Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 9

Problem 1 *(2 points)* Show Parseval's Theorem

$$\int_{t=-\infty}^{t=\infty}[x(t)]^2dt = \int_{\omega=-\infty}^{\omega=\infty}|X(j\omega)|^2\tfrac{d\omega}{2\pi}$$

where

$$|X(j\omega)|^2 = X(j\omega)X^*(j\omega)$$

$x(t)$ is real, and $X^*(j\omega)$ is the complex conjugate of $X(j\omega)$.
Hint:

$$X^*(j\omega) = \int_{t=-\infty}^{t=\infty}x(t)e^{j\omega t}dt$$

Solution:

$$\begin{aligned}
\int_{\omega=-\infty}^{\omega=\infty}|X(j\omega)|^2\tfrac{d\omega}{2\pi} &= \int_{\omega=-\infty}^{\omega=\infty}X(j\omega)X^*(j\omega)\tfrac{d\omega}{2\pi} \\
&= \int_{\omega=-\infty}^{\omega=\infty}X(j\omega)\int_{t=-\infty}^{t=\infty}x(t)e^{j\omega t}dt\tfrac{d\omega}{2\pi} \\
&= \int_{\omega=-\infty}^{\omega=\infty}\int_{t=-\infty}^{t=\infty}X(j\omega)x(t)e^{j\omega t}dt\tfrac{d\omega}{2\pi} \\
&= \int_{t=-\infty}^{t=\infty}\int_{\omega=-\infty}^{\omega=\infty}X(j\omega)x(t)e^{j\omega t}\tfrac{d\omega}{2\pi}dt \\
&= \int_{t=-\infty}^{t=\infty}x(t)\underbrace{\int_{\omega=-\infty}^{\omega=\infty}X(j\omega)e^{j\omega t}\tfrac{d\omega}{2\pi}}_{x(t)}dt \\
&= \int_{t=-\infty}^{t=\infty}x(t)x(t)dt \\
\int_{\omega=-\infty}^{\omega=\infty}|X(j\omega)|^2\tfrac{d\omega}{2\pi} &= \int_{t=-\infty}^{t=\infty}[x(t)]^2dt
\end{aligned}$$

Problem 2 *(4 points)* A non-ideal filter or communication channel $H(j\omega)$ with finite *transition band* Δf between the *passband* (where $H(j\omega) \neq 0$) and *stopband* (where $H(j\omega) = 0$) can be modeled as $H(j\omega) = H_0(j\omega) * H_\Delta(j\omega)$ where

$$H_0(j\omega) = \begin{cases} 1 & -2\pi f_0 < \omega < 2\pi f_0 \\ 0 & \text{otherwise} \end{cases} \qquad H_\Delta(j\omega) = \begin{cases} \frac{1}{2\pi\Delta f} & -2\pi\Delta f/2 < \omega < 2\pi\Delta f/2 \\ 0 & \text{otherwise} \end{cases}$$

A. Find an expression for $h_0(t)$, the impulse response of the ideal channel.

Solution:

$$h_0(t) = 2f_0 \text{sinc}(2\pi f_0 t)$$

B. Find an expression for $H(j\omega)$ and sketch it.

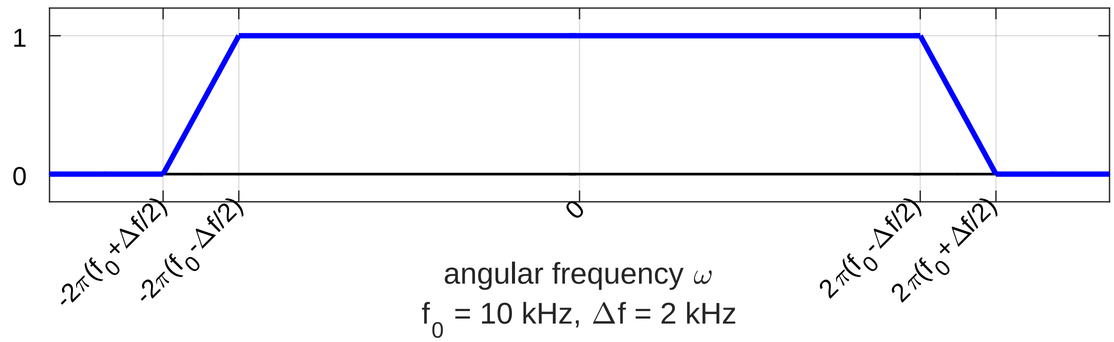
Solution:

$$H(j\omega) = H_0(j\omega) * H_\Delta(j\omega) = \int_{-\infty}^{\infty} H_0(\omega')H_\Delta(\omega - \omega')d\omega'$$

$$\begin{aligned} H(j\omega) = & \int_{-\infty}^{-2\pi(f_0+\Delta f/2)} \cancelto{0}{H_0(\omega')} H_\Delta(\omega - \omega')d\omega' \\ & + \int_{-2\pi(f_0+\Delta f/2)}^{-2\pi(f_0-\Delta f/2)} H_0(\omega')H_\Delta(\omega - \omega')d\omega' \\ & + \int_{-2\pi(f_0-\Delta f/2)}^{2\pi(f_0-\Delta f/2)} \cancelto{1}{H_0(\omega')} H_\Delta(\omega - \omega')d\omega' \\ & + \int_{2\pi(f_0-\Delta f/2)}^{2\pi(f_0+\Delta f/2)} H_0(\omega')H_\Delta(\omega - \omega')d\omega' \\ & + \int_{2\pi(f_0+\Delta f/2)}^{\infty} \cancelto{0}{H_0(\omega')} H_\Delta(\omega - \omega')d\omega' \end{aligned}$$

$$H(j\omega) = \begin{cases} 0, & -\infty < \omega < -2\pi(f_0 + \Delta f/2) \\ \int_{-2\pi f_0}^{\omega+2\pi\Delta f/2} \frac{1}{2\pi\Delta f} d\omega', & -2\pi(f_0 + \Delta f/2) < \omega < -2\pi(f_0 - \Delta f/2) \\ \int_{\omega-2\pi\Delta f/2}^{\omega+2\pi\Delta f/2} \frac{1}{2\pi\Delta f} d\omega', & -2\pi(f_0 - \Delta f/2) < \omega < 2\pi(f_0 - \Delta f/2) \\ \int_{\omega-2\pi\Delta f/2}^{2\pi f_0} \frac{1}{2\pi\Delta f} d\omega', & 2\pi(f_0 - \Delta f/2) < \omega < 2\pi(f_0 + \Delta f/2) \\ 0, & 2\pi(f_0 + \Delta f/2) < \omega < \infty \end{cases}$$

$$H(j\omega) = \begin{cases} 0, & -\infty < \omega < -2\pi(f_0 + \Delta f/2) \\ \frac{\omega + 2\pi\Delta f/2 + 2\pi f_0}{2\pi\Delta f}, & -2\pi(f_0 + \Delta f/2) < \omega < -2\pi(f_0 - \Delta f/2) \\ 1, & -2\pi(f_0 - \Delta f/2) < \omega < 2\pi(f_0 - \Delta f/2) \\ \frac{2\pi f_0 + 2\pi\Delta f/2 - \omega}{2\pi\Delta f}, & 2\pi(f_0 - \Delta f/2) < \omega < 2\pi(f_0 + \Delta f/2) \\ 0, & 2\pi(f_0 + \Delta f/2) < \omega < \infty \end{cases}$$



C. Find an expression for the impulse response $h(t)$ of the non-ideal channel.

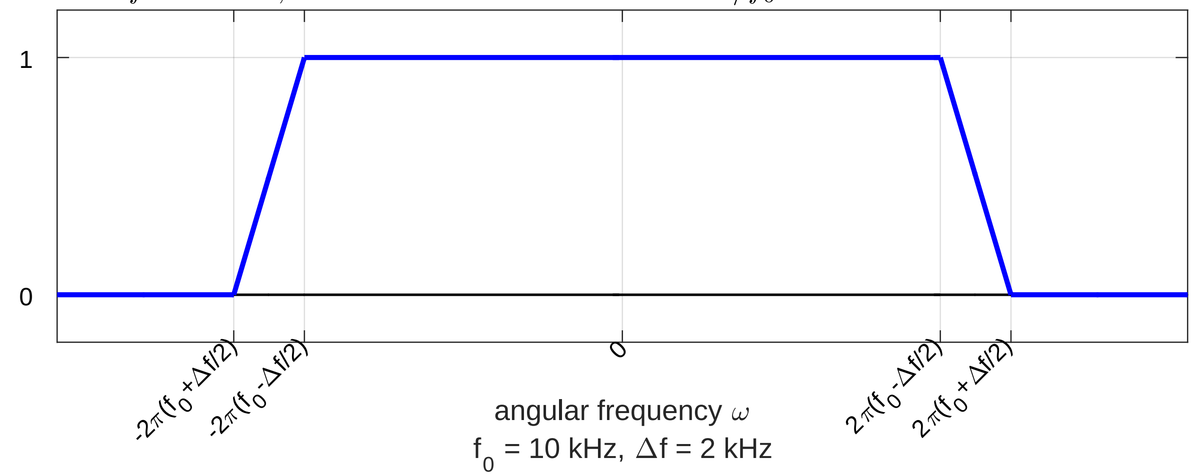
Solution:

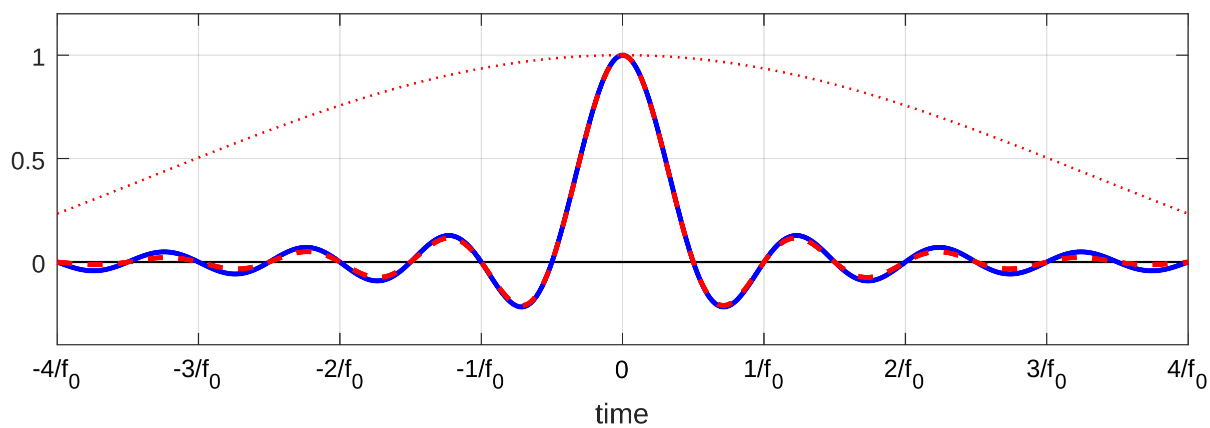
$$h(t) = 2\pi \mathscr{F}^{-1}\{H_0(j\omega)\}\mathscr{F}^{-1}\{H_\Delta(j\omega)\}$$
$$h(t) = 2\pi h_0(t) h_\Delta(t)$$
$$h(t) = \cancel{2\pi} \cdot 2f_0 \text{sinc}(2\pi f_0 t) \cdot \frac{1}{\cancel{2\pi\Delta f}} \cancel{\Delta f} \text{sinc}(\pi \Delta f t)$$
$$h(t) = 2f_0 \text{sinc}(2\pi f_0 t) \cdot \text{sinc}(\pi \Delta f t)$$

D. Plot $h(t)$ and $h_0(t)$ on the same axes when $f_0 = 10$ kHz and $\Delta f = 2$ kHz and sketch the associated $H(j\omega)$. Repeat the plot for the same values when $\Delta f = 5$ kHz. Compare the plots. For what values of Δf is the effect of the non-ideal transition band noticeable? Is the effect what you would expect? Explain.

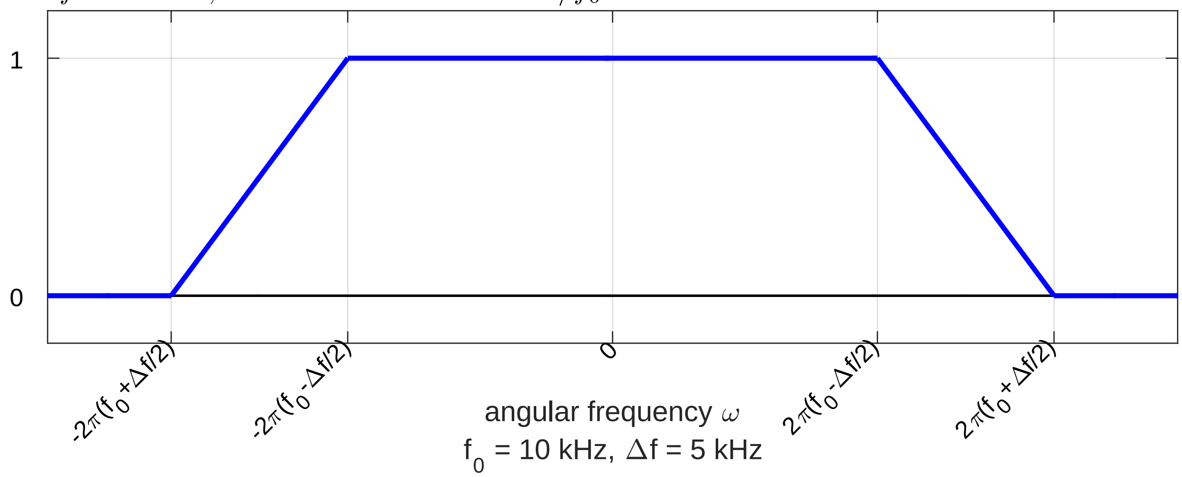
Solution:

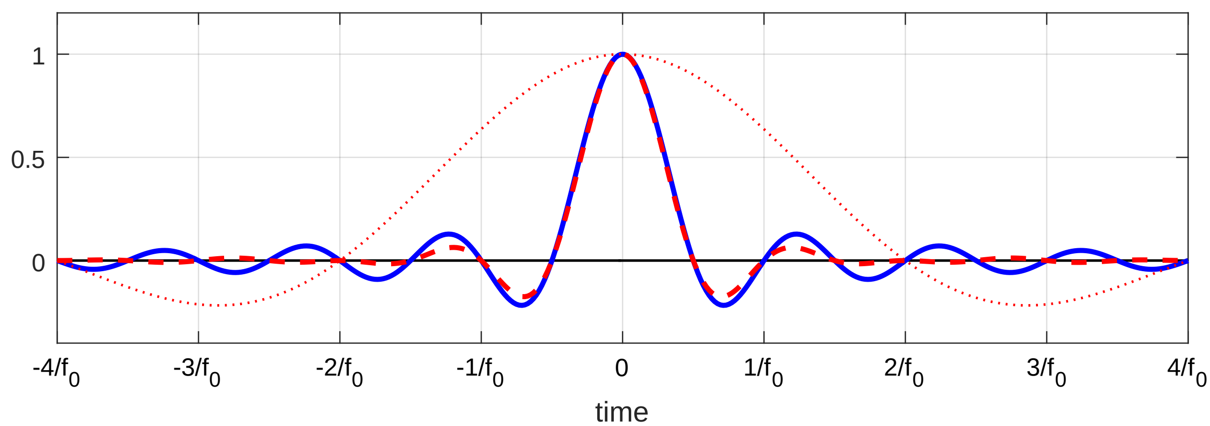
When $\Delta f = 2\ kHz$, the effect is noticeable around $2/f_0$.





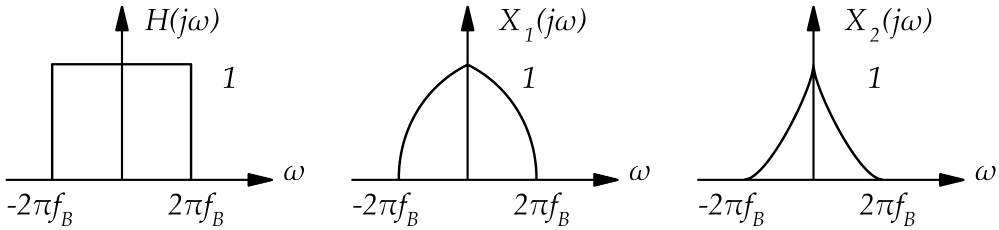
If $\Delta f = 5\ kHz$, the effect is clear after $1/f_0$.

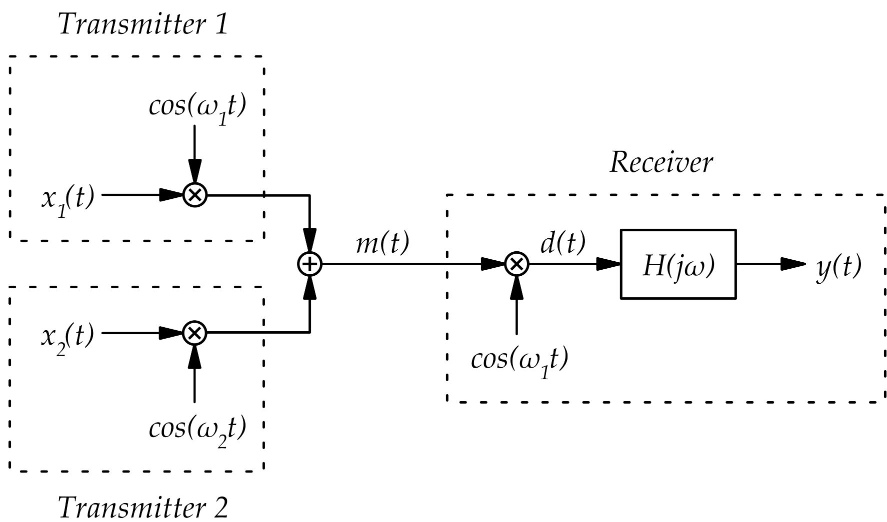




As expected, a more gradual transition in the frequency domain reduces the amount of oscillations, or “ringing” in the time domain.

Problem 3 *(4 points)* The system shown below represents a basic communication system where two messages $x_1(t)$ and $x_2(t)$ share a common communication channel. Signals $x_1(t)$ and $x_2(t)$ are bandlimited to f_B and have a frequency content as shown below. The receiver has an ideal low-pass filter $H(j\omega)$ with a cutoff frequency of f_B as shown below.





A. What would happen if $\omega_1 = \omega_2 = 0$? Find $y(t)$ in terms of $x_1(t)$ and/or $x_2(t)$, and show its frequency content.

Solution:
If $\omega_1 = \omega_2 = 0$, the output of Transmitter 1 is

$$x_1(t)\cos(\omega_1 t) = x_1(t)\cos(0t) = x_1(t)$$

and the output of Transmitter 2 is

$$x_2(t)\cos(\omega_2 t) = x_2(t)\cos(0t) = x_2(t)$$

Therefore,

$$m(t) = x_1(t) + x_2(t)$$

Multiplying by $\cos(\omega_1(t))$,

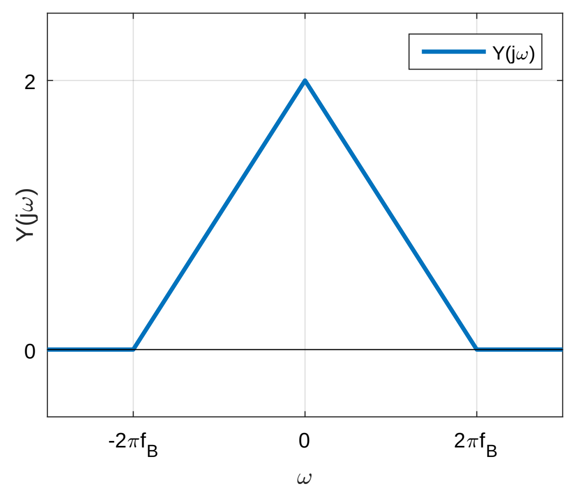
$$d(t) = m(t)\cos(\omega_1(t)) = m(t)\cos(0t) = m(t) = x_1(t) + x_2(t)$$

Note that the filter produced by $H(j\omega)$ removes all frequencies above $2\pi f_b$ and below $-2\pi f_b$; however, neither $x_1(t)$ nor $x_2(t)$ have any frequency content above $2\pi f_b$ or below $-2\pi f_b$. Therefore, the low-pass filter does not have any effect on $d(t)$, and

$$y(t) = d(t) = x_1(t) + x_2(t)$$

Therefore, the frequency content is merely the sum of the frequency content $x_1(t)$ and the frequency content of $x_2(t)$; $Y(j\omega)$ is shown below.

$$Y(j\omega) = X_1(j\omega) + X_2(j\omega)$$

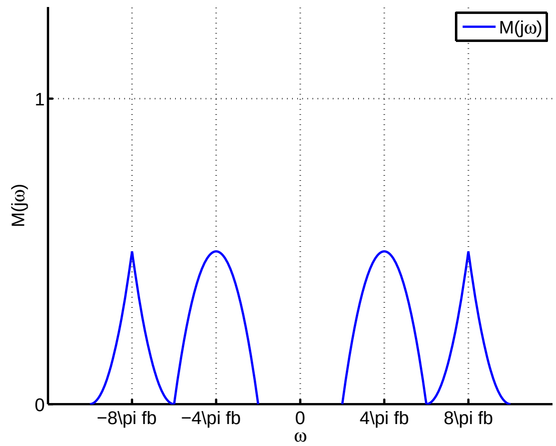


B. Find constraints on ω_1 and ω_2 such that there is no frequency interference (aliasing). Show the frequency content of $m(t)$ and $d(t)$ under these constraints. *Note: There may be multiple solutions; just find one that works.*

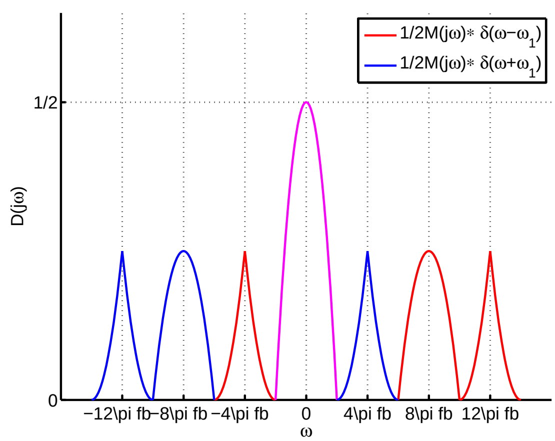
Solution:
A number of frequency constraints on ω_1 and ω_2 are possible - in particular, the output $M(j\omega)$ will consist of bandlimited peaks of width $4\pi f_b$ at frequencies $-\omega_1$, ω_1, $-\omega_2$, and ω_2. Algebraically,

$$\begin{aligned}
m(t) &= x_1(t)\cos(\omega_1 t) + x_2(t)\cos(\omega_2 t) \\
M(j\omega) &= \frac{1}{2\pi}\left(\mathscr{F}\{x_1(t)\} * \mathscr{F}\{\cos(\omega_1 t)\}\right) + \frac{1}{2\pi}\left(\mathscr{F}\{x_2(t)\} * \mathscr{F}\{\cos(\omega_2 t)\}\right) \\
M(j\omega) &= \frac{1}{2\pi}\left(X_1(j\omega) * [\pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1)]\right) \\
&\quad + \frac{1}{2\pi}\left(X_2(j\omega) * [\pi\delta(\omega-\omega_2) + \pi\delta(\omega+\omega_2)]\right)
\end{aligned}$$

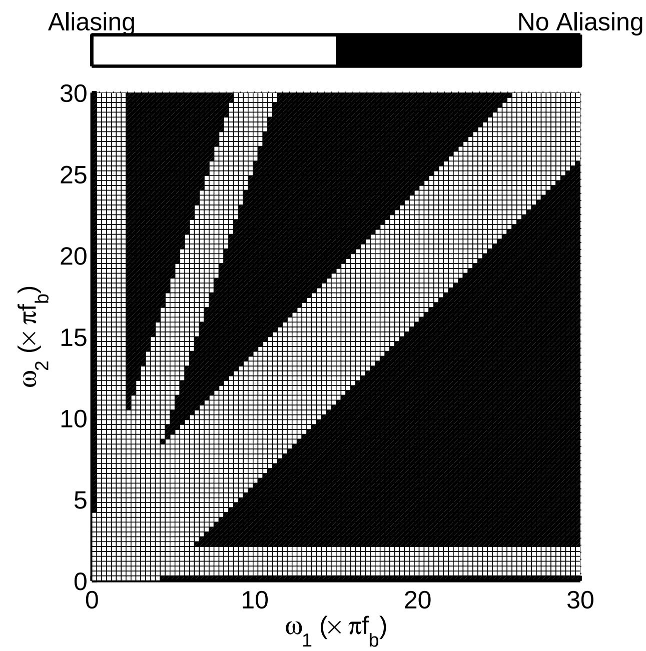
We could choose $\omega_1 = 4\pi f_b$ and $\omega_2 = 8\pi f_b$, which would avoid aliasing in $M(j\omega)$, as shown below.



This choice of ω_1 and ω_2 also avoids aliasing in $\mathscr{F}\{d(t)\} = D(j\omega)$, as shown below: the frequency content of $d(t)$ consists of the frequency content of $m(t)$ shifted by ω_1 and $-\omega_1$. $D(j\omega)$ for $\omega_1 = 4\pi f_b$ and $\omega_2 = 8\pi f_b$ is shown below:



Indeed, any ω_1 and ω_2 will generate peaks of $X_1(j\omega)$ at $\omega = 0, 2\omega_1, 0$, and $-2\omega_1$, as well as peaks of $X_2(j\omega)$ at $\omega = \omega_1 - \omega_2, \omega_1 + \omega_2, -\omega_1 + \omega_2$, and $-\omega_1 - \omega_2$. Peaks must be separated by at least $4\pi f_b$ (the width of a single bandlimited signal peak). Therefore, any ω_1 and ω_2 that generate $X_1(j\omega)$ and $X_2(j\omega)$ as given previously that either overlap completely constructively or do not overlap will suffice. For example, in the $D(j\omega)$ shown above, two $X_1(j\omega)$ peaks combine at the origin, but no overlap is present between $X_1(j\omega)$ and $X_2(j\omega)$ peaks and no partial overlap is present. Either of these situations would constitute aliasing. To find the constraints on aliasing, we can enforce these constraints computationally and search a space of ω_1 and ω_2; the resulting regions of aliasing and no aliasing are shown below.



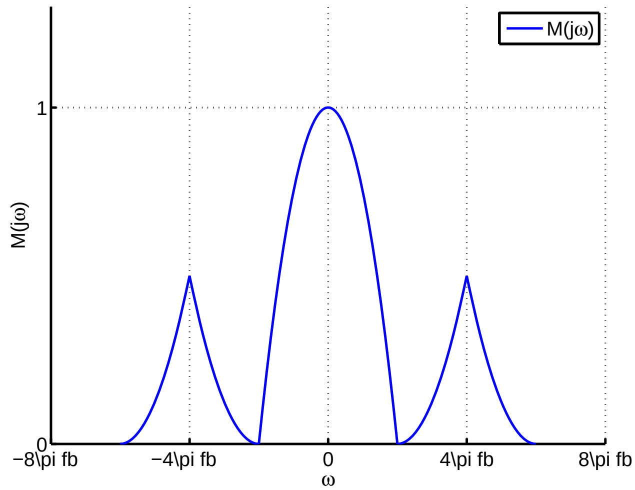
A full set of constraints for ω_1 and ω_2 is provided by:

$$
\begin{aligned}
|\omega_1 - \omega_2| &\geq 4\pi f_b \\
|4\omega_1 - \omega_2| &\geq 6\pi f_b \\
(\omega_1 = 0 \text{ or } \omega_1 &\geq 4\pi f_b) \\
(\omega_2 = 0 \text{ or } \omega_2 &\geq 4\pi f_b)
\end{aligned}
$$

One possible simple frequency constraint on ω_1 and ω_2 is $\omega_1 = 0$ and $\omega_2 \geq 4\pi f_b$; in this case, the output of Transmitter 1 will be $x_1(t)$ (since $x_1(t) \cdot \cos(0t) = x_1(t)$), and the frequency content of Transmitter 2 will be located at two peaks that do not intersect those of the output of Transmitter 1. Then, to find the frequency content of $m(t)$, we observe that

$$
\begin{aligned}
m(t) &= x_1(t) + x_2(t)\cos(\omega_2 t) \\
M(j\omega) &= \mathscr{F}\{x_1(t)\} + \mathscr{F}\{x_2(t)\cos(\omega_2 t)\} \\
M(j\omega) &= X_1(j\omega) + \frac{1}{2\pi}\left(X_2(j\omega) * (\pi\delta(\omega + \omega_2) + \pi\delta(\omega - \omega_2))\right)
\end{aligned}
$$

A graphical representation of $M(j\omega)$ is shown below. Note that when $\omega_1 = 0$, the height of the "peak" at $\omega_1 = 0$ is 1; otherwise, its height is 1/2.



Since $\cos(\omega_1 t) = 1$, $d(t) = m(t)\cos(\omega_1 t) = m(t)\cos(0t) = m(t)$, so the frequency content of $d(t)$ is identical to the frequency content of $m(t)$.

C. Show the frequency content and find an algebraic expression for $y(t)$ in terms of $x_1(t)$ and/or $x_2(t)$ assuming the constraints of part B.

Solution:
If $H(j\omega)$ is the ideal low-pass filter given, it will remove all frequencies above $2\pi f_b$ and all frequencies below $-2\pi f_b$. Therefore, the two peaks of $X_2(j\omega)$ in $d(t)$ will be eliminated. If $\omega_1 = 0$, $x_1(t)$ will simply pass through and the resulting $Y(j\omega)$ will simply be $X_1(j\omega)$, and therefore, $y(t) = x_1(t)$. However, if $x_1(t)$ has been modulated with any nonzero frequency ω_1, the filter will cut the high frequency components of the demodulated $x_1(t)$ such that $Y(j\omega) = \frac{1}{2}X_1(j\omega)$ and $y(t) = \frac{1}{2}x_1(t)$.