Case Study III: Finite Difference Solutions of Linear Boundary Value Problems
MTH 3150: Numerical Methods and Scientific Computing
Franklin W. Olin College of Engineering

Overview

We have already explored using the shooting method to solve boundary value ordinary differential equations (BVODEs). We will now explore another class of numerical solution techniques called finite difference (FD) methods. FD methods are a class of methods that translate the bounded continuum problem to a set of simultaneous algebraic equations based on a nodalization of the domain. In other words, the continuum, which contains an infinite number of points, is represented by a finite set of discrete points separated in space. The translation of calculus to algebra is based on numerical differentiation – replacing the differential operators by algebraic approximations.

This case study has two parts: i) solving the BVODE from Case Study II using the FD method and ii) applying FD method to a problem in radial coordinates with a variety of boundary conditions. Both parts of this case study involve the solution of second order ODEs in which derivative terms are represented by central differences. Such formulations often give rise to tridiagonal matrices that need to be inverted in order to find the nodal solution values. Numerical algorithms especially designed to invert tridiagonal matrices are more computationally efficient than general matrix solvers.

Part 1 - Finite Difference Solution of Case Study II BVODE

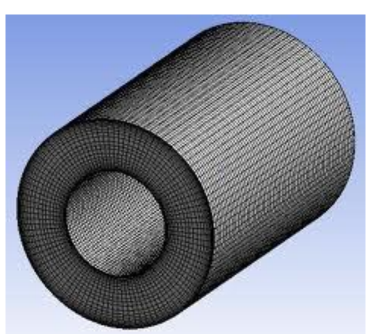
Solve the following equation with the specified boundary conditions using a FD method.

$$v'' + 2xv' - x^2 v = x^2 \qquad v(0) = 1 \qquad v(1) = 0 \tag{1}$$

Before solving this particular problem, develop a general tridiagonal solver (sometimes called the Thomas algorithm) that you will use when solving the various problems in this case study. When solving this equation, present your solution for at least two different nodalizations using equally spaced nodes in both cases. Compare your answers to your results from Case Study II.

Part 2 – All Problems Are Not Cartesian and All Boundary Conditions Are Not Dirichlet

A common problem in heat transfer is the heating or cooling of an annular tube. The tube may have an internal distributed heat source (e.g. electrically heated furnace, nuclear fuel rod, etc.). Consider the following cross-section.



The differential equation that needs to be solved to compute the temperature (T) distribution in this geometry is

$$\frac{d}{dr} r \frac{dT}{dr} = -Sr \tag{2}$$

Let's fix the temperature at the inner radius to 0 → $T(r_{in}) = 0$ and set S=1000, r_{in}=0.5, and r_{out}=1.0.

Up until now we have only solved problems with fixed boundary values, like the boundary condition at the inner radius of this problem. This is called a Dirichlet boundary condition or boundary of the "first kind". In this problem we will explore three types of boundary conditions at the outer radius location:

a) Fixed value (or Dirichlet) $\quad T(r_{out}) = 0 \quad$ (3)

b) Fixed gradient (or Neumann) $\quad \left.\frac{dT}{dr}\right|_{r_{out}} = 0 \quad$ (4)

c) Mixed boundary (or Robin) $\quad \left.\frac{dT}{dr}\right|_{r_{out}} = -bT(r_{out}) \qquad b = 1 \quad$ (5)

In this part of the case study, you should:

i) Develop a finite difference solution for this problem using 20 equally spaced (radial) nodes including appropriate treatment of boundary conditions.
ii) Solve the problem using the three conditions. Validate your code against the analytical solution to the fixed value boundary condition case.
iii) Using the mixed boundary condition case, find the value of b that yields the temperature at the outer radius of 10, i.e. $T(r_{out}) = 10$.

<u>Objectives</u>:
The objectives of this case study are:

- Become familiar with the finite difference method for the solution of BVODEs
- Understand the relation between matrix structure and finite difference formulation
- Develop a tridiagonal matrix inversion solver
- Gain experience using non-Cartesian geometries
- Implement different kinds of boundary conditions
- Explore the relationship between accuracy and efficiency as it relates to mesh density.

Report:

Prepare a report in which you:

- review the finite difference method as a solution method for BVODEs,
- describe your implementation strategies for the both parts of the case study, including mathematical basis, textual explanations and code snippets,
- present your computational results with figures and analysis, and
- investigate the literature and report briefly on i) solution algorithms that involve iterative solutions (versus inverting a matrix directly) and ii) implications of using non-uniform grids, and
- cite references as appropriate.