Olin College of Engineering
Linearity 1, 2016

Studio 1 Problems
Due Wednesday, January 27, at the end of class (10:40 am)

(1) Vector Arithmetic

Use the vectors and scalars below for all problems.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 1.5 \\ 1 \\ 2 \end{bmatrix}, \quad a = 4, \quad b = -2$$

For each expression below, compute if possible. If not possible, say why not.

(a) $a\mathbf{y}$
(b) $\mathbf{y} + \mathbf{x}$
(c) $\mathbf{x} \cdot \mathbf{y}$
(d) $\mathbf{x} \cdot \mathbf{z}$
(e) $\mathbf{x} \cdot a$
(f) $a\mathbf{x} + b\mathbf{z}$

(2) Matrices act on vectors

Part I: Algebra of linear combinations and dot products

Below are three more vectors used in the remaining problems.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

(a) Compute the vector

$$\begin{bmatrix} \mathbf{u} \cdot \mathbf{y} \\ \mathbf{v} \cdot \mathbf{y} \\ \mathbf{w} \cdot \mathbf{y} \end{bmatrix}$$

and check that it the same as $a\mathbf{x} + b\mathbf{z}$ which you computed above. *Why* are they the same? Write down the reason as precisely as you can.
What you have computed (twice) is the **matrix multiplication**

$$\begin{bmatrix} 1 & 1.5 \\ 1 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -16 \end{bmatrix}$$

(b) Find a 4×4 matrix that will swap the second and fourth entry of any vector. More concretely find a matrix A so that

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \end{bmatrix}$$

(c) When the vector on the left hand side contains unknowns instead of numbers the result of a matrix multiplication is a **system of linear equations**. Write down the system of linear equations that corresponds to the matrix multiplication

$$\begin{bmatrix} -3 & 6 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(d) Write down the matrix-vector equation that corresponds to the system of linear equations below. You are not (yet) being asked to solve these equations for x_1, x_2, etc.

$$\begin{aligned} -3x_1 + 6x_2 - x_3 + x_4 - 7x_5 &= 0, \\ x_1 - 2x_2 + 2x_3 + 3x_4 - x_5 &= 0, \\ 2x_1 - 4x_2 + 5x_3 + 8x_4 - 4x_5 &= 0. \end{aligned}$$

(3) Systems of Equations Solve each of the following systems of linear equations. You may use methods from high school algebra. These exercises may appear boring. The point of the question is to think about how many solutions a system of equations has.

(a) $$\begin{aligned} x + y &= 3 \\ 2x + y &= 5 \end{aligned}$$

(b) $$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$

(c) $$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 5 \end{aligned}$$

(d) $$\begin{aligned} x + y + z &= 3 \\ y + z &= 2 \\ x + z &= 1 \end{aligned}$$

(e) $$\begin{aligned} 2x + y + z &= 4 \\ x - y + z &= 1 \\ x - y - z &= -1 \end{aligned}$$

(f) $$\begin{aligned} 2x + y + z &= 4 \\ x - y + z &= 1 \\ 3x + 2z &= 5 \end{aligned}$$

(g) $$\begin{aligned} 2x + y + z &= 4 \\ x - y + z &= 1 \\ 3x + 2z &= 6 \end{aligned}$$

Now for the interesting part

(h) Consider a system of linear equations $\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta \end{aligned}$ with constants $a, b, c, d, \alpha, \beta$ to be solved for variables x and y. What conditions on the constants guarantee that there is one solution? zero solutions? infinitely many solutions?

* (Optional exploration) What about three equations with three unknowns? If you can do that, try n equations with m unknowns?

(4) Vector Geometry

The equations below define the **magnitude** of a vector(written $|\mathbf{q}|$), the **angle** between two vectors (written $\theta(\mathbf{v},\mathbf{u})$) and the **projection** of one vector $\mathbf{u}$ onto another vector $\mathbf{v}$ (written $\mathrm{Proj}_{\mathbf{v}}\mathbf{u}$):

$$|\mathbf{q}|^2 = \mathbf{q}\cdot\mathbf{q}$$

and

$$\mathbf{v}\cdot\mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos\theta(\mathbf{v},\mathbf{u})$$

and

$$\mathrm{Proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{v}}{|\mathbf{v}|}\cdot\mathbf{u}\right)\frac{\mathbf{v}}{|\mathbf{v}|}$$

For the exercises below use the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

that you saw earlier in this studio.

(a) Compute $|\mathbf{v}|^2$, $|\mathbf{u}|^2$ and $|\mathbf{w}|^2$.
(b) Compute $\theta(\mathbf{v},\mathbf{u})$, $\theta(\mathbf{v},\mathbf{w})$ and $\theta(\mathbf{u},\mathbf{w})$.
(c) Compute $\mathrm{Proj}_{\mathbf{u}}\mathbf{v}$ and $\mathrm{Proj}_{\mathbf{u}}\mathbf{w}$.
(d) *Why* are these quantities called magnitude, angle and projection? Sketch a figure to illustrate your answer.

(5) Coordinate Systems

A coordinate system is a way to represent vectors. When solving a problem, the choice of coordinate system matters — not in terms of the answer — but in terms of how hard it is to get there. We will use the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

that you used to learn the definitions of magnitude, angle and projection.

(a) In this problem you will think about the **coordinate system** $\{\mathbf{u},\mathbf{w}\}$. Note that $\theta(\mathbf{u},\mathbf{w}) = \pi/2$
   i. Write $\mathbf{v}$ as a **linear combination** of $\mathbf{u}$ and $\mathbf{w}$. In other words: find scalars c_1 and c_2 so that $\mathbf{v} = c_1\mathbf{u} + c_2\mathbf{w}$.
   ii. Check that $|\mathbf{v}|^2 = c_1^2|\mathbf{u}|^2 + c_2^2|\mathbf{w}|^2$. *Why* is this?
   iii. Check that $c_1 = \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|^2}$ and $c_2 = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}$. *Interpret* this in the context of projection.

(b) In this problem you will think about the **coordinate system** $\{\mathbf{v},\mathbf{w}\}$. Note that $\theta(\mathbf{v},\mathbf{w}) \neq \pi/2$.
   i. Find scalars k_1 and k_2 so that $\mathbf{u} = k_1\mathbf{v} + k_2\mathbf{w}$.
   ii. Check that $|\mathbf{u}|^2 \neq k_1^2|\mathbf{v}|^2 + k_2^2|\mathbf{w}|^2$. Check also that k_1 and k_2 can not be obtained by formulas analgous to those above for c_1 and c_2. *Why* is this?