Olin College of Engineering
Linearity 1, 2017

Studio 6 Problems

Due Wednesday, March 1, at the end of class (10:40 am)

Lessons developed in or extrapolated from Studio 5

- Eigenvalues and eigenvectors determine the long-time behavior of solutions to the difference equation $\mathbf{x}_{n+1} = A\mathbf{x}_n$.
- Finding eigenvalues is equivalent to determining whether or not a collection of vectors is linearly dependent.
- The eigenvalues of A are those values of λ for which the matrix-vector equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has a non-zero solution $\mathbf{v}$.
- Equivalently, the eigenvalues of A are those values of λ for which the matrix $A - \lambda I$ has linearly dependent columns.
- The **determinant** is a gadget which tells you whether or not a (square) matrix has linearly dependent columns.
- The **characteristic polynomial** of a matrix A is the polynomial $p(\lambda) = \det(A - \lambda I)$. Its roots are the eigenvalues of the matrix.

(1) Transpose

Given an $n \times m$ matrix A its **transpose**, written A^T is the $m \times n$ matrix obtained by swapping the rows and columns. For example

$$\begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 0 & 4 \end{bmatrix}$$

(a) In the true statements below, $\mathbf{v}$ and $\mathbf{w}$ are $n \times 1$ column vectors, B is an $m \times n$ matrix and let A is an $n \times m$ matrix. Verify these statements for specific $\mathbf{v}$, $\mathbf{w}$, A and B. *Challenge:* verify them for general n and m.

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w} = \mathbf{w}^T\mathbf{v}$
- $\mathbf{v}^T(AB)\mathbf{w} = \mathbf{w}^T(AB)^T\mathbf{v}$
- $(AB)^T = B^TA^T$

(2) Matrices as Images

Thus far we have concerned ourselves with how matrices act on vectors. We will now think of matrices as nouns rather than as verbs (in programming jargon we think of matrices now as objects rather than as functions).

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

and the corresponding matrices $\mathbf{v}_j\mathbf{v}_j^T$. For example

$$\mathbf{v}_3\mathbf{v}_3^T = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

A convenient way to think of matrices as objects is by letting them encode images. We let 1 correspond to a black pixel, -1 correspond to a white pixel and numbers between -1 and 1 correspond to pixels with appropriate grayscale.

A very symmetric image, such as the one corresponding to the matrix

$$A = \frac{1}{16}\begin{bmatrix} 7 & 3 & -13 & 7 \\ 3 & 7 & 7 & -13 \\ -13 & 7 & 7 & 3 \\ 7 & -13 & 3 & 7 \end{bmatrix}$$

can always be written as a linear combination of the matrices $\mathbf{v}_1\mathbf{v}_1^T, \mathbf{v}_2\mathbf{v}_2^T, \mathbf{v}_3\mathbf{v}_3^T$ and $\mathbf{v}_4\mathbf{v}_4^T$. In this case

$$A = \frac{1}{16}\mathbf{v}_1\mathbf{v}_1^T - \frac{1}{4}\mathbf{v}_2\mathbf{v}_2^T + \frac{1}{4}\mathbf{v}_3\mathbf{v}_3^T + \frac{3}{8}\mathbf{v}_4\mathbf{v}_4^T$$

In some sense, we are thinking of the matrices $\mathbf{v}_j\mathbf{v}_j^T$ as **coordinates** for the matrix A. By writing A as a linear combination of coarser and finer checkerboards, we are extracting the **frequency components** of the matrix A. By looking at the coefficients, we can see that the pattern $\mathbf{v}_4\mathbf{v}_4^T$ is dominant while the contribution of the pattern $\mathbf{v}_1\mathbf{v}_1^T$ is very small. From the perspective of data compression, we could get away with approximating A by $-\frac{1}{4}\mathbf{v}_2\mathbf{v}_2^T - \frac{1}{4}\mathbf{v}_3\mathbf{v}_3^T + \frac{3}{8}\mathbf{v}_4\mathbf{v}_4^T$.

We'd like to address the following questions:

- Given a matrix A, how do we know whether or not A can be written as a linear combination of the $\mathbf{v}_j\mathbf{v}_j^T$?
- How do we find the coefficients?
- If A can't be written as a linear combination of the $\mathbf{v}_j\mathbf{v}_j^T$, can we find some other good coordinates $\mathbf{w}_j\mathbf{w}_j^T$? If so, how?

(a) Check that $\mathbf{v}_j$ is an eigenvector of A.

(b) Check that the coefficient c_j multiplying $\mathbf{v}_j\mathbf{v}_j^T$ is given by the eigenvalue of A corresponding to the eigenvector $\mathbf{v}_j$ divided by $|\mathbf{v}_j|^2$.

(c) Write the matrix $\begin{bmatrix} 0.75 & 0.75 & 0.25 & 0.25 \\ 0.75 & 0.75 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.75 & 0.75 \\ 0.25 & 0.25 & 0.75 & 0.75 \end{bmatrix}$ as a linear combination of the $\mathbf{v}_j\mathbf{v}_j^T$.

(d) Show that the matrix $B = \begin{bmatrix} 0.5 & 0.4 & 0 & 0 \\ 0.4 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.1 \\ 0 & 0 & 0.9 & 0.5 \end{bmatrix}$ cannot be written as a linear combination of the $\mathbf{v}_j\mathbf{v}_j^T$ given at the beginning of this problem.

(e) Find the eigenvectors $\mathbf{w}_j$ of the matrix B and use them to write B as a linear combination of the matrices $\mathbf{w}_j\mathbf{w}_j^T$

(f) Describe the method outlined above in your own words. For it to work, it is crucial that the eigenvectors be orthogonal. *Why*?

(g) When the eigenvectors are not orthogonal this method breaks, but it can be fixed by doing **singular value decomposition**. Do the readings on singular value decomposition.

(3) ORTHOGONAL MATRICES Let $\mathbf{v}_1, \ldots \mathbf{v}_n$ be mutually orthogonal ($\mathbf{v}_j \cdot \mathbf{v}_k = 0$ for $j \neq k$) unit vectors ($\mathbf{v}_j \cdot \mathbf{v}_j = 1$).

The following mathematical statements are all different ways to say the same thing.

- $$\mathbf{w} = c_1\mathbf{v}_1 + \ldots c_n\mathbf{v}_n \text{ with } c_j = \mathbf{v}_j \cdot \mathbf{w}.$$

- $$\begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{w} \text{ has solution } \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \ldots & \mathbf{v_1}^T & \ldots \\ & \vdots & \\ \ldots & \mathbf{v}_n^T & \ldots \end{bmatrix} \mathbf{w}$$

- $$\begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix}^{-1} = \begin{bmatrix} \ldots & \mathbf{v_1}^T & \ldots \\ & \vdots & \\ \ldots & \mathbf{v}_n^T & \ldots \end{bmatrix}$$

(a) Use the information above to find the inverse of the matrix

$$\frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

(b) Use the information above to find the inverse of the matrix

$$\frac{1}{5}\begin{bmatrix} 0 & 0 & 3 & 4 \\ 0 & 0 & -4 & 3 \\ 4 & 3 & 0 & 0 \\ 4 & -3 & 0 & 0 \end{bmatrix}$$

(c) *What* are the statements above saying and *why* are they equivalent?