Olin College of Engineering
ENGR2410 – Signals and Systems

Overview and Tables

Main concepts

1. Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\omega t} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(j\omega)e^{j\omega t}$$

2. Most functions can be expressed as an infinitely dense sum (i.e., an integral) of exponentials. If the sum is discrete, the function must be periodic.

$$x(t) = \int_{\omega=-\infty}^{\omega=\infty} \underbrace{X(j\omega)}_{\text{frequency content}} e^{j\omega t}\frac{d\omega}{2\pi}$$

3. The frequency content of the output is the frequency content of the input multiplied by the transfer function. A *Bode plot* is a logarithmic plot of the transfer function.

$$y(t) = \int_{\omega=-\infty}^{\omega=\infty} \underbrace{H(j\omega)X(j\omega)}_{Y(j\omega)} e^{j\omega t}\frac{d\omega}{2\pi} \Longleftrightarrow \begin{tikzcd} x(t) \arrow[r] \arrow[d, "\mathscr{F}"'] & \boxed{\text{LTI}} \arrow[r] & y(t) \\ X(j\omega) \arrow[r] & \boxed{H(j\omega)} \arrow[r] & Y(j\omega) = X(j\omega)H(j\omega) \arrow[u, "\mathscr{F}^{-1}"'] \end{tikzcd}$$

4. The frequency content of an impulse is constant over all frequencies. As a result, the frequency content of the impulse response is given by the transfer function.

$$\begin{tikzcd} \delta(t) \arrow[r] \arrow[d, "\mathscr{F}"'] & \boxed{h(t)} \arrow[r] & h(t) \\ 1 \arrow[r] & \boxed{H(j\omega)} \arrow[r] & H(j\omega) \arrow[u, "\mathscr{F}^{-1}"'] \end{tikzcd}$$

5. Convolution in time is equivalent to multiplication in frequency.

Thus, in the time domain, the output of a system is the input convolved with the impulse response. But, the *transform of the* output is the *transform of the* input ~~convolved~~ **multiplied with the *transform of the* impulse response.**

$$\begin{tikzcd} x(t) \arrow[r] \arrow[d, "\mathscr{F}"'] & \boxed{h(t)} \arrow[r] & y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)h(t-\tau)d\tau \\ X(j\omega) \arrow[r] & \boxed{H(j\omega)} \arrow[r] & Y(j\omega) = X(j\omega)H(j\omega) \arrow[u, "\mathscr{F}^{-1}"'] \end{tikzcd}$$

6. Exponentials are more generalized eigenfunctions of LTI systems (than complex exponentials).

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

7. The eigenvalue of the eigenfunction e^{st} is the Laplace transform of $h(t)$ since

$$H(s)e^{st} = \underbrace{e^{st}}_{\text{input}} * \underbrace{h(t)}_{\text{system}} = \int_{\tau=-\infty}^{\tau=\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st} \underbrace{\int_{\tau=-\infty}^{\tau=\infty} h(\tau)e^{-s\tau}d\tau}_{H(s)}$$

8. A *pole-zero diagram* is a plot of the complex plane showing the values of s where the transfer function is zero (*zeroes*) or infinity (*poles*). A system is stable iff all the poles are in the left half-plane ($\text{Re}\{s\} > 0$).

Useful equations

The underdamped second-order system

$\ddot{v}_{out} + 2\alpha\dot{v}_{out} + \omega_0^2 v_{out} = \omega_0^2 v_{in}, \quad \omega_0 > \alpha$

has poles at $s = -\alpha \pm j\omega_d, \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}, \quad Q = \omega_0/2\alpha$

and step response

$v_{out}(t) = 1 - \sqrt{1 + (\alpha/\omega_d)^2}e^{-\alpha t}\cos(\omega_d t - \tan^{-1}\alpha/\omega_d), \quad t > 0$

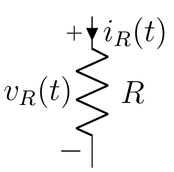
Sinusoidal steady-state

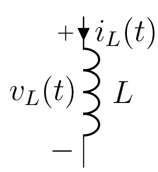
$\cos\omega t \rightarrow \boxed{H(j\omega)} \rightarrow |H(j\omega)|\cos(\omega t + \angle H(j\omega))$

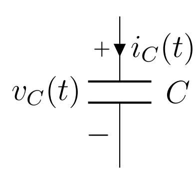
Euler's equation

$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \qquad \cos(\omega t) = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin(\omega t) = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$

Impedance







$v_R(t) = Ri_R(t) \qquad v_L(t) = L\dfrac{di_L}{dt} \qquad i_C(t) = C\dfrac{dv_C}{dt}$

$Z_R(s) = R \qquad Z_L(s) = Ls \qquad Z_C(s) = \dfrac{1}{Cs}$

Continuous time Fourier transform

$$\text{If}\quad x(t)=\int_{\omega=-\infty}^{\omega=\infty} X(j\omega)e^{j\omega t}\tfrac{d\omega}{2\pi}\qquad\text{then}\qquad X(j\omega)=\int_{t=-\infty}^{t=\infty} x(t)e^{-j\omega t}dt \triangleq \mathscr{F}\{x(t)\}$$

Alternatively, $\quad x(t) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad X(j\omega)$

System representation

$$\begin{array}{ccccc}
x(t) & \longrightarrow & \boxed{h(t)} & \longrightarrow & y(t) = x(t)*h(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)h(t-\tau)d\tau \\
\mathscr{F}\big\downarrow & & & & \big\uparrow \mathscr{F}^{-1} \\
X(j\omega) & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & Y(j\omega) = X(j\omega)H(j\omega)
\end{array}$$

Properties and transforms

$$\begin{aligned}
\mathscr{F}\{ax_1(t)+bx_2(t)\} &= aX_1(j\omega)+bX_2(j\omega)\\
\mathscr{F}\{x(t+T)\} &= X(j\omega)e^{j\omega T}\\
\mathscr{F}\{X(t)\} &= 2\pi x(-\omega)\\
\mathscr{F}\{x(at)\} &= \frac{1}{|a|}X(\frac{j\omega}{a})\\
\mathscr{F}\{\delta(t)\} &= 1\\
\mathscr{F}\{1\} &= 2\pi\delta(\omega)\\
\mathscr{F}\{e^{j\omega_0 t}\} &= 2\pi\delta(\omega-\omega_0)\\
\mathscr{F}\{\cos(\omega_0 t)\} &= \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)\\
\mathscr{F}\{\sin(\omega_0 t)\} &= \frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)\\
\mathscr{F}\left\{\frac{1}{2}\delta(t-T)+\frac{1}{2}\delta(t+T)\right\} &= \cos(\omega T)\\
\mathscr{F}\{e^{-t/\tau}u(t)\} &= \frac{1}{j\omega+1/\tau}\\
\mathscr{F}\{x(t)*y(t)\} &= X(j\omega)Y(j\omega)\\
\mathscr{F}\{x(t)y(t)\} &= \frac{1}{2\pi}X(j\omega)*Y(j\omega)\\
\mathscr{F}\,\{\Pi(\mathrm{t}/\mathrm{T}_1)\} &= 2T_1\frac{\sin(\omega T_1)}{\omega T_1}\ \text{ where } \Pi(\mathrm{x}) = \begin{cases}1 & -1\le x\le 1\\ 0 & \text{otherwise}\end{cases}\\
\mathscr{F}\left\{\frac{\omega_0}{\pi}\cdot\frac{\sin(\omega_0 t)}{\omega_0 t}\right\} &= \Pi(\omega/\omega_0)\ \text{ where } \Pi(\mathrm{x}) = \begin{cases}1 & -1\le x\le 1\\ 0 & \text{otherwise}\end{cases}\\
\mathscr{F}\left\{\sum_{n=-\infty}^{\infty}\delta(t-nT)\right\} &= \sum_{k=-\infty}^{\infty}\frac{2\pi}{T}\delta\left(\omega-k\frac{2\pi}{T}\right)\ \text{ where } n \text{ and } k \text{ are integers}\\
\mathscr{F}\left\{e^{t^2/\sigma^2}\right\} &= |\sigma|\sqrt{\pi}e^{-\omega^2\sigma^2/4}
\end{aligned}$$

Discrete time Fourier transform ($n \in \mathbb{Z}$)

$$\text{If} \quad x[n] = \int_{2\pi} X(\Omega) e^{j\Omega n} \frac{d\Omega}{2\pi} \quad \text{then} \quad X(\Omega) = \sum_n x[n] e^{-j\Omega n}$$

Alternatively, $\quad x[n] \overset{\mathscr{F}}{\Longleftrightarrow} X(\Omega)$

System representation

$$\begin{array}{ccccc} x[n] & \longrightarrow & \boxed{h[n]} & \longrightarrow & y[n] = x[n] * h[n] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k] \\ \mathscr{F}\big\downarrow & & & & \big\uparrow \mathscr{F}^{-1} \\ X(\Omega) & \longrightarrow & \boxed{H(\Omega)} & \longrightarrow & Y(\Omega) = X(\Omega)H(\Omega) \end{array}$$

Properties and transforms

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad 1$$

$$x[n-n_0] \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad X(\Omega)e^{-j\Omega n_0}$$

$$e^{j\Omega_0 n} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \sum_k 2\pi\delta(\Omega - \Omega_0 - 2\pi k), \quad k \in \mathbb{Z}$$

$$a^n u[n] \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \frac{1}{1 - ae^{-j\Omega}}, \quad |a| < 1$$

DT processing of CT signals

$$x_c(t) \longrightarrow \underset{\substack{\uparrow \\ f_S}}{\boxed{\text{C/D}}} \xrightarrow{x_d[n] = x_c\left(\frac{n}{f_S}\right)} \boxed{H_d(\Omega)} \xrightarrow{y_d[n]} \underset{\substack{\uparrow \\ f_S}}{\boxed{\text{D/C}}} \xrightarrow{y_c(t) = y_d[n], \text{ if } t = \frac{n}{f_S}} y_c(t)$$

Sampling at f_S constrains Ω such that $\frac{\omega}{f_S} = \Omega$. If $x_c(t)$ is bandlimited by f_{max} such that the sampling frequency $f_S > 2f_{max}$, the system above is equivalent to an LTI system $H_c(j\omega)$, where

$$H_c(j\omega) = \begin{cases} H_d(\omega/f_S) & -2\pi\frac{f_S}{2} \le \omega \le 2\pi\frac{f_S}{2} \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, $\quad Y_c(j\omega) = H_d(\omega/f_S)X_c(j\omega)$

Laplace Transform

$$X(s) = \int_{t=-\infty}^{t=\infty} x(t)e^{-st}dt \equiv \mathscr{L}\{x(t)\} \quad \text{with region of convergence } (ROC) : R$$

$$\begin{aligned}
\mathscr{L}\{ax_1(t) + bx_2(t)\} &= aX_1(s) + bX_2(s) && ROC: \text{At least } R_1 \cap R_2 \\
\mathscr{L}\{x(t-T)\} &= X(s)e^{-sT} && ROC: R \\
\mathscr{L}\{x(t) * y(t)\} &= X(s)Y(s) && ROC: \text{At least } R_1 \cap R_2 \\
\mathscr{L}\{\delta(t)\} &= 1 && ROC: \text{All } s \\
\mathscr{L}\{u(t)\} &= \frac{1}{s} && ROC: \text{Re}\{s\} > 0 \\
\mathscr{L}\{e^{-t/\tau}u(t)\} &= \frac{1}{s + 1/\tau} && ROC: \text{Re}\{s\} > -\frac{1}{\tau} \\
\mathscr{L}\{e^{-\alpha t}\sin(\omega_d t)u(t)\} &= \frac{\omega_d}{(s+\alpha)^2 + \omega_d^2} && ROC: \text{Re}\{s\} > -\alpha \\
\mathscr{L}\{e^{-\alpha t}\cos(\omega_d t)u(t)\} &= \frac{s+\alpha}{(s+\alpha)^2 + \omega_d^2} && ROC: \text{Re}\{s\} > -\alpha
\end{aligned}$$

Fourier series

If $x(t+T) = x(t)$, then

$$x(t) = \Sigma_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}, \qquad c_n = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}nt}$$

Alternatively,

$$c_n = \frac{1}{T}X_T\left(j\frac{2\pi}{T}n\right), \qquad X_T(j\omega) = \int_T x(t)e^{-j\omega t}dt$$