# Olin College of Engineering
# ENGR2410 – Signals and Systems

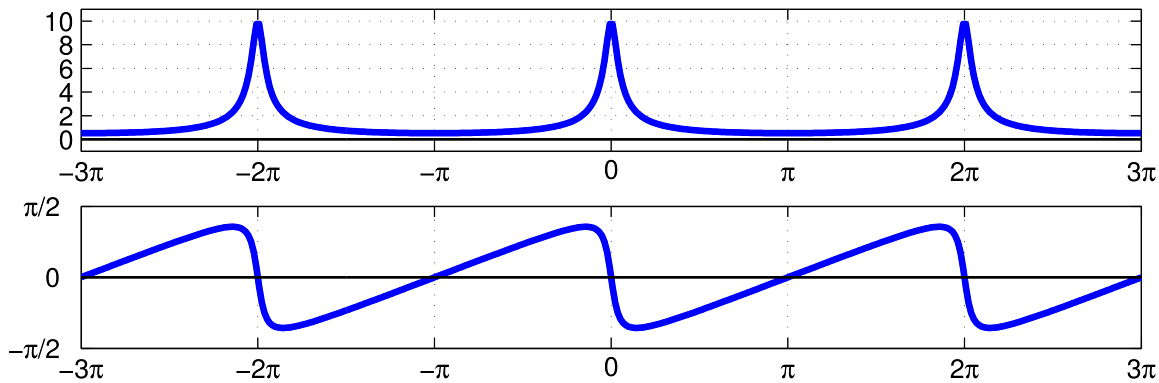
## Assignment 8

**Problem 1** In this problem, you will analyze several discrete time filters. Recall that the transfer function of the first-order difference equation

$$y[n] - ay[n-1] = x[n] \qquad \text{is} \qquad H(\Omega) = \frac{1}{1 - ae^{-j\Omega}}.$$

A. Plot the magnitude and phase of $H(\Omega)$ from $-3\pi$ to $3\pi$ when $a = 0.9$.

**Solution:**

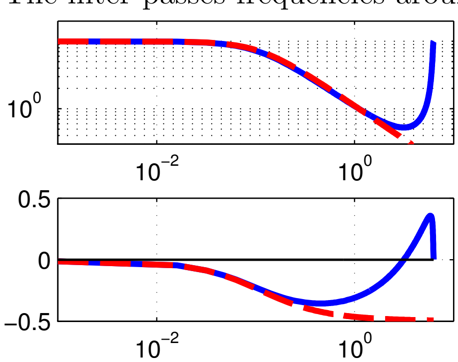



B. Since $e^x \approx 1 + x$ when $x \ll 1$, we can examine the behavior of the filter when $\Omega \approx 2\pi n$.

$$H(\Omega) \approx \frac{1}{1 - a(1 - j\Omega)} = \frac{\frac{1}{a}}{\frac{1-a}{a} + j\Omega} = H_{approx}(\Omega)$$

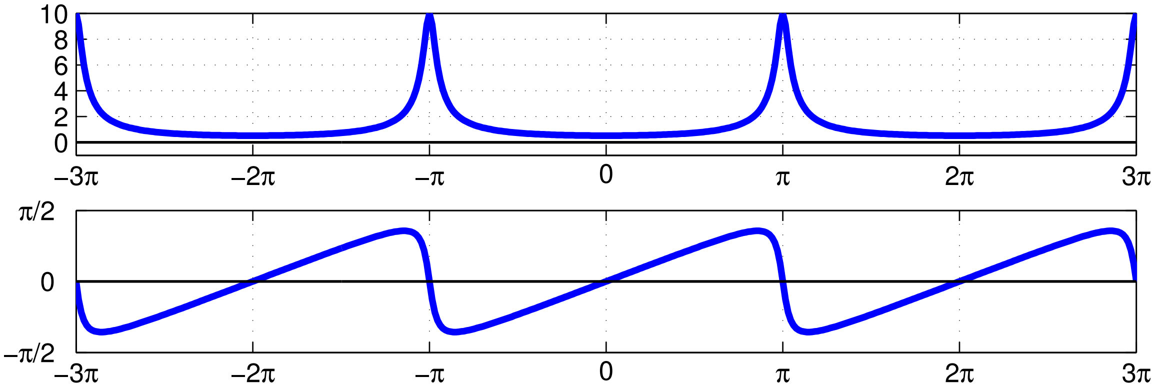
Make a Bode plot of both $H(\Omega)$ and $H_{approx}(\Omega)$ when $a = 0.9$ and $10^{-3} < \Omega < 2\pi$. What kind of filter is this?

**Solution:**
The filter passes frequencies around 0, so it is a low pass filter.



C. Redo the part A when $a = -0.9$. What kind of filter is this? Explain clearly.

**Solution:**
The filter passes frequencies around $\pi$. This is the highest frequency a discrete system can have (any higher frequencies will be aliased down), so it is a high pass filter.



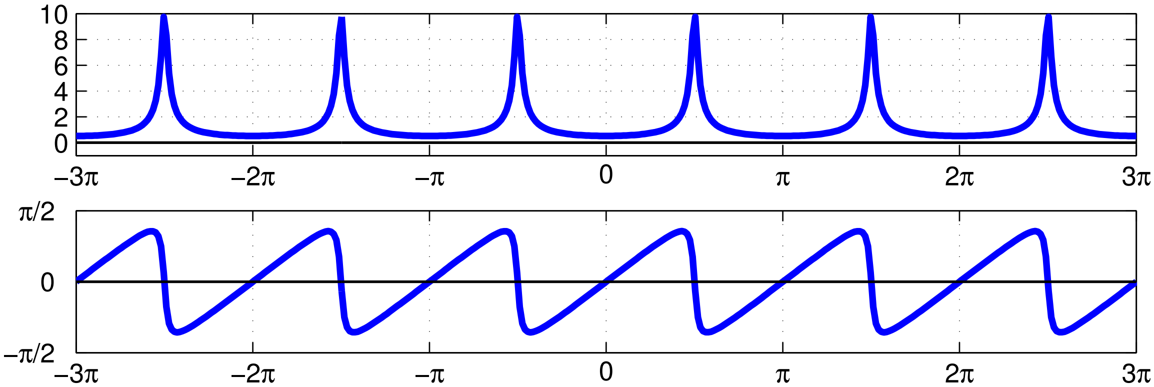

D. Find the transfer function for the difference equation below and plot it as in part A. What kind of filter is this? Explain clearly.

$$y[n] + 0.9y[n-2] = x[n]$$

**Solution:**

$$H(\Omega) = \frac{1}{1 + 0.9e^{-2j\Omega}}$$

The filter passes frequencies around $\pi/2$ and rejects both frequencies around 0 and $\pi$. $\pi/2$ is the center frequency between the lowest discrete frequency (0) and the highest ($\pi$), so it is a band pass filter.

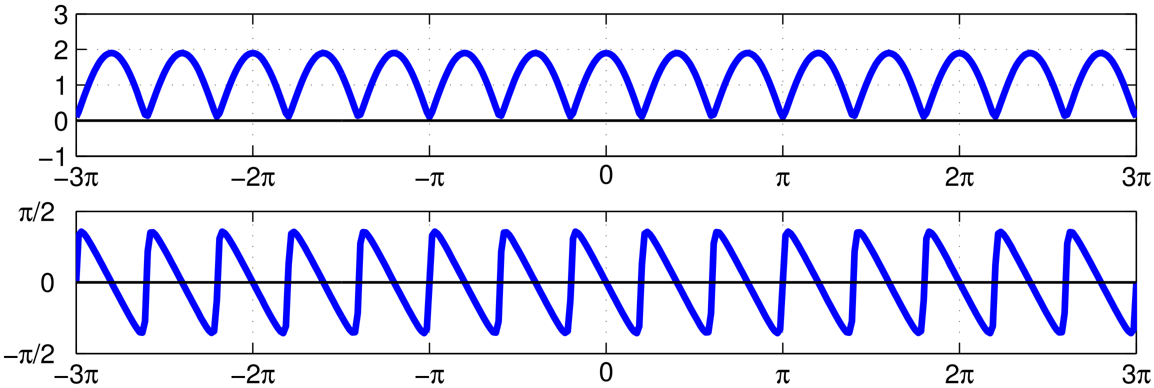


E. Find the transfer function for the difference equation below and plot it as in part A. This is called a *comb filter*.

$$y[n] = x[n] - 0.9x[n-5]$$

**Solution:**

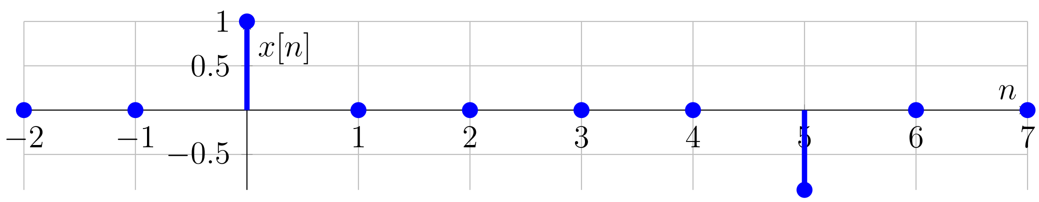
$$H(\Omega) = 1 - 0.9e^{-5j\Omega}$$




F. Find and sketch the impulse response for the comb filter of part E. *This type of filter is a* finite impulse response *(FIR) filter. The first order difference equation of part A is an* infinite impulse response *(IIR) filter.*

**Solution:**

$$h[n] = \mathscr{F}^{-1}\{H(\Omega)\} = \mathscr{F}^{-1}\{1 - 0.9e^{-5j\Omega}\} = \delta[n] - 0.9\delta[n-5]$$

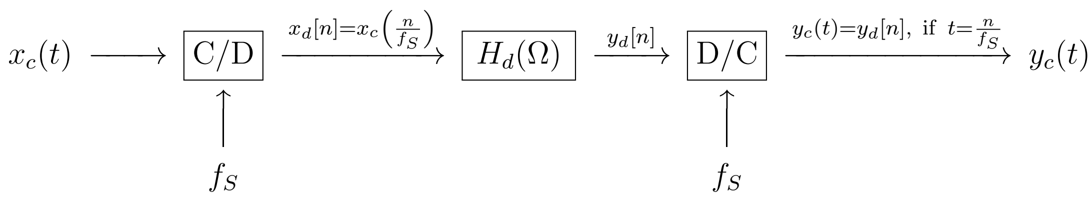


**Problem 2** In this problem, you will find the impulse response of an analog delay filter implemented using a digital filter when the delay is smaller than the sampling frequency of the digital filter.

A. Find and sketch the transfer function $H_c(j\omega)$ such that

$$y_c(t) = x_c\left(t - \frac{1}{3f_S}\right)$$

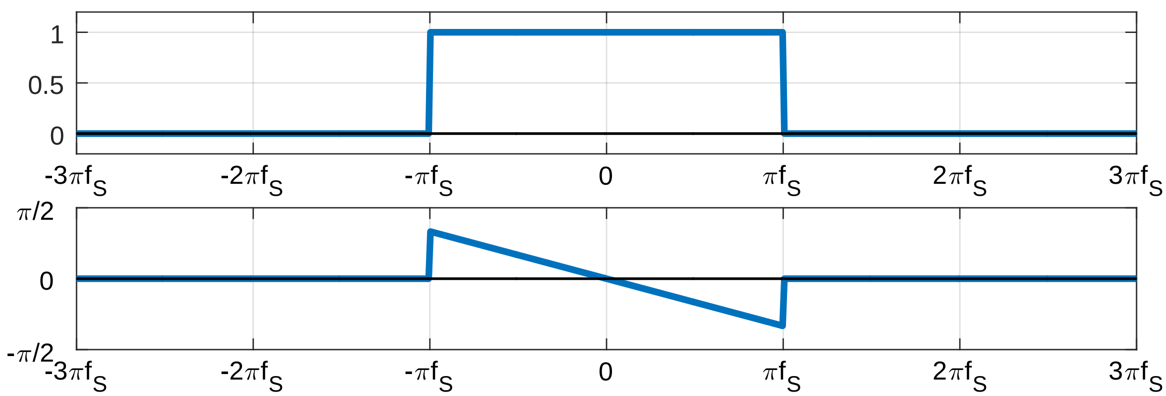
in the system shown below, assuming $x_c(t)$ is bandlimited by $f_{max}$ such that the sampling frequency $f_S > 2f_{max}$.




**Solution:**

$$Y_c(j\omega) = X_c(j\omega)e^{-j\frac{\omega}{3f_S}}$$

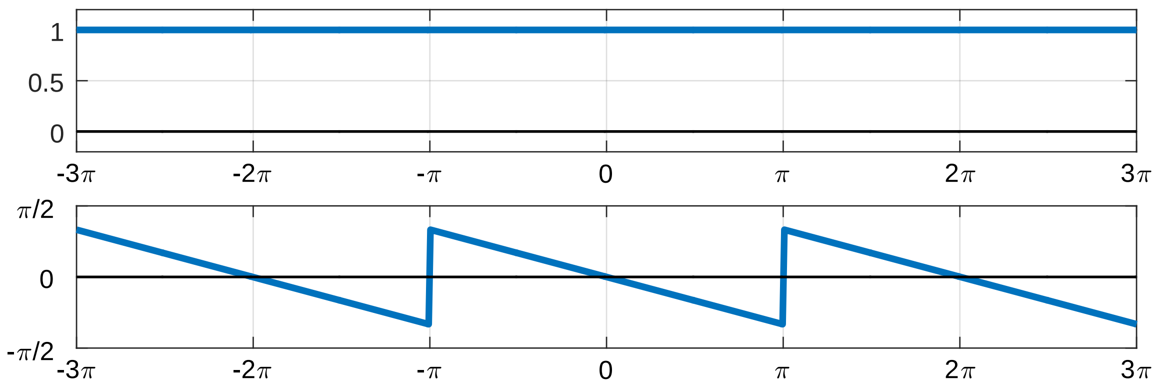
$$H_c(j\omega) = \begin{cases} e^{-j\frac{\omega}{3f_S}} & -2\pi\frac{f_S}{2} \leq \omega \leq 2\pi\frac{f_S}{2} \\ 0 & \text{otherwise} \end{cases}$$



B. Find and sketch $H_d(\Omega)$.

**Solution:**

$$H_d(\Omega) = e^{-j\frac{\Omega - 2\pi k}{3}}, \quad -\pi + 2\pi k \le \Omega \le \pi + 2\pi k, \quad k \in \mathbb{Z}$$




C. Find the naive expression for $y_d[n]$ in terms of $x_d[n]$ by transforming $H_d(\Omega)$. Note that while your result is technically true, it cannot be applied literally! The next two parts give us the actual answer.

**Solution:**

$$Y_d(\Omega) = X_d(\Omega)e^{-j\frac{\Omega}{3}}$$

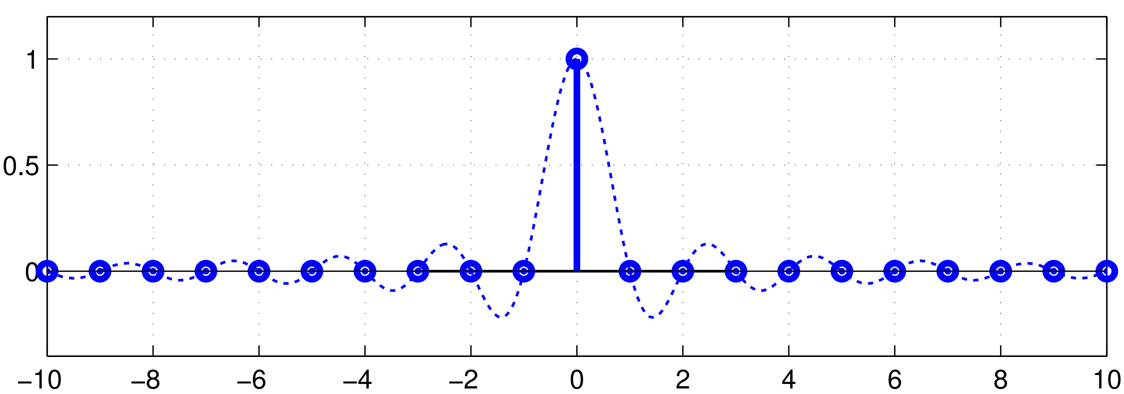
$$y_d[n] = x_d[n - 1/3]$$

$y_d[n]$ is nonsensical, since $n$ must be an integer!

D. Assume $x_c(t) = \text{sinc}(\pi f_s t)$. Verify that $x_d[n] = \delta[n]$. Combine both plots in the same set of axes.

**Solution:**

$$x_d[n] = x_c(n/f_S) = \text{sinc}(\pi f_S n/f_S) = \text{sinc}(\pi n) = \delta[n]$$

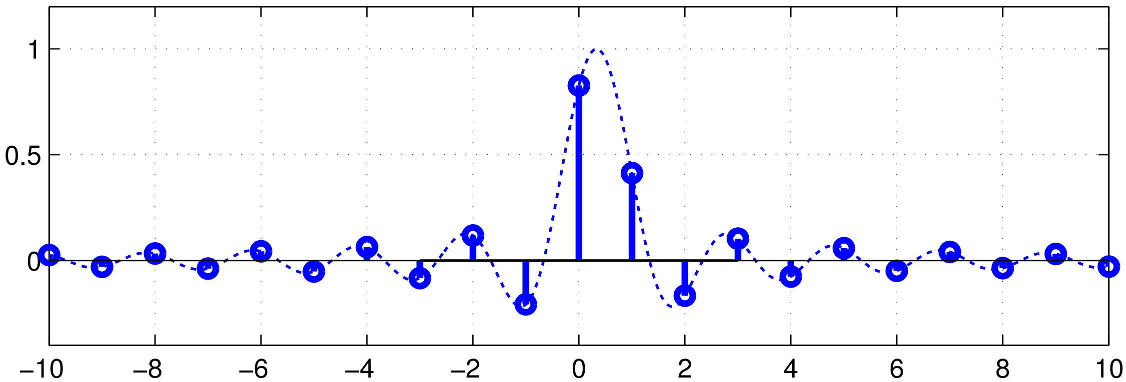


E. Find $y_c(t)$ and $y_d[n]$. Explain why $y_d[n] = h_d[n]$. Combine both plots in the same set of axes.

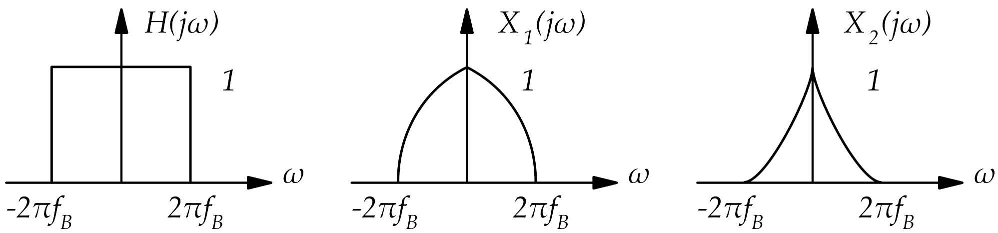
**Solution:**
Since $H_c(j\omega)$ is a delay of $\frac{1}{3f_S}$, then $y_c(t) = \text{sinc}\left[\pi f_S\left(t - \frac{1}{3f_S}\right)\right]$.
We can invoke again that $y_d[n] = y_c(n/f_S)$ to find $y_d[n]$:

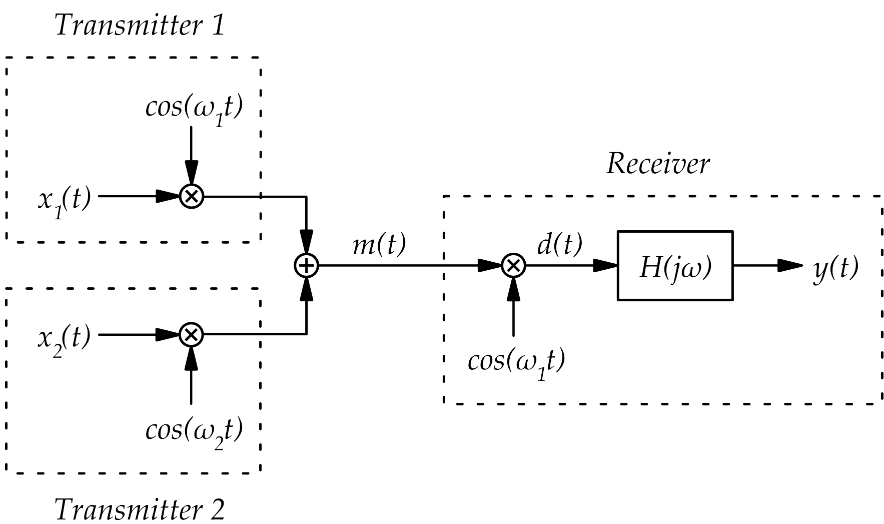
$$y_d[n] = \text{sinc}\left[\pi\left(n - 1/3\right)\right]$$

Regardless of the definition of $x_c(t)$, since $x_d[n] = \delta[n]$, $y_d$ must be the impulse response $h_d[n]$.



**Problem 3** The system shown below represents a basic communication system where two messages $x_1(t)$ and $x_2(t)$ share a common communication channel. Signals $x_1(t)$ and $x_2(t)$ are bandlimited to $f_B$ and have a frequency content as shown below. The receiver has an ideal low-pass filter $H(j\omega)$ with a cutoff frequency of $f_B$ as shown below.




A. What would happen if $\omega_1 = \omega_2 = 0$? Find $y(t)$ in terms of $x_1(t)$ and/or $x_2(t)$, and show its frequency content.

**Solution:**
If $\omega_1 = \omega_2 = 0$, the output of Transmitter 1 is

$$x_1(t)\cos(\omega_1 t) = x_1(t)\cos(0t) = x_1(t)$$

and the output of Transmitter 2 is

$$x_2(t)\cos(\omega_2 t) = x_2(t)\cos(0t) = x_2(t)$$

Therefore,

$$m(t) = x_1(t) + x_2(t)$$

Multiplying by $\cos(\omega_1(t))$,

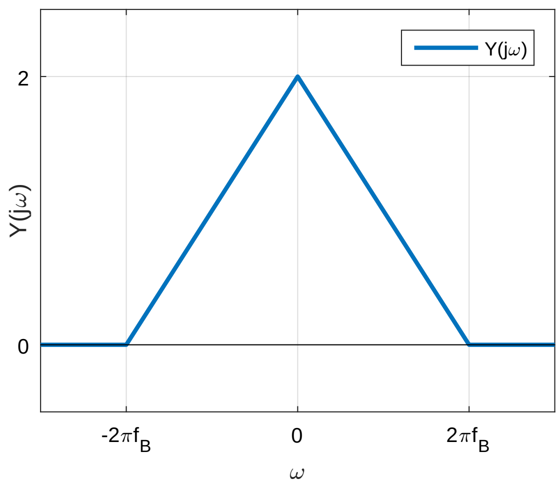
$$d(t) = m(t)\cos(\omega_1(t)) = m(t)\cos(0t) = m(t) = x_1(t) + x_2(t)$$

Note that the filter produced by $H(j\omega)$ removes all frequencies above $2\pi f_b$ and below $-2\pi f_b$; however, neither $x_1(t)$ nor $x_2(t)$ have any frequency content above $2\pi f_b$ or below $-2\pi f_b$. Therefore, the low-pass filter does not have any effect on $d(t)$, and

$$y(t) = d(t) = x_1(t) + x_2(t)$$

Therefore, the frequency content is merely the sum of the frequency content $x_1(t)$ and the frequency content of $x_2(t)$; $Y(j\omega)$ is shown below.

$$Y(j\omega) = X_1(j\omega) + X_2(j\omega)$$

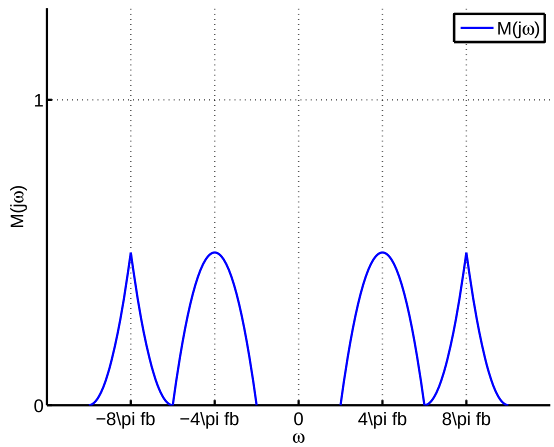



B. Find constraints on $\omega_1$ and $\omega_2$ such that there is no frequency interference (aliasing). Show the frequency content of $m(t)$ and $d(t)$ under these constraints. *Note: There may be multiple solutions; just find one that works.*

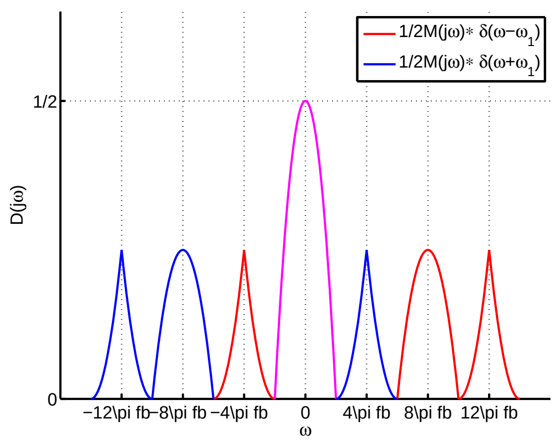
**Solution:**
A number of frequency constraints on $\omega_1$ and $\omega_2$ are possible - in particular, the output $M(j\omega)$ will consist of bandlimited peaks of width $4\pi f_b$ at frequencies $-\omega_1$, $\omega_1$, $-\omega_2$, and $\omega_2$. Algebraically,

$$\begin{aligned}
m(t) &= x_1(t)\cos(\omega_1 t) + x_2(t)\cos(\omega_2 t) \\
M(j\omega) &= \frac{1}{2\pi}\left(\mathscr{F}\{x_1(t)\} * \mathscr{F}\{\cos(\omega_1 t)\}\right) + \frac{1}{2\pi}\left(\mathscr{F}\{x_2(t)\} * \mathscr{F}\{\cos(\omega_2 t)\}\right) \\
M(j\omega) &= \frac{1}{2\pi}\left(X_1(j\omega) * [\pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1)]\right) \\
&\quad + \frac{1}{2\pi}\left(X_2(j\omega) * [\pi\delta(\omega-\omega_2) + \pi\delta(\omega+\omega_2)]\right)
\end{aligned}$$

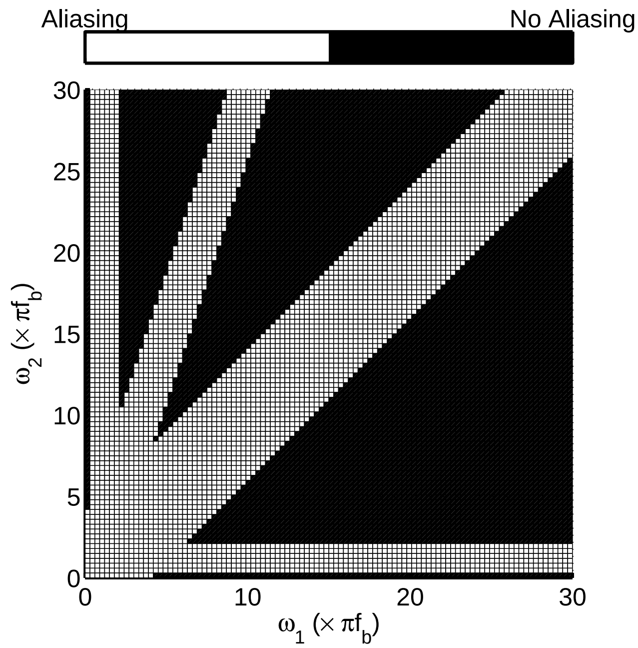
We could choose $\omega_1 = 4\pi f_b$ and $\omega_2 = 8\pi f_b$, which would avoid aliasing in $M(j\omega)$, as shown below.




This choice of $\omega_1$ and $\omega_2$ also avoids aliasing in $\mathscr{F}\{d(t)\} = D(j\omega)$, as shown below: the frequency content of $d(t)$ consists of the frequency content of $m(t)$ shifted by $\omega_1$ and $-\omega_1$. $D(j\omega)$ for $\omega_1 = 4\pi f_b$ and $\omega_2 = 8\pi f_b$ is shown below:




Indeed, any $\omega_1$ and $\omega_2$ will generate peaks of $X_1(j\omega)$ at $\omega = 0, 2\omega_1, 0$, and $-2\omega_1$, as well as peaks of $X_2(j\omega)$ at $\omega = \omega_1 - \omega_2, \omega_1 + \omega_2, -\omega_1 + \omega_2$, and $-\omega_1 - \omega_2$. Peaks must be separated by at least $4\pi f_b$ (the width of a single bandlimited signal peak). Therefore, any $\omega_1$ and $\omega_2$ that generate $X_1(j\omega)$ and $X_2(j\omega)$ as given previously that either overlap completely constructively or do not overlap will suffice. For example, in the $D(j\omega)$ shown above, two $X_1(j\omega)$ peaks combine at the origin, but no overlap is present between $X_1(j\omega)$ and $X_2(j\omega)$ peaks and no partial overlap is present. Either of these situations would constitute aliasing. To find the constraints on aliasing, we can enforce these constraints computationally and search a space of $\omega_1$ and $\omega_2$; the resulting regions of aliasing and no aliasing are shown below.



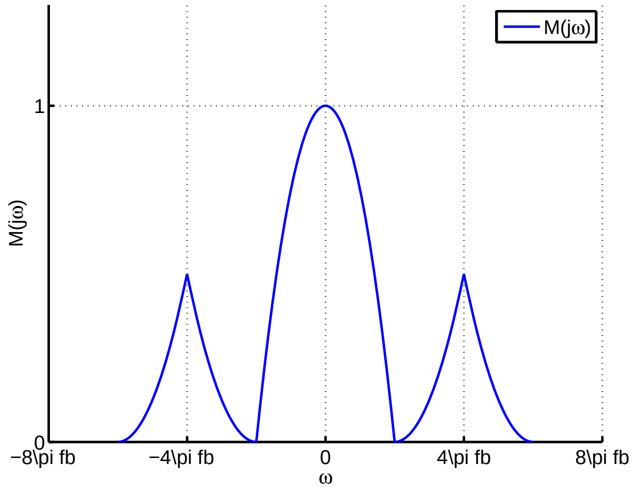

A full set of constraints for $\omega_1$ and $\omega_2$ is provided by:

$$\begin{aligned} |\omega_1 - \omega_2| &\geq 4\pi f_b \\ |4\omega_1 - \omega_2| &\geq 6\pi f_b \\ (\omega_1 = 0 \text{ or } \omega_1 &\geq 4\pi f_b) \\ (\omega_2 = 0 \text{ or } \omega_2 &\geq 4\pi f_b) \end{aligned}$$

One possible simple frequency constraint on $\omega_1$ and $\omega_2$ is $\omega_1 = 0$ and $\omega_2 \geq 4\pi f_b$; in this case, the output of Transmitter 1 will be $x_1(t)$ (since $x_1(t) \cdot \cos(0t) = x_1(t)$), and the frequency content of Transmitter 2 will be located at two peaks that do not intersect those of the output of Transmitter 1. Then, to find the frequency content of $m(t)$, we observe that

$$\begin{aligned} m(t) &= x_1(t) + x_2(t)\cos(\omega_2 t) \\ M(j\omega) &= \mathscr{F}\{x_1(t)\} + \mathscr{F}\{x_2(t)\cos(\omega_2 t)\} \\ M(j\omega) &= X_1(j\omega) + \frac{1}{2\pi}\left(X_2(j\omega) * (\pi\delta(\omega+\omega_2) + \pi\delta(\omega-\omega_2))\right) \end{aligned}$$

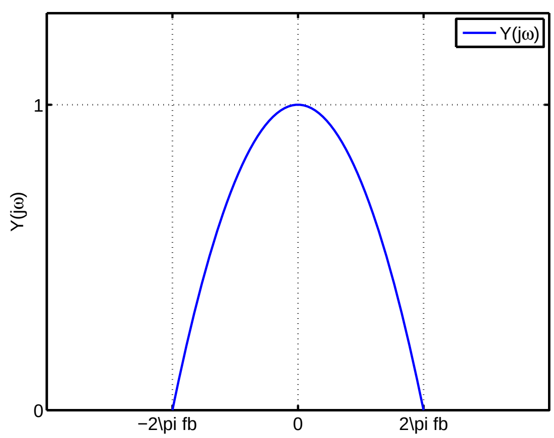
A graphical representation of $M(j\omega)$ is shown below. Note that when $\omega_1 = 0$, the height of the "peak" at $\omega_1 = 0$ is 1; otherwise, its height is 1/2.




Since $\cos(\omega_1 t) = 1$, $d(t) = m(t)\cos(\omega_1 t) = m(t)\cos(0t) = m(t)$, so the frequency content of $d(t)$ is identical to the frequency content of $m(t)$.

C. Show the frequency content and find an algebraic expression for $y(t)$ in terms of $x_1(t)$ and/or $x_2(t)$ assuming the constraints of part B.

**Solution:**
If $H(j\omega)$ is the ideal low-pass filter given, it will remove all frequencies above $2\pi f_b$ and all frequencies below $-2\pi f_b$. Therefore, the two peaks of $X_2(j\omega)$ in $d(t)$ will be eliminated. If $\omega_1 = 0$, $x_1(t)$ will simply pass through and the resulting $Y(j\omega)$ will simply be $X_1(j\omega)$, and therefore, $y(t) = x_1(t)$. However, if $x_1(t)$ has been modulated with any nonzero frequency $\omega_1$, the filter will cut the high frequency components of the demodulated $x_1(t)$ such that $Y(j\omega) = \frac{1}{2}X_1(j\omega)$ and $y(t) = \frac{1}{2}x_1(t)$.



**Course feedback**

Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?