Olin College of Engineering
ENGR2410 – Signals and Systems

Reference 3

Standard forms

When working with symbolic expressions, pattern recognition plays a major role. Force yourself to always follow standard forms, where highest order derivate or term in a polynomial does not have a coefficient, for example,

$$\ddot{v}_{out} + 2\alpha\dot{v}_{out} + \omega_0^2 v_{out} = \omega_0^2 v_{in}$$

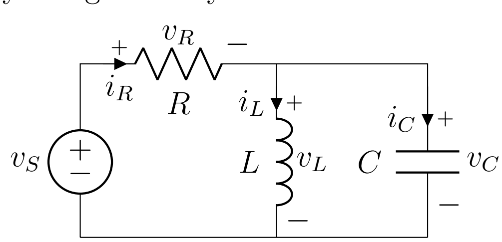
$$H(s) = \frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2}$$

$$H(s) = \frac{1/\tau}{s + 1/\tau}$$

Identify known terms (especially time constants and frequencies), group and keep them together, and write them in the typical order: RC, LC, L/R.

Circuits as LTI systems

So far we have only used circuit diagrams to derive a differential equation. We can use them directly to our advantage by using linearity.



Circuit diagrams are graphical ways to represent differential equations. For example, the circuit above satisfies the conditions:

KVL	KCL	Device laws
$v_S(t) = v_R(t) + v_L(t)$	$i_R(t) = i_L(t) + i_C(t)$	$v_R(t) = Ri_R(t)$
$v_L(t) = v_C(t)$		$v_L(t) = L\dfrac{di_L}{dt}$
		$i_C(t) = C\dfrac{dv_C}{dt}$

Linearity implies that if any one of

$$v_R(t) = V_R e^{st} \qquad v_L(t) = V_L e^{st} \qquad v_C(t) = V_C e^{st}$$

is true, then all the others must be true also. As a result, exponential amplitudes satisfy KVL and KCL:

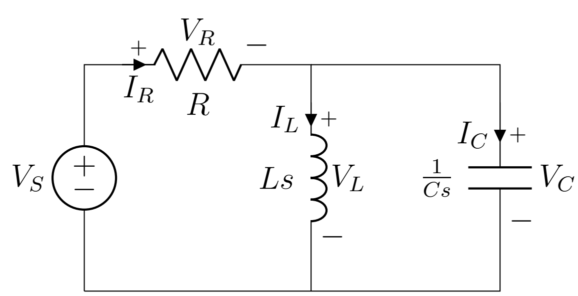
$$\begin{aligned} v_S(t) &= V_R(t) + v_L(t) \\ V_S \cancel{e^{st}} &= V_R \cancel{e^{st}} + V_L \cancel{e^{st}} \\ V_S &= V_R + V_L \end{aligned}$$

Exponential amplitudes satisfy device laws as *algebraic* equations:

$$v_C(t) = V_C e^{st}$$

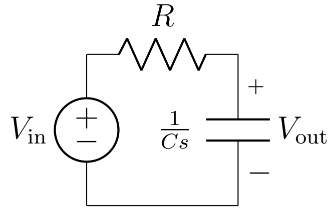
$$\begin{aligned} i_C(t) &= I_C e^{st} \\ i_C &= C\dot{v}_C(t) \\ I_C \cancel{e^{st}} &= C V_C s \cancel{e^{st}} \\ Z_C &= \frac{V_C}{I_C} = \frac{1}{Cs} \end{aligned}$$

Z_C is called *impedance*. The impedance of a resistor R is $Z_R = R$. The impedance of an inductor is $Z_L = Ls$. Therefore, linear circuits can be written in terms of impedances and exponential amplitudes.



Manipulating impedances

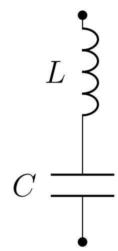
Impedances can be treated as resistors, including all the usual resistor techniques: series, parallel, voltage dividers, current dividers and so forth.



$$V_{out} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_{in}$$

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

Impedances can also be computed for combinations of elements.



$$Z_{eq} = Ls + \frac{1}{Cs} = \frac{LCs^2 + 1}{Cs} = \frac{s^2 + \frac{1}{LC}}{\frac{s}{L}}$$

In particular, the impedance at the resonant frequency $\omega_0 = 1/\sqrt{LC}$ of each element is

$$Z_L = jL\omega = jL\frac{1}{\sqrt{LC}} = j\sqrt{\frac{L}{C}}$$

$$Z_C = -j\frac{1}{C\omega} = -j\frac{\sqrt{LC}}{C} = -j\sqrt{\frac{L}{C}}$$

and $Z_L + Z_C = 0$ at resonance.