Assignment 7

Problem 1 *(6 points)* In this problem, you will consolidate most of the transforms you already know. The purpose is two-fold: first, while these tables exist everywhere, you should be able to decompose problems into pieces that will have transforms. More importantly, a lot of the transforms have been derived as needed in a particular context. In this exercise, focus on the relationships between different transforms. *For every subproblem, find both an algebraic expression and sketch the function well enough to convince the grader you understand the expression.*

A. You might have noticed that both the Fourier and inverse Fourier integrals are essentially the same with some sign and factor differences. This means that transforms come in pairs. Show the *duality* property of Fourier transforms,

If $f(t) \overset{\mathscr{F}}{\longleftrightarrow} g(\omega)$ then $g(t) \overset{\mathscr{F}}{\longleftrightarrow} 2\pi f(-\omega)$.

Solution:

$$\mathscr{F}\{f(t)\} = g(\omega) \qquad \text{Given}$$

$$g(\omega) = \int_{t=-\infty}^{t=\infty} f(t)e^{-j\omega t}dt \qquad \text{By definition}$$

$$g(t) = \int_{\omega=-\infty}^{\omega=\infty} f(\omega)e^{-j\omega t}d\omega \qquad \text{Switch } \omega \text{ and } t$$

$$g(-t) = \int_{\omega=-\infty}^{\omega=\infty} f(\omega)e^{j\omega t}d\omega \qquad \text{Substitute } -t \text{ for } t$$

$$g(-t) = \int_{\omega=-\infty}^{\omega=\infty} 2\pi f(\omega)e^{j\omega t}\frac{d\omega}{2\pi} \qquad \text{Multiply by } \frac{2\pi}{2\pi}$$

$$g(-t) = \mathscr{F}^{-1}\{2\pi f(\omega)\} \qquad \text{By definition}$$

$$\mathscr{F}\{g(-t)\} = 2\pi f(\omega) \qquad \text{Fourier transform property}$$
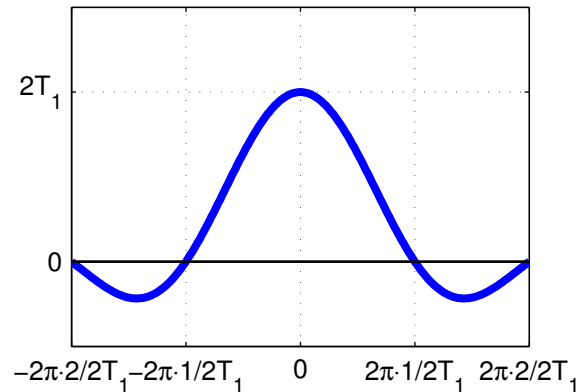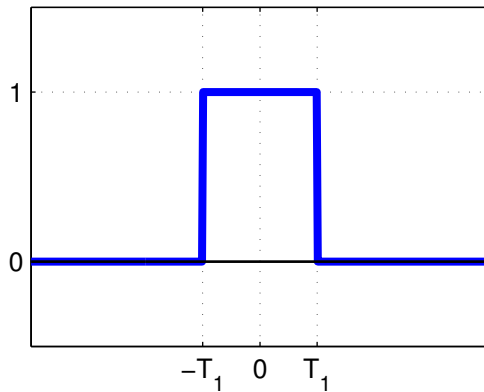
Alternatively,

$$\mathscr{F}\{f(t)\} = g(\omega) \qquad\qquad \text{Given}$$

$$g(\omega) = \mathscr{F}^{-1}\{f(t)\} \qquad\qquad \text{Fourier transform property}$$

$$f(t) = \int_{\omega=-\infty}^{\omega=\infty} g(\omega)e^{j\omega t}\frac{d\omega}{2\pi} \qquad\qquad \text{By definition}$$

$$f(\omega) = \int_{t=-\infty}^{t=\infty} g(t)e^{j\omega t}\frac{dt}{2\pi} \qquad\qquad \text{Switch } \omega \text{ and } t$$

$$f(\omega) = \int_{t=\infty}^{t=-\infty} g(-t)e^{-j\omega t}\frac{-dt}{2\pi} \qquad\qquad \text{Substitute } -t \text{ for } t$$

$$f(\omega) = \int_{t=-\infty}^{t=\infty} g(-t)e^{-j\omega t}\frac{dt}{2\pi} \qquad\qquad \text{Reverse integration limits}$$

$$2\pi f(\omega) = \int_{t=-\infty}^{t=\infty} g(-t)e^{-j\omega t}dt \qquad\qquad \text{Multiply by } 2\pi$$

$$\mathscr{F}\{g(-t)\} = 2\pi f(\omega) \qquad\qquad \text{By definition}$$

B. Recall and sketch the transform for a box in the time domain of unit height, from $-T_1$ to T_1. Find an expression and sketch the inverse transform of a box of unit height in the frequency domain from $-W$ to W. *This is known as a low-pass "brick" filter. Note that the impulse response starts at $-\infty$. Typical systems in the real world are causal which means their response cannot extend to $t = -\infty$.*
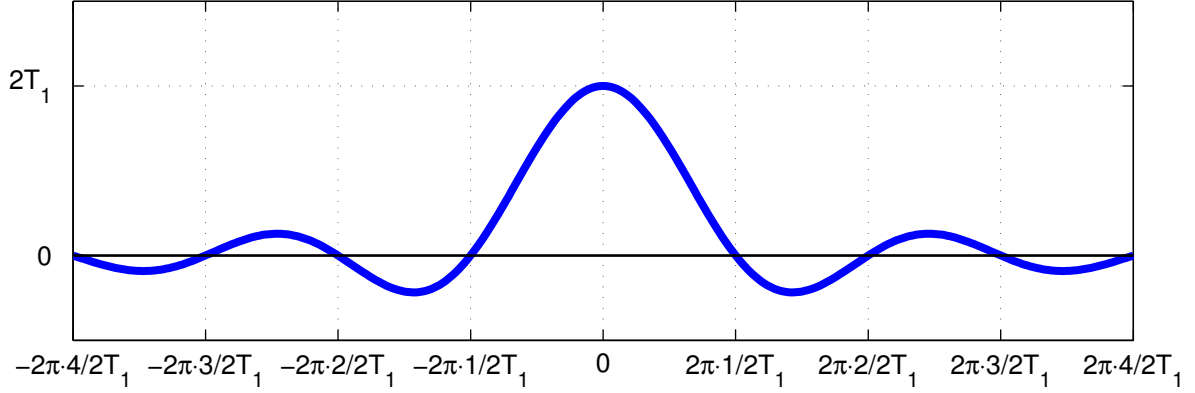
Solution:

sinc in frequency

$$\mathscr{F}\{\Pi(t/T_1)\} = 2T_1\frac{\sin(\omega T_1)}{\omega T_1} \quad \text{where } \Pi(x) = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$
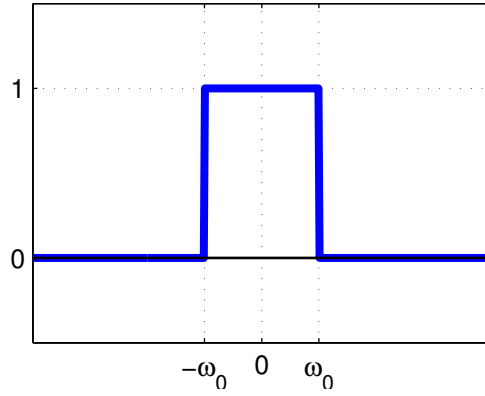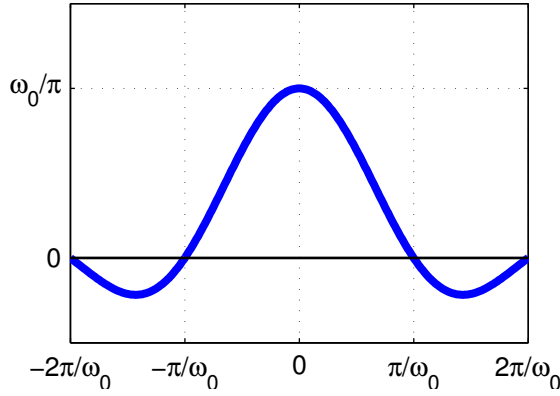
Exploded view of sinc in frequency

$-2\pi\cdot4/2T_1$ $\quad -2\pi\cdot3/2T_1$ $\quad -2\pi\cdot2/2T_1$ $\quad -2\pi\cdot1/2T_1$ $\qquad 0 \qquad$ $2\pi\cdot1/2T_1$ $\quad 2\pi\cdot2/2T_1$ $\quad 2\pi\cdot3/2T_1$ $\quad 2\pi\cdot4/2T_1$

sinc in time

$$\mathscr{F}\left\{\frac{\omega_0}{\pi}\cdot\frac{\sin(\omega_0 t)}{\omega_0 t}\right\} = \Pi(\omega/\omega_0) \text{ where } \Pi(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$-2\pi/\omega_0 \qquad -\pi/\omega_0 \qquad 0 \qquad \pi/\omega_0 \qquad 2\pi/\omega_0$

$-\omega_0 \quad 0 \quad \omega_0$

C. Scale the unit boxes in both time and frequency such that the area inside the box is constant. Take the limits as both $T_1 \to 0$ and $W \to 0$ such that the boxes approach impulses. Recall or rederive the transforms for impulses in both time and frequency domains and show that you approach this result in both limits. Use sketches as necessary to make your explanation clear.

Solution:

$$\Pi(t/T_1) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad 2T_1\frac{\sin(\omega T_1)}{\omega T_1}$$

Scale the box.

$$\frac{1}{2T_1}\Pi(t/T_1) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \frac{\sin(\omega T_1)}{\omega T_1}$$

Take $T_1 \to 0$.

$$\delta(t) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad 1$$

$$\frac{\omega_0}{\pi} \cdot \frac{\sin(\omega_0 t)}{\omega_0 t} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \Pi(\omega/\omega_0)$$

Scale the box.

$$\frac{1}{2\pi} \cdot \frac{\sin(\omega_0 t)}{\omega_0 t} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \frac{1}{2\omega_0}\Pi(\omega/\omega_0)$$

Take $\omega_0 \to 0$.

$$\frac{1}{2\pi} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \delta(\omega)$$

D. Recall the time shift property of the Fourier transform, where you expressed the transform of $x(t+T)$ in term of the transform of $x(t)$. Find the dual property, the frequency shift property where you express the transform of $X(j\omega - j\omega_0)$ in terms of $X(j\omega)$.
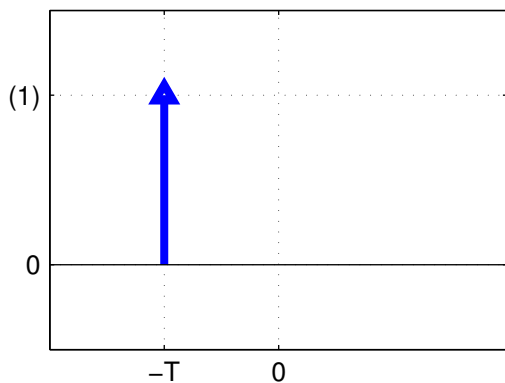
Solution:

$$\mathscr{F}\{x(t+T)\} = X(j\omega)e^{j\omega T}$$
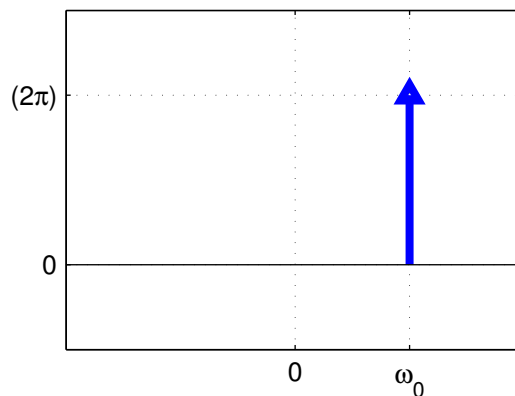$$\mathscr{F}\{x(t)e^{j\omega_0 t}\} = 2\pi X(\omega - \omega_0)$$

E. Rederive and sketch (only the functions that involve impulses) the dual transforms for complex exponentials in both time and frequency by using the shift properties for the case where $x(t) = \delta(t)$ and $X(j\omega) = \delta(\omega)$ as appropriate.

Solution:

$$\mathscr{F}\{\delta(t)\} = 1 \qquad\qquad \mathscr{F}\{1\} = 2\pi\delta(\omega)$$
$$\mathscr{F}\{\delta(t+T)\} = e^{j\omega T} \qquad\qquad \mathscr{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$$
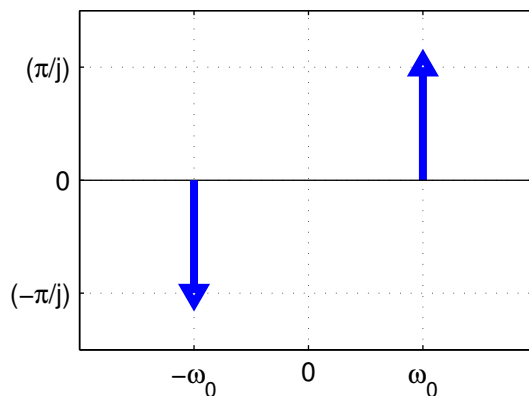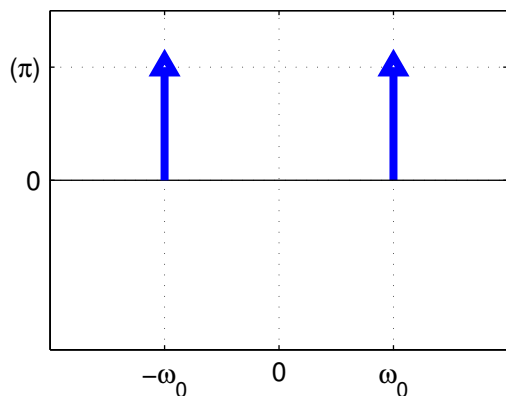

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F. Rederive and sketch (only the functions that involve impulses) for both $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$.

Solution:

$$\mathscr{F}\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$
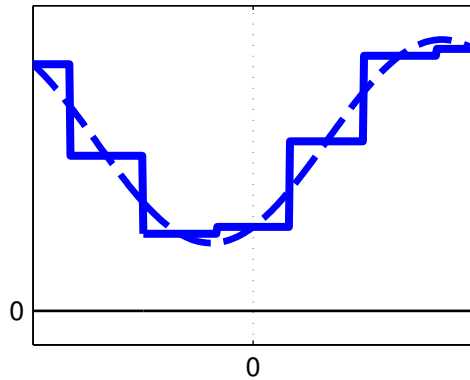
$$\mathscr{F}\{\sin(\omega_0 t)\} = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

Problem 2 *(4 points)* Real sampling requires finite width pulses. In fact, the most common sampling method is called *zero-order hold*, where the value of the sampled function is held constant until the next sample is taken [1].

   A. Sketch a smooth signal $x(t)$ and a function $x_Z(t)$ reconstructed from equally spaced samples at frequency f_S with zero-order hold.

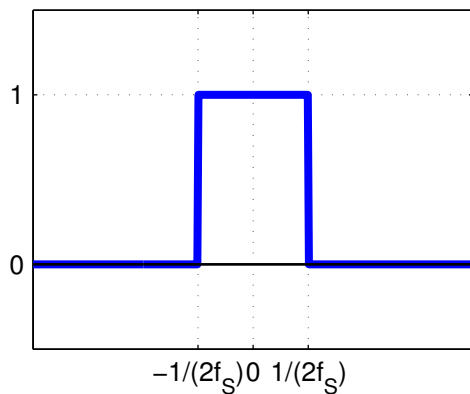   **Solution:**

   B. Zero-order hold can be modeled as taking the sampled function $x_S(t)$ and convolving it with a pulse of the same width as the sampling period, $1/f_S$. Sketch the pulse. Find an expression and sketch the frequency content of $x_Z(t)$. Assume $x(t)$ is flat and bandlimited to f_{max}.
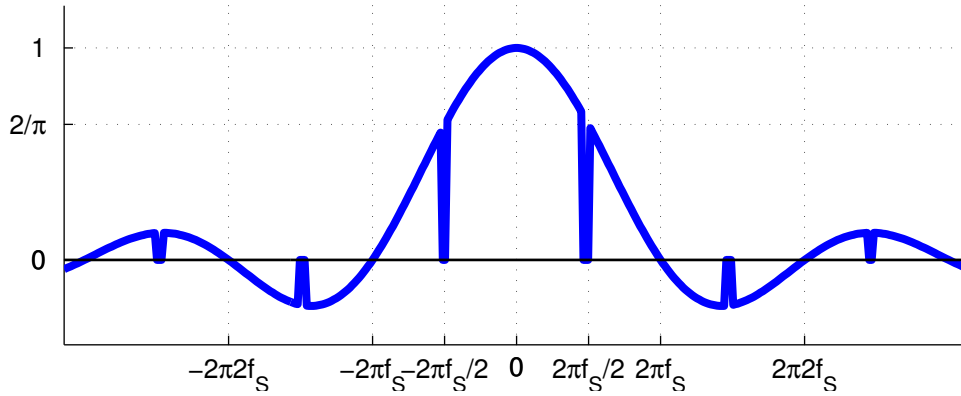
      Solution:

      In the frequency domain, $X_S(j\omega)$ is multiplied by the transform of a single pulse of width $1/f_S$.

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[1]http://eetimes.com/electronics-news/4197022/Switched-Capacitor-Filters-Beat-Active-Filters-at-Their-Own-Game

The transform of the pulse is $\frac{1}{f_S}\mathrm{sinc}\left(\frac{\omega}{2f_S}\right)$. The resulting transform is

$$X_Z(j\omega) = \frac{1}{\cancel{f_S}}\mathrm{sinc}\left(\frac{\omega}{2f_S}\right)\cancel{f_S}\sum_{n=-\infty}^{+\infty} X[j(\omega - 2\pi f_S n)]$$

C. Assuming no aliasing, what is the attenuation at the highest possible signal frequency?

Solution:

If we are sampling at $1/f_S$, the maximum frequency $X(j\omega)$ can have before we see aliasing is $f_S/2$. At this frequency, the gain is

$$\mathrm{sinc}\left(\frac{1}{2\cancel{f_S}}\cancel{2\pi}\frac{\cancel{f_S}}{2}\right) = \frac{2}{\pi} \approx 0.64$$