Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 1

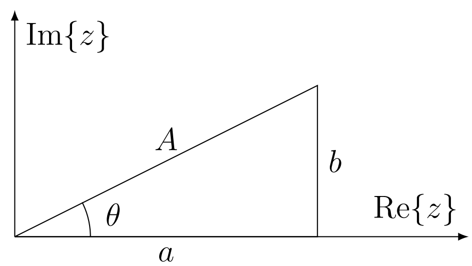
Problem 1 LTI analysis relies heavily on manipulations with complex numbers. The solutions to this problem will be very useful throughout the course.

A. Find explicit algebraic expressions for A and θ in terms of a and b such that

$$Ae^{j\theta} = a + jb$$

where $j^2 = -1$ and $e^{j\theta} = \cos\theta + j\sin\theta$. Use the complex plane to illustrate.

Solution:



$$A = \sqrt{a^2 + b^2} \qquad\qquad \theta = \tan^{-1}\frac{b}{a}$$

B. Invert the transformation of part A and find explicit algebraic expressions for a and b in terms of A and θ. Again, use the complex plane to illustrate.

Solution:
Using the same geometry as above, $\cos\theta = \frac{a}{A}$ and $\sin\theta = \frac{b}{A}$. As a result:

$$a = A\cos\theta \qquad\qquad b = A\sin\theta$$

C. Find explicit algebraic expressions for A and θ such that

$$Ae^{j\theta} = \frac{jb_1}{a_2 + jb_2} \cdot \frac{a_3 + jb_3}{-a_4}$$

Solution:

$$\begin{aligned}
\frac{jb_1}{a_2 + jb_2} \cdot \frac{a_3 + jb_3}{-a_4} &= \frac{b_1 e^{j\frac{\pi}{2}}}{\sqrt{a_2^2 + b_2^2} e^{j\tan^{-1}\left(\frac{b_2}{a_2}\right)}} \cdot \frac{\sqrt{a_3^2 + b_3^2} e^{j\tan^{-1}\left(\frac{b_3}{a_3}\right)}}{a_4 e^{j\pi}} \\
&= \frac{b_1}{\sqrt{a_2^2 + b_2^2}} \cdot \frac{\sqrt{a_3^2 + b_3^2}}{a_4} e^{j\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{b_2}{a_2}\right) + \tan^{-1}\left(\frac{b_3}{a_3}\right) - \pi\right)} \\
A &= \frac{b_1}{\sqrt{a_2^2 + b_2^2}} \cdot \frac{\sqrt{a_3^2 + b_3^2}}{a_4} \\
\theta &= \frac{\pi}{2} - \tan^{-1}\left(\frac{b_2}{a_2}\right) + \tan^{-1}\left(\frac{b_3}{a_3}\right) - \pi
\end{aligned}$$

There are many other equivalent solutions, found at various points of simplification.

D. Simplify these expressions:

(i) $|e^{j\pi/2}| + |e^{j\pi}|$

Solution:
2

(ii) $|e^{j\pi/2} + e^{j\pi}|$

Solution:
$\sqrt{2}$

E. Write $je^{j\pi}$ in Cartesian form (i.e., $a + jb$).

Solution:
$-j$

F. Write j in polar form (i.e., $Ae^{j\phi}$). What is the effect of multiplying any number by j?

Solution:
$e^{j\pi/2}$. Rotation by $\pi/2$.

Problem 2 This problems uses a *similarity transformation* to solve a system of coupled, first-order differential equations. This transformation is necessary to arrive at the general solution! You will need to find *(or guess correctly)* eigenvalues and eigenvectors.

A. Solve the easy problem: show that the solution for the system of equations

$$\begin{bmatrix} \dot{x_1}(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

Solution:
The first equation can be separated into two independent equations: $\dot{x_1} = 4x_1$ and $\dot{x_2} = 2x_2$. We can solve these equations separately.

$$\begin{aligned} \frac{dx_1}{dt} &= 4x_1 \\ \frac{dx_1}{x_1} &= 4dt \\ \int \frac{dx_1}{x_1} &= \int 4dt \\ \ln x_1 &= 4t + C \\ e^{\ln x_1} &= e^{4t+C} \\ x_1 &= e^C e^{4t} \end{aligned}$$

Since we know $x_1(0) = x_{0,1}$,

$$x_1 = x_{0,1} e^{4t}$$

Similarly, $x_2 = x_{0,2} e^{2t}$. Rewriting these two equations with matrices we get:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

B. Solve the hard problem: show that the solution for the system of equations

$$\begin{bmatrix} \dot{x_1}(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

Solution:
By the definition of eigenvectors and eigenvalues, we know that for some vector v,

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ (A - \lambda I)\vec{v} &= \vec{0} \end{aligned}$$

Thus, we find the values of λ that make $A - \lambda I$ singular.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} &= 0 \\ &= 9 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

Therefore, $\lambda_1 = 4$ and $\lambda_2 = 2$. For $\lambda_1 = 4$,

$$\begin{aligned} \begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \vec{v_1} &= \vec{0} \\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v_1} &= \vec{0} \\ \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

For $\lambda_1 = 2$,

$$\begin{aligned} \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \vec{v_2} &= \vec{0} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{v_2} &= \vec{0} \\ \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

We can rewrite the two equations $A\vec{v_1} = \lambda_1\vec{v_1}$ and $A\vec{v_2} = \lambda_2\vec{v_2}$ using matrices.

$$A[\vec{v_1}\vec{v_2}] = [\vec{v_1}\vec{v_2}]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Right-multiplying both sides by the inverse of the eigenvector matrix, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$, yields

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

Now we substitute for A in our original equation:

$$\dot{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Left-multiplying both sides by the inverse of the eigenvector matrix we get

$$\dot{\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right)} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Substituting $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ for $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ we obtain an equation that is in the form of the easy problem from part A.

$$\dot{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$\vec{y}$ is a transformation of $\vec{x}$,

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where $\vec{x}$ is expressed in the basis of the eigenvectors of the original matrix A.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Using this technique, we have transformed the problem into something we can solve. From here, we solve for $\vec{y}$,

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} y_{0,1} \\ y_{0,2} \end{bmatrix}$$

then solve back for $\vec{x}$.

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} \\ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} \end{aligned}$$

C. Generalize. Follow the same steps to show that the solution to

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x} = Ve^{\Lambda t}V^{-1}\mathbf{x}_0$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}, \quad AV = V\Lambda, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

and V is an invertible matrix.

$A = V\Lambda V^{-1}$ *is a* similarity transformation *such that A and Λ are* similar *matrices. Moreover, since Λ is a diagonal matrix and V is invertible, A is a* diagonalizable *matrix. Check out* http://en.wikipedia.org/wiki/Diagonalizable_matrix *and* http://planetmath.org/diagonalization.

Solution:
We are presented with the equation $\dot{\vec{x}} = A\vec{x}$ where A can be any matrix. If A was in the form $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, we would be just fine because we know how to solve that problem (you solved it in part A). Unfortunately, we don't know that A looks like that, so we are going to try and rewrite the equation $\dot{\vec{x}} = A\vec{x}$ in such a way that it looks like this $\dot{\vec{y}} = B\vec{y}$ where B is an array in the form $\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$. It seems reasonable that to do this we'll need to substitute something in for the matrix A and manipulate the equation.

Since A is a diagonalizable matrix, it has two eigenvectors (v_1, v_2) and two eigenvalues (λ_1, λ_2). By the definition of eigenvectors and eigenvalues,

$$A\vec{v_1} = \lambda_1\vec{v_1}, \; A\vec{v_1} = \lambda_1\vec{v_1}$$

We can write these equations using matrices:

$$\begin{aligned} A[\vec{v_1} \; \vec{v_2}] &= [\lambda_1\vec{v_1} \; \lambda_2\vec{v_2}] \\ A[\vec{v_1} \; \vec{v_2}] &= [\vec{v_1} \; \vec{v_2}] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ AV &= V\Lambda \end{aligned}$$

Now we can solve for A.

$$\begin{aligned} AV &= V\Lambda \\ AVV^{-1} &= V\Lambda V^{-1} \\ A &= V\Lambda V^{-1} \end{aligned}$$

By substitution into our original equation $\dot{\vec{x}} = A\vec{x}$,

$$\dot{\vec{x}} = V\Lambda V^{-1}\vec{x}$$

So now what? Well, the matrix Λ is in the same form as the matrix B so wouldn't it be cool if we could substitute $V^{-1}\vec{x}$ for $\vec{y}$. First, though, we need to make the left side of the equation look like $\dot{\vec{y}}$. We can do this by multiplying both sides by V^{-1} (V^{-1} is a matrix of constants so we can stick it inside the derivative).

$$\dot{\left(V^{-1}\vec{x}\right)} = \Lambda\left(V^{-1}\vec{x}\right)$$

Let $\vec{y} = V^{-1}\vec{x}$,

$$\dot{\vec{y}} = \Lambda\vec{y}$$

We now have the equation in a solvable form. We know the solution to this problem from part A.

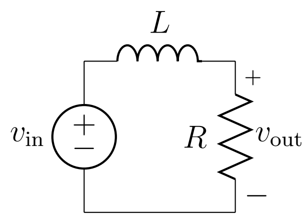
$$\vec{y} = e^{\Lambda t}\vec{y}_0$$

Since $\vec{y} = V^{-1}\vec{x}$, $\vec{x} = V\vec{y}$. Similarly, we also solve for $\vec{y}_0$: $\vec{y}_0 = V^{-1}\vec{x}_0$. Putting it all together:

$$\begin{aligned} \vec{x} &= V\vec{y} \\ \vec{x} &= Ve^{\Lambda t}\vec{y}_0 \\ \vec{x} &= Ve^{\Lambda t}V^{-1}\vec{x}_0 \end{aligned}$$

Problem 3 This problem is essentially identical to the RC circuit derived in class. You can write most of the answer by inspection.

A. Write a differential equation relating v_{in} and v_{out}. Remember that for an inductor, $v_L = L\frac{di_L}{dt}$.



Solution:
$\dot{v}_{out} + \frac{1}{L/R}v_{out} = \frac{1}{L/R}v_{in}$

B. If $v_{in} = V$ for $t > 0$, write the particular solution for v_{out}.

Solution:
$v_{out,particular} = V, t > 0$

C. Write the homogeneous solution for v_{out}.

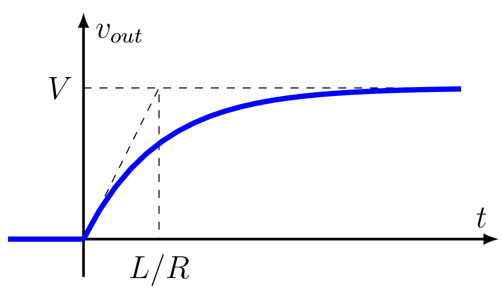
Solution:
$v_{out,homogeneous} = Ae^{\frac{t}{L/R}}, t > 0$

D. Write the specific solution for v_{out} assuming $v_{out}(0) = 0$.

Solution:
$v_{out} = V(1 - e^{-\frac{t}{L/R}}), t > 0$

E. Make a clear, neat sketch of v_{out} for $t > 0$. Show the initial value, asymptote and time constant.

Solution:



Problem 4 This problem is hard, but it steps you through the derivation of the second order underdamped response, one of the most useful equations in engineering. Be neat, and work with someone else. You will likely need to retrace your steps. Remember that this system models all sorts of decaying oscillations, e.g., a pendulum, a car after a bump, electromagnetic waves moving through space. (In fact, check out the examples in https://en.wikipedia.org/wiki/Oscillation.) Totally worth it!

A. Show that the general solution for the underdamped second order differential equation

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0, \qquad \omega_0 > \alpha$$

is

$$x = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}), \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Solution:
We start by guessing that the solution is an exponential, Ae^{st}. We aim to find values of s which make that guess correct.

Take the derivative and second derivative and plug these expressions into the differential equation.

$$x = Ae^{st} \qquad \dot{x} = sAe^{st} \qquad \ddot{x} = s^2 Ae^{st}$$

$$s^2 Ae^{st} + 2\alpha s Ae^{st} + \omega_0^2 Ae^{st} = 0$$

Ae^{st} is a common factor, so we can remove it to simplify the expression.

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Next, we need to solve for s as a function of α and ω_0. We can use the quadratic formula to solve for the roots of this function.

$$s = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$$

Both of these values give us solutions to the differential equation in the form of our initial guess, Ae^{st}. If we add these two solutions together (since a linear combination of solutions is also a solution) we get the following expression, the general solution:

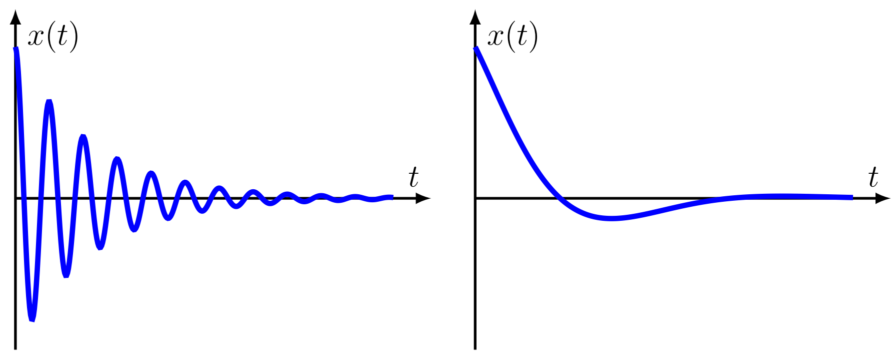
$$x = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$
$$x = e^{-\alpha t}(A_1 e^{jw_d t} + A_2 e^{-jw_d t})$$

B. The specific solution with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ is

$$x = x_0\sqrt{1 + (\alpha/\omega_d)^2}e^{-\alpha t}\cos(\omega_d t - \tan^{-1}\alpha/\omega_d)$$

Assume $x_0 = \alpha = 1$. Use a computer to graph this solution for $\omega_d = 1$ and $\omega_d = 10$. For each value of ω_d, verify the initial conditions graphically and zoom out to show the asymptotic behavior.

Solution:
This is the solution for ten seconds, which shows the asymptotic behavior for both values of ω_d.



If you are curious, the complete derivation of the solution follows. This is painful algebra!

Let's first restate the problem.

$$\ddot{x} + 2\alpha\dot{x} + \omega_0 x = 0, \quad x(0) = x_0, \quad \dot{x}(0) = 0, \quad \omega_0 > \alpha \text{ (underdamped)}$$

As usual, assume $x = Ae^{st}$. Let's carefully define the complex conjugates values.

$$s = -\alpha + j\underbrace{\sqrt{\omega_0^2 - \alpha^2}}_{\omega_d}, \text{ and } s^* = -\alpha - j\underbrace{\sqrt{\omega_0^2 - \alpha^2}}_{\omega_d}$$

The general solution is

$$x = A_1e^{st} + A_2e^{s^*t}$$

Since x must be real, A_2 must be the complex conjugate of A_1

$$A_2 = A_1^* \text{ for } \mathrm{Im}\{x\} = 0$$

Substituting and taking the derivative,

$$x = Ae^{st} + A^*e^{s^*t} \qquad\qquad \dot{x} = sAe^{st} + s^*A^*e^{s^*t}$$

The initial condition $x(0) = x_0$ yields the real part of A

$$x(0) = A + A^* = x_0 \Rightarrow \mathrm{Re}\{A\} = x_0/2$$

The initial condition $\dot{x}(0) = 0$ yields the imaginary part of A

$$\dot{x}(0) = sA + s^* A^* = 2\mathrm{Re}\{sA\}$$

$$\begin{aligned}
&sA = (-\alpha + j\omega_d)(x_0/2 + j\mathrm{Im}\{A\}) \\
&\mathrm{Re}\{sA\} = -\alpha x_0/2 - \omega_d \mathrm{Im}\{A\} = 0 \\
&\Rightarrow \mathrm{Im}\{A\} = -\frac{\alpha}{\omega_d}\frac{x_0}{2}
\end{aligned}$$

Since we know the real and imaginary parts of A,

$$A = \frac{x_0}{2} - j\frac{\alpha}{\omega_d}\frac{x_0}{2} = \frac{x_0}{2}\left(1 - j\frac{\alpha}{\omega_d}\right)$$

We know express A in terms of magnitude and phase,

$$A = |A|e^{j\angle A} = \frac{x_0}{2}\sqrt{1 + (\alpha/\omega_d)^2}e^{-j\tan^{-1}\alpha/\omega_d}$$

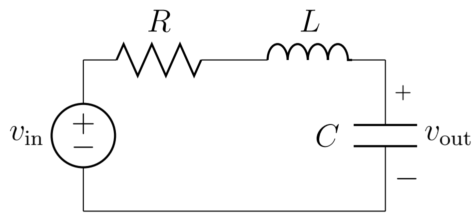
Since x is the sum of two complex conjugates, it can be written in terms of a real-valued cosine,

$$x = 2|A|e^{-\alpha t}\cos(\omega_d t + \angle A)$$

Substituting the magnitude and phase of A finally yields

$$x = x_0\sqrt{1 + (\alpha/\omega_d)^2}e^{-\alpha t}\cos(\omega_d t - \tan^{-1}\alpha/\omega_d)$$

C. Find the step response of the circuit shown below. Assume that $\frac{R}{2} < \sqrt{\frac{L}{C}}$ and let $v_{in} = Vu(t)$. Find an expression for v_{out} and graph it. *Hint 1: Since the circuit was at rest for* $t < 0$*, you can assume* $v_{out}(0) = 0$ *and* $\dot{v}_{out}(0) = 0$*. Hint 2: Use a linear transformation of the solution from the previous part to avoid deriving the solution again.*



Solution:
The step response is the response of a system to an instantaneous change in input from 0 to some constant. The electrical analogy is simply turning the power on.

The equation for this system is best expressed in terms of the voltage drop across each component (inductor first, then resistor, then capacitor) which, by Kirchoff's Voltage Law, must sum to the voltage increase generated by the battery.

$$\begin{aligned}
v_{in} &= L\frac{di}{dt} + Ri + v_{out} \\
i &= C\frac{dv_{out}}{dt} \\
v_{in} &= L\frac{d}{dt}\left(C\frac{dv_{out}}{dt}\right) + RC\frac{dv_{out}}{dt} + v_{out} \\
v_{in} &= LC\frac{d^2v_{out}}{dt^2} + RC\frac{dv_{out}}{dt} + v_{out} \\
\frac{d^2v_{out}}{dt^2} &+ \frac{R}{L}\frac{dv_{out}}{dt} + \frac{1}{LC}v_{out} = \frac{1}{LC}v_{in}
\end{aligned}$$

This is almost identical to the expression in parts A and B, but rather than summing to 0 they sum to v_{in}. We can still use the solution from the previous parts if we make this expression sum to 0 by linearly shifting the entire system by the amplitude of the unit step, V. After t=0, v_{in} equals V. We can define a new variable, v_2, that represents a shifted voltage and insert it into our equation.

$$v_2 = v_{out} - V \qquad \Rightarrow \qquad v_{out} = v_2 + V$$

This V is a constant, so when taking the first and second derivative of v_2, it is eliminated, leaving only a single V term.

$$\begin{aligned}
&\frac{d^2v_2}{dt^2} + \frac{R}{L}\frac{dv_2}{dt} + \frac{1}{LC}v_2 + \frac{1}{LC}V = \frac{1}{LC}V \\
&\frac{d^2v_2}{dt^2} + \frac{R}{L}\frac{dv_2}{dt} + \frac{1}{LC}v_2 = 0
\end{aligned}$$

This is now in the same form as in part A. Additionally, the initial conditions become

$$\begin{aligned}
v_2(0) &= v_{out}(0) - V = -V \\
\dot{v}_2(0) &= \dot{v}_{out}(0) = 0
\end{aligned}$$

Using part B, the solution becomes

$$v_2(t) = -V\sqrt{1 + (\alpha/\omega_d)^2}e^{-\alpha t}\cos(\omega_d t - \tan^{-1}\alpha/\omega_d)$$

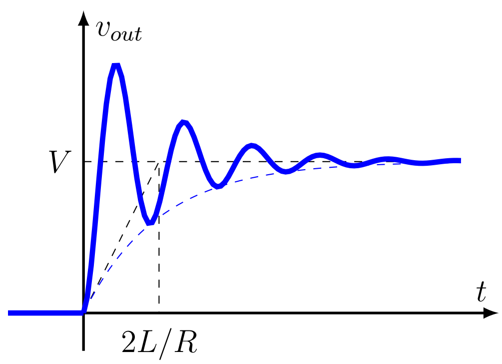
where

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

Finally, we can shift back to solve for v_{out}.

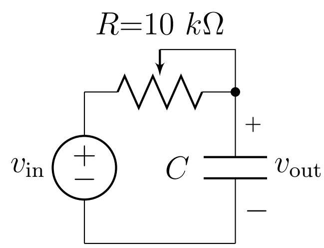
$$v_{out}(t) = V - V\sqrt{1 + (\alpha/\omega_d)^2}e^{-\alpha t}\cos(\omega_d t - \tan^{-1}\alpha/\omega_d)$$

Here's a sample graph. Unsurprisingly, the step response is is similar to the previous problem–the voltage oscillates towards the voltage supplied by the battery. Electrically, this makes sense, as an RLC circuit, given enough time, will reach a steady state where there is no current at all. The voltage supplied by the battery will eventually equal the voltage across the capacitor.

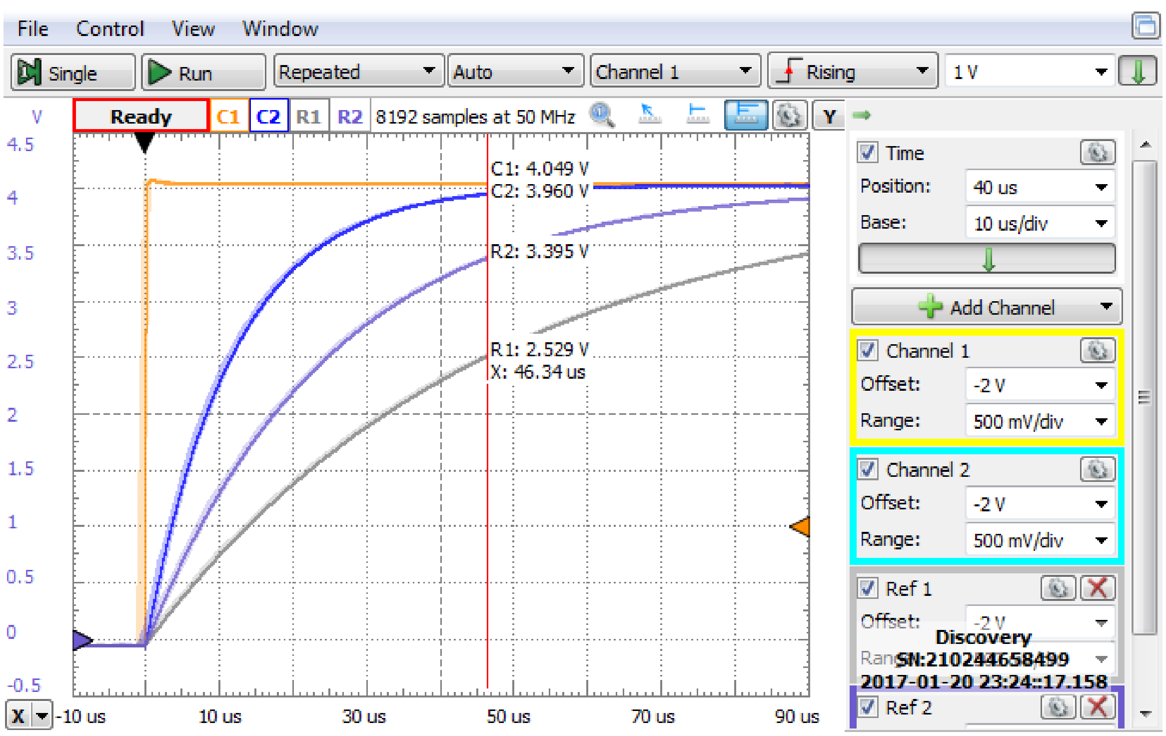


Problem 5 Engineering is not just about understanding phenomena, but also about connecting theory and measurement. Additionally, use the potentiometers to play with these circuits keeping in mind the math behind it. Be neat and keep your circuits! We will use them several times throughout the course. The course webpage links to a few videos that should be useful, in particular the one about *circuit hygiene*!

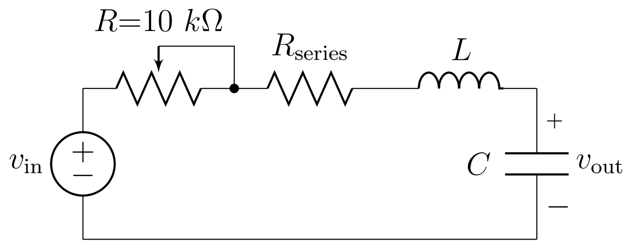
A. Build the RC circuit shown below using a $R = 10$ kΩ potentiometer. Calculate the time constant and measure it using the step response. Measure two more responses, one at two thirds and one at a third of the original time constant, and include them in your plot. *Be careful to use values where you can see the time constant, and verify the values of your components! (My first two potentiometers were more than 10% off.)*



I show my results below, but you should use your own capacitor value. I used a $C = 4.7$ nF capacitor, for time constant $RC = 47$ μs. I drove my circuit with a $v_{in} = 4$ V step. After one time constant, the model predicts $1 - e^{-1} = 63.2\%$ of the final voltage, or 4 V $\times$ 63.2% $=$ 2.53 V. I measured this voltage at 46.3 μs, as shown below, a difference of $(46.3\ \mu\text{s} - 47\ \mu\text{s})/47\ \mu\text{s} = -1.5\%$.



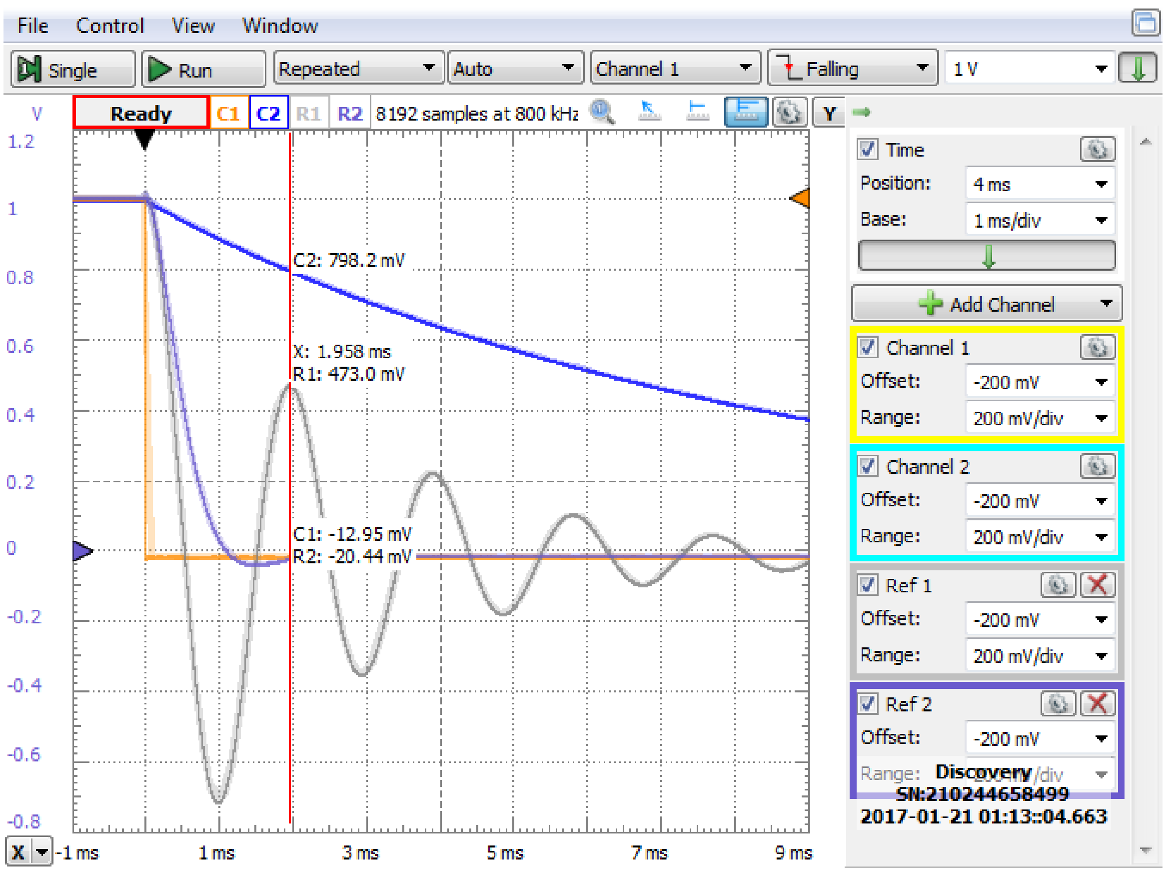
B. Build the RLC circuit shown below using a 10 kΩ potentiometer. The resistance R_{series} represents the *internal series resistance* of the inductor, which you can measure using a multimeter. Verify the frequency ω_0 and decay α when the potentiometer is shorted. Measure two more responses, one close to critical damping, and another overdamped.



I used a $C = 1\ \mu$F capacitor and a $L = 0.1$ H inductor with a series resistance $R_{\text{series}} = 69\ \Omega$. I drove the circuit with a $v_{in} = 1$ V step. The model predicts a period of $2\pi\sqrt{LC} = 1.99$ ms. The measured period shown below is 1.96 ms, a difference of -1.5%. After one period, the voltage should fall to

$$1\text{ V} \times \mathrm{e}^{-\alpha\frac{2\pi}{\omega_0}} = 1\text{ V} \times \mathrm{e}^{-\pi\frac{\mathrm{R_{series}}}{\sqrt{\mathrm{L/C}}}} = 0.504\text{ V}.$$

The measured voltage is 473 mV, a -6.2% difference.



Course feedback

Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?