Olin College of Engineering
ENGR2410 – Signals and Systems

Reference 9

Modulation

$$
\begin{tikzcd}
x(t) \arrow[r] & \bigotimes \arrow[r, "x_{mod}(t)"] & \bigotimes \arrow[r, "y_{demod}(t)"] & \boxed{H(j\omega)} \arrow[r] & y(t) \\
 & \cos(\omega_0 t) \arrow[u] & \cos(\omega_0 t) \arrow[u] & &
\end{tikzcd}
$$

Assume $x(t) \overset{\mathscr{F}}{\Longleftrightarrow} X(j\omega)$, and $X(j\omega)$ is bandlimited to $2\pi f_{max}$, where $2\pi f_{max} \ll \omega_0$.

$$x_{mod}(t) = x(t)\cos(\omega_0 t)$$

$$X_{mod}(j\omega) = \frac{1}{2\pi}X(j\omega) * [\pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega0)] = \frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$$

$$y_{demod} = x(t)\cos^2(\omega_0 t) = x(t)\left[\frac{1}{2} + \frac{1}{2}\cos(2\omega_0 t)\right]$$

$$Y_{demod}(j\omega) = \frac{1}{2\pi}\left[\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)\right] * [\pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)]$$

$$Y_{demod}(j\omega) = \frac{1}{4}X(\omega+2\omega_0) + \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega-2\omega_0)$$

If $H(j\omega)$ is an ideal low pass filter with cutoff frequency at $2\pi f_{max}$, $Y(j\omega) = \frac{1}{2}X(j\omega)$ and $y(t) = \frac{1}{2}x(t)$.

Laplace transform

In general, e^{st} is an eigenfunction of LTI systems and $H(s)$ is the associated eigenvalue.

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

$$H(s)e^{st} = e^{st} * h(t) = \int_{-\infty}^{\infty} h(t')e^{s(t-t')}dt' = e^{st} \underbrace{\int_{-\infty}^{\infty} h(t')e^{-st'}dt'}_{H(s)}$$

$H(s)$ is the *Laplace transform* of $h(t)$,

$$H(s) \triangleq \mathscr{L}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

The Fourier transform is the Laplace transform when $s = j\omega$,

$$H(s)\big|_{s=j\omega} = \mathscr{F}\{h(t)\}$$

For example, if $h(t) = e^{-t/\tau}u(t)$,

$$H(s) = \int_0^{\infty} e^{-t/\tau}e^{-st}dt = \frac{1}{-(s+1/\tau)}\left[e^{-(s+1/\tau)t}\right]_0^{\infty} = \frac{1}{s+1/\tau}, \quad \mathrm{Re}\{s\} > -1/\tau$$

Since the integral only converges if $\mathrm{Re}\{s\} > -1/\tau$, the Laplace transform exists only if this condition holds. This condition can be represented as a region in the *s-plane* of all possible complex values of s, known as *region of convergence (ROC)*.

In general, $H(s)$ can be represented in the s-plane. Points where $H(s) = 0$ are zeros, usually labeled as "o", and points where $H(s) \to \infty$ are poles, usually labeled as "x". In our examples, $H(s)$ has a single pole at $s = -1/\tau$, and the ROC is the half-plane to the right of this pole. The boundary of any ROC always has at least one pole.

The vertical (imaginary) axis is where $s = j\omega$. Thus, the Fourier transform is a "slice" of $H(s)$ along this line. In our example, if the pole is in the left-half plane (LHP), where $\mathrm{Re}\{s\} < 0$, the Fourier transform exists since it is inside the ROC. In this case, $h(t)$ approaches 0 as $t \to \infty$. On the other hand, if the pole is on the right-half plane (RHP), $\tau < 0$ and $h(t)$ approaches ∞ as $t \to \infty$. In this case, the Laplace transform does not converge when $s = j\omega$ and the Fourier transform does not exist.

Given the close relationship between the Laplace and Fourier transforms, most properties of the Fourier transform are also true for the Laplace transform. In particular, the Laplace transform is also linear, $Y(s) = H(s)X(s)$, impedances are as expected if $s = j\omega$.

The page of Peter Mathys at
http://ecee.colorado.edu/~mathys/ecen2420/notes/FilterPlots.html
shows the relationship between the frequency response of filters and their associated pole-zero diagram using the s-plane.