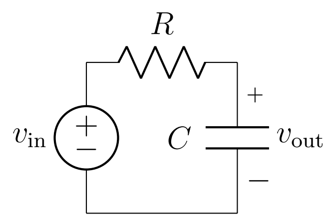
Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 2

Problem 1 Consider the RC circuit shown below.



A. Find a differential equation that relates v_{in} and v_{out}.

Solution:
$v_{in} = v_R + v_{out}$
$v_{in} = RC\dot{v_{out}} + v_{out}$
$\dot{v_{out}} + \frac{1}{RC}v_{out} = \frac{1}{RC}v_{in}$
Let $RC = \tau$ so that
$\dot{v_{out}} + \frac{1}{\tau}v_{out} = \frac{1}{\tau}v_{in}$

B. Derive an expression for the transfer function from v_{in} to v_{out}.

Solution:
Assume $v_{in} = e^{j\omega t}$ find $H(j\omega)$ such that $v_{out} = H(j\omega)e^{j\omega t}$
$j\omega H(j\omega)e^{j\omega t} + \frac{1}{\tau}H(j\omega)e^{j\omega t} = \frac{1}{\tau}e^{j\omega t}$
$H(j\omega) = \dfrac{1/\tau}{j\omega + 1/\tau}$ so $|H(j\omega)| = \dfrac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}}$ and $\angle H(j\omega) = -\tan^{-1}(\omega\tau)$

C. Find $v_{out}(t)$ when $v_{in} = V\sin\omega t$. Assume the system is in sinusoidal steady state (i.e., all transients have disappeared).

Solution:
Scale and shift the input
$v_{out} = |H(j\omega)|V\sin(\omega t + \angle H(j\omega))$
or

$$\boxed{v_{out} = V\frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}}\sin[\omega t - \tan^{-1}(\omega\tau)]}$$

D. Sketch the Bode plot of the circuit using asymptotic approximations.

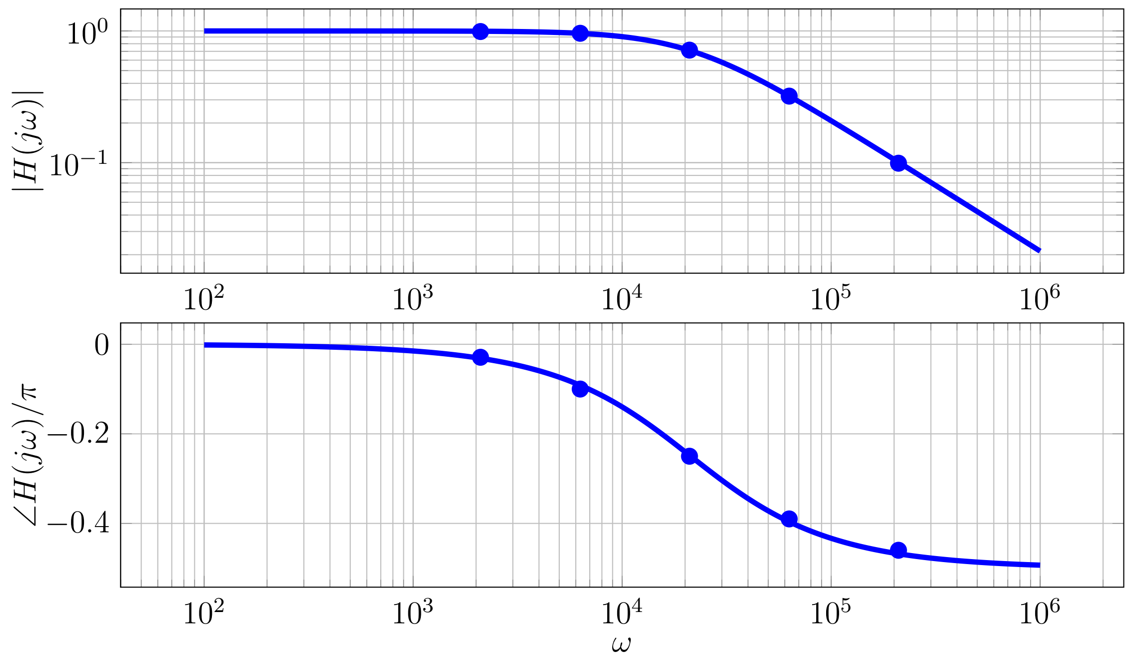
My solution follows as an example.

$$H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}$$

If $\omega \to 0$, then $H(j\omega) \approx \dfrac{1/\tau}{1/\tau} = 1$ and $|H(j\omega)| = 1$, $\angle H(j\omega) = 0$

If $\omega \to \infty$, then $H(j\omega) \approx \dfrac{1/\tau}{j\omega}$ and $|H(j\omega)| = \dfrac{1/\tau}{\omega}$, $\angle H(j\omega) = -\dfrac{\pi}{2}$

Intersection at $\omega = 1/\tau$. Since this system is first order, the Bode plot transitions smoothly at the intersection. In the plot, $1/\tau = 21,300$ rad/s, and the dots are data from the next part.



E. Build the circuit and verify your Bode plot with your own values *using at least five separate sinusoidal steady-state measurements* (that is, don't simply use an automated Bode plot function). Adjust your plot according to your own element values, and make sure to choose frequencies that capture the behavior of the circuit (e.g., choose the natural frequency and a couple frequencies above and below). Plot your measurements on top of your Bode plot sketch.

My solution follows.
$R = 10\ \mathrm{k}\Omega, \mathrm{C} = 4.7\ \mathrm{nF}, \tau = \mathrm{RC} = 47\ \mu\mathrm{s}, \frac{1}{\mathrm{RC}} = 21,300\ \mathrm{rad/s}, \frac{1}{2\pi\mathrm{RC}} = 3.4\ \mathrm{kHz}$

f (kHz)	$\omega = 2\pi f$ (krad/s)	v_{in} (mVACrms)	v_{out} (mVACrms)	v_{out}/v_{in}	ϕ (°)	ϕ (rad/π)
0.34	2.1	752	741	0.99	– 5.2	– 0.029
1	6.3	673	645	0.96	– 18.3	– 0.10
3.4	21	706	505	0.72	– 45.6	– 0.25
10	63	719	231	0.32	– 70.5	– 0.39
34	210	723	71.3	0.099	– 82.3	– 0.46

F. Assume an input $v_{in} = V\sin(\omega t)u(t)$ so that the circuit is initially at rest. Find an expression for $v_{out}(t)$ when $t > 0$.

Solution:
Step 1: Find particular solution
The sinusoidal steady state from the previous part is the particular solution for $t > 0$.

Step 2: Find homogeneous solution
Solve $\dot{v_{out}} + \frac{1}{\tau}v_{out} = 0$
Integrate to obtain $v_{out,homogeneous} = Ae^{-\frac{t}{\tau}}$

Step 3: Add the particular solution to the homogeneous solution and use the initial condition to determine the constant of integration A

$$v_{out} = V\frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}}\sin[\omega t - \tan^{-1}(\omega\tau)] + Ae^{-\frac{t}{\tau}}, t > 0$$

$$v_{out}(0) = 0$$

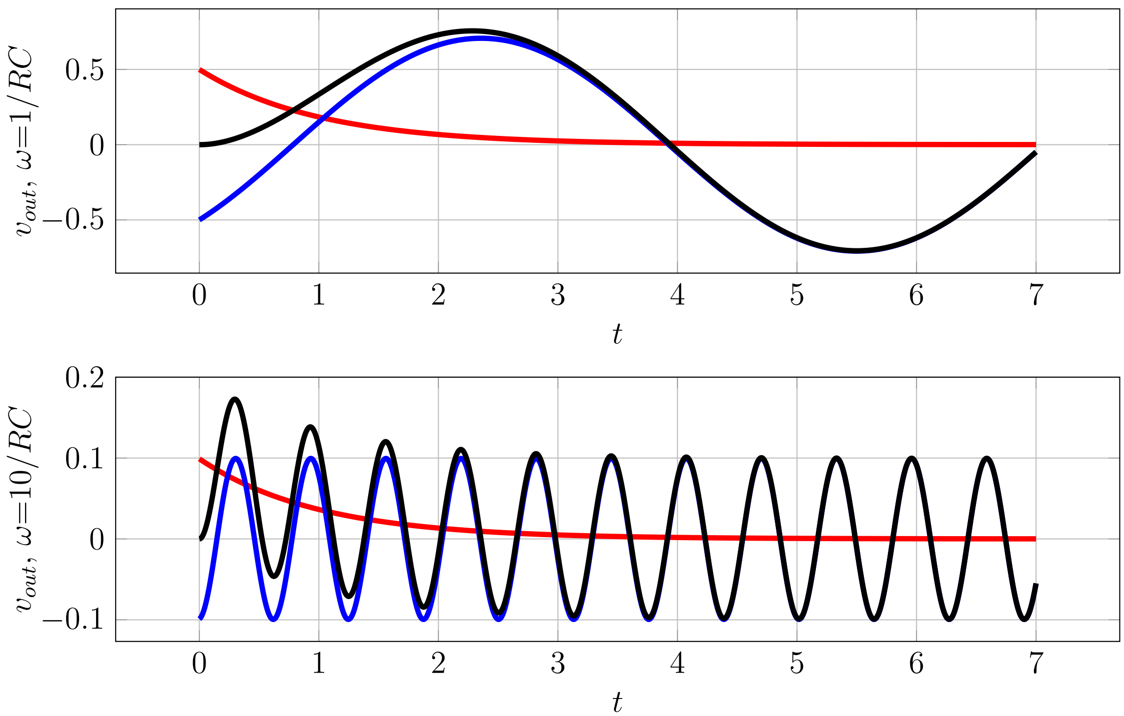
$$0 = -V\frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}}\sin[\tan^{-1}(\omega\tau)] + A$$

$$A = V\frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}}\frac{\omega}{\sqrt{\omega^2 + 1/\tau^2}}$$

$$\boxed{v_{out} = V\frac{1/\tau}{\sqrt{\omega^2 + 1/\tau^2}}\left(\sin[\omega t - \tan^{-1}(\omega\tau)] + \frac{\omega}{\sqrt{\omega^2 + 1/\tau^2}}e^{-\frac{t}{\tau}}\right), t > 0}$$

G. Plot the solution in the case when the driving frequency $\omega = 1/RC$ as well as when the driving frequency $\omega = 10/RC$.

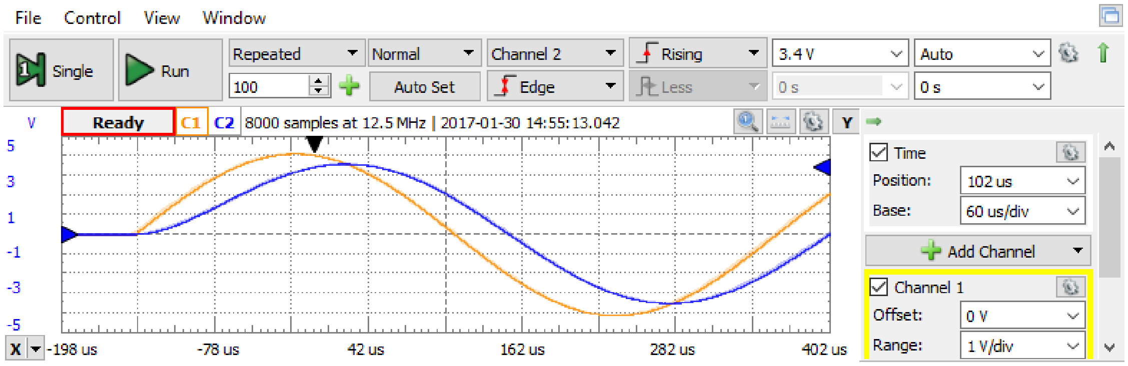
Solution:
In the following plots, $V = 1$ and $RC = 1$ s.



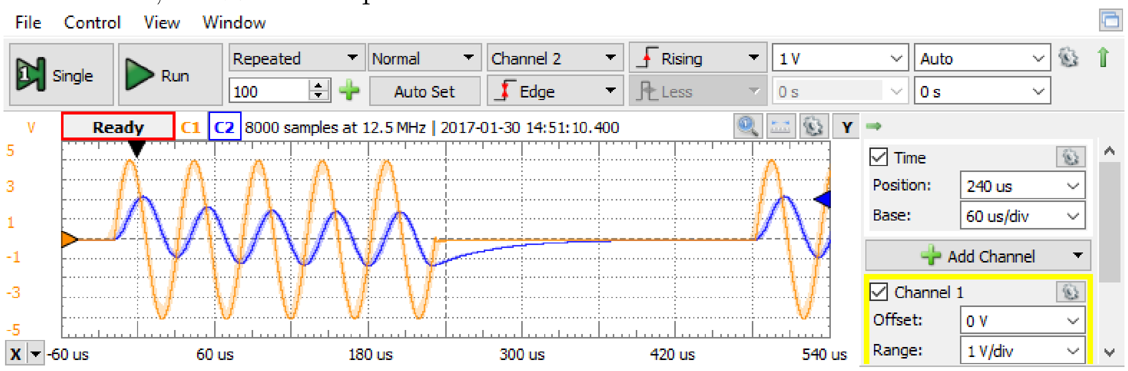
H. Drive your circuit with a sinusoid that turns on and off and show data qualitatively similar to both of your plots. You don't have to make quantitative measurements on the data.

My solution follows:
$R = 10\ \text{k}\Omega, \text{C} = 4.7\ \text{nF}, \tau = \text{RC} = 47\ \mu\text{s}$

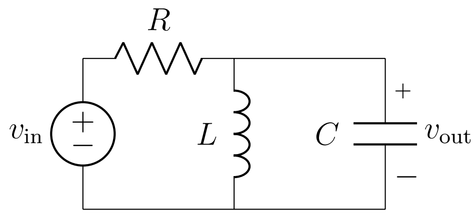
Drive: AM modulation
Carrier: 2 kHz, 2 V sine wave, phase 1° (to avoid spike)
AM: 500 Hz, 100% index square wave



Drive: AM modulation
Carrier: 20 kHz, 2 V sine wave, phase 1° (to avoid spike)
AM: 2 kHz, 100% index square wave



Problem 2 Consider the parallel RLC circuit shown below.



A. Find a differential equation that relates v_{in} and v_{out}.

Solution:

$$\frac{v_{in} - v_{out}}{R} = C\dot{v}_{out} + \int \frac{v_{out}}{L} dt$$

$$\frac{\dot{v}_{in} - \dot{v}_{out}}{RC} = \ddot{v}_{out} + \frac{v_{out}}{LC}$$

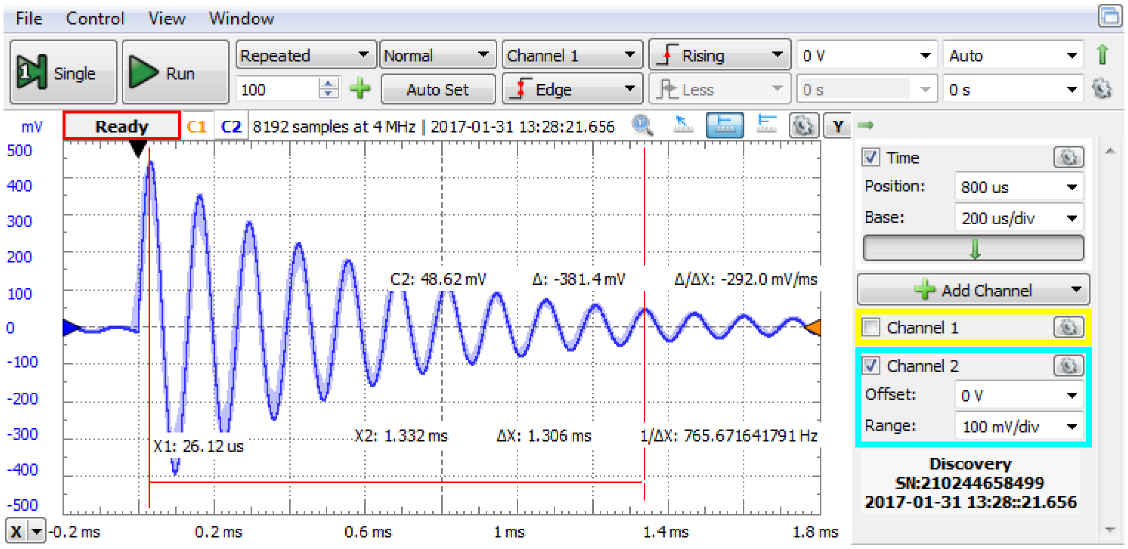
$$\ddot{v}_{out} + \frac{\dot{v}_{out}}{RC} + \frac{v_{out}}{LC} = \frac{\dot{v}_{in}}{RC}$$

$$\boxed{\ddot{v}_{out} + 2\alpha\dot{v}_{out} + \omega_0^2 v_{out} = 2\alpha\dot{v}_{in}}$$

$$\boxed{\alpha = \frac{1}{2RC} \qquad w_0 = \frac{1}{\sqrt{LC}}}$$

B. Build the circuit and verify the resonant frequency ω_d and the decay constant α from your own element values using the step response.

Solution:
$R = 100\ \mathrm{k}\Omega, \mathrm{L} = 0.1\ \mathrm{H}, \mathrm{C} = 4.7\ \mathrm{nF}$
$2\pi/\omega_0 = 2\pi\sqrt{LC} = 136\ \mu\mathrm{s}, 1/\alpha = 2\mathrm{RC} = 940\ \mu\mathrm{s}$ (α is negligible.)
Drive: 100 Hz, 5 V amplitude square wave



From the plot, 10 cycles are 1.306 ms, resulting in a measured $2\pi/\omega_0$ of 131 μs. However, in 10 cycles, the voltages decays from 430 mV to 49 mV. This decrease corresponds to

$$1/\alpha = -1.306\ \mathrm{ms}/\ln(49/430) = 600\ \mu\mathrm{s}$$

This is a more advanced analysis beyond what is expected! Assuming the value of our capacitance is correct, this $1/\alpha$ implies a resistance of 63 kΩ. This implies an additional resistance in parallel with the capacitor and inductor of 170 kΩ. If this is correct, at resonance, the gain should be $170/(100+170) = 0.63$. The measured gain at resonance is 458 mV/718 mV = 0.638!

C. Derive an expression for the transfer function from v_{in} to v_{out}.

Solution:
Assume $v_{in} = e^{j\omega t}$ find $H(j\omega)$ such that $v_{out} = H(j\omega)e^{j\omega t}$

$$\ddot{v}_{out} + 2\alpha\dot{v}_{out} + \omega_0^2 v_{out} = 2\alpha\dot{v}_{in}$$
$$\alpha = \frac{1}{2RC} \qquad w_0 = \frac{1}{\sqrt{LC}}$$
$$-\omega^2 H(j\omega)e^{j\omega t} + 2\alpha j\omega H(j\omega)e^{j\omega t} + \omega_0^2 H(j\omega)e^{j\omega t} = 2\alpha j\omega e^{j\omega t}$$
$$\boxed{H(j\omega) = \frac{2\alpha j\omega}{-\omega^2 + 2\alpha j\omega + \omega_0^2}}$$

D. Sketch the Bode plot for the transfer function from v_{in} to v_{out} using asymptotic approximations in the underdamped case ($Q \gg 1$, or $\alpha \ll \omega_0$).

Solution:

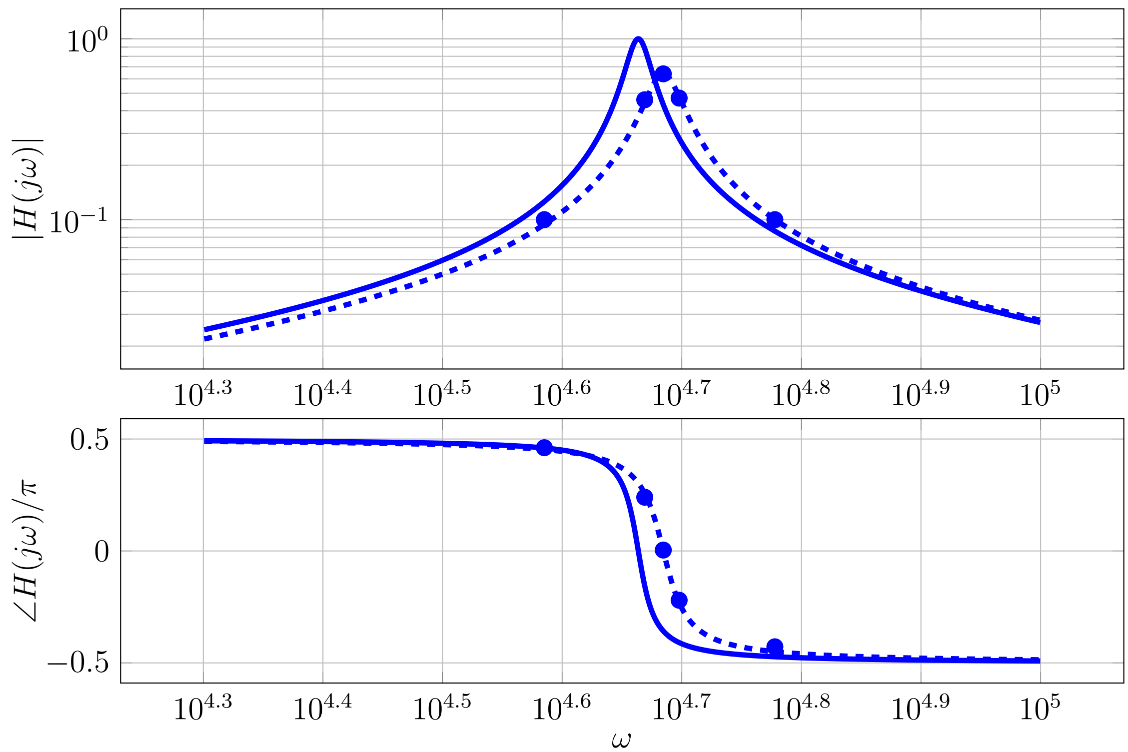
$$H(j\omega) = \frac{2\alpha j\omega}{-\omega^2 + 2\alpha j\omega + \omega_0^2}$$

$$\omega \to 0 : H(j\omega) \approx \frac{2\alpha j\omega}{\omega_0^2} \quad \Rightarrow \quad |H(j\omega)| = \frac{2\alpha\omega}{\omega_0^2}, \quad \angle H(j\omega) = \pi/2$$

$$\omega \to \infty : H(j\omega) \approx -\frac{2\alpha j\omega}{\omega^2} = -\frac{2\alpha j}{\omega} \quad \Rightarrow \quad |H(j\omega)| = \frac{2\alpha}{\omega}, \quad \angle H(j\omega) = -\pi/2$$

$$\text{Intersection} : \frac{2\alpha\omega}{\omega_0^2} = \frac{2\alpha}{\omega} \Rightarrow \omega = \omega_0; \quad \text{Asymptote value at intersection} : \frac{2\alpha}{\omega_0} = \frac{1}{Q}$$

$$\text{but the actual value is} \quad H(j\omega_0) = 1 \quad \Rightarrow \quad |H(j\omega_0)| = 1, \quad \angle H(j\omega_0) = 0$$



More advanced analysis: The dashed plot is centered at the measured resonant frequency of 7.70 kHz, as opposed to the theoretical of 7.35 kHz (a 4.5% error). Additionally, the damping term α in the denominator is multiplied by 1/0.64 to adjust for the extra parallel resistance across the capacitor and inductor.

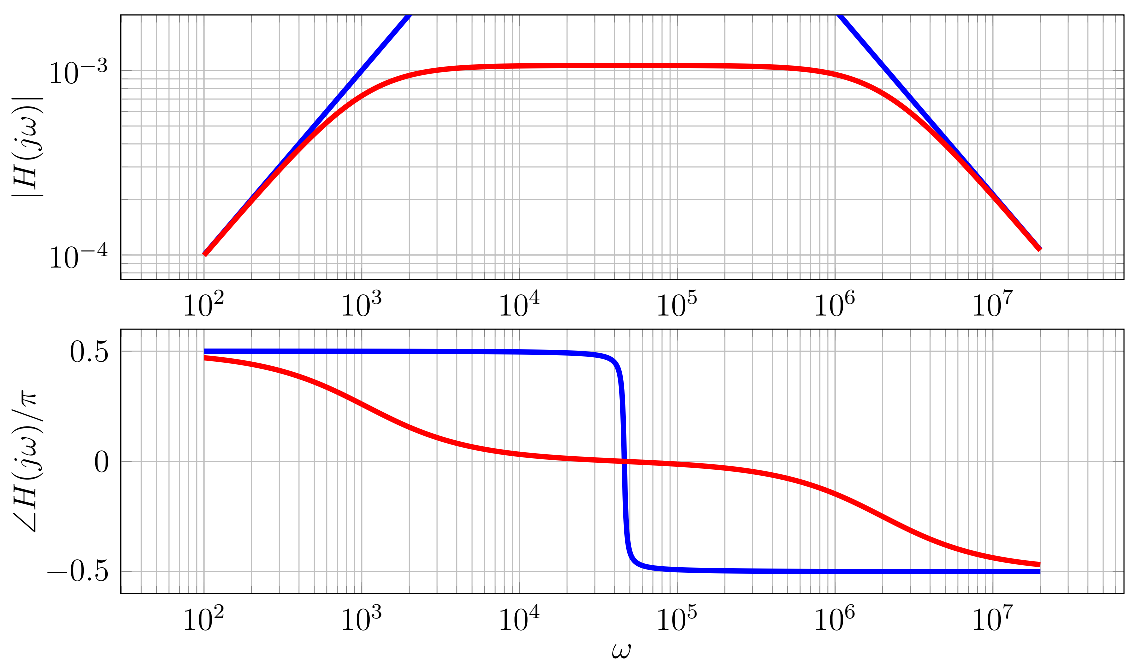
E. Build the circuit and verify your Bode plot with your own values *using at least five separate sinusoidal steady-state measurements* (that is, don't simply use an automated Bode plot function). Adjust your plot according to your own element values, and make sure to choose frequencies that capture the behavior of the circuit (e.g., choose the natural frequency and a couple frequencies above and below). Plot your measurements on top of your Bode plot sketch.

Solution:
$R = 100\ \text{k}\Omega, \text{L} = 0.1\ \text{H}, \text{C} = 4.7\ \text{nF}$
$2\pi/\omega_0 = 2\pi\sqrt{LC} = 136\ \mu\text{s} = \frac{1}{7.35\ \text{kHz}}, 1/\alpha = 2\text{RC} = 940\ \mu\text{s}$ (α is negligible.)

f (kHz)	$\omega = 2\pi f$ (krad/s)	v_{in} (mVACrms)	v_{out} (mVACrms)	v_{out}/v_{in}	ϕ (°)	ϕ (rad/π)
6.13	38.5	713	72.3	0.1	83	0.461
7.43	46.7	735	339	0.46	44	0.24
7.70	48.4	731	468	0.64	0.7	0.004
7.94	49.9	720	338	0.47	− 40	− 0.22
9.55	60.0	729	73.3	0.1	− 77	− 0.428

F. Repeat your analysis and sketch the Bode plot in the overdamped case ($Q \ll 1$, or $\alpha \gg \omega_0$). You don't have to build and measure the behavior for this case.

Solution:
This system behaves as two first order systems together. Plotted below in red with the underdamped case in blue.



Course feedback

Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?