Olin College of Engineering
ENGR2410 – Signals and Systems

Reference 7

Multiplication in the time domain

The effect of multiplication in the time domain is convolution in the frequency domain. The derivation below is almost identical to the one previously shown. We start by expressing the two functions of time in the frequency domain.

$$x(t)p(t) = \int_{\omega'=-\infty}^{\omega'=\infty} X(j\omega')e^{-j\omega' t}\frac{d\omega'}{2\pi} \cdot \int_{\Omega=-\infty}^{\Omega=\infty} P(j\Omega)e^{-j\Omega t}\frac{d\Omega}{2\pi}$$

Combine the integrals.

$$x(t)p(t) = \int_{\Omega=-\infty}^{\Omega=\infty} \left[\int_{\omega'=-\infty}^{\omega'=\infty} X(j\omega')P(j\Omega)e^{-j(\omega'+\Omega)t}\frac{d\omega'}{2\pi}\right]\frac{d\Omega}{2\pi}$$

Let $\omega = \omega' + \Omega$. In the inner integral, Ω is a constant, so that $\omega' = \omega - \Omega$ and $d\omega' = d\omega$.

$$x(t)p(t) = \int_{\Omega=-\infty}^{\Omega=\infty} \left[\int_{t=-\infty}^{t=\infty} X(j\omega - j\Omega)P(j\Omega)e^{-j\omega t}\frac{d\omega}{2\pi}\right]\frac{d\Omega}{2\pi}$$

Change the order of integration.

$$x(t)p(t) = \int_{\omega=-\infty}^{\omega=\infty} \left[\int_{\Omega=-\infty}^{\Omega=\infty} X(j\omega - j\Omega)P(j\Omega)e^{-j\omega t}\frac{d\Omega}{2\pi}\right]\frac{d\omega}{2\pi}$$

Since $e^{-j\omega t}$ does not depend on Ω, we can take it outside the integral. If we also take out $1/2\pi$, the remaining expression is the convolution of $X(j\omega)$ and $P(j\omega)$.

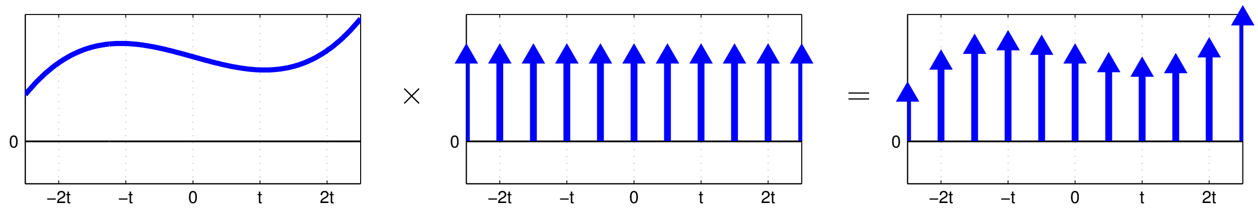
$$x(t)p(t) = \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{2\pi}\underbrace{\left[\int_{\Omega=-\infty}^{\Omega=\infty} X(j\omega - j\Omega)P(j\Omega)d\Omega\right]}_{X(j\omega)*P(j\omega)} e^{-j\omega t}\frac{d\omega}{2\pi}$$

Therefore, we know how to transform multiplication into the frequency domain

$$x(t)p(t) \quad \stackrel{\mathscr{F}}{\Longleftrightarrow} \quad \frac{1}{2\pi}X(j\omega) * P(j\omega)$$

Sampling

If we multiply an impulse train in the time domain with a function $x(t)$, we will *sample* the value of the function in every impulse.



At first glance, it appears that we lose information, but frequency analysis shows that this is not true if the original function $x(t)$ does change too quickly and its frequency content is *band-limited* to f_{max}. Let's find the frequency content of $x_S(t)$, the sampled signal,

$$x_S(t) = x(t) \cdot p(t)$$

where $p(t)$ is the unit impulse train,

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n/f_S)$$

Since $x(t)$ and $p(t)$ are multiplied in the time domain, their transforms convolve in the frequency domain.

$$X_S(j\omega) = \frac{1}{\cancel{2\pi}} X(j\omega) * \cancel{2\pi} f_S \sum_{n=-\infty}^{+\infty} \delta(\omega - 2\pi f_S n)$$

This convolution creates an infinite number of copies of $X(j\omega)$ in frequency.

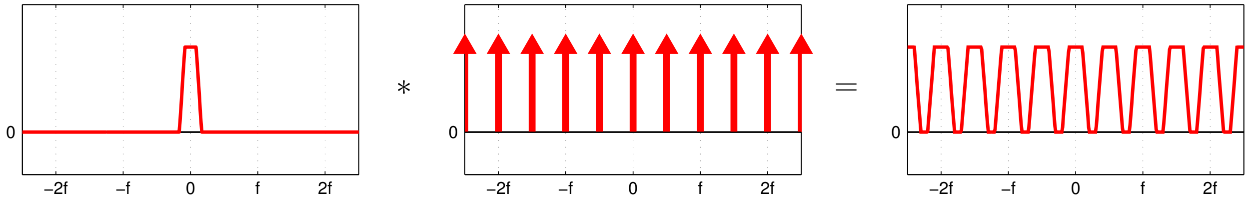
$$X_S(j\omega) = f_S \sum_{n=-\infty}^{+\infty} X[j(\omega - 2\pi f_S n)]$$

In order to avoid *aliasing* the convolved copies cannot overlap. Therefore,

$$f_{max} \leq f_S - f_{max}$$

$$f_S \geq 2f_{max}$$

This minimum frequency $2f_{max}$ is known as the *Nyquist rate*.



If $x_S(t)$ goes through an ideal low-pass filter with a cutoff at f_{max}, the output will be the original signal $x(t)$.