Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 1

Problem 1 LTI analysis relies heavily on manipulations with complex numbers. The solutions to this problem will be very useful throughout the course.

A. Find explicit algebraic expressions for A and θ in terms of a and b such that

$$Ae^{j\theta} = a + jb$$

where $j^2 = -1$ and $e^{j\theta} = \cos\theta + j\sin\theta$. Use the complex plane to illustrate.

B. Invert the transformation of part A and find explicit algebraic expressions for a and b in terms of A and θ. Again, use the complex plane to illustrate.

C. Find explicit algebraic expressions for A and θ such that

$$Ae^{j\theta} = \frac{jb_1}{a_2 + jb_2} \cdot \frac{a_3 + jb_3}{-a_4}$$

D. Simplify these expressions:

(i) $|e^{j\pi/2}| + |e^{j\pi}|$

(ii) $|e^{j\pi/2} + e^{j\pi}|$

E. Write $je^{j\pi}$ in Cartesian form (i.e., $a + jb$).

F. Write j in polar form (i.e., $Ae^{j\phi}$). What is the effect of multiplying any number by j?

Problem 2 This problems uses a *similarity transformation* to solve a system of coupled, first-order differential equations. This transformation is necessary to arrive at the general solution! You will need to find *(or guess correctly)* eigenvalues and eigenvectors.

A. Solve the easy problem: show that the solution for the system of equations

$$\begin{bmatrix} \dot{x_1}(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

B. Solve the hard problem: show that the solution for the system of equations

$$\begin{bmatrix} \dot{x_1}(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

C. Generalize. Follow the same steps to show that the solution to

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x} = Ve^{\Lambda t}V^{-1}\mathbf{x}_0$$

where

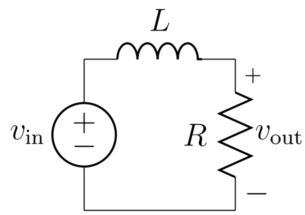
$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}, \quad AV = V\Lambda, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

and V is an invertible matrix.

$A = V\Lambda V^{-1}$ is a similarity transformation *such that A and Λ are* similar *matrices. Moreover, since Λ is a diagonal matrix and V is invertible, A is a* diagonalizable *matrix. Check out* http://en.wikipedia.org/wiki/Diagonalizable_matrix *and* http://planetmath.org/diagonalization*.*

Problem 3 This problem is essentially identical to the RC circuit derived in class. You can write most of the answer by inspection.

A. Write a differential equation relating v_{in} and v_{out}. Remember that for an inductor, $v_L = L\frac{di_L}{dt}$.



B. If $v_{in} = V$ for $t > 0$, write the particular solution for v_{out}.

C. Write the homogeneous solution for v_{out}.

D. Write the specific solution for v_{out} assuming $v_{out}(0) = 0$.

E. Make a clear, neat sketch of v_{out} for $t > 0$. Show the initial value, asymptote and time constant.

Problem 4 This problem is hard, but it steps you through the derivation of the second order underdamped response, one of the most useful equations in engineering. Be neat, and work with someone else. You will likely need to retrace your steps. Remember that this system models all sorts of decaying oscillations, e.g., a pendulum, a car after a bump, electromagnetic waves moving through space. (In fact, check out the examples in https://en.wikipedia.org/wiki/Oscillation.) Totally worth it!

A. Show that the general solution for the underdamped second order differential equation

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0, \qquad \omega_0 > \alpha$$

is

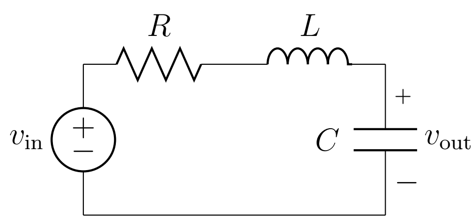
$$x = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}), \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

B. The specific solution with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ is

$$x = x_0\sqrt{1 + (\alpha/\omega_d)^2} e^{-\alpha t} \cos(\omega_d t - \tan^{-1} \alpha/\omega_d)$$

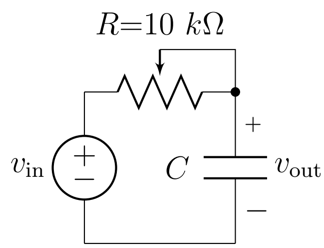
Assume $x_0 = \alpha = 1$. Use a computer to graph this solution for $\omega_d = 1$ and $\omega_d = 10$. For each value of ω_d, verify the initial conditions graphically and zoom out to show the asymptotic behavior.

C. Find the step response of the circuit shown below. Assume that $\frac{R}{2} < \sqrt{\frac{L}{C}}$ and let $v_{in} = Vu(t)$. Find an expression for v_{out} and graph it. *Hint 1: Since the circuit was at rest for $t < 0$, you can assume $v_{out}(0) = 0$ and $\dot{v}_{out}(0) = 0$. Hint 2: Use a linear transformation of the solution from the previous part to avoid deriving the solution again.*

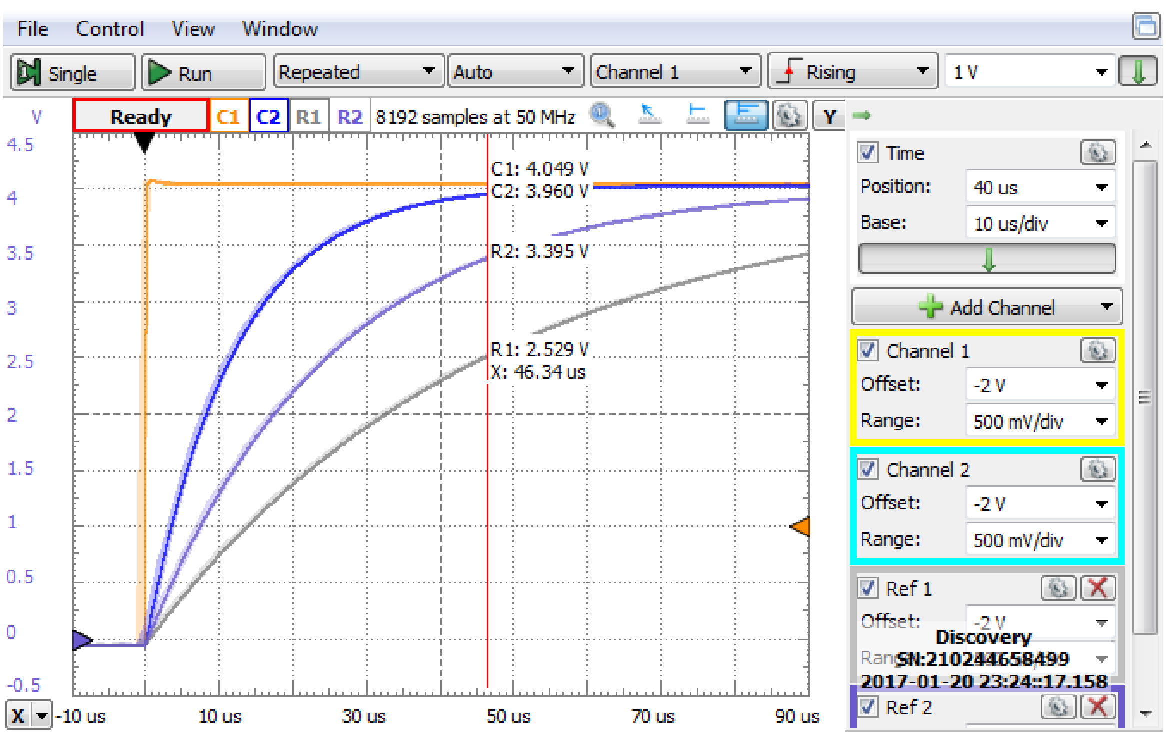


Problem 5 Engineering is not just about understanding phenomena, but also about connecting theory and measurement. Additionally, use the potentiometers to play with these circuits keeping in mind the math behind it. Be neat and keep your circuits! We will use them several times throughout the course. The course webpage links to a few videos that should be useful, in particular the one about *circuit hygiene*!

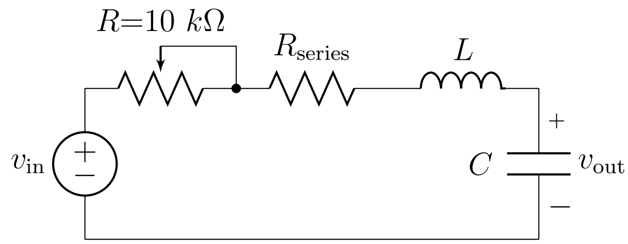
A. Build the RC circuit shown below using a $R = 10$ kΩ potentiometer. Calculate the time constant and measure it using the step response. Measure two more responses, one at two thirds and one at a third of the original time constant, and include them in your plot. *Be careful to use values where you can see the time constant, and verify the values of your components! (My first two potentiometers were more than 10% off.)*



I show my results below, but you should use your own capacitor value. I used a $C = 4.7$ nF capacitor, for time constant $RC = 47$ μs. I drove my circuit with a $v_{in} = 4$ V step. After one time constant, the model predicts $1 - e^{-1} = 63.2\%$ of the final voltage, or 4 V $\times$ 63.2% = 2.53 V. I measured this voltage at 46.3 μs, as shown below, a difference of $(46.3\ \mu\text{s} - 47\ \mu\text{s})/47\ \mu\text{s} = -1.5\%$.



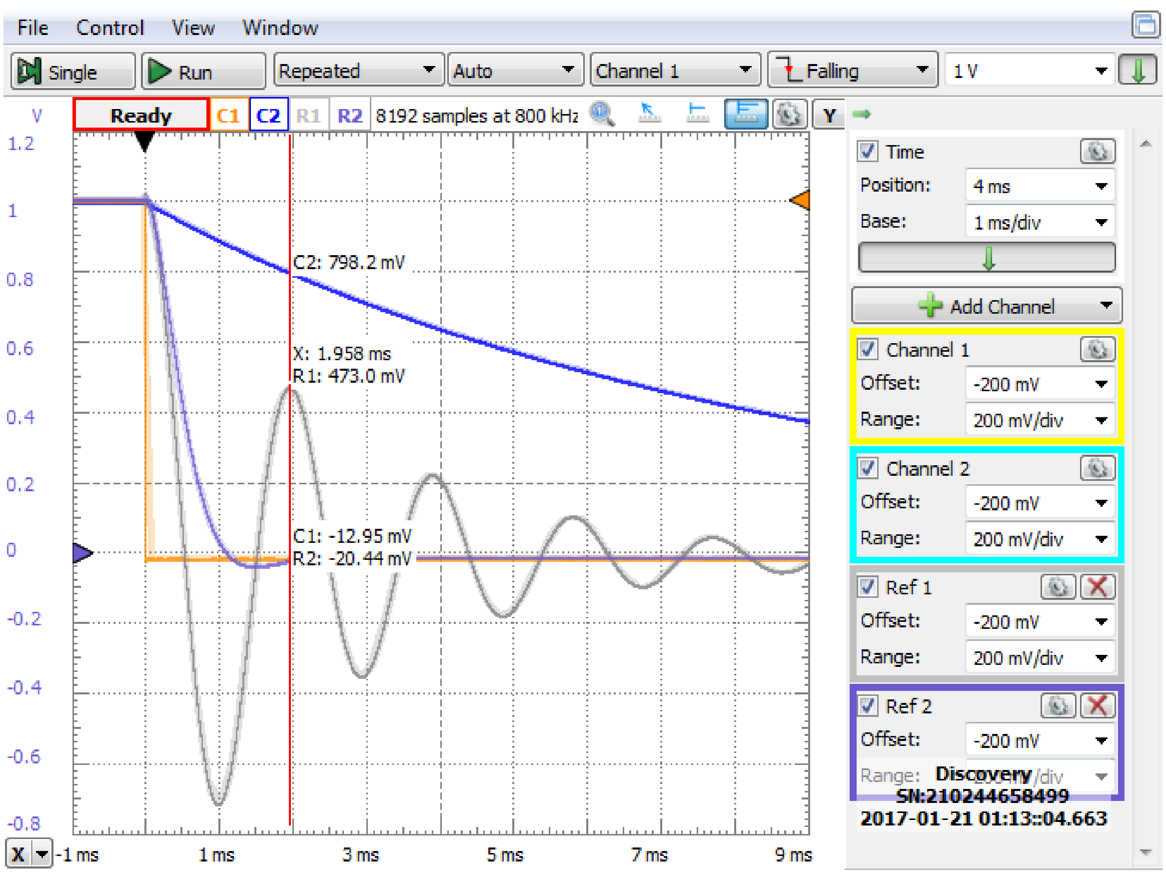
B. Build the RLC circuit shown below using a 10 kΩ potentiometer. The resistance R_{series} represents the *internal series resistance* of the inductor, which you can measure using a multimeter. Verify the frequency ω_0 and decay α when the potentiometer is shorted. Measure two more responses, one close to critical damping, and another overdamped.



I used a $C = 1\ \mu$F capacitor and a $L = 0.1$ H inductor with a series resistance $R_{\text{series}} = 69\ \Omega$. I drove the circuit with a $v_{in} = 1$ V step. The model predicts a period of $2\pi\sqrt{LC} = 1.99$ ms. The measured period shown below is 1.96 ms, a difference of -1.5%. After one period, the voltage should fall to

$$1\text{ V} \times \mathrm{e}^{-\alpha\frac{2\pi}{\omega_0}} = 1\text{ V} \times \mathrm{e}^{-\pi\frac{\mathrm{R_{series}}}{\sqrt{\mathrm{L/C}}}} = 0.504\text{ V}.$$

The measured voltage is 473 mV, a -6.2% difference.



Course feedback
Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?