Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 1 Solutions

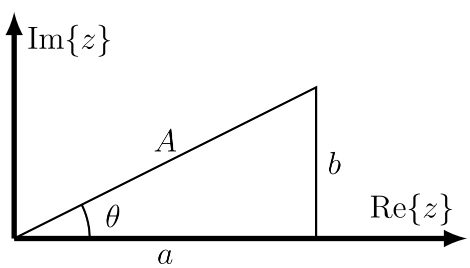
Problem 1: *(2 points)*

A. Find explicit algebraic expressions for A and θ in terms of a and b such that

$$Ae^{j\theta} = a + jb$$

where $j^2 = -1$ and $e^{j\theta} = \cos\theta + j\sin\theta$. Use the complex plane to illustrate.

Solution:



$$A = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a}$$

B. Invert the transformation of part A and find explicit algebraic expressions for a and b in terms of A and θ. Again, use the complex plane to illustrate.

Solution:
Using the same geometry as above, $\cos\theta = \frac{a}{A}$ and $\sin\theta = \frac{b}{A}$. As a result:

$$a = A\cos\theta \qquad b = A\sin\theta$$

C. Find explicit algebraic expressions for A and θ such that

$$Ae^{j\theta} = \frac{jb_1}{a_2 + jb_2} \cdot \frac{a_3 + jb_3}{-a_4}$$

Solution:

$$\begin{aligned}
\frac{jb_1}{a_2 + jb_2} \cdot \frac{a_3 + jb_3}{-a_4} &= \frac{b_1 e^{j\frac{\pi}{2}}}{\sqrt{a_2^2 + b_2^2} e^{j \tan^{-1}\left(\frac{b_2}{a_2}\right)}} \cdot \frac{\sqrt{a_3^2 + b_3^2} e^{j \tan^{-1}\left(\frac{b_3}{a_3}\right)}}{a_4 e^{j\pi}} \\
&= \frac{b_1}{\sqrt{a_2^2 + b_2^2}} \cdot \frac{\sqrt{a_3^2 + b_3^2}}{a_4} e^{j\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{b_2}{a_2}\right) + \tan^{-1}\left(\frac{b_3}{a_3}\right) - \pi\right)} \\
A &= \frac{b_1}{\sqrt{a_2^2 + b_2^2}} \cdot \frac{\sqrt{a_3^2 + b_3^2}}{a_4} \\
\theta &= \frac{\pi}{2} - \tan^{-1}\left(\frac{b_2}{a_2}\right) + \tan^{-1}\left(\frac{b_3}{a_3}\right) - \pi
\end{aligned}$$

There are many other equivalent solutions, found at various points of simplification.

Problem 2: *(4 points)*

A. Solve the easy problem: show that the solution for the system of equations

$$\begin{bmatrix} \dot{x_1}(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

Solution:
The first equation can be separated into two independent equations: $\dot{x_1} = 4x_1$ and $\dot{x_2} = 2x_2$. We can solve these equations separately.

$$\begin{aligned}
\frac{dx_1}{dt} &= 4x_1 \\
\frac{dx_1}{x_1} &= 4dt \\
\int \frac{dx_1}{x_1} &= \int 4dt \\
\ln x_1 &= 4t + C \\
e^{\ln x_1} &= e^{4t+C} \\
x_1 &= e^C e^{4t}
\end{aligned}$$

Since we know $x_1(0) = x_{0,1}$,

$$x_1 = x_{0,1}e^{4t}$$

Similarly, $x_2 = x_{0,2}e^{2t}$. Rewriting these two equations with matrices we get:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

B. Solve the hard problem: show that the solution for the system of equations

$$\begin{bmatrix} \dot{x_1}(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

Solution:
By the definition of eigenvectors and eigenvalues, we know that for some vector v,

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ (A - \lambda I)\vec{v} &= \vec{0} \end{aligned}$$

Thus, we find the values of λ that make $A - \lambda I$ singular.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} &= 0 \\ &= 9 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

Therefore, $\lambda_1 = 4$ and $\lambda_2 = 2$. For $\lambda_1 = 4$,

$$\begin{aligned} \begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \vec{v_1} &= \vec{0} \\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v_1} &= \vec{0} \\ \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

For $\lambda_1 = 2$,

$$\begin{aligned}
\begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \vec{v_2} &= \vec{0} \\
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{v_2} &= \vec{0} \\
\vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

We can rewrite the two equations $A\vec{v_1} = \lambda_1 \vec{v_1}$ and $A\vec{v_2} = \lambda_2 \vec{v_2}$ using matrices.

$$A[\vec{v_1}\vec{v_2}] = [\vec{v_1}\vec{v_2}] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Right-multiplying both sides by the inverse of the eigenvector matrix, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$, yields

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

Now we substitute for A in our original equation:

$$\begin{bmatrix} \dot{x_1}(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Left-multiplying both sides by the inverse of the eigenvector matrix we get

$$\overset{\cdot}{\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right)} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Substituting $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ for $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ we obtain an equation that is in the form of the easy problem from part A.

$$\begin{bmatrix} \dot{y_1}(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$\vec{y}$ is a transformation of $\vec{x}$,

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where $\vec{x}$ is expressed in the basis of the eigenvectors of the original matrix A.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Using this technique, we have transformed the problem into something we can solve. From here, we solve for $\vec{y}$,

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} y_{0,1} \\ y_{0,2} \end{bmatrix}$$

then solve back for $\vec{x}$.

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} \\ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} \end{aligned}$$

Voila!

C. Generalize. Follow the same steps to show that the solution to

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x} = Ve^{\Lambda t}V^{-1}\mathbf{x}_0$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}, \quad AV = V\Lambda, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

and V is an invertible matrix.

$A = V\Lambda V^{-1}$ is a similarity transformation *such that A and Λ are* similar *matrices. Moreover, since Λ is a diagonal matrix and V is invertible, A is a* diagonalizable *matrix. Check out* *http://en.wikipedia.org/wiki/Diagonalizable_matrix* *and* *http://planetmath.org/diagonalization*.

Solution:
We are presented with the equation $\dot{\vec{x}} = A\vec{x}$ where A can be any matrix. If A was in the form $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, we would be just fine because we know how to solve that problem (you solved it in part A). Unfortunately, we don't know that A looks like that, so we are going to try and rewrite the equation $\dot{\vec{x}} = A\vec{x}$ in such a way that it looks like this $\dot{\vec{y}} = B\vec{y}$ where B is an array in the form $\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$. It seems reasonable that to do this we'll need to substitute something in for the matrix A and manipulate the equation.

Since A is a diagonalizable matrix, it has two eigenvectors (v_1, v_2) and two eigenvalues (λ_1, λ_2). By the definition of eigenvectors and eigenvalues,

$$A\vec{v_1} = \lambda_1\vec{v_1}, \; A\vec{v_1} = \lambda_1\vec{v_1}$$

We can write these equations using matrices:

$$\begin{aligned} A[\vec{v_1}\ \vec{v_2}] &= [\lambda_1\vec{v_1}\ \lambda_2\vec{v_2}] \\ A[\vec{v_1}\ \vec{v_2}] &= [\vec{v_1}\ \vec{v_2}]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ AV &= V\Lambda \end{aligned}$$

Now we can solve for A.

$$\begin{aligned} AV &= V\Lambda \\ AVV^{-1} &= V\Lambda V^{-1} \\ A &= V\Lambda V^{-1} \end{aligned}$$

By substitution into our original equation $\dot{\vec{x}} = A\vec{x}$,

$$\dot{\vec{x}} = V\Lambda V^{-1}\vec{x}$$

So now what? Well, the matrix Λ is in the same form as the matrix B so wouldn't it be cool if we could substitute $V^{-1}\vec{x}$ for $\vec{y}$. First, though, we need to make the left side of the equation look like $\dot{\vec{y}}$. We can do this by multiplying both sides by V^{-1} (V^{-1} is a matrix of constants so we can stick it inside the derivative).

$$\dot{\left(V^{-1}\vec{x}\right)} = \Lambda\left(V^{-1}\vec{x}\right)$$

Let $\vec{y} = V^{-1}\vec{x}$,

$$\dot{\vec{y}} = \Lambda\vec{y}$$

We now have the equation in a solvable form. We know the solution to this problem from part A.

$$\vec{y} = e^{\Lambda t}\vec{y}_0$$

Since $\vec{y} = V^{-1}\vec{x}$, $\vec{x} = V\vec{y}$. Similarly, we also solve for $\vec{y}_0$: $\vec{y}_0 = V^{-1}\vec{x}_0$. Putting it all together:

$$\begin{aligned}\vec{x} &= V\vec{y}\\ \vec{x} &= Ve^{\Lambda t}\vec{y}_0\\ \vec{x} &= Ve^{\Lambda t}V^{-1}\vec{x}_0\end{aligned}$$

Problem 3: *(4 points)*

A. Show that the general solution for the underdamped second order differential equation

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0, \qquad \omega_0 > \alpha$$

is

$$x = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}), \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Solution:
We start by guessing that the solution is an exponential, Ae^{st}. We aim to find values of s which make that guess correct.

Take the derivative and second derivative and plug these expressions into the differential equation.

$$x = Ae^{st} \qquad \dot{x} = sAe^{st} \qquad \ddot{x} = s^2 Ae^{st}$$

$$s^2 Ae^{st} + 2\alpha s Ae^{st} + \omega_0^2 Ae^{st} = 0$$

Ae^{st} is a common factor, so we can remove it to simplify the expression.

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Next, we need to solve for s as a function of α and ω_0. We can use the quadratic formula to solve for the roots of this function.

$$s = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$$

Both of these values give us solutions to the differential equation in the form of our initial guess, Ae^{st}. If we add these two solutions together (since a linear combination of solutions is also a solution) we get the following expression, the general solution:

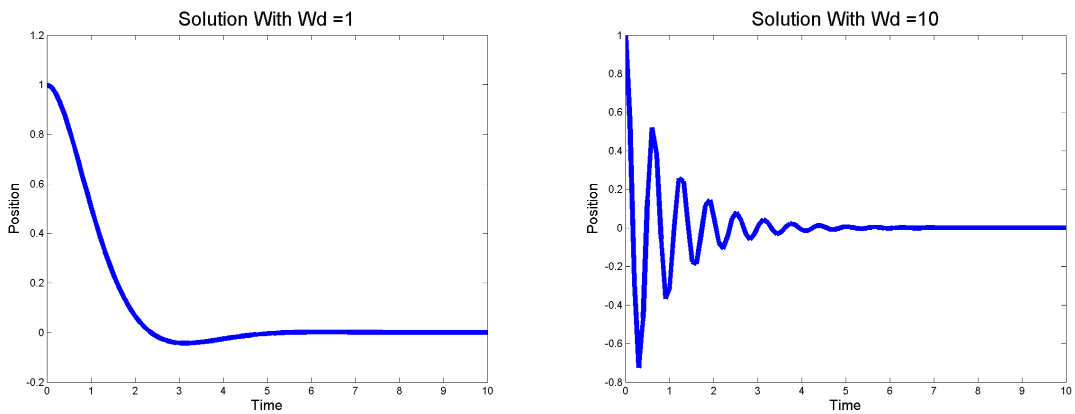
$$x = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$
$$x = e^{-\alpha t}(A_1 e^{jw_d t} + A_2 e^{-jw_d t})$$

B. The specific solution with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ is

$$x = x_0 \sqrt{1 + (\alpha/\omega_d)^2} e^{-\alpha t} \cos(\omega_d t - \tan^{-1} \alpha/\omega_d)$$

Assume $x_0 = \alpha = 1$. Use a computer to graph this solution for $\omega_d = 1$ and $\omega_d = 10$. For each value of ω_d, verify the initial conditions graphically and zoom out to show the asymptotic behavior.

Solution:
This is the solution for ten seconds, which shows the asymptotic behavior for both values of ω_d.



Here's the code to generate these graphs.

```
%returns the solution at a particular instance in time
function x = solve_diffeq(wd, t)
    alpha = 1;
    x0 = alpha;
    left = x0 * sqrt(1 + (alpha/wd)^2) * exp(-alpha*t);
    right = cos(wd*t - atan(alpha/wd));
    x = left .* right;
end

%plots the solution to 3b.
function plot_sol(wd, end_time)
    time = linspace(0, end_time,100);
    clf;
    h = figure();
    plot(time, solve_diffeq(wd, time));
    title(strcat('Solution With Wd = ',num2str(wd)), 'fontsize',20);
    xlabel('Time','fontsize',15);
    ylabel('Position','fontsize',15);
end
```

If you are curious, the complete derivation of the solution follows. This is painful algebra!

Let's first restate the problem.

$$\ddot{x} + 2\alpha\dot{x} + \omega_0 x = 0, \quad x(0) = x_0, \quad \dot{x}(0) = 0, \quad \omega_0 > \alpha \text{ (underdamped)}$$

As usual, assume $x = Ae^{st}$. Let's carefully define the complex conjugates values.

$$s = -\alpha + j\underbrace{\sqrt{\omega_0^2 - \alpha^2}}_{\omega_d}, \text{ and } s^* = -\alpha - j\underbrace{\sqrt{\omega_0^2 - \alpha^2}}_{\omega_d}$$

The general solution is

$$x = A_1 e^{st} + A_2 e^{s^* t}$$

Since x must be real, A_2 must be the complex conjugate of A_1

$$A_2 = A_1^* \text{ for } \mathrm{Im}\{x\} = 0$$

Substituting and taking the derivative,

$$x = Ae^{st} + A^* e^{s^* t} \qquad \dot{x} = sAe^{st} + s^* A^* e^{s^* t}$$

The initial condition $x(0) = x_0$ yields the real part of A

$$x(0) = A + A^* = x_0 \Rightarrow \mathrm{Re}\{A\} = x_0/2$$

The initial condition $\dot{x}(0) = 0$ yields the imaginary part of A

$$\dot{x}(0) = sA + s^* A^* = 2\mathrm{Re}\{sA\}$$

$$sA = (-\alpha + j\omega_d)(x_0/2 + j\mathrm{Im}\{A\})$$

$$\mathrm{Re}\{sA\} = -\alpha x_0/2 - \omega_d \mathrm{Im}\{A\} = 0$$

$$\Rightarrow \mathrm{Im}\{A\} = -\frac{\alpha}{\omega_d}\frac{x_0}{2}$$

Since we know the real and imaginary parts of A,

$$A = \frac{x_0}{2} - j\frac{\alpha}{\omega_d}\frac{x_0}{2} = \frac{x_0}{2}\left(1 - j\frac{\alpha}{\omega_d}\right)$$

We know express A in terms of magnitude and phase,

$$A = |A|e^{j\angle A} = \frac{x_0}{2}\sqrt{1 + (\alpha/\omega_d)^2}\, e^{-j\tan^{-1}\alpha/\omega_d}$$

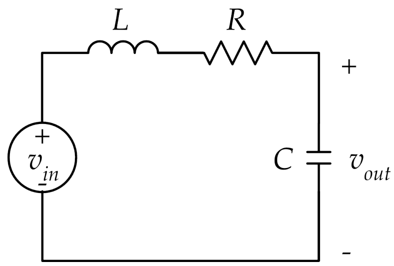
Since x is the sum of two complex conjugates, it can be written in terms of a real-valued cosine,

$$x = 2|A|e^{-\alpha t}\cos(\omega_d t + \angle A)$$

Substituting the magnitude and phase of A finally yields

$$x = x_0\sqrt{1 + (\alpha/\omega_d)^2}\, e^{-\alpha t}\cos(\omega_d t - \tan^{-1}\alpha/\omega_d)$$

C. Find the step response of the circuit shown below. Assume that $\frac{R}{2} < \sqrt{\frac{L}{C}}$ and let $v_{in} = Vu(t)$. Find an expression for v_{out} and graph it. *Hint 1: Since the circuit was at rest for* $t < 0$*, you can assume* $v_{out}(0) = 0$ *and* $\dot{v}_{out}(0) = 0$*. Hint 2: Use a linear transformation of the solution from the previous part to avoid deriving the solution again.*



Solution:
The step response is the response of a system to an instantaneous change in input from 0 to some constant. The electrical analogy is simply turning the power on.

The equation for this system is best expressed in terms of the voltage drop across each component (inductor first, then resistor, then capacitor) which, by Kirchoff's Voltage Law, must sum to the voltage increase generated by the battery.

$$\begin{aligned}
v_{in} &= L\frac{di}{dt} + Ri + v_{out} \\
i &= C\frac{dv_{out}}{dt} \\
v_{in} &= L\frac{d}{dt}\left(C\frac{dv_{out}}{dt}\right) + RC\frac{dv_{out}}{dt} + v_{out} \\
v_{in} &= LC\frac{d^2v_{out}}{dt^2} + RC\frac{dv_{out}}{dt} + v_{out} \\
\frac{d^2v_{out}}{dt^2} + \frac{R}{L}\frac{dv_{out}}{dt} &+ \frac{1}{LC}v_{out} = \frac{1}{LC}v_{in}
\end{aligned}$$

This is almost identical to the expression in parts A and B, but rather than summing to 0 they sum to v_{in}. We can still use the solution from the previous parts if we make this expression sum to 0 by linearly shifting the entire system by the amplitude of the unit step, V. After t=0, v_{in} equals V. We can define a new variable, v_2, that represents a shifted voltage and insert it into our equation.

$$v_2 = v_{out} - V \qquad \Rightarrow \qquad v_{out} = v_2 + V$$

This V is a constant, so when taking the first and second derivative of v_2, it is eliminated, leaving only a single V term.

$$\frac{d^2v_2}{dt^2} + \frac{R}{L}\frac{dv_2}{dt} + \frac{1}{LC}v_2 + \frac{1}{LC}V = \frac{1}{LC}V$$
$$\frac{d^2v_2}{dt^2} + \frac{R}{L}\frac{dv_2}{dt} + \frac{1}{LC}v_2 = 0$$

This is now in the same form as in part A. Additionally, the initial conditions become

$$v_2(0) = v_{out}(0) - V = -V$$
$$\dot{v}_2(0) = \dot{v}_{out}(0) = 0$$

Using part B, the solution becomes

$$v_2(t) = -V\sqrt{1+(\alpha/\omega_d)^2}e^{-\alpha t}\cos(\omega_d t - \tan^{-1}\alpha/\omega_d)$$

where

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

Finally, we can shift back to solve for v_{out}.

$$v_{out}(t) = V - V\sqrt{1+(\alpha/\omega_d)^2}e^{-\alpha t}\cos(\omega_d t - \tan^{-1}\alpha/\omega_d)$$

Here's a sample graph. Unsurprisingly, the step response is is similar to the previous problem–the voltage oscillates towards the voltage supplied by the battery. Electrically, this makes sense, as an RLC circuit, given enough time, will reach a steady state where there is no current at all. The voltage supplied by the battery will eventually equal the voltage across the capacitor.