Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 4

Problem 1: *(2 points)* Assume that n is an integer. Show that if the complex coefficients $c_{-n} = c_n^*$, then

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} = c_0 + \sum_{n=1}^{\infty} 2\mathrm{Re}\{c_n\}\cos\left(\tfrac{2\pi}{T}nt\right) + \sum_{n=1}^{\infty}(-2)\mathrm{Im}\{c_n\}\sin\left(\tfrac{2\pi}{T}nt\right)$$

Solution:
Let's begin by expanding the summation:

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} \quad = \quad ... + c_{-2}e^{-j\frac{4\pi}{T}t} + c_{-1}e^{-j\frac{2\pi}{T}t} + c_0 + c_1 e^{j\frac{2\pi}{T}t} + c_2 e^{j\frac{4\pi}{T}t} + ...$$

We can see a pattern emerging. For each $c_{-n}e^{-j\frac{2\pi}{T}nt}$ term there is a corresponding $c_n e^{-j\frac{2\pi}{T}nt}$ term. Now let's rewrite the complex exponentials with sines and cosines using Euler's formula:

$$... + c_{-n}\cos\left(\tfrac{2\pi}{T}nt\right) - c_{-n}j\sin\left(\tfrac{2\pi}{T}nt\right) + ... + c_0 + ... + c_n\cos\left(\tfrac{2\pi}{T}nt\right) + c_n j\sin\left(\tfrac{2\pi}{T}nt\right) + ...$$

If we group similar cosine and sine terms, we can factor and simplify:

$$\begin{aligned} c_0 \quad &+ \quad ... + c_n\cos\left(\tfrac{2\pi}{T}nt\right) + c_{-n}\cos\left(\tfrac{2\pi}{T}nt\right) + c_n j\sin\left(\tfrac{2\pi}{T}nt\right) - c_{-n}j\sin\left(\tfrac{2\pi}{T}nt\right) + ... \\ c_0 \quad &+ \quad ... + (c_n + c_{-n})\cos\left(\tfrac{2\pi}{T}nt\right) + (c_n - c_{-n})\,j\sin\left(\tfrac{2\pi}{T}nt\right) + ... \end{aligned}$$

We know that c_{-n} and c_n are complex conjugates. Thus, if c_n is in the form $\mathrm{Re}\{c_n\}+j\mathrm{Im}\{c_n\}$, c_{-n} is in the form $\mathrm{Re}\{c_n\} - j\mathrm{Im}\{c_n\}$. In other words,

$$\begin{aligned} c_n + c_{-n} \quad &= \quad \mathrm{Re}\{c_n\} + j\mathrm{Im}\{c_n\} + \mathrm{Re}\{c_n\} - j\mathrm{Im}\{c_n\} = 2\mathrm{Re}\{c_n\} \\ c_n - c_{-n} \quad &= \quad \mathrm{Re}\{c_n\} + j\mathrm{Im}\{c_n\} - \mathrm{Re}\{c_n\} + j\mathrm{Im}\{c_n\} = 2j\mathrm{Im}\{c_n\} \end{aligned}$$

Substituting these values into our expression we obtain:

$$c_0 \quad + \quad ... + 2\mathrm{Re}\{c_n\}\cos\left(\tfrac{2\pi}{T}nt\right) - 2\mathrm{Im}\{c_n\}\sin\left(\tfrac{2\pi}{T}nt\right) + ...$$

Finally, we rewrite the expanded terms in separate sums:

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} = c_0 + \sum_{n=1}^{\infty} 2\mathrm{Re}\{c_n\}\cos\left(\tfrac{2\pi}{T}nt\right) + \sum_{n=1}^{\infty}(-2)\mathrm{Im}\{c_n\}\sin\left(\tfrac{2\pi}{T}nt\right)$$

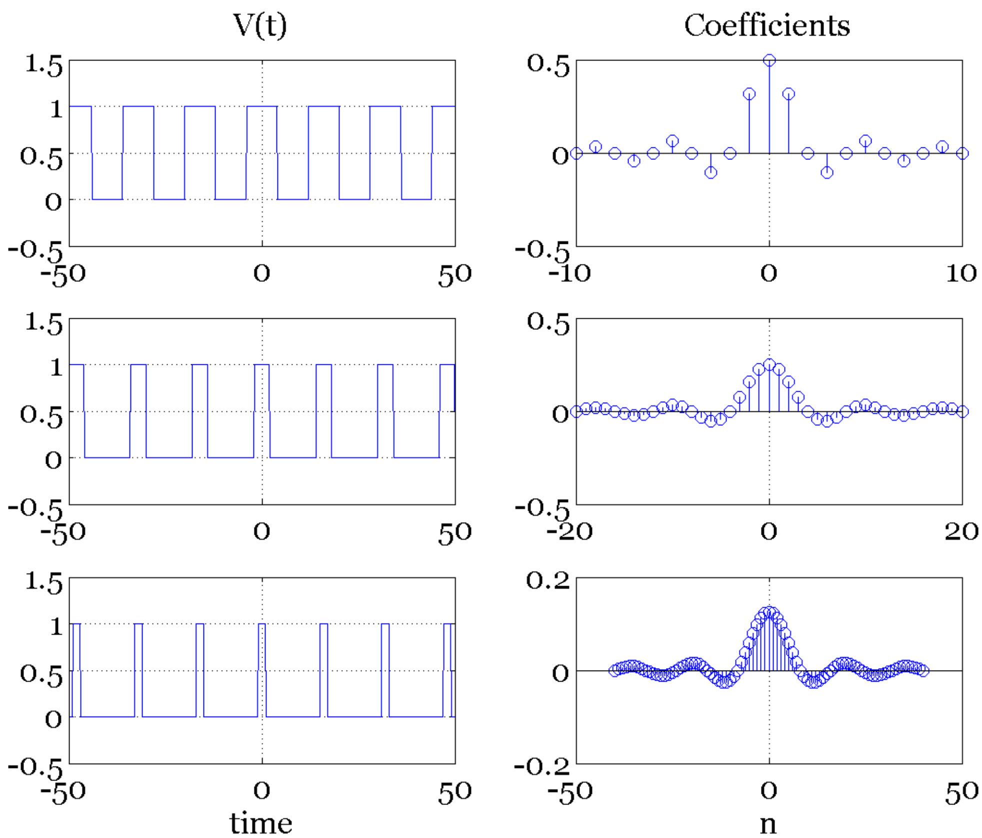
Problem 2: *(4 points)* In lecture we found out that if we try to represent the function

$$v(t) = \begin{cases} V & -T_1 + nT < t < T_1 + nT, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

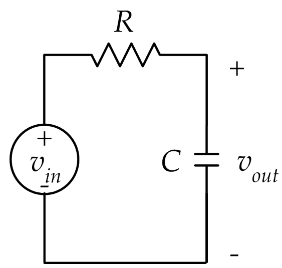
as $v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$ then $c_n = 2V\frac{T_1}{T}\text{sinc}\left(\frac{2\pi}{T}nT_1\right)$.

A. Plot the original $v(t)$ and the coefficients c_n when $T_1 = T/4$, $T_1 = T/8$, and $T_1 = T/16$. Use the *stem* command in Matlab. *Careful: the definition of sinc in Matlab is different than the one we used in class.*

Solution:



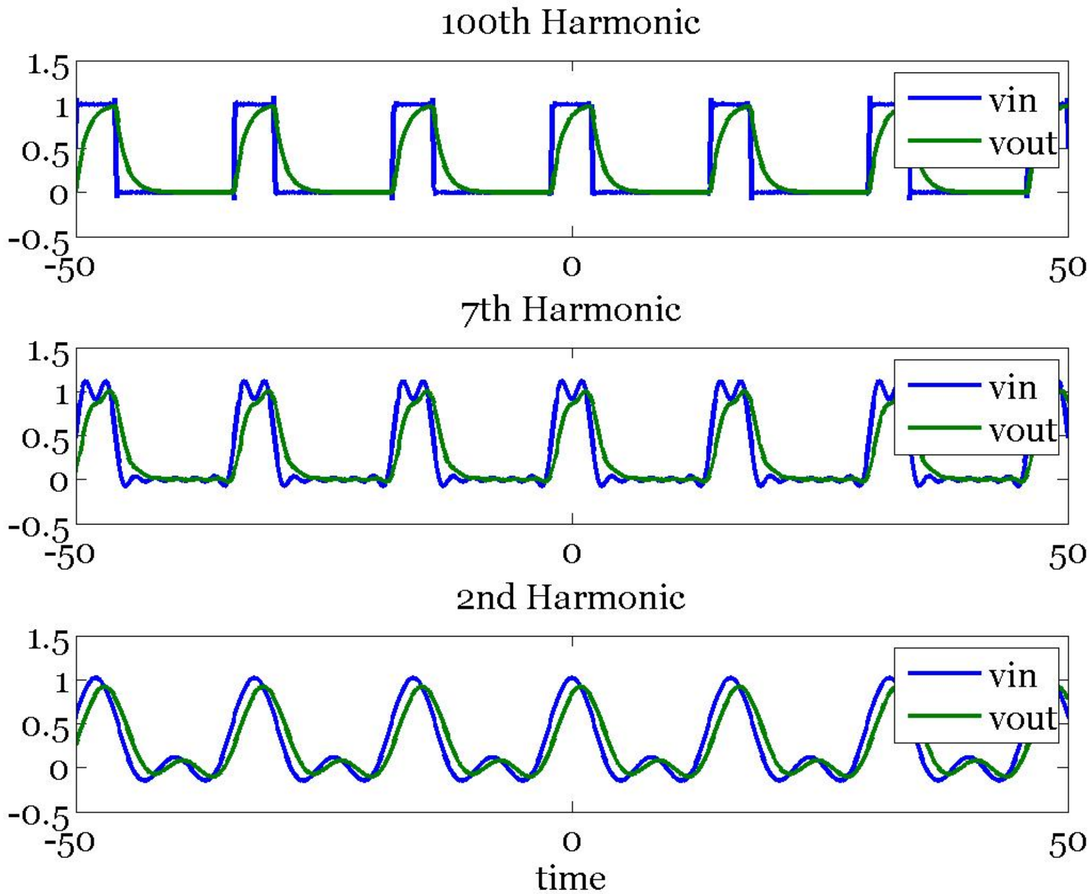
B. Assume $v_{in}(t) = v(t)$ and $T_1 = T/8$. Plot the response of the circuit shown below using Fourier decomposition with complex exponentials. Plot both the input and the output using 2, 7, and 100 harmonics. Note how similar the responses are even though you are using a very crude approximation.



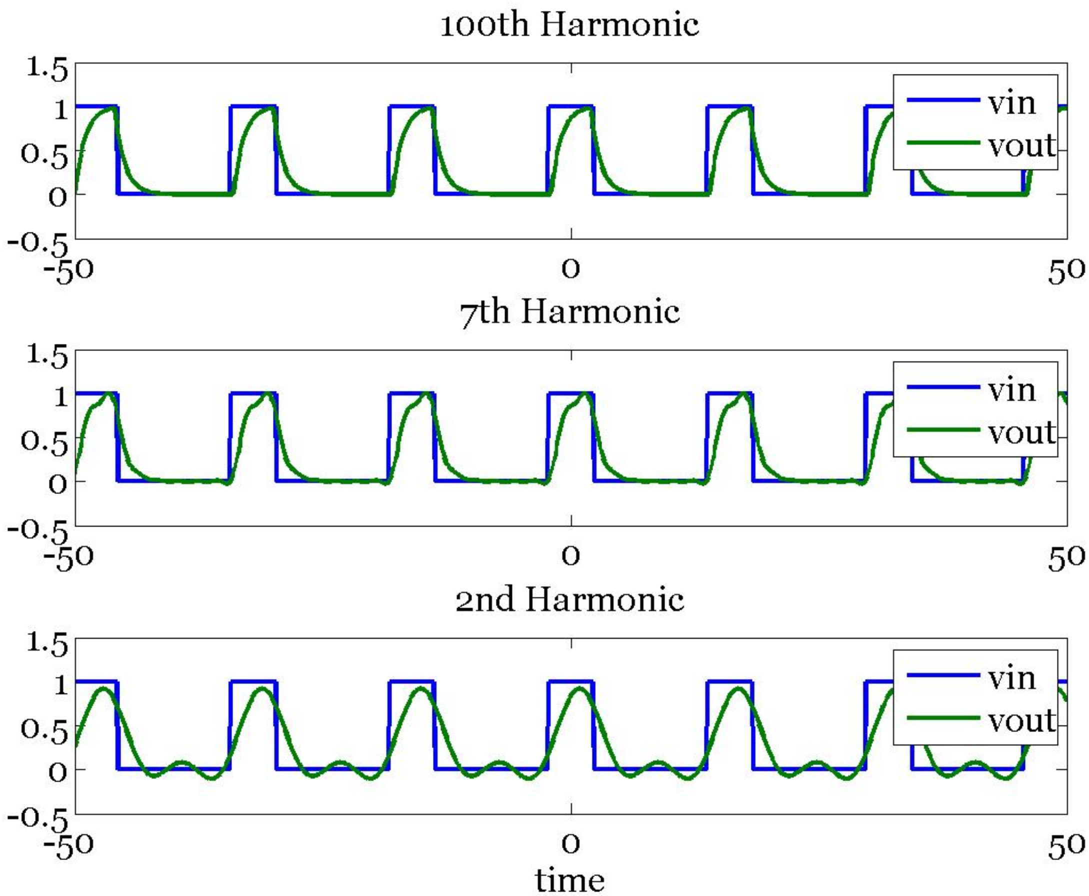
Solution:
The transfer function of the RC circuit is $H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}, \tau = RC$. If we represent our input as the continuous-time Fourier series, we can apply the transfer function to each exponential in the series as follows.

$$v_{out}(t) = \sum_{n=-\infty}^{\infty} H\left(j\frac{2\pi}{T}n\right) c_n e^{j\frac{2\pi}{T}nt} = \sum_{n=-\infty}^{\infty} \frac{1/\tau}{j\frac{2\pi}{T}n + 1/\tau} c_n e^{j\frac{2\pi}{T}nt}$$

Both the input and the output become coarse as we sum a decreasing amount of harmonics to reconstruct them. The title of each subplot indicates the largest harmonic that was used in the sum.

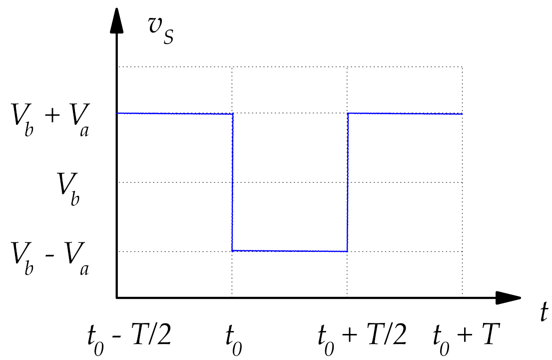


Compare that to the response below, which shows the output voltage when the input function is a well-constructed square wave. The first subplot, which shows the output summing only the first 100 harmonics, is indistinguishable from the actual response of the circuit. However, note that just using the first 7 harmonics, the response is quite close to its peak value and shows a reasonable exponential curve. Again, the title of each subplot indicates the largest harmonic that was used in the sum.



Problem 3: *(4 points)* This problem will introduce the concept of *average power* in a signal, and will explore the relationships across the frequency and time domains.

A. The voltage signal shown below is a square wave of amplitude V_a and average value V_b.



The *average power* of a signal is defined as

$$< P >= \frac{1}{T} \int_{t_0}^{t_0+T} |v_S|^2 dt$$

where $|v_S|^2 = v_S v_S^*$ is the square magnitude of the signal and the asterisk denotes complex conjugation. Show that the average power for v_S is $V_a^2 + V_b^2$.

Solution:

$$\frac{1}{T}\left[\int_{t_0}^{t_0+\frac{T}{2}} (V_b - V_a)^2 dt + \int_{t_0+\frac{T}{2}}^{t_0+T} (V_b + V_a)^2 dt\right]$$
$$\frac{1}{2}(V_b - V_a)^2 + \frac{1}{2}(V_b + V_a)^2$$
$$V_b^2 + V_a^2$$

B. Show that the average power of a complex exponential $c_n e^{j\omega t}$ is its squared magnitude, that is, $< P >= |c_n|^2$.

Solution:

$$
\begin{aligned}
< P > &= \frac{1}{T}\int_{t_0}^{t_0+T} |v_s|^2 dt \\
|v_s|^2 &= v_s \cdot v_s^* = c_n e^{j\omega t} \cdot c_n e^{-j\omega t} = |c_n|^2 \\
< P > &= \frac{1}{T}\int_{t_0}^{t_0+T} |c_n|^2 dt \\
&= \frac{T}{T}|c_n|^2 \\
&= |c_n|^2
\end{aligned}
$$

C. Find the Fourier decomposition of v_S assuming $t_0 = T/4$ (*note that this choice is irrelevant for this problem; think about why*) and using complex exponentials such that

$$
v_S(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}
$$

Solution:
In order to find the Fourier decomposition of v_S, we need to find an expression for c_n using the equation:

$$
c_n = \frac{1}{T}\int_T v_S(t) e^{-j\frac{2\pi}{T}nt} dt
$$

We simplify this integral by realizing that the DC offset only changes c_0. In other words, we can shift the square wave up or down to obtain the easiest integral and then add the difference (DC) as a constant term to the integral. So what square wave will yield the simplest integral? Shift the square wave down by $V_b - V_a$ so that half the integral is zero. Now we can solve for c_n using the following integral and just add

$V_a - V_b$ to c_0 to obtain the Fourier decomposition of our original square wave:

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} 2V_a e^{j\frac{2\pi}{T}nt}dt \\
&= \frac{2V_a}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{j\frac{2\pi}{T}nt}dt \\
&= \frac{2V_a}{T}\cdot\frac{T}{j2\pi n}\cdot\left(e^{j\frac{2\pi n}{T}\cdot\frac{T}{4}} - e^{-j\frac{2\pi n}{T}\cdot\frac{T}{4}}\right) \\
&= \frac{2V_a}{\pi n}\cdot\frac{1}{2j}\left(e^{j\frac{\pi n}{2}} - e^{-j\frac{\pi n}{2}}\right) \\
&= \frac{2V_a}{\pi n}\sin\left(\tfrac{\pi n}{2}\right),\ n \neq 0
\end{aligned}
$$

We can simplify this expression because sine is zero when n is an even integer, and ± 1 when n is an odd integer.

$$
\begin{aligned}
\text{If } n \text{ is even:} &\qquad c_n = 0 \\
\text{If } n \text{ is odd:} &\qquad c_n = \frac{2V_a(-1)^{\frac{n-1}{2}}}{\pi n}
\end{aligned}
$$

Thus, the Fourier decomposition of the square wave can be written:

$$
\begin{aligned}
v_S(t) &= \underbrace{V_a - V_b}_{\text{shifted amount}} + \underbrace{V_a}_{\text{DC offset}} + \sum_{n\in[1,3,5\ldots\infty)} \frac{2V_a(-1)^{\frac{n-1}{2}}}{\pi n} e^{j\frac{2\pi}{T}nt} \\
v_S(t) &= V_b + \sum_{n\in[1,3,5\ldots\infty)} \frac{2V_a(-1)^{\frac{n-1}{2}}}{\pi n} e^{j\frac{2\pi}{T}nt}
\end{aligned}
$$

D. Show that the average power of v_S can be separated into a sum of a term that depends on V_a and a term that depends on V_b. When computing the average power, you might find useful to know that

$$\sum_{n\in[1,3,5\ldots\infty)} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Solution:
Average power is average DC power plus average AC power:

$$\begin{aligned}
< P > &= < P_{DC} > + < P_{AC} > \\
< P_{DC} > &= c_n(0)^2 = V_b^2 \\
\text{Let } \widetilde{Y}(n) &= \frac{2V_a(-1)^{\frac{n-1}{2}}}{\pi n} e^{j\frac{2\pi}{T}nt} \\
< P_{AC} > &= \sum_{n\in[1,3,5\ldots\infty)} |\widetilde{Y}(n)|^2 \\
&= \sum_{n\in[1,3,5\ldots\infty)} \widetilde{Y}(n)\widetilde{Y}^*(n) \\
&= \sum_{n\in[1,3,5\ldots\infty)} \left(\frac{2V_a}{n\pi}\right)^2 \\
&= \frac{4V_a^2}{\pi^2} \sum_{n\in[1,3,5\ldots\infty)} \frac{1}{n^2} \\
&= \frac{4V_a^2}{\pi^2} \cdot 2\left(\frac{\pi^2}{8}\right) \\
&= V_a^2
\end{aligned}$$

E. The average power of v_S that depends on V_a is called the AC, or alternating current power, while the power that depends on V_b is called the DC, or direct current power. For the square wave, show that the fundamental (frequency 1/T) contains $8/\pi^2 \approx 80\%$ of the total AC power.

Solution:

$$\begin{aligned}
\text{Let } \widetilde{Y}(n) &= \frac{2V_a(-1)^{\frac{n-1}{2}}}{\pi n} e^{j\frac{2\pi}{T}nt} \\
< P_{AC} > &= \sum_{n\in[1,3,5\ldots\infty)} |\widetilde{Y}(n)|^2 \\
< P_{AC} >|_{n=-1,n=1} &= |\widetilde{Y}(-1)|^2 + |\widetilde{Y}(1)|^2 \\
&= \widetilde{Y}(-1)\widetilde{Y}^*(-1) + \widetilde{Y}(1)\widetilde{Y}^*(1) \\
&= \left(\frac{2V_a}{\pi}\right)^2 + \left(\frac{2V_a}{\pi}\right)^2 \\
&= \frac{8V_a^2}{\pi^2} \\
&= \frac{8}{\pi^2} < P_{AC} >
\end{aligned}$$