Olin College of Engineering
ENGR2410 – Signals and Systems

Reference 10

Laplace transform

Since e^{st} is an eigenfunction of LTI systems,

$$e^{st} \longrightarrow \boxed{\text{h(t)}} \longrightarrow H(s)e^{st}$$

Therefore,

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

and

$$\mathscr{F}\{h(t)\} = H(s)|_{s=j\omega}$$

if $s = j\omega$ is in the region of convergence (ROC) of $H(s)$. Last time, we found that

$$e^{-at}u(t) \quad \stackrel{\mathscr{L}}{\Longleftrightarrow} \quad \frac{1}{s+a}, \quad \text{Re}\{s\} > -a$$

If $a = 0$,

$$u(t) \quad \stackrel{\mathscr{L}}{\Longleftrightarrow} \quad \frac{1}{s}, \quad \text{Re}\{s\} > 0$$

As a result,

$$\mathscr{L}^{-1}\left\{\frac{X(s)}{s}\right\} = u(t) * x(t) = \int_{-\infty}^{\infty} x(t')u(t-t')dt' = \int_{-\infty}^{t} x(t')dt'$$

Therefore,

$$\int_{-\infty}^{t} x(t')dt' \quad \stackrel{\mathscr{L}}{\Longleftrightarrow} \quad \frac{X(s)}{s}$$

Similarly,

$$\dot{x}(t) \quad \stackrel{\mathscr{L}}{\Longleftrightarrow} \quad sX(s)$$

Analysis of proper rational transfer functions

Systems of the form

$$\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = \ddot{x} + b_1\dot{x} + b_0 x$$

can be transformed to

$$s^3Y + a_2s^2Y + a_1sY + a_0Y = s^2X + b_1sX + b_0X$$

$$H(s) = \frac{Y}{X} = \frac{s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)(s+p_3)}$$

Note that if $H(s)$ is real, any poles or zeros that are not real must have their complex conjugate. Finally, $H(s)$ can be expanded into several fractions such that

$$H(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3}$$

Therefore,

$$h(t) = A_1 e^{-p_1 t}u(t) + A_2 e^{-p_2 t}u(t) + A_3 e^{-p_3 t}u(t)$$

Thus, *the poles of a system correspond to its natural response.* Also, the frequency response can be interpreted using vectors from each zero and pole to the $j\omega$ axis,

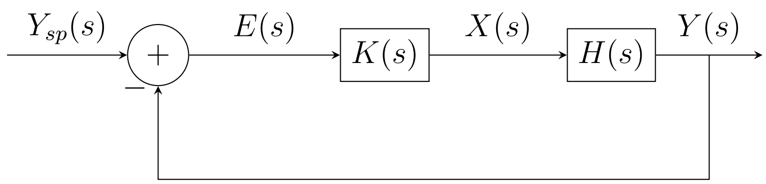
$$H(j\omega) = \frac{(j\omega + z_1)(j\omega + z_2)}{(j\omega + p_1)(j\omega + p_2)(j\omega + p_3)}$$

As a rule of thumb, real poles decrease the slope by 1 when $\omega = p$ as ω increases, decrease the phase by $\pi/2$ at the same time. Real zeros have increase the slope and phase correspondingly. Complex poles have "double" effect: the slope decreases by 2 and the phase by π when $\omega = \mathrm{Im}\{p\}$.

Control

Control systems are used typically for *tracking*, where some output y must "follow" some set point y_{sp}, or to *stabilize* the dynamics of a system by moving any poles from the left-half plane to the right-half plane such that $h(t)$ is bounded as $t \to \infty$.

In the system below, the set point y_{sp} is compared to the output y, and the resulting error e is fed into a controller $K(s)$ that then drives the plant $H(s)$ with x.



The overall transfer function is

$$K(\underbrace{Y_{sp} - HX}_{error}) = X$$

such that

$$\frac{X}{Y_{sp}} = \frac{K}{1 + KH}$$

Since $Y = HX$,

$$\frac{Y}{Y_{sp}} = \frac{KH}{1 + KH}$$

This is *Black's formula.*

In control systems, we typically care about the step response, since it shows the response of the system when the set point is changed. In particular, the step response has a *settling time* until it reaches the new set point, the final value might have an *offset*, or a *DC gain* not equal to one, and some *overshoot* beyond the set point. We typically want to decrease all these as much as possible without making the system unstable or sensitive to external disturbances and system variations.

The *final value theorem* states that

$$\lim_{s \to 0} sX(s) = x(\infty)$$

Similarly, the *initial value theorem* states that

$$\lim_{s \to \infty} sX(s) = x(0)$$

For any system $H(s)$, the step response is $u(t) * h(t) = \mathscr{F}^{-1}\left\{\frac{1}{s}H(s)\right\}$. We can find the DC gain using the final value theorem such that

$$\text{DC gain} = \lim_{s\to 0} \cancel{s} \cdot \underbrace{\frac{1}{\cancel{s}}H(s)}_{\textit{step response}} = \lim_{s\to 0} H(s)$$

Proportional control

Proportional control is letting $K(s) = K_p$. For example, assume $H(s)$ is a first order system

$$H(s) = \frac{1/\tau}{s + 1/\tau}$$

If we let $K(s) = K_p$, the overall transfer function becomes

$$\frac{Y(s)}{Y_{sp}(s)} = \frac{K_p \frac{1/\tau}{s+1/\tau}}{1 + K_p \frac{1/\tau}{s+1/\tau}} = \frac{K_p/\tau}{s + (K_p + 1)/\tau}$$

The DC gain is $K_p/(K_p + 1)$ and the equivalent time constant is $\tau/(K_p + 1)$. It seems that choosing an arbitrarily high K_p would reduce the DC gain and the settling time. However, any real system will have some delay. This can be modeled as a system with step response $\delta(t - t_0)$. The transfer function of this delay system is e^{-st_0}. Including this delay in the system yields

$$\frac{Y(s)}{Y_{sp}(s)} = \frac{K_p e^{-st_0}/\tau}{s + (K_p e^{-st_0} + 1)/\tau}$$

Since $t_0 \ll 1$, we can approximate the delay as $e_{-st_0} \approx 1 - st_0$. Substituting this approximation into the transfer function yields

$$\frac{Y(s)}{Y_{sp}(s)} \approx \frac{K_p(1 - st_0)/\tau}{s + [K_p(1 - st_0) + 1]/\tau} = \frac{K_p(1 - st_0)/\tau}{s(1 - K_p t_0/\tau) + (K_p + 1)/\tau}$$

The resulting pole is

$$s = \frac{-(K_p + 1)/\tau}{(1 - K_p t_0/\tau)}$$

which means the system will become unstable if $s > 0$, or

$$1 < K_p t_0/\tau \quad \Rightarrow \quad K_p > \tau/t_0$$