Olin College of Engineering
ENGR2410 – Signals and Systems

Lecture 5 Reference

From Fourier series to Fourier transform

The integral of $f(x)$ is the limit of the sum of the area of rectangles under the function as they become narrower ($\Delta x \to 0$) and denser ($n \to \infty$),

$$\int_{x_a}^{x_b} f(x)dx = \lim_{\Delta x \to 0} \sum_{n=x_a/\Delta x}^{x_b/\Delta x} f(n\Delta x)\Delta x$$

As derived in the last lecture, an even square wave $v(t)$ with period T and pulse width $2T_1$ can be expressed as sum of complex exponentials such that

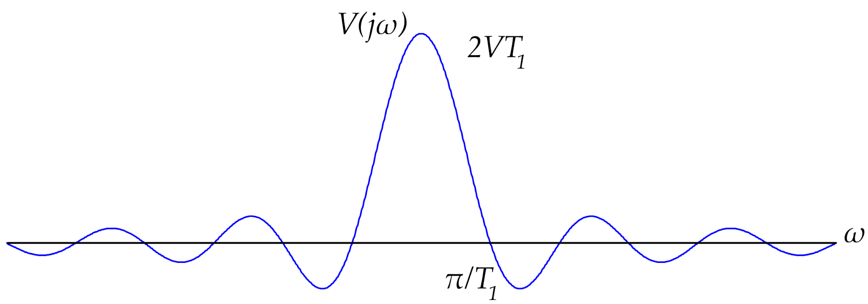
$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}, \qquad c_n = 2V\frac{T_1}{T}\text{sinc}\left(\frac{2\pi}{T}nT_1\right)$$

with each n corresponding to a frequency $\omega = \frac{2\pi}{T}n$. Define $\Delta\omega \triangleq \frac{2\pi}{T}$ such that if $T \to \infty$, then $\Delta\omega \to 0$. The original $v(t)$ becomes

$$v(t) = \sum_{n=-\infty}^{\infty} \underbrace{2VT_1\text{sinc}\left(\Delta\omega n T_1\right)}_{c_n} e^{j\Delta\omega n t} \underbrace{\frac{\Delta\omega}{2\pi}}_{1/T}$$

The first zero of the sinc function occurs when $\Delta\omega n T_1 = \pi$ such that $n = \frac{\pi}{\Delta\omega T_1}$. As $\Delta\omega \to 0$, $n \to \infty$, but the first zero does not change from $\Delta\omega n = \pi/T_1$. In general, $\omega = \Delta\omega n$. As $\Delta\omega \to 0$, the summation converges to

$$v(t) = \int_{\omega=-\infty}^{\infty} \underbrace{2VT_1\text{sinc}\left(\omega T_1\right)}_{V(j\omega)} e^{j\omega t}\frac{d\omega}{2\pi}$$



LTI system response in the frequency domain

In general, the Fourier transform of $x(t)$ is $X(j\omega) = \mathscr{F}\{x(t)\}$, where

$$\mathscr{F}\{x(t)\} = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Similarly, the inverse Fourier transform of $X(j\omega)$ is $x(t) = \mathscr{F}^{-1}\{X(j\omega)\}$, where

$$\mathscr{F}^{-1}\{X(j\omega)\} = \int_{\omega=-\infty}^{\infty} X(j\omega)e^{j\omega t}\frac{d\omega}{2\pi}$$

Equivalently, we can write $x(t) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad X(j\omega)$.

The response to an exponential $e^{j\omega t}$ of any LTI system with transfer function $H(j\omega)$ is given by

$$e^{j\omega t} \rightarrow \boxed{LTI} \rightarrow H(j\omega)e^{j\omega t}$$

If we can express the input $x(t)$ to a system as an integral of complex exponentials, the output $y(t)$ is equal to **the same integral of complex exponentials scaled by the transfer function**. Make sure you understand this last sentence clearly.

$$x(t) = \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}dt \rightarrow \boxed{H(j\omega)} \rightarrow y(t) = \int_{-\infty}^{\infty} \underbrace{H(j\omega)X(j\omega)}_{Y(j\omega)} e^{j\omega t}dt$$

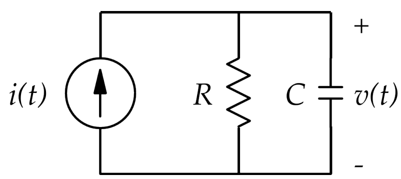
In system notation,

$$\begin{array}{ccccc} x(t) & \longrightarrow & \boxed{\text{LTI}} & \longrightarrow & y(t) \\ {\scriptstyle\mathscr{F}}\big\downarrow & & & & \big\uparrow{\scriptstyle\mathscr{F}^{-1}} \\ X(j\omega) & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & Y(j\omega)= X(j\omega)H(j\omega) \end{array}$$

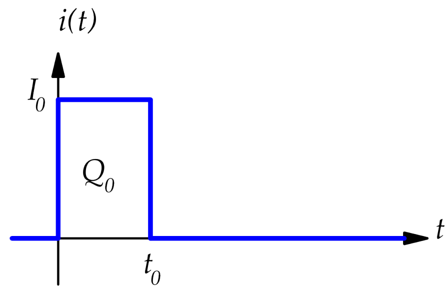
Impulse and impulse response

The impulse response, as we will see shortly, is the inverse Fourier transform of a given system ($h(t) = \mathscr{F}^{-1}\{H(j\omega)\}$). We will now show that the impulse response depends only of the area of the impulse, regardless of its shape. In particular, the differential equation describing the behavior of the circuit below is given by

$$\dot{v} + \frac{1}{RC}v = \frac{1}{C}i(t)$$



The input current $i(t)$ is given by the piecewise function $i(t) = \begin{cases} I_0 & 0 < t < t_0 \\ 0 & \text{otherwise} \end{cases}$ shown below.



The output voltage is $v(t) = \begin{cases} I_0 R(1 - e^{-t/\tau}) & 0 < t < t_0 \\ I_0 R(1 - e^{-t_0/\tau})e^{-(t-t_0)/\tau} & t > t_0 \end{cases}, \qquad \tau = RC$

The integral of $i(t)$ is equal to the charge $Q_0 = I_0 t_0$. If we scale the current I_0 such that the charge Q_0 remains constant, the solution becomes

$$v(t) = \begin{cases} \dfrac{Q_0}{t_0}R(1 - e^{-t/\tau}) & 0 < t < t_0 \\ \dfrac{Q_0}{t_0}R(1 - e^{-t_0/\tau})e^{-(t-t_0)/\tau} & t > t_0 \end{cases}$$

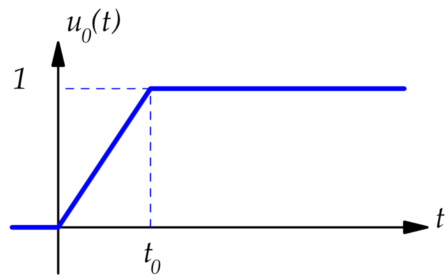
If we let $t_0 \to 0$, $i(t) = Q_0\delta(t)$, where $\delta(t)$ is the unit impulse. In the same limit, the output voltage $v(t)$ is the *impulse response*, equal to

$$v(t) = \lim_{t_0 \to 0} \begin{cases} \dfrac{Q_0}{t_0}R(1 - e^{-t/\tau}) & 0 < t < t_0 \\ \dfrac{Q_0}{t_0}R(e^{t_0/\tau} - 1)e^{-t/\tau} & t > t_0 \end{cases}$$

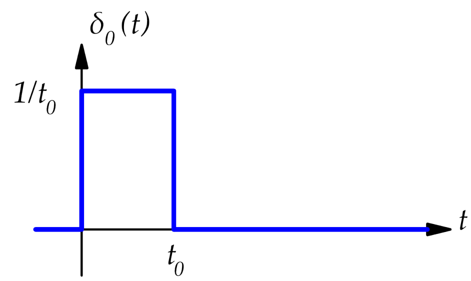
$$v(t) \approx \frac{Q_0}{t_0}R\left(1 + \frac{t_0}{\tau} - 1\right)e^{-t/\tau} = \frac{Q_0}{\cancel{t_0}}\cancel{R}\frac{\cancel{t_0}}{\cancel{R}C}e^{-t/\tau} = \frac{Q_0}{C}e^{-t/\tau}, \quad t > 0$$

Properties of the impulse

Most properties of the impulse can be shown by constructing appropriate functions and then taking the limit. For example, in order to show that the unit impulse is the derivative of the unit step, $\delta(t) = \frac{du(t)}{dt}$, use the function $u_0(t)$ shown below.



Its derivative is the function $\delta_0(t)$,



In the limit when $t_0 \to 0$, $u_0(t) \to u(t)$ and $\delta_0(t) \to \delta(t)$. The impulse $\delta(t)$ may be defined as

$$\delta = 0, (-\infty, 0^-) \cup (0^+, \infty); \qquad \int_{0^-}^{0^+} \delta(t) = 1$$

The impulse can be scaled and shifted, leading to the "picking" property,

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

This property allows us to compute the Fourier transform of the impulse,

$$\mathscr{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{j\omega(0)} = 1$$

Using the inverse Fourier transform, the previous transform implies that the impulse is the integral of all complex exponentials, equally weighted,

$$\delta(t) = \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t}\frac{d\omega}{2\pi}$$

Finally, since $\mathscr{F}\{\delta(t)\} = 1$, the frequency content of the impulse response is $1 \cdot H(j\omega)$. Therefore, $h(t) = \mathscr{F}^{-1}\{H(j\omega)\}$ is the impulse response of a system with transfer function $H(j\omega)$. In system notation,

$$
\begin{array}{ccccc}
\delta(t) & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & h(t) \\
{\scriptstyle \mathscr{F}}\big\downarrow & & & & \big\uparrow{\scriptstyle \mathscr{F}^{-1}} \\
1 & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & H(j\omega)
\end{array}
$$

The behavior of a system can be fully characterized equally by using a differential equation, a transfer function, a Bode plot, or the impulse response.