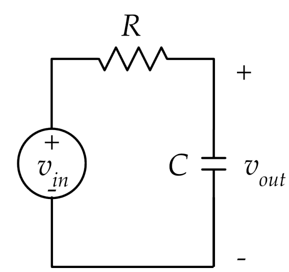
Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 2 Solutions

Problem 1: *(5 points)* Consider the RC circuit shown below.



A. Find $v_{out}(t)$ when $v_{in} = V \sin \omega t$. Assume the system is in sinusoidal steady state (i.e., all transients have disappeared).

Solution:
Step 1: Find the ODE
$v_{in} = v_R + v_{out}$
$v_{in} = RC\dot{v_{out}} + v_{out}$
$\dot{v_{out}} + \frac{1}{RC} v_{out} = \frac{1}{RC} v_{in}$
Let $RC = \tau$ so that
$\dot{v_{out}} + \frac{1}{\tau} v_{out} = \frac{1}{\tau} v_{in}$

Step 2: Find the transfer function
Assume $v_{in} = e^{j\omega t}$ find $H(j\omega)$ such that $v_{out} = H(j\omega)e^{j\omega t}$
$j\omega H(j\omega)e^{j\omega t} + \frac{1}{\tau} H(j\omega)e^{j\omega t} = \frac{1}{\tau} e^{j\omega t}$

$$H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau} \text{ so } |H(j\omega)| = \frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}} \text{ and } \angle H(j\omega) = -\tan^{-1}(\omega\tau)$$

Step 3: Write solution
$v_{out} = |H(j\omega)| V \sin(\omega t + \angle H(j\omega))$
or

$$\boxed{v_{out} = V \frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}} \sin[\omega t - \tan^{-1}(\omega\tau)]}$$

B. Assume an input $v_{in} = V \sin(\omega t)u(t)$ so that the circuit is initially at rest. Find an expression for $v_{out}(t)$ when $t > 0$.

Solution:
Step 1: Find particular solution
The sinusoidal steady state from the previous part is the particular solution for $t > 0$.

Step 2: Find homogeneous solution
Solve $\dot{v_{out}} + \frac{1}{\tau} v_{out} = 0$
Integrate to obtain $v_{out,homogeneous} = Ae^{-\frac{t}{\tau}}$

Step 3: Add the particular solution to the homogeneous solution and use the initial condition to determine the constant of integration A

$$v_{out} = V \frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}} \sin[\omega t - \tan^{-1}(\omega\tau)] + Ae^{-\frac{t}{\tau}}, t > 0$$

$$v_{out}(0) = 0$$

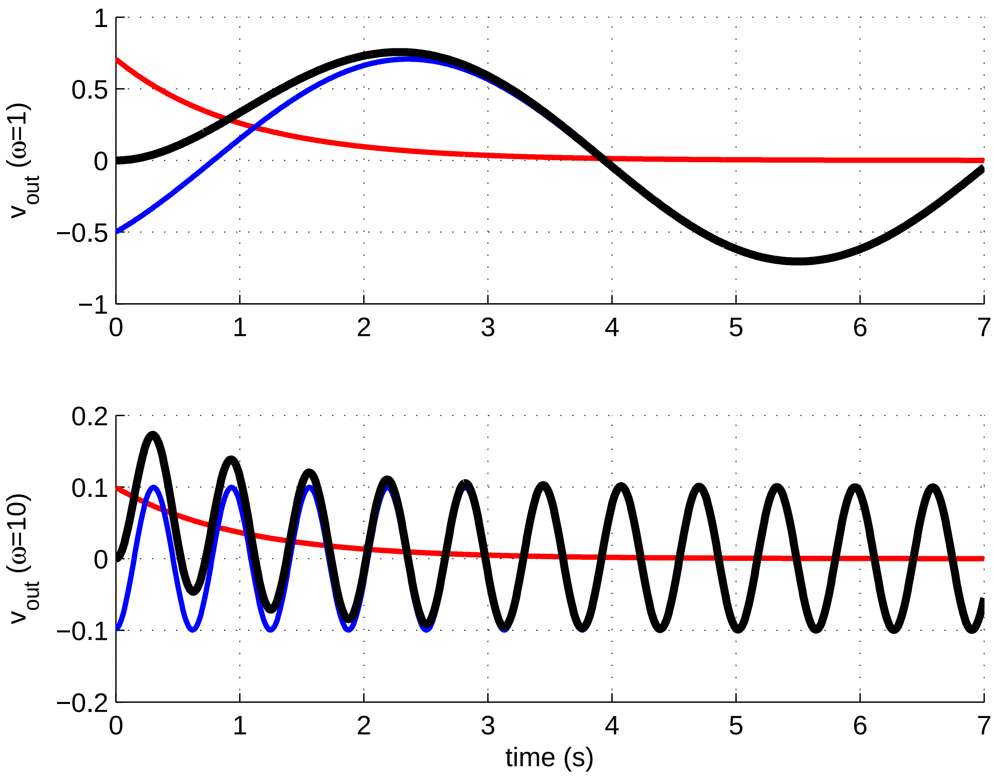
$$0 = -V \frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}} \sin[\tan^{-1}(\omega\tau)] + A$$

$$A = V \frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}} \frac{\omega}{\sqrt{\omega^2 + 1/\tau^2}}$$

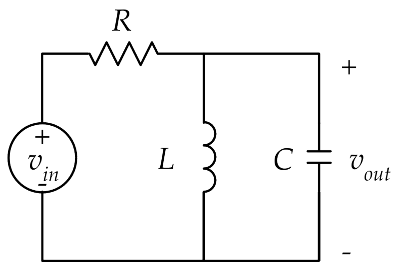
$$\boxed{v_{out} = V \frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}} \left(\sin[\omega t - \tan^{-1}(\omega\tau)] + \frac{\omega}{\sqrt{\omega^2 + 1/\tau^2}} e^{-\frac{t}{\tau}} \right), t > 0}$$

C. Plot the solution assuming $V = 1$, $\omega = 1$, and $RC = 1$ as well as $V = 1$, $\omega = 10$, and $RC = 1$.

Solution:



Problem 2: *(5 points)* Consider the RLC circuit shown below.



A. Find a differential equation that relates v_{in} and v_{out}.

Solution:

$$\frac{v_{in} - v_{out}}{R} = C\dot{v}_{out} + \int \frac{v_{out}}{L} dt$$
$$\frac{\dot{v}_{in} - \dot{v}_{out}}{RC} = \ddot{v}_{out} + \frac{v_{out}}{LC}$$
$$\ddot{v}_{out} + \frac{\dot{v}_{out}}{RC} + \frac{v_{out}}{LC} = \frac{\dot{v}_{in}}{RC}$$
$$\boxed{\ddot{v}_{out} + 2\alpha\dot{v}_{out} + \omega_0^2 v_{out} = 2\alpha\dot{v}_{in}}$$
$$\boxed{\alpha = \frac{1}{2RC} \qquad w_0 = \frac{1}{\sqrt{LC}}}$$

B. Derive an expression for the transfer function from v_{in} to v_{out}.

Solution:
Assume $v_{in} = e^{j\omega t}$ find $H(j\omega)$ such that $v_{out} = H(j\omega)e^{j\omega t}$

$$\ddot{v}_{out} + 2\alpha\dot{v}_{out} + \omega_0^2 v_{out} = 2\alpha\dot{v}_{in}$$
$$\alpha = \frac{1}{2RC} \qquad w_0 = \frac{1}{\sqrt{LC}}$$
$$-\omega^2 H(j\omega)e^{j\omega t} + 2\alpha j\omega H(j\omega)e^{j\omega t} + \omega_0^2 H(j\omega)e^{j\omega t} = 2\alpha j\omega e^{j\omega t}$$
$$\boxed{H(j\omega) = \frac{2\alpha j\omega}{-\omega^2 + 2\alpha j\omega + \omega_0^2}}$$

C. Sketch the Bode plot for the transfer function from v_{in} to v_{out} using asymptotic approximations.

Solution:

$$H(j\omega) = \frac{2\alpha j\omega}{-\omega^2 + 2\alpha j\omega + \omega_0^2}$$

If $\omega \to 0$, then $H(j\omega) \approx \dfrac{2\alpha j\omega}{\omega_0^2}$

$$|H(j\omega)| = \frac{2\alpha\omega}{\omega_0^2}, \quad \angle H(j\omega) = \pi/2$$

If $\omega \to \infty$, then $H(j\omega) \approx -\dfrac{2\alpha j\omega}{\omega^2} = -\dfrac{2\alpha j}{\omega}$

$$|H(j\omega)| = \frac{2\alpha}{\omega}, \quad \angle H(j\omega) = -\pi/2$$

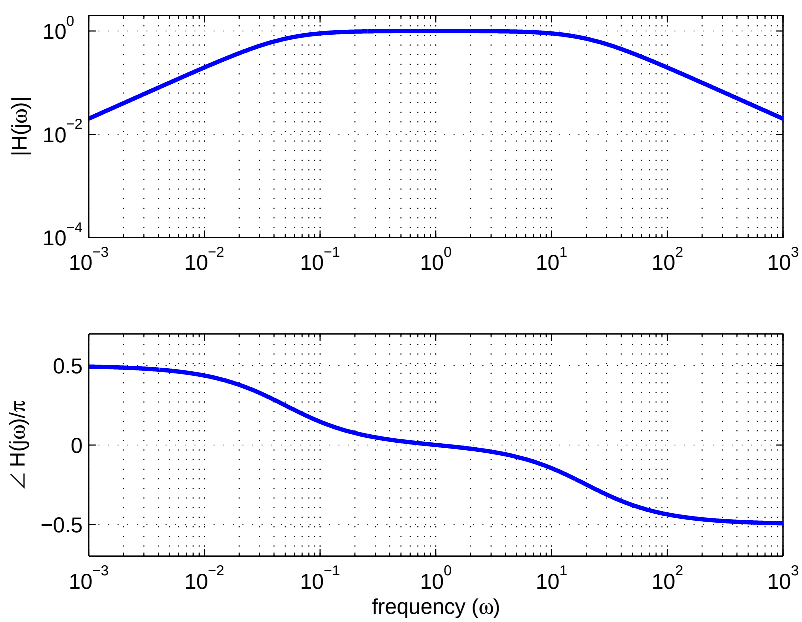
Intersection $\quad \dfrac{2\alpha\omega}{\omega_0^2} = \dfrac{2\alpha}{\omega} \Rightarrow \omega = \omega_0$

Asymptote value at intersection $\quad \dfrac{2\alpha}{\omega_0} = \dfrac{1}{Q}$

but the actual value is $\quad H(j\omega_0) = 1$

$$|H(j\omega_0)| = 1, \quad \angle H(j\omega_0) = 0$$

The magnitude of $1/Q$ determines whether the system resonates or not. If $Q \ll 1$, then $\alpha \gg \omega_0$. This system is overdamped and does not oscillates. It behaves as two first order systems together.



If $Q \gg 1$, then $\alpha \ll \omega_0$. This system is underdamped and resonates.