Olin College of Engineering
Linearity 1, 2016

Studio 3 Problems

Due Wednesday, February 8, at the end of class (10:40 am)

Lessons developed in or extrapolated from Studio 2

- Matrix multiplication can be used to perform **elementary row operations**. In particular matrix multiplication can represent the technique of isolating variables from a coupled system of equations by adding the equations together (for example given $x + y = 2$ and $5x - 2y = 2$ we can multiply the first by two and add to get $7x = 4$)
- A **coordinate system** or equivalently a **basis** for a vector space is a collection of vectors that are both linearly independent and span the space. Equivalently, a set of vectors form a basis (for a space) if any other vector (in that space) can be written in one and only one way as a linear combination of the basis vectors
- **Orthogonal** coordinate systems are relatively easy to use. You can get coefficients by using **orthogonal projection**.
- The process of **changing basis** can be represented by matrix multiplication. One matrix P maps a vector $\mathbf{v}$ to the vector of coefficients needed to write it in terms of a new basis while another matrix Q maps a vector of coefficients to the corresponding linear combination of basis vectors.
- Many matrices have inverses. The inverse of a matrix A is written A^{-1} and it undoes whatever A does.
- Matrices transform spaces. The fundamental building blocks of these transformations are rotation, reflection, contraction, dilation and shear.
- Order matters.
- Once you know what a matrix does to the **basis vectors** (equivalently **coordinate axes**) you know what the matrix does to everything.
- A matrix A can be **conjugated** by an invertible matrix P to form PAP^{-1}. The conjugation PAP^{-1} is a representation of A in the basis given by the columns of P. The matrices A and PAP^{-1} are said to be **similar**.
- $(PAP^{-1})^n = PA^nP^{-1}$.

(1) Reduced Echelon Form For the past two weeks you have solved systems of linear equations as part of your work. You have found solutions by using methods from high school. With a large number of equations, the likelihood of arithmetic errors and general tedium is high. The point of this exercise is to develop/learn an algorithm that you could code on a computer, thus eliminating arithmetic errors and tedium.

(a) The system of equations below is in **reduced row echelon form** because it can be solved by starting with the last equation and working your way up without ever having to solve for one variable in terms of others or ever having to add equations. Solve the system of equations below by starting with the last equation and working your way up.

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 5 \\ x_2 + 2x_3 + 3x_4 = 4 \\ x_3 - 2x_4 = 8 \\ x_4 = 9 \end{cases}$$

(b) Given a system of linear equations, you can always rewrite it in reduced row echelon form by following the psuedocode below

```
n = Total Number of Equations
m = Total Number of Variables
For Variable_Number = 1:m
    For Equation_Number = Variable_Number:n
        Get rid of variable Variable_Number in equation Equation_Number by
        adding a multiple of equation Variable_Number to equation Equation_Number
    End
End
```

Use this procedure to solve the system of equations

$$\begin{aligned} x + y + z &= 1 \\ 2x + y + z &= 2 \\ 3x + y + 2z &= 4 \end{aligned}$$

* (Optional) Write code in your favorite programming language which solves a system of n linear equations in m unknowns. Test your code.

(2) Matrices act on matrices

(a) Find a 3×3 matrix Z so that for any 3×3 matrix A, the matrix ZA looks like A except the second and third rows have been swapped.

(b) Find a 3×3 matrix Y so that for any matrix A, the matrix YA is the same as A except the third row has been replaced by the third row minus seven times the first row.

(c) Now find a matrix X that reverses these steps, that is $XYA = A$ for any A.

(d) Fact or fiction?[1] If the third column of a matrix W is all zero and A is a matrix for which AW is defined, then the third column of AW is all zero.

(e) Fact or fiction? If the third row of a matrix W is all zero and AW is defined then the third column of AW is all zero.

(f) Consider the linear system of equations

$$\begin{bmatrix} -3 & 6 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

[1] If fact, explain. If fiction, give counterexample

Find a matrix that replaces the second row with 3 times the second row plus the first row. Multiply both sides of the equation (on the left) by this matrix. Your new system of equations should be in reduced row echelon form.

(g) Reformulate the pseudocode from the previous problem in terms of matrix multiplication.

* Play with the MATLAB command `rref`.

(3) STARTING OFF WITH LINEAR DIFFERENCE EQUATIONS

(a) Solve the difference equation $x_{n+1} = \frac{1}{2}x_n$. To solve a difference equation means to write the nth time step x_n in terms of the initial condition x_0.

(b) Solve the difference equation $\mathbf{x}_{n+1} = D\mathbf{x}_n$ with $D = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

(c) Solve the difference equation $\mathbf{y}_{n+1} = R_{\pi/3}DR_{-\pi/3}\mathbf{y}_n$ (it might be easiest to give a geometric description here).

(4) MARKOV CHAINS

Consider the following (not very sophiticated) money laundering scheme. An billionaire has money split between a cayman islands shell company, real estate and lukoil shares. All of the money in the shell company stays there. Every day, half of the money in real estate is transferred to the shell company while the other half of the money stays in real estate. Two thirds of the money in lukoil is transferred to the caymans while a third of that money stays in lukoil.

Initially half of the money is in lukoil and half of it is in real estate. In other words, the **initial condition** is

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}.$$

The **transition matrix** is given by

$$A = \begin{pmatrix} 1 & 1/2 & 2/3 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

The entry in the ith row and jth column indicates the proportion of money transferred in state j which is transerred to state i. The **state vector** after n steps is called $\mathbf{x}_n$. Its ith entry is the proportion of money in the ith state after n steps.

(a) The rule for determining $\mathbf{x}_{n+1}$ from $\mathbf{x}_n$ is given by matrix multiplication $\mathbf{x}_{n+1} = A\mathbf{x}_n$. *Why?*

(b) The first state in this model is called **absorbing**. *Why?*

(c) As $n \to \infty$ the vector $\mathbf{x}_n$ approaches the limiting vector $\mathbf{x}_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *Why?*

(d) The vector $\mathbf{x}_n$ can be computed "directly" from the initial condition by $\mathbf{x}_n = A^n\mathbf{x}_0$. *Why* is this equation true? *Why* is the word "directly" in quotes?

(e) A convenient basis for this problem is

$$\mathbf{x}_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Write the initial condition $\mathbf{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ as a linear combination $c_\infty \mathbf{x}_\infty + c_v \mathbf{v} + c_w \mathbf{w}$.

Note that the basis is not orthogonal. This makes it harder to find the coefficients c_∞, c_v and c_w because we can't use orthogonal projection. We will be compensated later for this extra effort now.

The vectors $\mathbf{x}_\infty$, $\mathbf{v}$ and $\mathbf{w}$ are called **eigenvectors** because the equations

$$A\mathbf{x}_\infty = \mathbf{x}_\infty, \quad A\mathbf{v} = \frac{1}{2}\mathbf{v}, \quad A\mathbf{w} = \frac{1}{3}\mathbf{w}$$

hold. The numbers 1, $\frac{1}{2}$ and $\frac{1}{3}$ are called **eigenvalues**. These vectors are special to the matrix A because matrix multiplication by A acts on them like scalar multiplication.

(f) Write $\mathbf{x}_n$ as a linear combination of $\mathbf{x}_\infty$, $\mathbf{v}$ and $\mathbf{w}$ for general n.

(5) DYNAMICS WITH GEOMETRY

(a) Last time you saw the matrices $D = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ which act by dilating horizontally while contracting vertically and by rotating through an angle of θ respectively. You also saw the matrix $R_\theta D R_{-\theta}$ which acts by dilating the line at angle θ to the horizontal while contracting the line at angle θ to the vertical.

i. Find (Wolfram is okay here, but you'll have to think it out eventually) the eigenvectors and eigenvalues of the matrix product $R_\theta D R_{-\theta}$. *Interpret* geometrically.

ii. Write the vector $\mathbf{x_0} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ as a linear combination of the eigenvectors of $R_\theta D R_{-\theta}$.

iii. Solve the difference equation $\mathbf{x}_{n+1} = R_\theta D R_{-\theta} \mathbf{x}_n$ with initial condtion $\mathbf{x_0} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) Construct a matrix that acts geometrically by dilating the line $y = 2x$ by a factor of 3 and contracting the line $y = -\frac{1}{2}x$ by a factor of 2. Study the corresponding dynamics.

(c) Construct a matrix that acts geometrically by contracting the $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ direction by a factor of two, contracting the $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ direction by a factor of 3 and dilating the $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ direction by a factor of 4.

(d) *Challenge:* Construct a matrix that acts geometrically by contracting the line $y+z-2x = 0$ by a factor of three while rotating the plane orthogonal to it by $\pi/3$. What are its eigenvalues and eigenvectors?

(6) PageRank

Consider the following toy model: The internet consists of the following webpages: (i) The sticky note page, (ii) the blue foam page, (iii) the polymer extrusion page, (iv) the swimming robots page and (v) the flying robots page. The structure of the links is as follows: (i) links to (ii), (iii), (iv) and (v), (ii) links to (i) and (iii), (iii) links to (i) and (ii), (iv) links to (v) and (v) links to (iv).

Suppose that you start on page (i) and follow links randomly (each link from a given page has the same probability). After a long time has passed, at which page are you most likely to be?

To do this, set up a difference equation of the form $\mathbf{x}_{n+1} = A\mathbf{x}_n$, use a computer to find the eigenvalues and eigenvectors and use the eigenvectors and eigenvalues to determine the solution.