Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 4

Problem 1 In this problem, you will derive a more limited expression for Fourier series using sines and cosines. In particular, complex exponentials allow us to express any complex function in general. However, restricting to sines and cosines forces the resulting function to be purely real. In this case, corresponding complex coefficients must be complex conjugates[1] ($c_{-n} = c_n^*$), where n is any integer. Show that

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} = c_0 + \sum_{n=1}^{\infty} 2\mathrm{Re}\{c_n\}\cos\left(\tfrac{2\pi}{T}nt\right) + \sum_{n=1}^{\infty}(-2)\mathrm{Im}\{c_n\}\sin\left(\tfrac{2\pi}{T}nt\right)$$

Solution:
Let's begin by expanding the summation:

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} \quad = \quad ... + c_{-2}e^{-j\frac{4\pi}{T}t} + c_{-1}e^{-j\frac{2\pi}{T}t} + c_0 + c_1 e^{j\frac{2\pi}{T}t} + c_2 e^{j\frac{4\pi}{T}t} + ...$$

We can see a pattern emerging. For each $c_{-n}e^{-j\frac{2\pi}{T}nt}$ term there is a corresponding $c_n e^{-j\frac{2\pi}{T}nt}$ term. Now let's rewrite the complex exponentials with sines and cosines using Euler's formula:

$$... + c_{-n}\cos\left(\tfrac{2\pi}{T}nt\right) - c_{-n}j\sin\left(\tfrac{2\pi}{T}nt\right) + ... + c_0 + ... + c_n\cos\left(\tfrac{2\pi}{T}nt\right) + c_n j\sin\left(\tfrac{2\pi}{T}nt\right) + ...$$

If we group similar cosine and sine terms, we can factor and simplify:

$$c_0 \quad + \quad ... + (c_n + c_{-n})\cos\left(\tfrac{2\pi}{T}nt\right) + (c_n - c_{-n})\,j\sin\left(\tfrac{2\pi}{T}nt\right) + ...$$

We know that c_{-n} and c_n are complex conjugates. Thus, if c_n is in the form $\mathrm{Re}\{c_n\}+j\mathrm{Im}\{c_n\}$, c_{-n} is in the form $\mathrm{Re}\{c_n\} - j\mathrm{Im}\{c_n\}$. In other words,

$$\begin{aligned} c_n + c_{-n} &= \mathrm{Re}\{c_n\} + j\mathrm{Im}\{c_n\} + \mathrm{Re}\{c_n\} - j\mathrm{Im}\{c_n\} = 2\mathrm{Re}\{c_n\} \\ c_n - c_{-n} &= \mathrm{Re}\{c_n\} + j\mathrm{Im}\{c_n\} - \mathrm{Re}\{c_n\} + j\mathrm{Im}\{c_n\} = 2j\mathrm{Im}\{c_n\} \end{aligned}$$

Substituting these values into our expression we obtain:

$$c_0 \quad + \quad ... + 2\mathrm{Re}\{c_n\}\cos\left(\tfrac{2\pi}{T}nt\right) - 2\mathrm{Im}\{c_n\}\sin\left(\tfrac{2\pi}{T}nt\right) + ...$$

[1] Properties of complex conjugates: if $z = a + jb$, then $z^* = a - jb$; if $z = Ae^{j\theta}$, then $z^* = Ae^{-j\theta}$. For example, $(je^{j\theta})^* = -je^{-j\theta}$.

Finally, we rewrite the expanded terms in separate sums:

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} = c_0 + \sum_{n=1}^{\infty} 2\mathrm{Re}\{c_n\}\cos\left(\tfrac{2\pi}{T}nt\right) + \sum_{n=1}^{\infty}(-2)\mathrm{Im}\{c_n\}\sin\left(\tfrac{2\pi}{T}nt\right)$$

Problem 2 If we try to represent the function

$$v(t) = \begin{cases} V & -T_1 + nT < t < T_1 + nT, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

using Fourier series such that $v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$ then $c_n = 2V\frac{T_1}{T}\mathrm{sinc}\left(\frac{2\pi}{T}nT_1\right)$.

A. Let's explore the coefficients c_n as we make the pulses thinner while holding the period T constant. *When plotting coefficients, use the* stem *command in Matlab. Careful: the definition of sinc in Matlab is normalized to* $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$, *as opposed to the more common unnormalized version we use in class,* $\mathrm{sinc}(x) = \sin(x)/x$.

   (i) Plot $v(t)$ and the coefficients c_n when $T = 1$, $T_1 = 1/4$, $V = 1$ and check that all the values are correct. For clarity and consistency, plot $v(t)$ from -4 to 4, and enough coefficients to capture 6 zero crossings while keeping c_0 centered.

   Solution:
   See combined plot below.

   (ii) What does c_0 correspond to in $v(t)$? That is, what would you call c_0 in terms of $v(t)$?

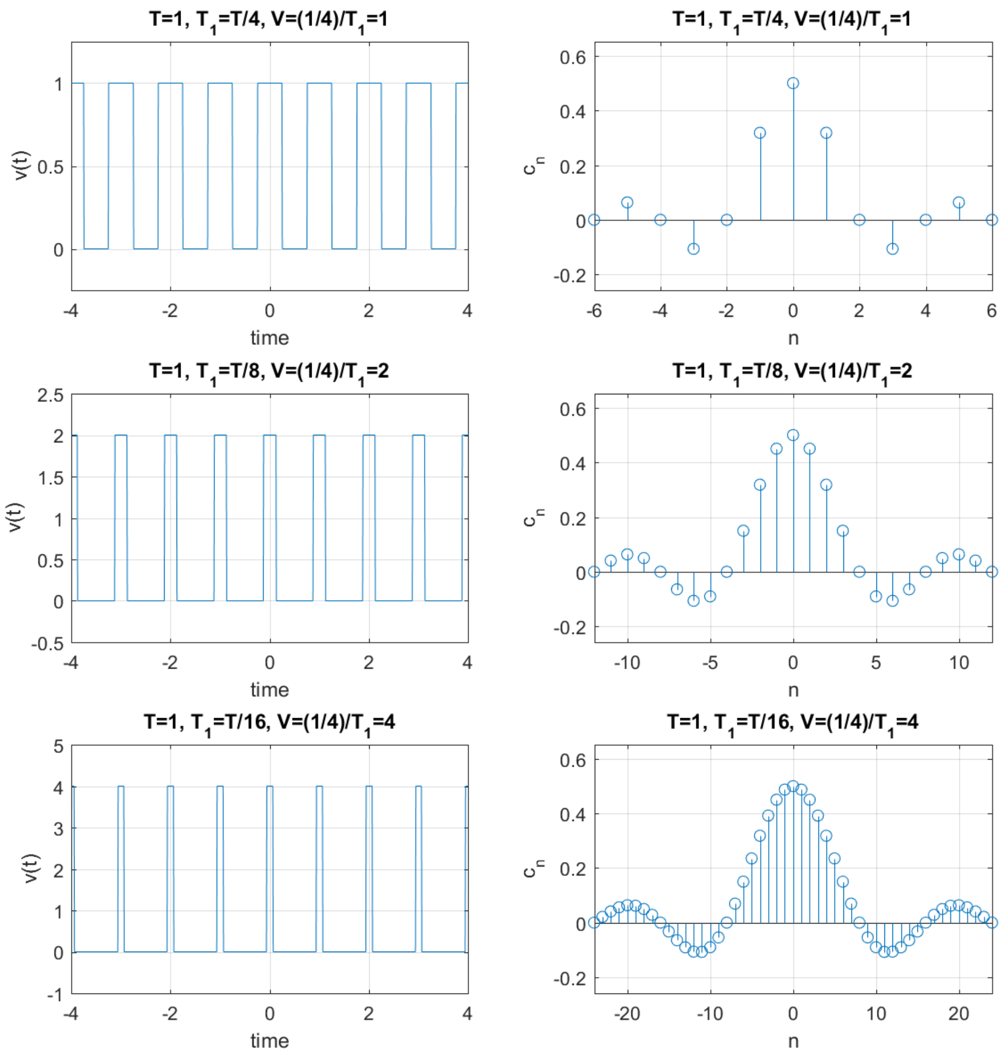
   Solution:
   c_0 is the average value of $v(t)$; in this case, 1/2.

   (iii) If you change T_1, how would scale V in order to keep c_0 constant?

   Solution:
   The average value of the square wave is $2T_1/T$. In order to keep this value constant, we would have to scale V inversely proportional to T_1. For example, if we decrease T_1 by a factor of 2, we need to multiply V by a factor of 2 to keep the average constant. Specifically, $V = (1/4)/T_1$.

(iv) Add to your plot the scaled versions of $v(t)$ and the coefficients c_n when $T_1 = T/8T$, and $T_1 = T/16$. Keep $T = 1$ and scale V so that c_0 remains constant. Again, for clarity and consistency, plot $v(t)$ from -4 to 4, and enough coefficients to capture 6 zero crossings while keeping c_0 centered.

Solution:



(v) Note how the number of complex exponentials under the first zero crossing increases as T_1 decreases. Intuitively, what do you think happens if $T_1 \to 0$, both in terms of $v(t)$ and the coefficients c_n?

Solution:
As $T_1 \to 0$, $v(t)$ approaches a sequence of infinitely thin and tall pulses with period T. The coefficients approach the continuous sinc envelope with its peak at 1/2. Furthermore, this also means that the average of $v(t)$ remains constant at 1/2!

B. Repeat the previous plot, but instead keep $T_1 = 1/4$ constant, and let T increase.

(i) Again, check that the base case where $T_1 = 1/4$, $T = 1$, and $V = 1$ is correct.

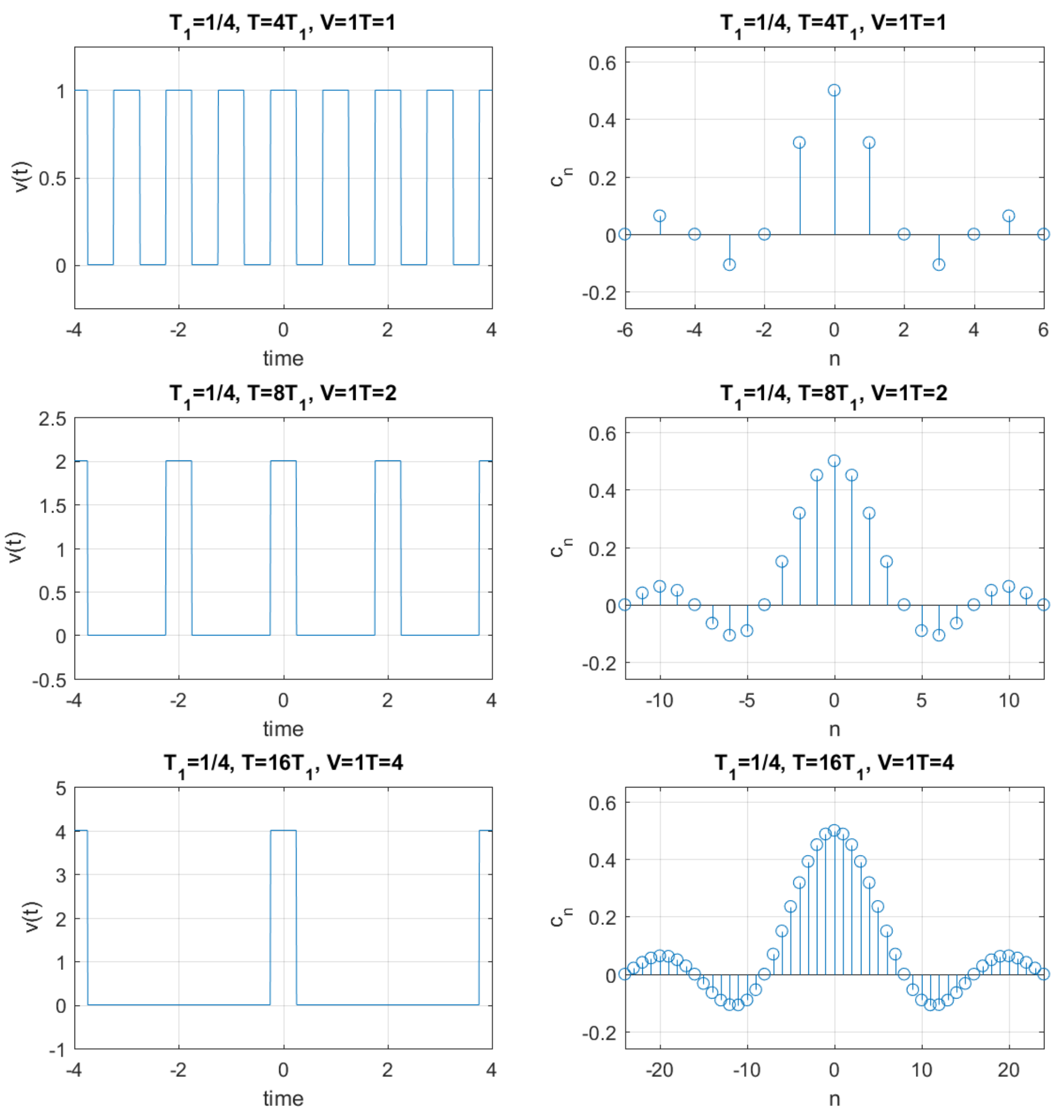
Solution:
See combined plot below.

(ii) How would you scale V in this case to keep c_0 constant?

Solution:
Again, the average value of the square wave is $2T_1/T$. We have to scale V in proportion to T. For example, if we increase T by a factor of 2, we need to multiply V by a factor of 2 to keep the average constant. Specifically, $V = 1 \cdot T$.

(iii) Add to your plot the scaled versions of $v(t)$ and the coefficients c_n when $T = 8T_1$, and $T = 16T_1$. Keep $T_1 = 1/4$ and scale V so that c_0 remains constant. Again, for clarity and consistency, plot $v(t)$ from -4 to 4, and enough coefficients to capture 6 zero crossings while keeping c_0 centered.

Solution:



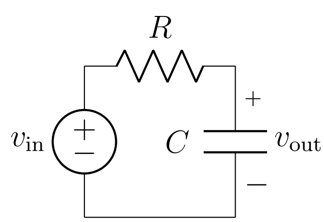
(iv) Intuitively, what do you think happens if $T \to \infty$, both in terms of $v(t)$ and the coefficients c_n?

Solution:
As $T \to \infty$, $v(t)$ approaches a single infinitely tall pulse with constant width, since all the other pulses tend to ∞. However, the coefficients still approach the continuous sinc envelope with its peak at 1/2. Even more strangely, this also means that the average of $v(t)$ remains constant at 1/2 in this case!

(v) Compare the coefficients in both cases, as well as $v(t)$ in both limits. You tell your friend your results and she is bothered by them. Why is she bothered, and how do you make sense of your results?

Solution:
She is bothered because the coefficients are identical, even though the functions are clearly different. In particular, the limits at infinity are particularly alarming! However, you realize that even though the coefficients are the same, they correspond to different frequencies! Specifically, c_1 is the coefficient of the exponential with frequency $2\pi/T$. In the first case, T is constant, so that all the coefficents of frequencies around 0 are approaching a constant 1/2. In the second case, the coefficients are packing increasingly closely under the sinc, but the zero crossings remain at the same frequency since they are inversely proportional to T_1. This intution will be formalized using the continuous time Fourier transform.

C. Assume $v_{in}(t) = v(t)$, $T = 1$, $T_1 = T/8$, and $RC = 0.1$. Plot the response of the circuit shown below using Fourier decomposition with complex exponentials for $-1.5 < t < 1.5$. Plot both the input and the output using 2, 7, and 20 harmonics[2]. Note how similar the responses are even though you are using a very crude approximation.

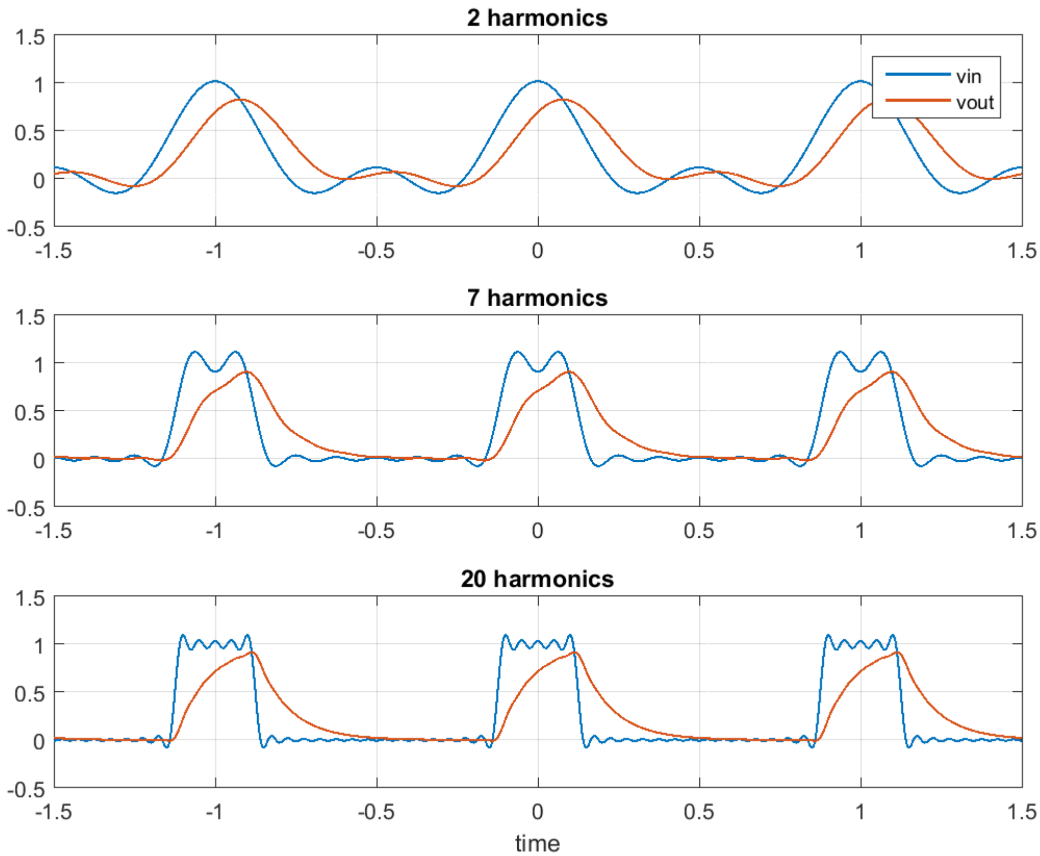


Solution:
The transfer function of the RC circuit is $H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}, \tau = RC$. If we represent our input as the continuous-time Fourier series, we can apply the transfer function to each exponential in the series as follows.

$$v_{out}(t) = \sum_{n=-\infty}^{\infty} H\left(j\frac{2\pi}{T}n\right) c_n e^{j\frac{2\pi}{T}nt} = \sum_{n=-\infty}^{\infty} \frac{1/\tau}{j\frac{2\pi}{T}n + 1/\tau} c_n e^{j\frac{2\pi}{T}nt}$$

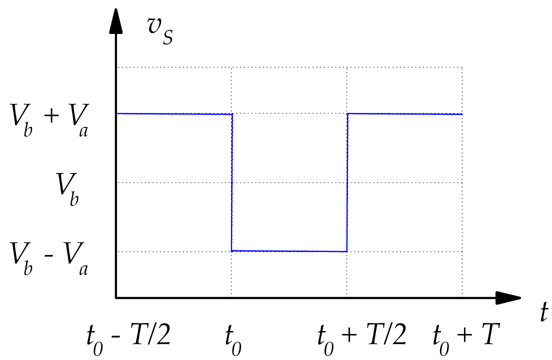
[2] Complex exponentials with frequencies $\pm 2\pi n/T$ are called the nth harmonics.

Both the input and the output become coarse as we sum a decreasing amount of harmonics to reconstruct them. The title of each subplot indicates the largest harmonic that was used in the sum. Note that just using the first 7 harmonics, the response is quite close to its peak value and shows a reasonable exponential curve. The output summing the first 20 harmonics is very close to the actual exponential response of the circuit.



Problem 3 This problem will introduce the concept of *average power* in a signal, and will explore the relationships across the frequency and time domains.

A. The voltage signal shown below is a square wave of amplitude V_a and average value V_b.



The *average power* of a signal is defined as

$$< P >= \frac{1}{T}\int_{t_0}^{t_0+T} |v_S|^2 dt$$

where $|v_S|^2 = v_S v_S^*$ is the square magnitude of the signal and the asterisk denotes complex conjugation. Show that the average power for v_S is $V_a^2 + V_b^2$.

Solution:

$$\frac{1}{T}\left[\int_{t_0}^{t_0+\frac{T}{2}} (V_b - V_a)^2 dt + \int_{t_0+\frac{T}{2}}^{t_0+T} (V_b + V_a)^2 dt\right]$$

$$\frac{1}{2}(V_b - V_a)^2 + \frac{1}{2}(V_b + V_a)^2$$

$$V_b^2 + V_a^2$$

B. Show that the average power of the sum of two complex exponentials,

$$c_1 e^{j\omega_1 t} + c_2 e^{j\omega_2 t}, \omega_1 \neq \omega_2,$$

is the square magnitude of each exponential, that is, $< P >= |c_1|^2 + |c_2|^2$. This is a profoundly important fact: the power at a particular frequency is independent of the power at any other frequency!

Solution:

$$\begin{aligned}
< P > &= \frac{1}{T}\int_{t_0}^{t_0+T} |v_s|^2 dt \\
|v_s|^2 &= v_s \cdot v_s^* = (c_1 e^{j\omega_1 t} + c_2 e^{j\omega_2 t})(c_1^* e^{-j\omega_1 t} + c_2^* e^{-j\omega_2 t}) \\
|v_s|^2 &= |c_1|^2 + c_1 c_2^* e^{j(\omega_1-\omega_2)t} + c_1^* c_2 e^{-j(\omega_1-\omega_2)t} + |c_2|^2 \\
< P > &= \frac{1}{T}\int_{t_0}^{t_0+T} |c_1|^2 + c_1 c_2^* e^{j(\omega_1-\omega_2)t} + c_1^* c_2 e^{-j(\omega_1-\omega_2)t} + |c_2|^2 dt \\
&= \frac{1}{T}(|c_1|^2 + |c_2|^2)\cdot T \\
&= |c_1|^2 + |c_2|^2
\end{aligned}$$

C. Find the Fourier decomposition of v_S assuming $t_0 = T/4$ (*note that this choice is irrelevant for this problem; think about why*) and using complex exponentials such that

$$v_S(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$$

Hint: Simplify the integral by shifting the square wave so that half of it is zero, that is, $v_s(t) = V_b - V_a + v_s'(t)$.

Solution:

In order to find the Fourier decomposition of v_S, we need to find an expression for c_n using the equation:

$$c_n = \frac{1}{T}\int_T v_S(t)e^{-j\frac{2\pi}{T}nt}dt$$

If we decompose the square wave as suggested, $v_s'(t)$ goes from 0 to $2V_a$. In this case, the integral becomes

$$\begin{aligned}
c_n &= \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} 2V_a e^{-j\frac{2\pi}{T}nt}dt \\
&= \frac{2V_a}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-j\frac{2\pi}{T}nt}dt \\
&= \frac{2V_a}{T}\cdot\frac{T}{j2\pi n}\cdot\left(e^{j\frac{2\pi n}{T}\cdot\frac{T}{4}} - e^{-j\frac{2\pi n}{T}\cdot\frac{T}{4}}\right) \\
&= \frac{2V_a}{\pi n}\cdot\frac{1}{2j}\left(e^{j\frac{\pi n}{2}} - e^{-j\frac{\pi n}{2}}\right) \\
&= \frac{2V_a}{\pi n}\sin\left(\frac{\pi n}{2}\right)
\end{aligned}$$

We can simplify this expression because sine is zero when n is an even integer, and ± 1 when n is an odd integer.

$$c_n = \begin{cases} V_a & n = 0 \\ \frac{2V_a(-1)^{\frac{n-1}{2}}}{\pi n} & n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Thus, the Fourier decomposition of the square wave can be written:

$$\begin{aligned}
v_S(t) &= \underbrace{V_b - V_a}_{\text{shifted amount}} + \underbrace{V_a}_{\text{DC offset}} + \sum_{n \text{ odd}} \frac{2V_a(-1)^{\frac{n-1}{2}}}{\pi n} e^{j\frac{2\pi}{T}nt} \\
v_S(t) &= V_b + \sum_{n \text{ odd}} \frac{2V_a(-1)^{\frac{n-1}{2}}}{\pi n} e^{j\frac{2\pi}{T}nt}
\end{aligned}$$

D. Show that the average power of v_S can be separated into a sum of a term that depends on V_a and a term that depends on V_b. When computing the average power, you might find useful to know that

$$\sum_{n\in[1,3,5\ldots\infty)} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Solution:

$$\begin{aligned} < P > &= V_b^2 + 2\sum_{n\in[1,3,5\ldots\infty)} \left| \frac{2V_a(-1)^{\frac{n-1}{2}}}{\pi n} e^{j\frac{2\pi}{T}nt} \right|^2 \\ &= V_b^2 + 2\frac{4V_a^2}{\pi^2} \sum_{n\in[1,3,5\ldots\infty)} \frac{1}{n^2} \\ &= V_b^2 + 2\frac{4V_a^2}{\pi^2} \cdot \left(\frac{\pi^2}{8}\right) \\ &= V_a^2 \end{aligned}$$

E. The average power of v_S that depends on V_a is called the AC, or alternating current power, while the power that depends on V_b is called the DC, or direct current power. For the square wave, show that the fundamental (frequency 1/T) contains $8/\pi^2 \approx 80\%$ of the total AC power.

Solution:
The fundamental corresponds to $n = 1$ and $n = -1$.

$$c_1 = c_{-1} = \frac{2V_a}{\pi}$$

The power in the fundamental is

$$|c_1|^2 + |c_{-1}|^2 = 2\frac{4V_a^2}{\pi^2}$$

This is $8/\pi^2$ of the total AC power, V_a^2.

Course feedback
Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?