Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 6

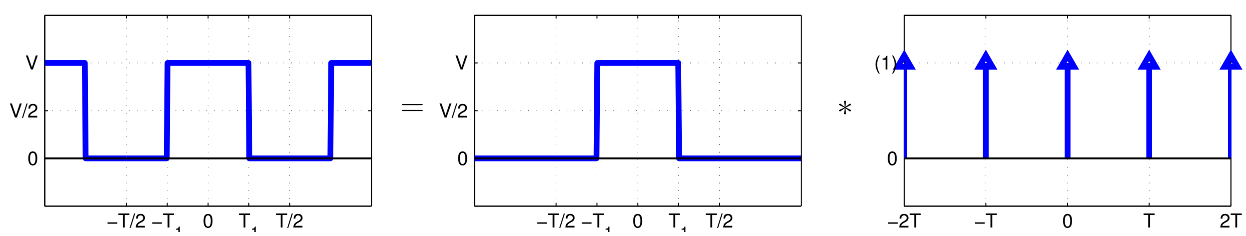
Problem 1 *(10 points)*

A. Find an expression for $\mathscr{F}\{v(t)\}$, where

$$v(t) = \begin{cases} V & -T_1 + nT < t < T_1 + nT, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

Hint: In lecture 6, we showed that convolution with a shifted impulse creates a copy of the original signal shifted by that amount. Use this to express $v(t)$ as the convolution of a pulse and an impulse train, as shown below.

$$v(t) = \left\{ \begin{matrix} V & -T_1 < t < T_1 \\ 0 & \text{otherwise} \end{matrix} \right\} * \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$



Solution:

In lecture 6, we showed that

$$\mathscr{F}\{x(t) * y(t)\} = X(j\omega)Y(j\omega)$$

Therefore,

$$\mathscr{F}\{v(t)\} = \mathscr{F}\left\{ \begin{matrix} V & -T_1 < t < T_1 \\ 0 & \text{otherwise} \end{matrix} \right\} \cdot \mathscr{F}\left\{ \sum_{k=-\infty}^{+\infty} \delta(t - kT) \right\}$$

In lecture 5, we showed that

$$v(t) = \begin{cases} V & -T_1 < t < T_1 \\ 0 & \text{otherwise} \end{cases} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad 2VT_1 \text{sinc}\,(\omega T_1)$$

In lecture 6, we showed that the Fourier transform of the impulse train $x(t)$ of period T and unit area is another impulse train of both period and area $2\pi/T$,

$$\sum_{k=-\infty}^{+\infty} \delta(t-kT) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T}n\right)$$

Therefore,

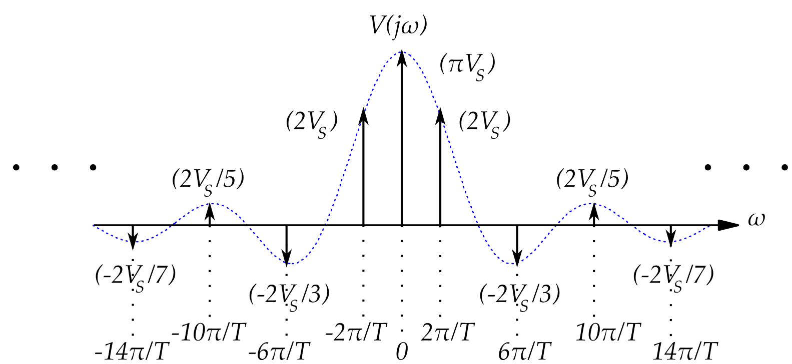
$$\mathscr{F}\{v(t)\} = 2VT_1 \text{sinc}\,(\omega T_1) \cdot \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T}n\right)$$

$$\mathscr{F}\{v(t)\} = 2VT_1 \cdot \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty} \text{sinc}\left(\frac{2\pi}{T}nT_1\right) \delta\left(\omega - \frac{2\pi}{T}n\right)$$

$$\mathscr{F}\{v(t)\} = 2\pi \cdot 2V\frac{T_1}{T}\sum_{n=-\infty}^{+\infty} \text{sinc}\left(2\pi n\frac{T_1}{T}\right) \delta\left(\omega - 2\pi n\frac{1}{T}\right)$$

B. Graph $\mathscr{F}\{v(t)\}$ for the case $T_1 = T/4$ and $V = V_S$.

Solution:



C. Show that the Fourier transform you found is equivalent to the coefficients for the even square wave with period T and pulse width $2T_1$,

$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}, \qquad c_n = 2V\frac{T_1}{T}\text{sinc}\left(2\pi n\frac{T_1}{T}\right)$$

Solution:

$$v(t) = \int_{-\infty}^{\infty} \mathscr{F}\{v(t)\}e^{j\omega t}\frac{d\omega}{2\pi}$$

$$v(t) = \int_{-\infty}^{\infty} \cancel{2\pi} \cdot 2V\frac{T_1}{T}\sum_{n=-\infty}^{+\infty} \text{sinc}\left(2\pi n\frac{T_1}{T}\right)\delta\left(\omega - 2\pi n\frac{1}{T}\right)e^{j\omega t}\frac{d\omega}{\cancel{2\pi}}$$

$$v(t) = \sum_{n=-\infty}^{+\infty} 2V\frac{T_1}{T}\text{sinc}\left(2\pi n\frac{T_1}{T}\right)\int_{-\infty}^{\infty}\delta\left(\omega - 2\pi n\frac{1}{T}\right)e^{j\omega t}d\omega$$

$$v(t) = \sum_{n=-\infty}^{+\infty} \underbrace{2V\frac{T_1}{T}\text{sinc}\left(2\pi n\frac{T_1}{T}\right)}_{c_n} e^{j\frac{2\pi}{T}nt}$$

D. *Optional:* In Quiz 5, you found that the frequency content of a period T function only exists in the harmonics of $2\pi/T$ (multiples of $\frac{2\pi}{T}$) and must be zero elsewhere. Can you generalize the transform of a periodic function $x(t) = x(t+T)$? For consistency of notation, define $x_T(t)$ as a single period of $x(t)$ and $X_T(j\omega) = \mathscr{F}\{x_T(t)\}$.

Solution:

$$\mathscr{F}\{x(t)\} = \mathscr{F}\{x_T(t)\} \cdot \mathscr{F}\left\{\sum_{k=-\infty}^{+\infty}\delta(t-kT)\right\}$$

$$\mathscr{F}\{x(t)\} = X_T(j\omega) \cdot \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi}{T}n\right)$$

$$\mathscr{F}\{x(t)\} = \sum_{n=-\infty}^{+\infty}\frac{2\pi}{T} \cdot X_T(j\omega)\delta\left(\omega - \frac{2\pi}{T}n\right)$$

$$\mathscr{F}\{x(t)\} = \sum_{n=-\infty}^{+\infty}\frac{2\pi}{T} \cdot X_T\left(j\frac{2\pi}{T}n\right)\delta\left(\omega - \frac{2\pi}{T}n\right)$$

E. *Optional:* Generalize the equivalence to series

$$c_n = \frac{1}{T} \cdot X_T\left(j\frac{2\pi}{T}n\right).$$

Solution:

$$x(t) = \int_{-\infty}^{\infty} \mathscr{F}\{x(t)\} e^{j\omega t} \frac{d\omega}{2\pi}$$

$$x(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{+\infty} \frac{\cancel{2\pi}}{T} \cdot X_T\left(j\frac{2\pi}{T}n\right) \delta\left(\omega - \frac{2\pi}{T}n\right) e^{j\omega t} \frac{d\omega}{\cancel{2\pi}}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T} \cdot X_T\left(j\frac{2\pi}{T}n\right) \int_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T}n\right) e^{j\omega t} d\omega$$

$$x(t) = \sum_{n=-\infty}^{+\infty} \underbrace{\frac{1}{T} \cdot X_T\left(j\frac{2\pi}{T}n\right)}_{c_n} e^{j\frac{2\pi}{T}nt}$$