & n = 4 \\ x_0 = 0 \end{cases}$$

Again, we can solve the equation directly. Since $x_0 = 0$, x_n stays at zero until the inhomogeneous term is added. In other words $x_4 = 0$ and $x_5 = 3$. At this point, the x value is halved at every time step, so the solution is

$$x_n = \begin{cases} 0 & n < 5 \\ 3\left(\frac{1}{2}\right)^{n-5} & n \geq 5 \end{cases}$$

(a) Find a linear combination of the solutions above that solves

$$\begin{cases} x_{n+1} = \frac{1}{2}x_n & n \neq 0, 4 \\ x_{n+1} = \frac{1}{2}x_n + 5 & n = 0 \\ x_{n+1} = \frac{1}{2}x_n + 1 & n = 4 \\ x_0 = 0 \end{cases}$$

(b) Solve $x_{n+1} = \frac{1}{2}x_n + \alpha^n$.

* *optional* Solve $\mathbf{v}_{n+1} = A\mathbf{v}_n + \mathbf{b}_n$ where $A = R_\theta D R_{-\theta}$ and $\mathbf{b}_n = \begin{bmatrix} \alpha^n \\ 0 \end{bmatrix}$.