# Olin College of Engineering
# ENGR2410 – Signals and Systems

## Continuous time Fourier transform

$$\text{If} \quad x(t) = \int_{\omega=-\infty}^{\omega=\infty} X(j\omega)e^{j\omega t}\frac{d\omega}{2\pi} \quad \text{then} \quad X(j\omega) = \int_{t=-\infty}^{t=\infty} x(t)e^{-j\omega t}dt \triangleq \mathscr{F}\{x(t)\}$$

Alternatively, $\quad x(t) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad X(j\omega)$

## System representation

$$\begin{array}{ccccc}
x(t) & \longrightarrow & \boxed{h(t)} & \longrightarrow & y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)h(t-\tau)d\tau \\
\mathscr{F}\big\downarrow & & & & \big\uparrow \mathscr{F}^{-1} \\
X(j\omega) & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & Y(j\omega) = X(j\omega)H(j\omega)
\end{array}$$

## Properties and transforms

$$\begin{aligned}
\mathscr{F}\{ax_1(t) + bx_2(t)\} &= aX_1(j\omega) + bX_2(j\omega) \\
\mathscr{F}\{x(t+T)\} &= X(j\omega)e^{j\omega T} \\
\mathscr{F}\{\delta(t)\} &= 1 \\
\mathscr{F}\{1\} &= 2\pi\delta(\omega) \\
\mathscr{F}\{e^{j\omega_0 t}\} &= 2\pi\delta(\omega - \omega_0) \\
\mathscr{F}\{\cos(\omega_0 t)\} &= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \\
\mathscr{F}\{\sin(\omega_0 t)\} &= \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0) \\
\mathscr{F}\left\{\frac{1}{2}\delta(t-T) + \frac{1}{2}\delta(t+T)\right\} &= \cos(\omega T) \\
\mathscr{F}\{e^{-t/\tau}u(t)\} &= \frac{1}{j\omega + 1/\tau} \\
\mathscr{F}\{x(t) * y(t)\} &= X(j\omega)Y(j\omega) \\
\mathscr{F}\{x(t)y(t)\} &= \frac{1}{2\pi}X(j\omega) * Y(j\omega) \\
\mathscr{F}\left\{\Pi(\mathrm{t}/\mathrm{T}_1)\right\} &= 2T_1\frac{\sin(\omega T_1)}{\omega T_1} \text{ where } \Pi(\mathrm{x}) = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \\
\mathscr{F}\left\{\frac{\omega_0}{\pi} \cdot \frac{\sin(\omega_0 t)}{\omega_0 t}\right\} &= \Pi(\omega/\omega_0) \text{ where } \Pi(\mathrm{x}) = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \\
\mathscr{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT)\right\} &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta\left(\omega - k\frac{2\pi}{T}\right) \text{ where } n \text{ and } k \text{ are integers}
\end{aligned}$$

**Discrete time Fourier transform ($n \in \mathbb{Z}$)**

$$\text{If} \quad x[n] = \int_{2\pi} X(\Omega) e^{j\Omega n} \frac{d\Omega}{2\pi} \quad \text{then} \quad X(\Omega) = \sum_n x[n] e^{-j\Omega n}$$

Alternatively, $\quad x[n] \overset{\mathscr{F}}{\Longleftrightarrow} X(\Omega)$

**System representation**

$$\begin{tikzcd}
x[n] \arrow[r] \arrow[d, "\mathscr{F}"'] & \boxed{h[n]} \arrow[r] & y[n] = x[n] * h[n] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k] \\
X(\Omega) \arrow[r] & \boxed{H(\Omega)} \arrow[r] & Y(\Omega) = X(\Omega)H(\Omega) \arrow[u, "\mathscr{F}^{-1}"']
\end{tikzcd}$$

**Properties and transforms**

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad 1$$

$$x[n - n_0] \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad X(\Omega) e^{-j\Omega n_0}$$

$$e^{j\Omega_0 n} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \sum_k 2\pi\delta(\Omega - \Omega_0 - 2\pi k), \quad k \in \mathbb{Z}$$

$$a^n u[n] \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \frac{1}{1 - ae^{-j\Omega}}, \quad |a| < 1$$

**DT processing of CT signals**

$$\begin{tikzcd}
x_c(t) \arrow[r] & \boxed{\text{C/D}} \arrow[r, "x_d[n]=x_c\left(\frac{n}{f_S}\right)"] & \boxed{H_d(\Omega)} \arrow[r, "{y_d[n]}"] & \boxed{\text{D/C}} \arrow[r, "{y_c(t)=y_d[n],\ \text{if}\ t=\frac{n}{f_S}}"] & y_c(t) \\
 & f_S \arrow[u] & & f_S \arrow[u] &
\end{tikzcd}$$

Sampling at $f_S$ constrains $\Omega$ such that $\frac{\omega}{f_S} = \Omega$. If $x_c(t)$ is bandlimited by $f_{max}$ such that the sampling frequency $f_S > 2f_{max}$, the system above is equivalent to an LTI system $H_c(j\omega)$, where

$$H_c(j\omega) = \begin{cases} H_d(\omega/f_S) & -2\pi\frac{f_S}{2} \leq \omega \leq 2\pi\frac{f_S}{2} \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, $\quad Y_c(j\omega) = H_d(\omega/f_S) X_c(j\omega)$

**Laplace Transform**

$$X(s) = \int_{t=-\infty}^{t=\infty} x(t)e^{-st}dt \equiv \mathscr{L}\{x(t)\} \quad \text{with region of convergence } (ROC) : R$$

$$\begin{aligned}
\mathscr{L}\{ax_1(t) + bx_2(t)\} &= aX_1(s) + bX_2(s) && ROC: \text{At least } R_1 \cap R_2 \\
\mathscr{L}\{x(t-T)\} &= X(s)e^{-sT} && ROC: R \\
\mathscr{L}\{x(t) * y(t)\} &= X(s)Y(s) && ROC: \text{At least } R_1 \cap R_2 \\
\mathscr{L}\{\delta(t)\} &= 1 && ROC: \text{All } s \\
\mathscr{L}\{u(t)\} &= \frac{1}{s} && ROC: \text{Re}\{s\} > 0 \\
\mathscr{L}\{e^{-t/\tau}u(t)\} &= \frac{1}{s + 1/\tau} && ROC: \text{Re}\{s\} > -\frac{1}{\tau} \\
\mathscr{L}\{e^{-\alpha t}\sin(\omega_d t)u(t)\} &= \frac{\omega_d}{(s+\alpha)^2 + \omega_d^2} && ROC: \text{Re}\{s\} > -\alpha \\
\mathscr{L}\{e^{-\alpha t}\cos(\omega_d t)u(t)\} &= \frac{s+\alpha}{(s+\alpha)^2 + \omega_d^2} && ROC: \text{Re}\{s\} > -\alpha
\end{aligned}$$