Olin College of Engineering
ENGR2410 – Signals and Systems

Reference 2

Definitions

Linear system:

$$\begin{aligned} &\text{If} \quad x_1 \to \boxed{L} \to y_1 \quad \text{and} \quad x_2 \to \boxed{L} \to y_2, \\ &\text{then} \quad ax_1 + bx_2 \to \boxed{L} \to ay_1 + by_2. \end{aligned}$$

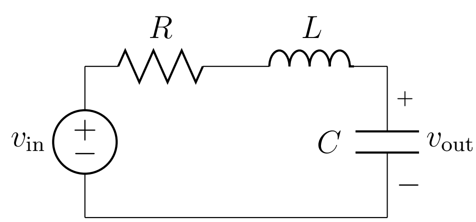
Time-invariant system:

$$\begin{aligned} &\text{If} \quad x_1(t) \to \boxed{TI} \to y_1(t) \\ &\text{then} \quad x_1(t-T) \to \boxed{TI} \to y_1(t-T). \end{aligned}$$

Eigenfunctions

$$e^{st} \to \boxed{LTI} \to \underbrace{H(s)}_{\text{transfer function}} \underbrace{e^{st}}_{\text{eigenfunction}}$$

Example:



If $v_{in} = e^{st}$ in the series RLC circuit above, when know that $v_{out} = H(s)e^{st}$. Find $H(s)$ by substituting in the differential equation:

$$\begin{aligned} &\ddot{v}_o + 2\alpha\dot{v}_o + \omega_0^2 v_o = \omega_0^2 v_i \\ &H(s)s^2\cancel{e^{st}} + 2\alpha H(s)s\cancel{e^{st}} + \omega_0^2 H(s)\cancel{e^{st}} = \omega_0^2\cancel{e^{st}} \\ &H(s)(s^2 + 2\alpha s + \omega_0^2) = \omega_0^2 \\ &H(s) = \frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2} \end{aligned}$$

Sinusoidal Steady State

A special (but important) case is when $s = j\omega$ such that the input is $e^{j\omega t}$, a complex exponential. By Euler's Equation,

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t).$$

We can "construct" the cosine function using a linear combination of two complex exponentials:

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t}).$$

By linearity,

$$\cos\omega t \to \boxed{LTI} \to \text{output}$$

is equivalent to

$$\begin{array}{ccc} \frac{1}{2}e^{j\omega t} \to \boxed{LTI} \to \frac{1}{2}H(j\omega)e^{j\omega t} \\ + \qquad\qquad\qquad\qquad + \\ \frac{1}{2}e^{-j\omega t} \to \boxed{LTI} \to \frac{1}{2}H(-j\omega)e^{-j\omega t} \end{array}$$

Therefore, the output is

$$\frac{1}{2}H(j\omega)e^{j\omega t} + \frac{1}{2}H(-j\omega)e^{-j\omega t}.$$

If H is *holomorphic*, then $H(-j\omega) = H^*(j\omega)$, where the asterisk denotes the complex conjugate. Likely all the functions you will use will be holomorphic.

In polar form,

$$\begin{aligned} H(j\omega) &= |H(j\omega)|e^{j\angle H(j\omega)} \\ H(-j\omega) &= H^*(j\omega) = |H(j\omega)|e^{-j\angle H(j\omega)} \end{aligned}$$

So the output of the system is

$$\begin{aligned} &\frac{1}{2}|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t} + \frac{1}{2}|H(j\omega)|e^{-j\angle H(j\omega)}e^{-j\omega t} \\ =&|H(j\omega)|\frac{e^{j(\omega t+\angle H(j\omega)} + e^{-j(\omega t+\angle H(j\omega)}}{2} \\ =&|H(j\omega)|\cos(\omega t + \angle H(j\omega)). \end{aligned}$$

The general solution for the sinusoidal steady state is

$$\boxed{\cos\omega t \to \boxed{LTI} \to |H(j\omega)|\cos(\omega t + \angle H(j\omega))}$$

Note:
(i) the output has the same frequency as the input
(ii) the output is scaled and shifted by $H(j\omega)$

For example, the transfer function of the series RLC circuit is

$$H(j\omega) = \frac{\omega_0^2}{-\omega^2 + j2\alpha\omega + \omega_0^2}$$

Its magnitude and phase are

$$|H(j\omega)| = \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\alpha\omega)^2}}, \quad \angle H(j\omega) = -\tan^{-1}\left(\frac{2\alpha\omega}{\omega_0^2 - \omega^2}\right).$$

Therefore, if

$$v_i = V\cos(\omega t),$$

then

$$v_o = V\frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\alpha\omega)^2}}\cos\left[\omega t - \tan^{-1}\left(\frac{2\alpha\omega}{\omega_0^2 - \omega^2}\right)\right].$$

Bode Plot

The plot of $|H(j\omega)|$ and $\angle H(j\omega)$ as a function of ω completely describes $H(j\omega)$. A Bode plot shows $\log|H(j\omega)|$ as a function of $\log\omega$ for the magnitude and $\angle H(j\omega)$ as a function of $\log\omega$ for the angle.

In order to sketch a Bode plot, we consider the asymptotes at both extremes. For example, the transfer function of the series RLC circuit is

$$H(j\omega) = \frac{\omega_0^2}{-\omega^2 + j2\alpha\omega + \omega_0^2}$$

$$\text{If } \omega \to 0, \text{ then } H(j\omega) \approx \frac{\omega_0^2}{\omega_0^2} = 1 \qquad \text{and} \qquad |H(j\omega)| = 1, \quad \angle H(j\omega) = 0$$

$$\text{If } \omega \to \infty, \text{ then } H(j\omega) \approx -\frac{\omega_0^2}{\omega^2} \qquad \text{and} \qquad |H(j\omega)| = \frac{\omega_0^2}{\omega^2}, \quad \angle H(j\omega) = \pm\pi$$

In the magnitude plot, the two lines intersect when $1 = \frac{\omega_0^2}{\omega^2}$, or $\omega = \omega_0$, the resonant frequency. If we substitute the resonant frequency into the transfer function, we find

$$H(j\omega_0) = \frac{\omega_0^2}{j2\alpha\omega_0} = -j\frac{\omega_0}{2\alpha} \qquad \text{and} \qquad |H(j\omega)| = \frac{\omega_0}{2\alpha} \triangleq Q, \quad \angle H(j\omega) = -\frac{\pi}{2}.$$

The variable Q is called the quality factor and is widely used to characterize 2nd order systems. The Bode plot is shown below.