Olin College of Engineering
ENGR2410 – Signals and Systems

Lecture 1 Reference

Change of Basis and Diagonalization

The system

$$\begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} = \begin{bmatrix} 3x_{in} + y_{in} \\ x_{in} + 3y_{in} \end{bmatrix}$$

takes the $[x_{in}\ y_{in}]^T$ vector[1] as input and outputs the vector $[x_{out}\ y_{out}]^T$,

$$\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}} \rightarrow \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix}.$$

The eigenvectors and eigenvalues for this matrix are

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can express any $[x_{in}\ y_{in}]^T$ as a linear combination of these eigenvectors so that

$$\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Note that the eigenvectors are the columns of the matrix.

If we express $[x_{in}\ y_{in}]^T$ as a linear combination of the eigenvectors, finding the output becomes easier since

$$\begin{aligned} \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix} &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2 \right) \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_1 + \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2 \\ &= 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_1 + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2. \end{aligned}$$

[1] v^T is the transpose of vector v.

Using more compact matrix notation,

$$\begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

However, in order to use this equation, we have to transform the input from $[x_{in}\ y_{in}]^T$ into $[a_1\ a_2]^T$,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}.$$

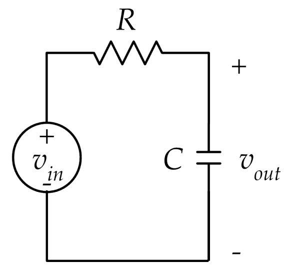
Combining these last two equations,

$$\begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}.$$

Using the same system notation as before,

$$\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \rightarrow \underbrace{\boxed{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}}_{\text{transform input into eigenvector space}} \rightarrow \boxed{\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}} \rightarrow \underbrace{\boxed{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}}_{\text{inverse transform}} \rightarrow \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix}.$$

RC circuit
We can express the behavior of the RC circuit shown below as a differential equation in terms of v_{out}:



$$
\begin{aligned}
v_{in} &= v_R + v_{out} \\
v_{in} &= Ri + v_{out} \\
v_{in} &= RC\dot{v}_{out} + v_{out} \\
\dot{v}_{out} + \frac{1}{RC}v_{out} &= \frac{1}{RC}v_{in} \qquad \text{(standard form)}
\end{aligned}
$$

Consider an input signal that is constant for $t > 0$,

$$v_{in}(t) = V, t > 0$$

The *particular solution* solves the full differential equation:

$$\dot{v}_{out,p} + \frac{1}{RC}v_{out,p} = \frac{1}{RC}V, t > 0$$

Because v_{in} is constant for $t > 0$, it is reasonable to guess the constant function K for $v_{out,p}$

$$
\begin{aligned}
v_{out,p}(t) &= K \\
\dot{v}_{out,p}(t) &= 0
\end{aligned}
$$

Substituting,

$$
\begin{aligned}
&0 + \frac{1}{RC}K = \frac{1}{RC}V \\
&\Rightarrow K = V \\
&\Rightarrow v_{out,p}(t) = V
\end{aligned}
$$

This output satisfies the differential equation, but we have to add *homogeneous solutions* that solve the homogeneous equation:

$$\dot{v_{out,h}} + \frac{1}{RC}v_{out,h} = 0$$

If these solutions exist, then $v_{out} = v_{out,h} + v_{out,p}$ is the *general solution* for the original equation.

Solving the homogeneous equation,

$$
\begin{aligned}
\frac{dv_{out,h}}{v_{out,h}} &= -\frac{1}{RC}dt \\
\ln v_{out,h} &= -\frac{t}{RC} + A' \\
v_{out,h} &= e^{A'}e^{-t/RC} \\
v_{out,h} &= Ae^{-t/RC}
\end{aligned}
$$

The general solution is

$$
\begin{aligned}
v_{out} &= v_{out,p} + v_{out,h} \\
&= V + Ae^{-t/RC}
\end{aligned}
$$

This is certainly a solution to the original equation, but it still contains the arbitrary constant A. We must also consider the initial conditions.

If we assume that the system was at rest for $t < 0$, then $v_{out}(0) = 0$. In this case,

$$v_{out}(0) = V + A = 0 \Rightarrow A = -V$$

so that the final solution is

$$v_{out}(t) = V(1 - e^{-t/RC}), t > 0$$

In system notation,

$$Vu(t) \rightarrow \boxed{\text{RC circuit}} \rightarrow V(1 - e^{-t/RC})u(t)$$

where $u(t)$ is the unit step function. The output is the *step response* of the system.