Olin College of Engineering
Linearity 1, 2016

Studio 5 Problems

Due Wednesday, February 22 at the end of class (10:40 am)

Lessons developed in or extrapolated from Studio 4

- Eigenvalues and eigenvectors determine the long-time behavior of solutions to the difference equation $\mathbf{x}_{n+1} = A\mathbf{x}_n$.
- Finding eigenvalues is equivalent to determining whether or not a collection of vectors is linearly dependent.
- The eigenvalues of A are those values of λ for which the matrix-vector equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has a non-zero solution $\mathbf{v}$.
- Equivalently, the eigenvalues of A are those values of λ for which the matrix $A - \lambda I$ has linearly dependent columns.
- Each square matrix has a characteristic polynomial. Its roots are the eigenvalues of the matrix.

(1) Row Reduced Form Find all solutions to the equation $A\mathbf{x} = \mathbf{b}$ for the following matrices A and vectors $\mathbf{b}$. Each of the matrices A is in row-reduced echelon form.

(a) $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \mathbf{0}$.

(b) $A = \begin{bmatrix} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \mathbf{0}$.

(c) $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Which of the systems above is **underdetermined**? Which is **overdetermined**? Why are these words used.

(2) Finding Eigenvalues

(a) Find the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

(b) Find the eigenvalues of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

(c) The matrix $D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ 0 & \dots & d_{n-1} & 0 \\ 0 & 0 & \dots & d_n \end{bmatrix}$ is **diagonal**. Find the eigenvalues and corresponding eigenvectors of D by solving the equation $(D - \lambda I)\mathbf{v} = \mathbf{0}$.

(d) The matrix $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix}$ is **block diagonal**. Find the eigenvalues and eigenvectors A by solving the equation $(A - \lambda I)v = 0$ via row reduction.

(e) What is the relationship between the eigenvalues of a block diagonal matrix and the eigenvalues of its diagonal blocks? *Why* does this relationship hold?

(3) Interpretation of eigenvectors

(a) A disgruntled electorate cannot decide which branch of government to blame for its woes. The blame shifts over time among the judiciary, the legislature and the executive. The vector $\mathbf{x}_n = \begin{bmatrix} j_n \\ l_n \\ e_n \end{bmatrix}$ quantifies the proportion of blame given to each branch. For example, if $\mathbf{x}_n = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, then $\frac{1}{4}$ of the blame goes to the judiciary while $\frac{1}{3}$ of the blame goes to the legislature and $\frac{5}{12}$ of the blame is goes to the executive. The vector $\mathbf{x}_n$ is updated by the rule $\mathbf{x}_{n+1} = A\mathbf{x}_n$ via the transition matrix $A = \begin{bmatrix} \frac{5}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{24} & \frac{21}{24} & \frac{13}{24} \end{bmatrix}$. Your teaching team has thoughtfully provided you with the eigenvectors and eigenvalues:

$$A\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad A\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

In the long run, how much blame does the executive endure?

(b) A tree grows by the following branching process: At every time step, each branch produces a twig. After one time step a twig matures into a branch. This process is encoded in the difference equation

$$\begin{bmatrix} b_{n+1} \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_n \\ t_n \end{bmatrix}$$

The eigenvalues and eigenvectors are

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{\phi} \end{bmatrix} = \phi \begin{bmatrix} 1 \\ \frac{1}{\phi} \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ \phi \end{bmatrix} = \frac{-1}{\phi} \begin{bmatrix} -1 \\ \phi \end{bmatrix}$$

where $\phi = \frac{1}{2}\left(\sqrt{5}+1\right)$ is the **golden ratio**.
In the long run, what is the ratio of branches to twigs?

(4) Phase plane
Consider the branching process described above:

$$\begin{bmatrix} b_{n+1} \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_n \\ t_n \end{bmatrix}$$

Pick an initial condition (b_0, t_0) and plot the **trajectory** (b_n, t_n) for $n = 1, \ldots 50$. Repeat for another initial condition. You should use a computer (and choose your scale appropriately). What is meant by the term **dominant eigenvector**? What is the dominant eigenvector for this system?

(5) Change of Basis
Recall the disgruntled electorate difference equation $\mathbf{x}_{n+1} = A\mathbf{x}_n$ with

$$A = \begin{bmatrix} \frac{5}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{24} & \frac{21}{24} & \frac{13}{24} \end{bmatrix}$$

Solving this equation with a given initial condition $\mathbf{x}_0$ requires three steps

- The **analysis** step in which the initial condition $\mathbf{x}_0$ is broken down into eigenvectors by finding coefficients c_1, c_2 and c_3 so that $\mathbf{x}_0 = c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.
- The **modification** step in which the individual components (eigenvectors) are scaled by the eigenvalues. After n time steps the $c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ component gets scaled to $c_2 \left(\frac{1}{2}\right)^n \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, etc.
- The **synthesis** step in which x_n is synthesized by taking a linear combination of eigenvectors with coefficients found in the analysis step and then scaled in the modification step.

Each of these steps is given by a matrix multiplication. Find the synthesis matrix P, the modification matrix D and the analysis matrix P^{-1}.