Case Study IV: Multivariable Root Finding and Non-Linear Boundary Value Problems
MTH 3150: Numerical Methods and Scientific Computing
Franklin W. Olin College of Engineering

Overview

We investigated the iterative solution of single non-linear equations in Case Study II and the finite difference solution of second order boundary value problems in Case Study III. This case study builds upon those two sets of experiences to allow us to tackle more complex problems involving several, mulitivariable simultaneous non-linear equations. Such sets of equations occur commonly in physical problems involving non-linear phenomena (e.g. circuit analysis with non-linear resistances) or evolve from numerical solution formulations (e.g. finite difference) to non-linear differential equations.

Part 1 – Multivariable Root Finding

Suppose we needed to find a particular set of roots for the following set of simultaneous equations.

$$\begin{aligned} x^2 + y^2 + z^2 &= 9 \\ xyz &= 1 \\ x + y - z^2 &= 0 \end{aligned}$$

Note that we could rewrite the above system in the form

$$\begin{bmatrix} x^2+y^2+z^2-9 \\ xyz-1 \\ x+y-z^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \underline{f}(\underline{x}) = \underline{0} \text{ where } \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We are going to develop two solution methods for this type of problem in which we have N equations in N unknowns (N=3) in the above case.

The two methods we are going to investigate are a form of successive substitution called a Picard (No, he was not the captain of the Enterprise on Star Trek.) iteration and Newton's Method for a non-linear system. The first part of this case study is to implement both solution algorithms in modular MATLAB scripts that can be easily generalized to equations of N variables and use your algorithms to find a set of roots for the above system. Explore issues of convergence behavior and computational efficiency.

Part 2 – Root Finding and Non-Linear BVODE Problems

Consider the following BVODE

$$U_{xx} = 3U + 10U^3 + x^2,\ U(0) = U(1) = 0.$$

Is this ODE linear or non-linear and why?

The second part of this case study is to develop a finite difference solution for this problem using second-order central differencing. This will lead to a set of simultaneous equations requiring numerical solution. Find the numerical solution to this problem employing both a Picard iteration approach and a Newton's Method approach.

Objectives:

The objectives of this case study are:

- Become familiar with root finding techniques for multivariable problems
- Develop MATLAB code for the general solution of N non-linear equations in N-unknowns suing either the Picard iteration or Newton's method
- Explore convergence behavior of iterative solution methods
- Appreciate the mathematical nature of linear versus non-linear BVODEs
- Combine multivariable root finding and finite difference methods to solve non-linear BVODEs.

Report:

Prepare a report in which you:

- review the Picard iteration and Newton's solution of simultaneous, multivariable equations,
- describe your implementation strategies for the both parts of the case study, including mathematical basis, textual explanations and code snippets,
- present your computational results with figures and analysis; in Part 1 report on convergence behavior and in Part II report on the changes in the solution accuracy and computation time as you increase the number of nodes,
- investigate the literature and report briefly on i) methods of solution particularly appropriate to simultaneous polynomials and ii) techniques fir accelerating solution convergence of iterative techniques, and
- cite references as appropriate.