Olin College of Engineering
Linearity 1, 2016

Studio 2 Problems

Due Wednesday, February 1, at the end of class (10:40 am)

Lessons developed in or extrapolated from Studio 1

- Vectors are geometric objects: they have a magnitude, there is an angle between two vectors and you can project one vector onto another vector.
- Matrix multiplication can be defined in two equivalent ways: linear combinations of the columns or projection onto the rows.
- Systems of linear equations can be written as $A\mathbf{x} = \mathbf{b}$.
- Redundant equations (i.e. **linearly dependent rows**) can be removed from the matrix A without changing the solutions.
- The system of linear equations $A\mathbf{x} = \mathbf{b}$ either
  - has exactly one solution for every $\mathbf{b}$ or,
  - depending on the choice of $\mathbf{b}$ has either
    - zero solutions or
    - infinitely many solutions.

  In mathematical circles this is called the **Fredholm Alternative**.

(1) Particular and homogeneous

A system of equations of the form $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \neq \mathbf{0}$ is called **inhomogeneous**. A system of equations of the form $A\mathbf{x} = \mathbf{0}$ is called **homogeneous**. The equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ are closely related as you will see.

In this exercise we will use the matrices

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & 2 \end{bmatrix} \text{ and } E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

(a) Last week you solved the equation $B \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ (Studio 1, #3b) as well as several other systems of equations involving the matrices B, C and D (the matrix E is new on this studio).

If you solved for x in terms of y you got $\begin{bmatrix} 3 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with y any real number.

If you solved for y in terms of x you got $\begin{bmatrix} 0 \\ 3 \end{bmatrix} + x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with x any real number.

Make sense of this statement:

> In both cases, the general solution can be written as the sum of a **particular** solution to the inhomogeneous equation and the general solution to the **homogeneous** equation.

(b) Solve the homogeneous equation $C \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and relate its solutions to the solutions of the inhomogeneous equation $C \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ you found in studio 1.

(c) Solve the homogeneous equation $D \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and relate its solutions to the solutions of the inhomogeneous equation $D \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$ that you found in studio 1.

(d) Solve the homogeneous equation $E \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$ and the inhomogeneous equation $E \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{b}$ for general $\mathbf{b}$.

(e) Make sense of the following statement, referring to the network flow exploration as appropriate:

> The solution to a homogeneous network flow problem is spanned by its closed loops.

(2) Matrices Are Transformations, Part I: Geometry in the plane

The first part of this section is about the matrices

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) Play with the matrix R and describe what it is doing geometrically. In this context the directive: "play with R" means "choose any vector $\mathbf{v}$ in $\mathbb{R}^2$, plot $\mathbf{v}$ and $\mathbf{Rv}$ on the same set of coordinate axes and compare them, then repeat with a new $\mathbf{v}$ and loop until you can explain what R does geometrically."

(b) Play with the matrix L and describe what it is doing geometrically.

(c) Check that your answers above include the words in this footnote[1]

(d) The expression R^2 means "do R twice." Describe what each of the following matrices is doing geometrically:

$$\text{(i) } R^2, \text{ (ii) } R^3, \text{ (iii) } R^4, \text{ (iv) } L^2, \text{ (v) } LRL, \text{ (vi) } RLR$$

(e) The expression $R^{1/2}$ means "do this twice and you'll get R." Not every matrix has a square root.

i. Does R have zero, one or two square roots?
ii. Describe what each of them does geometrically.
iii. Write down $R^{1/2}$ as a matrix (if you can!)
iv. Repeat (i)-(iii) above for L.

[1] `rotation, reflection`

(f) Define $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

i. What does R_θ do geometrically?

ii. Do the matrix multiplication for R_θ^2. Where have you seen the entries in this matrix before?

iii. What is $R_\theta R_{-\theta}$?

iv. True or False (if false, salvage; if true, *why?*): $R_\theta \mathbf{v} \cdot R_\theta \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ for any vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$.

v. True or False (if false, salvage; if true, *why?*): $R_\theta \mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot R_\theta \mathbf{w}$

vi. Solve $R_\theta \mathbf{x} = \mathbf{b}$ for an unknown $\mathbf{x}$ given a known $\mathbf{b}$.

(g) Describe how the matrix $D = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ acts geometrically.

(h) Describe how the matrix $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ acts geometrically.

(3) Matrices Are Changes of Coordinates

Use the matrices L, R_θ, D and S defined above. Give a geometric description of each of the following matrix products:

(a) $LR_\theta L$, (b) $R_{-\theta}DR_\theta$, (c) $R_\theta SR_{-\theta}$(c) D^5, (d) $(R_{-\theta}DR_\theta)^5$, (e) $R_\theta SDR_{-\theta}$, (f) $(R_\theta SDR_{-\theta})^5$

(4) Change of Coordinates

(a) We will use the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

that you used to learn the definitions of magnitude, angle and projection.

i. Last week you wrote $\mathbf{v}$ as a linear combination of $\mathbf{u}$ and $\mathbf{w}$. Given an arbitrary vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ find coefficients c_1 and c_2 (depending on α and β of course) so that

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = c_1\mathbf{u} + c_2\mathbf{w}.$$

You can do this by solving a system of linear equations, but it gives more geometric intuition if you use projection.

ii. Write down a matrix Q so that $Q\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Interpret the columns of Q.

iii. Write down a matrix P so that $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Interpret the rows of P.

iv. The matrices P and Q are called **change of basis** or **change of coordinates** matrices. *Why?*

v. What is the matrix product PQ? *Why?*

(b) In this problem you will think about a coordinate system in $\mathbb{R}^3$. We will use the vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

i. Find scalars k_a, k_b and k_c so that $\mathbf{d} = k_a\mathbf{a} + k_b\mathbf{b} + k_c\mathbf{c}$.

ii. Fact or fiction: $|\mathbf{d}|^2 = k_a^2|\mathbf{a}|^2 + k_b^2|\mathbf{b}|^2 + k_c^2|\mathbf{c}|^2$. *Why?*

iii. Find a 3×3 matrix Q that maps a vector of coefficients $\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ to the corresponding linear combination $k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c}$.

iv. Find a 3×3 matrix P that maps a vector $\mathbf{u}$ to the vector of coefficients $\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ needed to represent it as a linear combination $\mathbf{u} = k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c}$.