# *Case Study V: Finite Element Methods for Boundary Value Problems*

*MTH 3150: Numerical Methods and Scientific Computing*

*Franklin W. Olin College of Engineering*

Figure 1: Boris Galerkin played an important role in the development of finite element methods.

## *Overview*

IN THIS CASE STUDY we begin to examine finite element methods for solving boundary value problems. While there are a number of ways to think about finite element methods, we will consider an implementation that often goes by the name of Galerkin, although this refers to a more general approach to solving a range of problems. Needless to say, there is disagreement in the literature over this, so don't be surprised if you find yourself confused when you start reading more broadly. We want you to develop a finite element method in order to solve a single boundary-value problem for which you can also compute an exact solution. After you validate your algorithm, we want you to study the accuracy of the method as a function of the number of elements. We also want you to read (but not implement) about methods of generating a mesh, rather than simply choosing an equally-spaced grid.

## *Galerkin's Method*

In order to introduce the method, let's consider the linear two-point boundary value problem,

$$v_{xx} + p(x)v_x + q(x)v = f(x) \; 0 \leq x \leq 1,$$

with boundary conditions

$$v(0) = 0, \; v(1) = 0$$

Let's look for an approximate solution of the form

$$v(x) = \sum_{i=1}^{I} c_i \phi_i(x),$$

where the basis functions $\phi_i(x)$ satisfy the boundary conditions

$$\phi_i(0) = \phi_i(1) = 0, \; i = 1, 2, \ldots, I$$

There are lots of choices for basis functions. For example, we could choose globally-defined functions like the trigonometric polynomials

$\sin(i\pi x)$ or the algebraic polynomials $x^i(1-x)$. However, for reasons that will become clear later, we will instead choose locally-defined functions such as the *hat* function, which can be defined piecewise as

$$\phi_i(x) = \begin{pmatrix} \frac{(x-x_{i-1})}{(x_i-x_{i-1})} & x_{i-1} \leq x \leq x_i, \\ -\frac{(x-x_{i+1})}{(x_{i+1}-x_i)} & x_i \leq x \leq x_{i+1}, \\ 0 & x < x_{i-1}, x > x_{i+1}. \end{pmatrix}$$

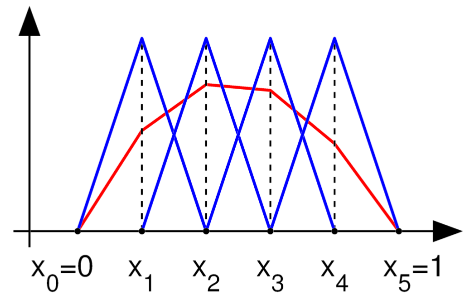

Figure 2: An approximation built from a set of hat functions

Irrespective of the choice of basis functions, Galerkin's method proceeds as follows. If the approximation was exact, the residual function

$$r(x) = v_{xx} + p(x)v_x + q(x)v - f(x) \; 0 \leq x \leq 1,$$

would be precisely zero. The idea behind Galerkin's method is that the approximation is chosen so that the residual function is orthogonal to each and every basis function,

$$\int_0^1 \left(v_{xx} + p(x)v_x + q(x)v - f(x)\right) \phi_i(x)\, dx = 0, i = 1, 2, \ldots, I.$$

Substituting in the guess and interchanging summation and integration leads to

$$\sum_{j=1}^{I} c_j \int_0^1 \left(\phi_j''(x) + p(x)\phi_j'(x) + q(x)\phi_j(x) - f(x)\right) \phi_i(x)\, dx = 0, i = 1, \ldots, I$$

This is just a system of linear equation $A\mathbf{c} = \mathbf{f}$ in the unknown coefficients $c_i$. The entries in the matrix are

$$a_{ij} = \int_0^1 \left(\phi_j''(x) + p(x)\phi_j'(x) + q(x)\phi_j(x)\right) \phi_i(x)\, dx$$

and the right hand side is

$$f_i = \int_0^1 f(x)\phi_i(x)\, dx, i = 1, \ldots, I$$

The terms in the matrix can be simplified by integrating by parts to give

$$a_{ij} = -\int_0^1 \phi_i'(x)\phi_j'(x)\, dx + \int_0^1 p(x)\phi_i(x)\phi_j'(x)\, dx + \int_0^1 q(x)\phi_i(x)\phi_j(x)\, dx$$

Although this may look horribly complicated, the right choice of basis functions results in a tri-diagonal matrix whose entries are pretty straight-forward to compute.

## *Objectives*

THIS CASE STUDY has several goals: we want you to use the *hat* basis functions and evaluate as many of these the terms by hand as possible; we want you to write a general implementation for a general linear boundary value problem; we want you to implement this method on the heat conduction problem from case study 3 with Dirichlet boundary conditions; and we want you to examine the accuracy of this method as a function of number of basis functions. Furthermore, since there is no need to define the basis functions at equally-spaced nodes, we want you to read the literature on methods of laying down nodes and meshes. Finally, we want you to write a report in which you put these various pieces together in a professionally-written document and in which you demonstrate your understanding of the methods and their context. In particular, we want you to

- Develop a set of functions in MATLAB in order to solve a general linear boundary value problem using a basis of hat functions.
- Validate your algorithm on the heat conduction problem from case study 3 subject to the Dirichlet boundary conditions, and explore the accuracy of your algorithm by examining the error as a function of the number of basis functions.

## *Report*

Prepare a brief typewritten report of roughly 4 pages in length in which you review the methods used, your implementation using code snippets as evidence, and your results. At a minimum your report should include:

- a section which briefly discusses the methods used, and places them in the broader context of methods for solving boundary value problems[1].
- a section which briefly discusses your implementation of the methods in general, using small pieces of code as evidence.
- a section which briefly discusses your validation of your algorithm on the heat conduction problem.
- a section which briefly discusses the results of your simulations, including an examination of the accuracy of the method.
- a section which briefly summarizes your findings, including questions you have about this case study and any reflection you care to provide.

[1] This means you need to **read** about methods for solving boundary value problems from a variety of sources, i.e. not just Wikipedia.