Olin College of Engineering
ENGR2410 – Signals and Systems

Reference 8

The properties of the discrete time Fourier transform are similar to the continuous time.

Continuous time

$$\begin{array}{ccccc} x(t) & \longrightarrow & \boxed{h(t)} & \longrightarrow & y(t) = x(t)*h(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)h(t-\tau)d\tau \\ \mathscr{F}\Big\downarrow & & & & \Big\uparrow\mathscr{F}^{-1} \\ X(j\omega) & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & Y(j\omega) = X(j\omega)H(j\omega) \end{array}$$

Discrete time[1] **($n \in \mathbb{Z}$)**

$$\begin{array}{ccccc} x[n] & \longrightarrow & \boxed{h[n]} & \longrightarrow & y[n] = x[n]*h[n] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k] \\ \mathscr{F}\Big\downarrow & & & & \Big\uparrow\mathscr{F}^{-1} \\ X(\Omega) & \longrightarrow & \boxed{H(\Omega)} & \longrightarrow & Y(\Omega) = X(\Omega)H(\Omega) \end{array}$$

Frequency content of a discrete signal

$$x[n] \longrightarrow \boxed{D/C} \longrightarrow x_S(t)$$

The discrete-to-continuous (D/C) transformation above defines $x_S(t)$ as a function of time sampled at an arbitrary f_S that corresponds to the discrete function $x[n]$ such that

$$x_S(t) = \sum_n x[n]\delta(t - n/f_S)$$

We can use the continuous time Fourier transform to find $X_S(j\omega)$:

$$X_S(j\omega) = \int_t \sum_n x[n]\delta(t - n/f_S)e^{-j\omega t}dt$$

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[1] In the same way $H(j\omega)$ is written to explicitly include the j in order to relate it to the Laplace transform, $H(\Omega)$ is sometimes written as $H(e^{j\Omega})$ to relate it to the Z transform.

Since we introduced f_S arbitrarily to define $x_S(t)$, we have to normalize the frequency such that $\Omega = \omega/f_S$. This is the *discrete time Fourier transform.*

$$\boxed{X_S(\Omega) = \sum_n x[n]e^{-j\Omega n}}$$

As in continuous time, $X_S(\Omega)$ is a complex function which has magnitude and phase. However, note that since $X_S(\Omega)$ is a linear combination of $e^{j\Omega n}$ basis functions, *it always has a period of* 2π.

We can use orthogonality to find the inverse discrete time transform, noting that since n is an integer, we only need to integrate $e^{j\Omega n}$ over any 2π period;

$$\int_{\omega_0}^{\omega_0+2\pi} e^{j\Omega n} d\Omega = \begin{cases} 2\pi & n = 0 \\ 0 & n \neq 0 \end{cases}, \quad n \in \mathbb{Z}, \quad \omega_0 \in \mathbb{R}$$

Expanding $X_S(\Omega)$,

$$X_S(\Omega) = ... + x[-1]e^{j\Omega} + x[0] + x[1]e^{-j\Omega} + ... + x[n]e^{-j\Omega n} + ...$$

$$X_S(\Omega)e^{j\Omega n} = ... + x[1]e^{j\Omega(n-1)} + ... + x[n] + ...$$

$$\int_{2\pi} X_S(\Omega)e^{j\Omega n} d\Omega = ... + \cancelto{0}{\int_{2\pi} x[1]e^{j\Omega(n-1)}} d\Omega + ... + \int_{2\pi} x[n] d\Omega + ...$$

$$\int_{2\pi} X_S(\Omega)e^{j\Omega n} d\Omega = 2\pi x[n]$$

$$\boxed{x[n] = \int_{2\pi} X_S(\Omega)e^{j\Omega n} \frac{d\Omega}{2\pi}}$$

A few properties and transforms

Discrete time impulse

$$\boxed{\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \sum_n \delta[n]e^{-j\Omega n} = 1}$$

Delay

$$\boxed{x[n-n_0] \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \sum_n x[\underbrace{n-n_0}_{m}]e^{-j\Omega n} = \underbrace{\sum_m x[m]e^{-j\Omega m}}_{X(\Omega)} e^{-j\Omega n_0} = X(\Omega)e^{-j\Omega n_0}}$$

In the case of a complex exponential $e^{j\Omega_0 n}$, we can verify that an impulse at Ω_0 works since

$$\int_{-\pi}^{\pi} \cancel{2\pi}\delta(\Omega - \Omega_0) e^{j\Omega n} \frac{d\Omega}{\cancel{2\pi}} = e^{j\Omega_0 n}, \quad -\pi \le \Omega_0 \le \pi$$

However, this must work for any 2π period, and we know that $X(\Omega)$ must be 2π periodic, so that the impulse can be shifted by an integer multiple of 2π:

$$\int_{-\pi+2\pi k}^{\pi+2\pi k} \cancel{2\pi}\delta(\Omega - \Omega_0 - 2\pi k) e^{j\Omega n} \frac{d\Omega}{\cancel{2\pi}} = e^{j\Omega_0 n} \cancelto{1}{e^{j2\pi k n}}, \quad -\pi \le \Omega_0 \le \pi, \quad k \in \mathbb{Z}$$

Since we are only integrating over one 2π period, we can add all these impulses and guarantee that there will be a single impulse in any 2π period:

$$\boxed{e^{j\Omega_0 n} = \int_{2\pi} \underbrace{\sum_k 2\pi\delta(\Omega - \Omega_0 - 2\pi k)}_{X(\Omega)} e^{j\Omega n} \frac{d\Omega}{2\pi} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \sum_k 2\pi\delta(\Omega - \Omega_0 - 2\pi k), \quad k \in \mathbb{Z}}$$

First-order difference equation

$$y[n] - ay[n-1] = x[n]$$

If $x[n] = \delta[n]$, then $y[n] = h[n]$, the impulse response.

$$\delta[n] \longrightarrow \boxed{h[n]} \longrightarrow h[n]$$

Since $\delta[n] = 0$ for all $n < 0$, $h[n] = 0$, for all $n < 0$. Specifically,

$$h[-1] = 0$$

Then,

$$h[0] - a\cancelto{0}{h[-1]} = \cancelto{1}{\delta[0]} \Rightarrow h[0] = 1$$

$$h[1] - a\cancelto{1}{h[0]} = \cancelto{0}{\delta[1]} \Rightarrow h[1] = a$$

$$h[2] - a\cancelto{a}{h[1]} = \cancelto{0}{\delta[2]} \Rightarrow h[2] = a^2$$

and so forth. In general,

$$h[n] = a^n u[n]$$

where

$$u[n] = \begin{cases} 1 & n = 0, 1, 2, 3... \\ 0 & \text{otherwise} \end{cases}$$

As in the continuous time case, complex exponentials are eigenfunctions of discrete time LTI systems such that

$$e^{j\Omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(\Omega)e^{j\Omega n}$$

We can use this to find $H(\Omega)$ for the first-order difference equation by letting $x[n] = e^{j\Omega n}$ and $y[n] = H(\Omega)e^{j\Omega n}$.

$$y[n] - ay[n-1] = x[n]$$

$$H(\Omega)\cancel{e^{j\Omega n}} - aH(\Omega)\cancel{e^{j\Omega n}}e^{-j\Omega} = \cancel{e^{j\Omega n}}$$

$$H(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

Note that we just found $h[n]$ and $H(\Omega)$ for the first-order difference equation. Since

$$\begin{array}{ccccc} \delta[n] & \longrightarrow & \boxed{h[n]} & \longrightarrow & h[n] \\ \mathscr{F}\big\downarrow & & & & \\ 1 & \longrightarrow & \boxed{H(\Omega)} & \longrightarrow & H(\Omega) \end{array}$$

we expect that

$$h[n] \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad H(\Omega)$$

Specifically,

$$\boxed{a^n u[n] \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \frac{1}{1 - ae^{-j\Omega}}}$$

In order to verify this, compute $H(\Omega)$ using the definition of the transform:

$$H(\Omega) = \sum_n h[n]e^{-j\Omega n} = \sum_n a^n u[n]e^{-j\Omega n} = \sum_{n=0}^{\infty}(ae^{-j\Omega})^n = \frac{1}{1 - ae^{-j\Omega}}$$

Note that $|a| < 1$ for the sum to converge.

DT processing of CT signals

Two subsystems that convert a continuous time signal $x_c(t)$ into a discrete time signal $x_d[n]$, (C/D), and vice versa, (D/C), can be used to process $x_c(t)$ using a discrete filter $H_d(\Omega)$.

$$\begin{tikzcd}
x_c(t) \arrow[r] & \boxed{\text{C/D}} \arrow[r, "x_d[n]=x_c\left(\frac{n}{f_S}\right)"] & \boxed{H_d(\Omega)} \arrow[r, "y_d[n]"] & \boxed{\text{D/C}} \arrow[r, "y_c(t)=y_d[n],\ \text{if}\ t=\frac{n}{f_S}"] & y_c(t) \\
& f_S \arrow[u] & & f_S \arrow[u] &
\end{tikzcd}$$

Keeping in mind that sampling at f_S constrains Ω such that

$$\boxed{\frac{\omega}{f_S} = \Omega}$$

If $x_c(t)$ is bandlimited by f_{max} such that the sampling frequency $f_S > 2f_{max}$, the system above is equivalent to an LTI system $H_c(j\omega)$, where

$$\boxed{H_c(j\omega) = \begin{cases} H_d(\omega/f_S) & -2\pi\frac{f_S}{2} \leq \omega \leq 2\pi\frac{f_S}{2} \\ 0 & \text{otherwise} \end{cases}}$$

Equivalently,

$$\boxed{Y_c(j\omega) = H_d(\omega/f_S)X_c(j\omega)}$$

In order to show this result, we need to define the C/D and D/C subsystems. The C/D subsystem samples $x_c(t)$ using a pulse train, and the resulting sampled $x_s(t)$ is redefined as a discrete sequence,

$$\begin{tikzcd}
x_c(t) \arrow[r] & \bigotimes \arrow[r, "x_s(t)"] & \boxed{\begin{array}{c}\text{sequence}\\ \text{to train}\end{array}} \arrow[r] & x_d[n] = x_c(n/f_S) \\
& \sum_n \delta\left(t-\frac{n}{f_S}\right) \arrow[u] & \frac{\omega}{f_S} = \Omega \arrow[u] &
\end{tikzcd}$$

Similarly, a continuous signal $y_c(t)$ can be reconstructed from a discrete signal $y_d[n]$ using the D/C system below.