Olin College of Engineering
Linearity 1, 2017

Studio 8 Problems
Due Wed. March 29, at the end of class (10:40 am)

(1) Complex Multiplication
A complex number is a number of the form $z = a + ib$ where a and b are real numbers and $i^2 = -1$. (Electrical engineers often use j for the square root of -1 to distinguish it from the current).

The real number a is called the **real part** of the complex number z. The real number b is called the **imaginary part** of the complex number z. The distance between 0 and z in the complex plane is called the **magnitude** of z and the angle in the counterclockwise direction from the positive real axis is called the **phase** of the z. More concisely, for $z = a + ib$ we have the following:

$$a = \mathrm{Re}(z), \quad b = \mathrm{Im}(z), \quad |z| = (a^2 + b^2)^{\frac{1}{2}}, \quad \mathrm{phase}(z) = \arctan(\frac{b}{a})$$

Check that the following identities hold:

$$|z_1 z_2| = |z_1||z_2| \qquad \mathrm{phase}(z_1 z_2) = \mathrm{phase}(z_1) + \mathrm{phase}(z_2)$$

(2) Complex Conjugates
Given a complex number $z = a + ib$, its complex conjugate is

$$\bar{z} = a - ib$$

Give a geometric reason why each of the following is true:

(a) $|\bar{z}| = |z|$
(b) $\mathrm{phase}(\bar{z}) = -\mathrm{phase}(z)$
(c) $z + \bar{z}$ is real.
(d) $z - \bar{z}$ is purely imaginary.

(3) Complex Eigenvalues

(a) Show that $R_{\frac{\pi}{2}}$ cannot have any eigenvalues or eigenvectors in the geometric sense that we have studied. More concretely, there is no non-trivial vector $\mathbf{v}$ in $\mathbb{R}^2$ so that $R_{\frac{\pi}{2}}\mathbf{v}$ is a scalar multiple of $\mathbf{v}$.
(b) Show that $R^2_{\frac{\pi}{2}}$ has -1 as an eigenvalue
(c) Show that the eigenvalues of the matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $\cos\theta \pm i\sin\theta$.

(d) Explain why (geometrically) the identity $(R_{\frac{\pi}{n}})^n = -I$ holds.

(e) Parts (c) and (d) above imply the identity $(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n})^n = -1$. *Why?*.

(4) Complex Exponentials

Euler's formula is

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

The identities

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \text{ and } \sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

will be convenient later. Check that these identities hold. Give a *geometric* reason as well as an *algebraic* one.

(5) Characteristic Polynomials

The differential equation $3\ddot{x} + 4\dot{x} + 5x = 0$ has the **characteristic polynomial** $3\lambda^2 + 4\lambda + 5$. More generally, the differential equation $a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1\frac{dx}{dt} + a_0$ has characteristic polynomial $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$.

(a) The roots of the characteristic polynomial are the values of λ for which $x(t) = e^{\lambda t}$ is a solution. Explain *why* is this true?

Find the roots of the characteristic polynomial for each of the following differential equations

(b) $RC\dot{V} = V$

(c) $\ddot{x} + 2Q\dot{x} + x = 0$

(a) Explain why RC is called the **time constant** for the equation $RC\dot{V} = V$.

(b) Explain why Q is called the **damping ratio** for the equation $\ddot{x} + 2Q\dot{x} + x = 0$.

(6) The Phase Plane

Our goal is to get a graphical representation of a family of functions for different values of x_0 and v_0. Our first example with be $x(t) = x_0\cos(2t) + \frac{v_0}{2}\sin(2t)$. There are (at least) two ways to do this: (i) What you are used to is to plot $x(t)$ against t. You will have to deal with the unspecified parameters x_0 and v_0, e.g. by choosing some representative examples and producing a plot for each choice. (ii) Another way to represent these functions is with a phase plane: plot $\dot{x}(t)$ against $x(t)$.

For each of the following family of functions, do both (i) and (ii). For (ii) you can use the same set of axes for many $(x, \dot{x})$ plots.

(a) $x(t) = x_0\cos(2t) + \frac{v_0}{2}\sin(2t)$

(b) $x(t) = x_0e^{-t}\cos(2t) + \frac{v_0}{2}e^{-t}\sin(2t)$

(c) $x(t) = \frac{2}{3}x_0(e^t + \frac{1}{2}e^{-2t}) + \frac{1}{3}v_0(e^t - e^{-2t})$

(d) The functions that you plotted above are all solutions to differential equations of the form $\ddot{x} + b\dot{x} + kx = 0$. Find b and k. (You will need to do this three times, once for each item above).

You can plot $\dot{x}$ against x for many different parameter values on the same set of axes without making the figure hard to read. This is because the trajectories do not cross each other.

(e) Check that trajectories don't cross in the examples you studied above.

(f) *Why not?*.