Olin College of Engineering
ENGR2410 – Signals and Systems

Reference 9

Natural frequencies of rational transfer functions

Exponentials are eigenfunction of LTI systems, and the associated eigenvalue is the transfer function,

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

We can find this transfer function $H(s)$ by direct substitution in a differential equation. In particular, systems described by differential equations of the form

$$\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = \ddot{x} + b_1\dot{x} + b_0 x$$

result in rational transfer functions whose polynomials can *always* be factored,

$$H(s) = \frac{Y}{X} = \frac{s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)(s+p_3)}$$

Note that if $H(s)$ is real, any poles or zeros that are not real must show in complex conjugate pairs. **The denominator of the transfer function is the characteristic polynomial of the homogeneous differential equation, and its roots are the exponents, or *natural frequencies* of the system represented by the differential equation**, such that the impulse response has the form

$$h(t) = A_1 e^{-p_1 t}u(t) + A_2 e^{-p_2 t}u(t) + A_3 e^{-p_3 t}u(t)$$

This analysis can be formalized using the *Laplace transform* of a system.

Laplace transform

Since we know that $H(s)e^{st}$ is the output of any LTI system when the input is e^{st}, we can use convolution to obtain an expression for the transfer function $H(s)$ in terms of the impulse response $h(t)$.

$$H(s)e^{st} = e^{st} * h(t) = \int_{-\infty}^{\infty} h(t')e^{s(t-t')}dt' = e^{st} \underbrace{\int_{-\infty}^{\infty} h(t')e^{-st'}dt'}_{H(s)}$$

$H(s)$ is the *Laplace transform* of $h(t)$,

$$H(s) \triangleq \mathscr{L}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

The Fourier transform is the Laplace transform when $s = j\omega$ *if the integral converges*,

$$H(s)|_{s=j\omega} = \mathscr{F}\{h(t)\}$$

For example, if $h(t) = e^{-t/\tau}u(t)$,

$$H(s) = \int_0^\infty e^{-t/\tau}e^{-st}dt = \frac{1}{-(s+1/\tau)}\left[e^{-(s+1/\tau)t}\right]_0^\infty = \frac{1}{s+1/\tau}, \quad \mathrm{Re}\{s\} > -1/\tau$$

Since the integral only converges if $\mathrm{Re}\{s\} > -1/\tau$, the Laplace transform exists only if this condition holds. This condition can be represented as a region in the *s-plane* of all possible complex values of s, known as *region of convergence (ROC)*.

In general, $H(s)$ can be represented as a *pole-zero map* in the complex *s-plane*. Points where $H(s) = 0$ are zeros, usually labeled as "o", and points where $H(s) \to \infty$ are poles, usually labeled as "x". In our examples, $H(s)$ has a single pole at $s = -1/\tau$, and the ROC is the half-plane to the right of this pole. The boundary of any ROC always has at least one pole.

The vertical (imaginary) axis is where $s = j\omega$. Thus, the Fourier transform is a "slice" of $H(s)$ along this line. In our example, if the pole is in the left-half plane (LHP), where $\mathrm{Re}\{s\} < 0$, the Fourier transform exists since it is inside the ROC. In this case, $h(t)$ approaches 0 as $t \to \infty$. On the other hand, if the pole is on the right-half plane (RHP), $\tau < 0$ and $h(t)$ approaches ∞ as $t \to \infty$. In this case, the Laplace transform does not converge when $s = j\omega$ and the Fourier transform does not exist.

Given the close relationship between the Laplace and Fourier transforms, most properties of the Fourier transform are also true for the Laplace transform. In particular, the Laplace transform is also linear, $Y(s) = H(s)X(s)$, impedances are as expected if $s = j\omega$.

The page of Peter Mathys at
http://ecee.colorado.edu/~mathys/ecen2420/notes/FilterPlots.html
shows the relationship between the frequency response of filters and their associated pole-zero diagram using the s-plane.

We can also find other transforms starting from the one we have already found,

$$e^{-t/\tau}u(t) \quad \stackrel{\mathscr{L}}{\Longleftrightarrow} \quad \frac{1}{s+1/\tau}, \quad \mathrm{Re}\{s\} > -1/\tau$$

If $1/\tau = 0$, we have the transform of the unit step,

$$u(t) \quad \stackrel{\mathscr{L}}{\Longleftrightarrow} \quad \frac{1}{s}, \quad \mathrm{Re}\{s\} > 0$$

We can use this property to find the Laplace transform of the integral of a signal, since

$$\mathscr{L}^{-1}\left\{\frac{X(s)}{s}\right\} = u(t) * x(t) = \int_{-\infty}^{\infty} x(t')u(t-t')dt' = \int_{-\infty}^{t} x(t')dt'$$

Therefore,

$$\int_{-\infty}^{t} x(t')dt' \quad \overset{\mathscr{L}}{\Longleftrightarrow} \quad \frac{X(s)}{s}$$

Similarly, the transform of the derivative of a signal is

$$\dot{x(t)} \quad \overset{\mathscr{L}}{\Longleftrightarrow} \quad sX(s)$$

Analysis of proper rational transfer functions using the Laplace transforms

Since we know how to take the Laplace transform of the derivative of a signal, we realize that our original analysis using substitution is equivalent to taking the Laplace transform of the differential equation,

$$\mathscr{L}\{\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = \ddot{x} + b_1\dot{x} + b_0 x\}$$

$$s^3Y + a_2s^2Y + a_1sY + a_0Y = s^2X + b_1sX + b_0X$$

$$H(s) = \frac{Y}{X} = \frac{s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)(s+p_3)}$$

The transfer function $H(s)$ can be expanded into several fractions such that

$$H(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3}$$

However, the inverse Laplace transform of the transfer function is the impulse response $h(t)$. Therefore,

$$h(t) = \mathscr{L}^{-1}\left\{H(s)\right\} = \mathscr{L}^{-1}\left\{\frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3}\right\}$$

$$h(t) = A_1e^{-p_1t}u(t) + A_2e^{-p_2t}u(t) + A_3e^{-p_3t}u(t)$$

Thus, **the poles of a system correspond to its natural frequencies**, as stated before.

The frequency response can also be interpreted using vectors from each zero and pole to the $j\omega$ axis,

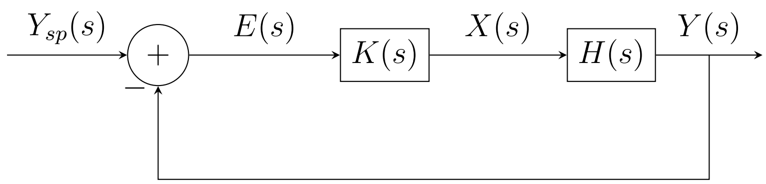
$$H(j\omega) = \frac{(j\omega+z_1)(j\omega+z_2)}{(j\omega+p_1)(j\omega+p_2)(j\omega+p_3)}$$

As a rule of thumb, real poles decrease the slope by 1 when $\omega = p$ as ω increases, decrease the phase by $\pi/2$ at the same time. Real zeros have increase the slope and phase correspondingly. Complex poles have "double" effect: the slope decreases by 2 and the phase by π when $\omega = \mathrm{Im}\{p\}$.

Control

Control systems are used typically for *tracking*, where some output y must "follow" some set point y_{sp}, or to *stabilize* the dynamics of a system by moving any poles from the left-half plane to the right-half plane such that $h(t)$ is bounded as $t \to \infty$.

In the system below, the set point y_{sp} is compared to the output y, and the resulting error e is fed into a controller $K(s)$ that then drives the plant $H(s)$ with x.



The overall transfer function can be computed by tracking the expression for $Y(s)$ as it travels around the loop such that

$$HK\underbrace{(Y_{sp} - Y)}_{E(s)} = Y$$

The resulting expression is *Black's formula*,

$$\frac{Y}{Y_{sp}} = \frac{HK}{1 + HK}$$

In control systems, we typically care about the step response, since it shows the response of the system when the set point is changed. In particular, the step response has a *settling time* until it reaches the new set point, the final value might have an *offset*, or a *DC gain* not equal to one, and some *overshoot* beyond the set point. We typically want to decrease all these as much as possible without making the system unstable or sensitive to external disturbances and system variations.

The *final value theorem* states that

$$\lim_{s \to 0} sX(s) = x(\infty)$$

Similarly, the *initial value theorem* states that

$$\lim_{s \to \infty} sX(s) = x(0)$$

For any system $H(s)$, the step response is $u(t) * h(t) = \mathscr{F}^{-1}\left\{\frac{1}{s}H(s)\right\}$. We can find the DC gain using the final value theorem such that

$$\text{DC gain} = \lim_{s \to 0} \cancel{s} \cdot \underbrace{\frac{1}{\cancel{s}} H(s)}_{\textit{step response}} = \lim_{s \to 0} H(s)$$

Proportional control

Proportional control is letting $K(s) = K_p$. For example, assume $H(s)$ is a first order system

$$H(s) = \frac{1/\tau}{s + 1/\tau}$$

If we let $K(s) = K_p$, the overall transfer function becomes

$$\frac{Y(s)}{Y_{sp}(s)} = \frac{K_p \frac{1/\tau}{s+1/\tau}}{1 + K_p \frac{1/\tau}{s+1/\tau}} = \frac{K_p/\tau}{s + (K_p + 1)/\tau}$$

The DC gain is $K_p/(K_p + 1)$ and the equivalent time constant is $\tau/(K_p + 1)$. It seems that choosing an arbitrarily high K_p would reduce the DC gain and the settling time. However, any real system will have some delay. This can be modeled as a system with step response $\delta(t - t_0)$. The transfer function of this delay system is e^{-st_0}. Including this delay in the system yields

$$\frac{Y(s)}{Y_{sp}(s)} = \frac{K_p e^{-st_0}/\tau}{s + (K_p e^{-st_0} + 1)/\tau}$$

Since $t_0 \ll 1$, we can approximate the delay as $e_{-st_0} \approx 1 - st_0$. Substituting this approximation into the transfer function yields

$$\frac{Y(s)}{Y_{sp}(s)} \approx \frac{K_p(1 - st_0)/\tau}{s + [K_p(1 - st_0) + 1]/\tau} = \frac{K_p(1 - st_0)/\tau}{s(1 - K_p t_0/\tau) + (K_p + 1)/\tau}$$

The resulting pole is

$$s = \frac{-(K_p + 1)/\tau}{(1 - K_p t_0/\tau)}$$

which means the system will become unstable if $s > 0$, or

$$1 < K_p t_0/\tau \quad \Rightarrow \quad K_p > \tau/t_0$$