Olin College of Engineering
ENGR2410 – Signals and Systems

Reference 6

Convolution

$$
\begin{tikzcd}
x(t) \arrow[r] \arrow[d, "\mathscr{F}"'] & \boxed{h(t)} \arrow[r] & y(t) \\
X(j\omega) \arrow[r] & \boxed{H(j\omega)} \arrow[r] & Y(j\omega) = X(j\omega)H(j\omega) \arrow[u, "\mathscr{F}^{-1}"']
\end{tikzcd}
$$

Our transform diagram above shows that

$$Y(j\omega) = H(j\omega)X(j\omega)$$

What is this operation in time? Write both $H(j\omega)$ and $X(j\omega)$ in terms of their respective Fourier transforms. *Note that it is customary to use τ, but in this case τ is *not* a time constant!*

$$Y(j\omega) = \int_{t'=-\infty}^{t'=\infty} h(t')e^{-j\omega t'}dt' \cdot \int_{\tau=-\infty}^{\tau=\infty} x(\tau)e^{-j\omega\tau}d\tau$$

Combine the integrals.

$$Y(j\omega) = \int_{\tau=-\infty}^{\tau=\infty}\left[\int_{t'=-\infty}^{t'=\infty} h(t')x(\tau)e^{-j\omega(t'+\tau)}dt'\right]d\tau$$

Let $t = t' + \tau$. In the inner integral, τ is a constant, so that $t' = t - \tau$ and $dt' = dt$.

$$Y(j\omega) = \int_{\tau=-\infty}^{\tau=\infty}\left[\int_{t=-\infty}^{t=\infty} h(t-\tau)x(\tau)e^{-j\omega t}dt\right]d\tau$$

Change the order of integration.

$$Y(j\omega) = \int_{t=-\infty}^{t=\infty}\left[\int_{\tau=-\infty}^{\tau=\infty} h(t-\tau)x(\tau)e^{-j\omega t}d\tau\right]dt$$

Since $e^{-j\omega t}$ does not depend on τ, we can take it outside the integral.

$$Y(j\omega) = \int_{t=-\infty}^{t=\infty}\left[\int_{\tau=-\infty}^{\tau=\infty} h(t-\tau)x(\tau)d\tau\right]e^{-j\omega t}dt$$

This expression has the form of a Fourier transform, so that the expression inside the brackets is the inverse transform of $Y(j\omega)$. Therefore,

$$Y(j\omega) = \int_{t=-\infty}^{t=\infty} \underbrace{\left[\int_{\tau=-\infty}^{\tau=\infty} h(t-\tau)x(\tau)d\tau \right]}_{y(t)} e^{-j\omega t} dt$$

This implies that $y(t)$ can be computed directly from $x(t)$ and $h(t)$. This operation is know as *convolution*.

$$y(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)h(t-\tau)d\tau \triangleq x(t) * h(t)$$

Completing the diagram,

$$\begin{array}{ccccc} x(t) & \longrightarrow & \boxed{h(t)} & \longrightarrow & y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)h(t-\tau)d\tau \\ \mathscr{F}\big\downarrow & & & & \big\uparrow \mathscr{F}^{-1} \\ X(j\omega) & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & Y(j\omega) = X(j\omega)H(j\omega) \end{array}$$

We know now the transform of multiplication into the time domain

$$x(t) * h(t) \quad \stackrel{\mathscr{F}}{\Longleftrightarrow} \quad X(j\omega)H(j\omega)$$

Convolution with an impulse

We used the frequency domain to find that

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$$

We can verify this result now using the convolution integral.

$$\delta(t) * h(t) = \int_{\tau=-\infty}^{\tau=\infty} \delta(\tau)h(t-\tau)d\tau = h(t-0) = h(t)$$

We can also verify that the convolution integral is time-invariant.

$$\delta(t-t_0) \longrightarrow \boxed{h(t)} \longrightarrow h(t-t_0)$$

You can also show that the convolution integral is commutative so that

$$h(t) \longrightarrow \boxed{\delta(t-t_0)} \longrightarrow h(t-t_0)$$

Can you see the difference between the two systems? In either case, the result shows that *convolution with a shifted impulse creates a copy of the original signal shifted by that amount.*

$$\delta(t-t_0) * h(t) = h(t-t_0)$$

Of course, the same is true in both the time or frequency domain. Remember that. It underlies many important results and applications.

LTI system response in the time domain

The expressions below means that $x(t)$ can be expressed as the sum of complex exponentials appropriately scaled. Likewise, the second expression means that $x(t)$ can also be expressed as the sum of shifted impulses appropriately scaled.

$$x(t) = \int_{\omega=-\infty}^{\infty} X(j\omega)e^{j\omega t}\frac{d\omega}{2\pi} = \lim_{\Delta\omega\to 0}\sum_{n=-\infty}^{\infty} X(jn\Delta\omega)e^{jn\Delta\omega t}\frac{\Delta\omega}{2\pi}$$

$$x(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = \lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty} x(n\Delta\tau)\delta(t-n\Delta\tau)\Delta\tau$$

Passing each scaled and shifted impulse results in a scaled and shifted impulse response

$$x(n\Delta\tau)\delta(t-n\Delta\tau)\Delta\tau \longrightarrow \boxed{h(t)} \longrightarrow x(n\Delta\tau)h(t-n\Delta\tau)\Delta\tau$$

To compute the output, we have to combine the responses as we did in the case of the input such that

$$y(t) = \lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty} x(n\Delta\tau)h(t-n\Delta\tau)\Delta\tau = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Using another parallel analysis, the Fourier transform can be constructed as a linear combination of exponentials.

$$\begin{array}{cccccc}
e^{j\omega_0 t} & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & H(j\omega_0)e^{j\omega_0 t} \\
X(j\omega_0)e^{j\omega_0 t} & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & H(j\omega_0)X(j\omega_0)e^{j\omega_0 t} \\
x(t) = \int_{\omega=-\infty}^{\infty} X(j\omega)e^{j\omega t}\frac{d\omega}{2\pi} & \longrightarrow & \boxed{H(j\omega)} & \longrightarrow & \int_{\omega=-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}\frac{d\omega}{2\pi} = y(t)
\end{array}$$

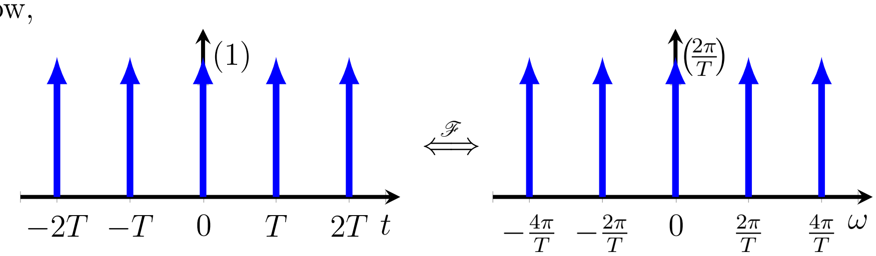
Convolution can be similarly constructed as a linear combination of impulses and impulse responses.

$$\begin{array}{ccccc}
\delta(t) & \longrightarrow & \boxed{h(t)} & \longrightarrow & h(t) \\
\delta(t-\tau) & \longrightarrow & \boxed{h(t)} & \longrightarrow & h(t-\tau) \\
x(\tau)\delta(t-\tau) & \longrightarrow & \boxed{h(t)} & \longrightarrow & x(\tau)h(t-\tau) \\
x(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau & \longrightarrow & \boxed{h(t)} & \longrightarrow & \int_{\tau=-\infty}^{\infty} x(\tau)x(\tau)h(t-\tau)d\tau = y(t)
\end{array}$$

Thus, both convolution and the Fourier transform express the input in their respective basis, and then combine the responses linearly to compute the final output.

The impulse train

The impulse train is important for sampling and finding the frequency content of periodic functions using the continuous time Fourier transform. The Fourier transform of the impulse train $x(t)$ of period T and unit area is another impulse train of both period and area $2\pi/T$, as shown below,



The unit impulse train of period T can be expressed as

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT).$$

Since $x(t)$ is has period T, we can express it in terms of all the complex exponentials of period $2\pi n/T$, where n is any integer.

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT) = \sum_{n=-\infty}^{+\infty} c_n e^{j\frac{2\pi}{T}nt}.$$

Using orthogonality, the coefficients are

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=-\infty}^{+\infty} \delta(t-kT)e^{-j\frac{2\pi}{T}nt}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-j\frac{2\pi}{T}nt}dt = \frac{1}{T}e^{-j\frac{2\pi}{T}n0} = \frac{1}{T},$$

and $x(t)$ can be represented as

$$x(t) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} e^{j\frac{2\pi}{T}nt}.$$

Since $\mathscr{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$[1],

$$\mathscr{F}\{x(t)\} = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T}n\right).$$

Therefore, the Fourier transform of an impulse train is still an impulse train.

[1] To show $\mathscr{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$, we'll show the inverse transform, $e^{j\omega_0 t} = \mathscr{F}^{-1}\{2\pi\delta(\omega - \omega_0)\}$ using the integral.

$$\mathscr{F}^{-1}\{2\pi\delta(\omega - \omega_0)\} = \int_{\omega=-\infty}^{\infty} \cancel{2\pi}\delta(\omega - \omega_0)e^{j\omega t}\frac{d\omega}{\cancel{2\pi}} = e^{j\omega_0 t}$$