Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 4

Problem 1 In this problem, you will derive a more limited expression for Fourier series using sines and cosines. In particular, complex exponentials allow us to express any complex function in general. However, restricting to sines and cosines forces the resulting function to be purely real. In this case, corresponding complex coefficients must be complex conjugates[1] ($c_{-n} = c_n^*$), where n is any integer. Show that

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} = c_0 + \sum_{n=1}^{\infty} 2\mathrm{Re}\{c_n\}\cos\left(\tfrac{2\pi}{T}nt\right) + \sum_{n=1}^{\infty}(-2)\mathrm{Im}\{c_n\}\sin\left(\tfrac{2\pi}{T}nt\right)$$

Problem 2 If we try to represent the function

$$v(t) = \begin{cases} V & -T_1 + nT < t < T_1 + nT, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

using Fourier series such that $v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$ then $c_n = 2V\frac{T_1}{T}\mathrm{sinc}\left(\frac{2\pi}{T}nT_1\right)$.

A. Let's explore the coefficients c_n as we make the pulses thinner while holding the period T constant. *When plotting coefficients, use the* stem *command in Matlab. Careful: the definition of sinc in Matlab is normalized to* $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$, *as opposed to the more common unnormalized version we use in class,* $\mathrm{sinc}(x) = \sin(x)/x$.

   (i) Plot $v(t)$ and the coefficients c_n when $T = 1$, $T_1 = 1/4$, $V = 1$ and check that all the values are correct. For clarity and consistency, plot $v(t)$ from -4 to 4, and enough coefficients to capture 6 zero crossings while keeping c_0 centered.

   (ii) What does c_0 correspond to in $v(t)$? That is, what would you call c_0 in terms of $v(t)$?

   (iii) If you change T_1, how would scale V in order to keep c_0 constant?

   (iv) Add to your plot the scaled versions of $v(t)$ and the coefficients c_n when $T_1 = T/8T$, and $T_1 = T/16$. Keep $T = 1$ and scale V so that c_0 remains constant. Again, for clarity and consistency, plot $v(t)$ from -4 to 4, and enough coefficients to capture 6 zero crossings while keeping c_0 centered.

[1]Properties of complex conjugates: if $z = a + jb$, then $z^* = a - jb$; if $z = Ae^{j\theta}$, then $z^* = Ae^{-j\theta}$. For example, $(je^{j\theta})^* = -je^{-j\theta}$.

(v) Note how the number of complex exponentials under the first zero crossing increases as T_1 decreases. Intuitively, what do you think happens if $T_1 \to 0$, both in terms of $v(t)$ and the coefficients c_n?

B. Repeat the previous plot, but instead keep $T_1 = 1/4$ constant, and let T increase.

(i) Again, check that the base case where $T_1 = 1/4$, $T = 1$, and $V = 1$ is correct.

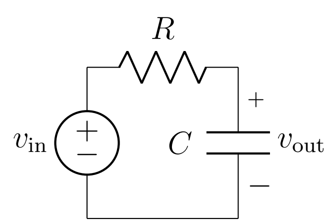
(ii) How would you scale V in this case to keep c_0 constant?

(iii) Add to your plot the scaled versions of $v(t)$ and the coefficients c_n when $T = 8T_1$, and $T = 16T_1$. Keep $T_1 = 1/4$ and scale V so that c_0 remains constant. Again, for clarity and consistency, plot $v(t)$ from -4 to 4, and enough coefficients to capture 6 zero crossings while keeping c_0 centered.

(iv) Intuitively, what do you think happens if $T \to \infty$, both in terms of $v(t)$ and the coefficients c_n?

(v) Compare the coefficients in both cases, as well as $v(t)$ in both limits. You tell your friend your results and she is bothered by them. Why is she bothered, and how do you make sense of your results?

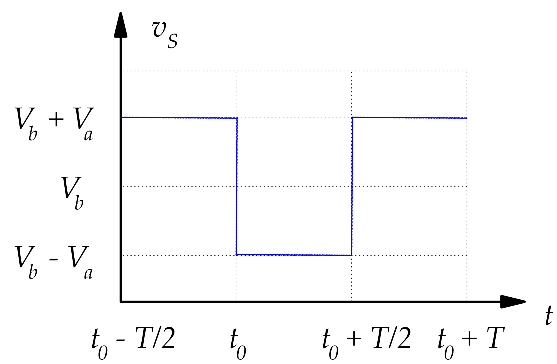
C. Assume $v_{in}(t) = v(t)$, $T = 1$, $T_1 = T/8$, and $RC = 0.1$. Plot the response of the circuit shown below using Fourier decomposition with complex exponentials for $-1.5 < t < 1.5$. Plot both the input and the output using 2, 7, and 20 harmonics[2]. Note how similar the responses are even though you are using a very crude approximation.



[2]Complex exponentials with frequencies $\pm 2\pi n/T$ are called the nth harmonics.

Problem 3 This problem will introduce the concept of *average power* in a signal, and will explore the relationships across the frequency and time domains.

A. The voltage signal shown below is a square wave of amplitude V_a and average value V_b.



The *average power* of a signal is defined as

$$< P >= \frac{1}{T} \int_{t_0}^{t_0+T} |v_S|^2 dt$$

where $|v_S|^2 = v_S v_S^*$ is the square magnitude of the signal and the asterisk denotes complex conjugation. Show that the average power for v_S is $V_a^2 + V_b^2$.

B. Show that the average power of the sum of two complex exponentials,

$$c_1 e^{j\omega_1 t} + c_2 e^{j\omega_2 t}, \omega_1 \neq \omega_2,$$

is the square magnitude of each exponential, that is, $< P >= |c_1|^2 + |c_2|^2$. This is a profoundly important fact: the power at a particular frequency is independent of the power at any other frequency!

C. Find the Fourier decomposition of v_S assuming $t_0 = T/4$ (*note that this choice is irrelevant for this problem; think about why*) and using complex exponentials such that

$$v_S(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$$

Hint: Simplify the integral by shifting the square wave so that half of it is zero, that is, $v_s(t) = V_b - V_a + v_s'(t)$.

D. Show that the average power of v_S can be separated into a sum of a term that depends on V_a and a term that depends on V_b. When computing the average power, you might find useful to know that

$$\sum_{n\in[1,3,5\ldots\infty)} \frac{1}{n^2} = \frac{\pi^2}{8}$$

E. The average power of v_S that depends on V_a is called the AC, or alternating current power, while the power that depends on V_b is called the DC, or direct current power. For the square wave, show that the fundamental (frequency 1/T) contains $8/\pi^2 \approx 80\%$ of the total AC power.

Course feedback

Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?