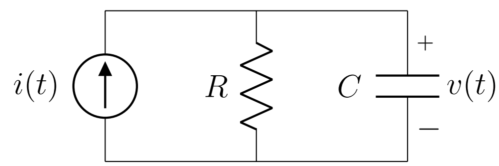
Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 5

Problem 1 In lecture, we showed that if the input current $i(t) = Q_0\delta(t)$ in the circuit below, then the output voltage $v(t) = \frac{Q_0}{C}e^{-t/\tau}u(t), \tau = RC$. Since $i(t)$ is an impulse input, $v(t)$ is the impulse response of the system.



A. Find the impedance of the circuit $Z(j\omega) = V(j\omega)/I(j\omega)$.

Solution:

$$Z(j\omega) = \frac{1}{C}\frac{1}{\left(j\omega + \frac{1}{RC}\right)}$$

B. Find the frequency content of the impulse response $v(t)$, $V(j\omega)$, by multiplying the Fourier transform of the input $i(t)$ with the impedance $Z(j\omega)$.

Solution:

$$I(j\omega) = Q_0$$
$$V(j\omega) = \frac{Q_0}{C}\frac{1}{\left(j\omega + \frac{1}{RC}\right)}$$

C. Take the Fourier transform of the impulse response $v(t)$ using the integral definition of the Fourier transform directly. Note that you have just found another important Fourier transform pair!

Solution:

$$V(j\omega) = \mathscr{F}\{v(t)\} = \frac{Q_0}{C}\mathscr{F}\{e^{-t/\tau}u(t)\} = \frac{Q_0}{C}\int_{t=0}^{\infty} e^{-t/\tau}e^{-j\omega t}dt = \frac{Q_0}{C}\int_{t=0}^{\infty} e^{-(1/\tau+j\omega)t}dt$$
$$= \frac{Q_0}{C}\frac{-1}{(1/\tau+j\omega)}\left. e^{-(1/\tau+j\omega)t}\right]_{t=0}^{\infty} = \frac{Q_0}{C}\frac{-1}{(1/\tau+j\omega)}[0-1] = \frac{Q_0}{C}\frac{1}{(1/\tau+j\omega)}, \tau = RC$$

Note that we have just found that:

$$\mathscr{F}\{e^{-t/\tau}u(t)\} = \frac{1}{1/\tau + j\omega}$$

This implies that:

$$e^{-t/\tau}u(t) = \mathscr{F}^{-1}\left\{\frac{1}{1/\tau + j\omega}\right\}$$

More generally, we write:

$$e^{-t/\tau}u(t) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \frac{1}{1/\tau + j\omega}$$

This transform will be useful in part A of problem 2.

D. Compare the answers to the previous two parts. Explain the relationship between them clearly.

Solution:
We have arrived at the Fourier transform of the impulse response by taking the Fourier transform directly and indirectly by multiplying the transfer function of the system times the transform of the impulse. In system notation,

$$\begin{array}{ccccc}
Q_0\delta(t) & \longrightarrow & \boxed{\text{RC ckt}} & \longrightarrow & \frac{Q_0}{C}e^{-t/\tau} \\
\mathscr{F}\big\downarrow & & & & \big\downarrow\mathscr{F} \\
Q_0 & \longrightarrow & \boxed{\dfrac{1}{C}\dfrac{1}{\left(j\omega + \frac{1}{RC}\right)}} & \longrightarrow & \frac{Q_0}{C}\frac{1}{\left(j\omega + \frac{1}{RC}\right)}
\end{array}$$

The remaining parts should convince you that since the impulse is the derivative of the step, the impulse response must be the derivative of the step response.

E. Find $v(t)$ if $i(t) = I_0u(t)$. Use either a differential equation, or circuit analysis and your knowledge of first-order systems.

Solution:

$$v(t) = I_0R(1 - e^{-t/\tau}) \qquad \tau = RC$$

F. Find the operator that transforms the input from part E into the input from part A. Be careful with any constants needed.

Solution:

$$Q_0\delta(t) = \underbrace{\left[\frac{Q_0}{I_0}\frac{d}{dt}\right]}_{\text{operator}} I_0u(t)$$

G. Apply the operator you found in the previous part to the step response you found in part E and compare it to the impulse response of the circuit. Explain your result.

Solution:

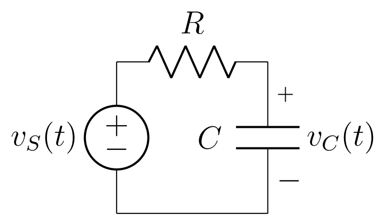
$$\underbrace{\left[\frac{Q_0}{I_0}\frac{d}{dt}\right]}_{\text{operator}} \underbrace{I_0R(1-e^{-t/\tau})}_{\text{step response}} = \frac{Q_0R}{\tau}e^{-t/\tau} = \frac{Q_0\cancel{R}}{\cancel{R}C}e^{-t/\tau} = \underbrace{\frac{Q_0}{C}e^{-t/\tau}}_{\text{impulse response}}$$

The derivative can be taken before or after finding the response of the system. In either case, we find the impulse response. In system notation,

$$\begin{tikzcd}
I_0u(t) \arrow[r] \arrow[d, swap, "\left[\frac{Q_0}{I_0}\frac{d}{dt}\right]"] & \boxed{\text{RC ckt}} \arrow[r] & I_0R(1-e^{-t/\tau}) \arrow[d, "\left[\frac{Q_0}{I_0}\frac{d}{dt}\right]"] \\
Q_0\delta(t) \arrow[r] & \boxed{\text{RC ckt}} \arrow[r] & \frac{Q_0}{C}e^{-t/\tau}
\end{tikzcd}$$

Problem 2 This problem emphasizes that distributions like the impulse are mathematical idealizations. Thus, the details of the distribution are not important as long as the effect is the same. In other words, if it behaves like an impulse, it is an impulse.

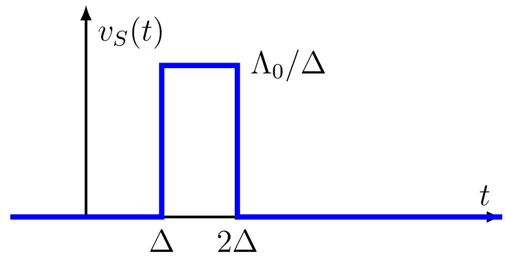
A. Find an algebraic expression for the impulse response $v_C(t)$ if $v_S(t) = \Lambda_0\delta(t)$ using frequency analysis. Λ_0 *is the product of voltage and time, with units of V·s (which turn out to be the units of magnetic flux for a really good reason if you're curious!)*



Solution:

$$
\begin{aligned}
v_S(t) &= \Lambda_0\delta(t) \\
v_s(j\omega) &= \mathscr{F}\left\{v_S(t)\right\} = \Lambda_0 \\
H(j\omega) &= \frac{\frac{1}{\tau}}{\frac{1}{\tau} + j\omega}, \qquad \tau = RC \\
v_C(j\omega) &= H(j\omega)v_s(j\omega) = \frac{\frac{\Lambda_0}{\tau}}{\frac{1}{\tau} + j\omega} \\
v_C(t) &= \mathscr{F}^{-1}\left\{\frac{\frac{\Lambda_0}{\tau}}{\frac{1}{\tau} + j\omega}\right\} = \frac{\Lambda_0}{\tau}e^{-\frac{t}{\tau}}u(t)
\end{aligned}
$$

B. For the same circuit as in part A, assume that $v_S(t)$ is as shown below and find an algebraic expression for $v_C(t)$. Let $\Delta \to 0$. Show that $v_C(t)$ approaches the impulse response you found in the previous part. *Hint: Note that in the limit only one part of the solution matters. Also, you will find it useful to remember that* $e^x \approx 1+x$ *if* $x \ll 1$.



Solution:

$$v_C(t) = \begin{cases} 0 & t < 0 \\ \frac{\Lambda_0}{\Delta}(1 - e^{-(t-\Delta)/\tau}) & \Delta < t < 2\Delta\ , \qquad \tau = RC \\ \frac{\Lambda_0}{\Delta}(1 - e^{-\Delta/\tau})e^{-(t-2\Delta)/\tau} & t > 2\Delta \end{cases}$$

Letting $\Delta \to 0$ we can show that $v_C(t)$ approaches the impulse response we found in the previous part.

$$\begin{aligned} \lim_{\Delta\to 0} v_C(t) &= \lim_{\Delta\to 0} \frac{\Lambda_0}{\Delta}(e^{2\Delta/\tau} - e^{\Delta/\tau})e^{-t/\tau}, \quad t > 2\Delta \\ &= \lim_{\Delta\to 0} \frac{\Lambda_0}{\Delta}\left(1 + \frac{2\Delta}{\tau} - 1 - \frac{\Delta}{\tau}\right) e^{-t/\tau}, \quad t > 2\Delta \\ &= \lim_{\Delta\to 0} \frac{\Lambda_0}{\cancel{\Delta}}\left(\frac{\cancel{\Delta}}{\tau}\right) e^{-t/\tau}, \quad t > 2\Delta \\ &= \frac{\Lambda_0}{\tau} e^{-t/\tau}, \quad t > 0 \end{aligned}$$

C. Since $v_C(t)$ approaches the impulse response, $v_S(t)$ must approach an impulse. However, what is the value of $v_S(0)$ for all values of Δ? Revisit the definition of the impulse and check that the limit is an impulse, regardless of the value of $v_S(0)$!

Solution:
Even though $v_S(0) = 0$ for all values of Δ, the remaining values of the function between 0^+ and 0^- are still undefined in the limit. Thus, as expected from an impulse,

$$v_S(t) = \begin{cases} 0 & -\infty < t < 0^- \\ \text{undefined} & 0^- < t < 0^+ \\ 0 & 0^+ < t < \infty \end{cases} \qquad \text{and} \qquad \frac{1}{\Lambda_0}\int_{0^-}^{0^+} \mathrm{v_S(t)} = 1$$

so that $\lim_{\Delta\to 0} v_S(t) = \Lambda_0 \delta(t)$.

Problem 3 This problem illustrates the inverse frequency-time relationship of the Fourier transform. This relationship illustrates fundamental tradeoffs in many real world applications.

For example, in quantum physics, the position and momentum of a particle are the result of applying a specific operator to the *wave function* describing the particle. The result is a probability distribution, and the quantity σ^2 in part C of this problem is equivalent to the variance of one these (say position).

The position is the Fourier transform of the momentum. In this problem, you will show that the variance of the momentum is proportional to $1/\sigma^2$. Thus, zero variance in either quantity implies infinite variance of the other! Furthermore, any other probability distribution other than Gaussians increases the product of the variances.

This trade-off in variance between momentum and position, a direct result if the inverse relation of the Fourier transform, is known as the *Heisenberg Uncertainty Principle.*

A. Using the fact that

$$\int_{-\infty}^{\infty} e^{-t^2} dt \; = \sqrt{\pi}$$

show that the Fourier transform of the Gaussian e^{-t^2} is the Gaussian $\sqrt{\pi}e^{-\frac{\omega^2}{4}}$. *Hint: Complete the square of the exponential and make a substitution.*

Solution:

$$\begin{aligned}
\mathscr{F}\left\{e^{-t^2}\right\} &= \int_t e^{-t^2} e^{-j\omega t} dt \\
&= \int_t e^{-t^2 - j\omega t} dt \\
&= \int_t e^{-t^2 - j\omega t + \frac{\omega^2}{4}} e^{-\frac{\omega^2}{4}} dt \\
&= \int_t e^{-(t - \frac{j\omega}{2})^2} e^{-\frac{\omega^2}{4}} dt \\
&= e^{-\frac{\omega^2}{4}} \int_t e^{-u^2} du \qquad u = t - \frac{j\omega}{2},\ du = dt \\
&= \sqrt{\pi} e^{-\frac{\omega^2}{4}}
\end{aligned}$$

B. Show the scaling property of the Fourier Transform, $\mathscr{F}\{x(at)\} = \frac{1}{|a|}X(\frac{j\omega}{a})$.

Solution:

$$
\begin{aligned}
\text{Given } X(j\omega) &= \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt, \\
\mathscr{F}\{x(at)\} &= \int_{t=-\infty}^{\infty} x(at)e^{-j\omega t}dt \\
&= \frac{1}{a}\int_{t'/a=-\infty}^{\infty} x(t')e^{-j\frac{\omega}{a}t'}dt', \qquad t' = at, dt' = adt \\
&= \begin{cases} \frac{1}{a}\int_{t'=-\infty}^{\infty} x(t')e^{-j\frac{\omega}{a}t'}dt' & a > 0 \\ \frac{1}{a}\int_{t'=\infty}^{-\infty} x(t')e^{-j\frac{\omega}{a}t'}dt' & a < 0 \end{cases} \\
&= \begin{cases} \frac{1}{a}\int_{t'=-\infty}^{\infty} x(t')e^{-j\frac{\omega}{a}t'}dt' & a > 0 \\ -\frac{1}{a}\int_{t'=-\infty}^{\infty} x(t')e^{-j\frac{\omega}{a}t'}dt' & a < 0 \end{cases} \\
&= \frac{1}{|a|}\int_{t'=-\infty}^{\infty} x(t')e^{-j\frac{\omega}{a}t'}dt' \\
&= \frac{1}{|a|}X(\frac{j\omega}{a})
\end{aligned}
$$

C. Use the scaling property to find the Fourier Transform of $e^{-\frac{t^2}{\sigma^2}}$.

Solution:

$$
\begin{aligned}
\mathscr{F}\{x(at)\} &= \frac{1}{|a|}X(\frac{j\omega}{a}) \\
\mathscr{F}\left\{e^{-t^2}\right\} &= \sqrt{\pi}e^{-\frac{\omega^2}{4}} \\
\mathscr{F}\left\{e^{-\frac{t^2}{\sigma^2}}\right\} &= |\sigma|\sqrt{\pi}e^{-\frac{\omega^2\sigma^2}{4}}
\end{aligned}
$$

D. Define the width of the Gaussian as the time or frequency where the function reaches $1/e$ of its peak value. Show that the width of the frequency Gaussian is inversely proportional to the width of the time Gaussian.

Solution:
The peak value of the function, $e^{-\frac{t^2}{\sigma^2}}$, occurs at $t = 0$.

$$\begin{aligned} e^{-\frac{\Delta t^2}{\sigma^2}} &= e^{-1} \\ -\frac{\Delta t^2}{\sigma^2} &= -1 \\ \Delta t^2 &= \sigma^2 \\ \Delta t &= \pm\sigma \end{aligned}$$

The peak value of the function, $|\sigma|\sqrt{\pi}e^{-\frac{\omega^2\sigma^2}{4}}$, occurs at $\omega = 0$.

$$\begin{aligned} |\sigma|\sqrt{\pi}e^{-\frac{\Delta\omega^2\sigma^2}{4}} &= |\sigma|\sqrt{\pi} \cdot e^{-1} \\ e^{-\frac{\Delta\omega^2\sigma^2}{4}} &= e^{-1} \\ \frac{\Delta\omega^2\sigma^2}{4} &= 1 \\ \Delta\omega^2 &= \frac{4}{\sigma^2} \\ \Delta\omega &= \pm\frac{2}{\sigma} \end{aligned}$$

Therefore, $|\Delta t| \cdot |\Delta\omega| = 2$. Thus, the width of the frequency Gaussian is inversely proportional to the width of the time Gaussian.

Problem 4

A. Show the time shift property of the Fourier transform, $\mathscr{F}\{x(t+T)\} = X(j\omega)e^{j\omega T}$ where $\mathscr{F}\{x(t)\} = X(j\omega)$ denotes that $X(j\omega)$ is the Fourier transform of $x(t)$. *Hint: Use the substitution $t' = t + T$.*

Solution:
We know $\mathscr{F}\{x(t)\} = X(j\omega) = \int_{t=-\infty}^{t=\infty} x(t)e^{-j\omega t}dt$. So let's try to write $\mathscr{F}\{x(t+T)\}$ in a form we recognize.

$$\begin{aligned}
\mathscr{F}\{x(t+T)\} &= \int_{t=-\infty}^{t=\infty} x(t+T)e^{-j\omega t}dt \\
\text{Let } t' &= t+T \\
t &= t' - T, \qquad dt = dt' + 0 = dt'
\end{aligned}$$

If t goes from $-\infty$ to ∞ in the integral before the substitution then $t' - T$ must also go from $-\infty$ to ∞.

$$\begin{aligned}
\mathscr{F}\{x(t+T)\} &= \int_{t'=-\infty}^{t'=\infty} x(t')e^{-j\omega(t'-T)}dt' \\
&= \int_{-\infty}^{\infty} x(t')e^{-j\omega t'}e^{j\omega T}dt'
\end{aligned}$$

Since the term $e^{j\omega T}$ is independent of t' we can bring it outside the integral.

$$\begin{aligned}
\mathscr{F}\{x(t+T)\} &= e^{j\omega T}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j\omega t'}dt'}_{X(j\omega)} \\
&= e^{j\omega T}X(j\omega)
\end{aligned}$$

B. A periodic function $x(t)$ has the property $x(t) = x(t+T)$. Show that the Fourier transform of this equation implies that the Fourier transform of $x(t)$ can only have non-zero frequency components at $\omega = \frac{2\pi k}{T}$, where k is any integer. *This closes the loop back to the Fourier series by showing that the frequency content of a period T function only exists in the harmonics of $2\pi/T$.*

Solution:

$$
\begin{aligned}
x(t) &= x(t+T) \\
\mathscr{F}\{x(t)\} &= \mathscr{F}\{x(t+T)\} \\
X(j\omega) &= X(j\omega)\underbrace{e^{j\omega T}}_{\text{delay}} \\
X(j\omega)[1-e^{j\omega T}] &= 0 \\
&\Rightarrow \begin{cases} 1-e^{j\omega T}=0 & \omega T = 2\pi k;\ k \in \mathbb{Z} \\ X(j\omega)=0 & \text{otherwise} \end{cases}
\end{aligned}
\tag{1}
$$

Thus, $X(j\omega)$ can only be non-zero when ω is an integer multiple of $\frac{2\pi}{T}$.

C. Show that

$$\cos(\omega_0 t) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad 2\pi\left[\frac{1}{2}\delta(\omega-\omega_0)+\frac{1}{2}\delta(\omega+\omega_0)\right]$$

Hint: Recall that $\cos(\omega t) = \frac{e^{j\omega t}+e^{-j\omega t}}{2}$ *and the "picking" property of the impulse,* $\int_{-\infty}^{\infty}\delta(x-x_0)f(x)dx = f(x_0)$.

Solution:
Since the Fourier transform is unique, we only need to show it in one direction. In particular,

$$\mathscr{F}\{\cos(\omega_0 t)\} = \int_{t=-\infty}^{t=\infty}\cos(\omega_0 t)e^{-j\omega t}dt$$

requires integration by parts. Rewriting this integral as

$$\mathscr{F}\{\cos(\omega_0 t)\} = \frac{1}{2}\int_{t=-\infty}^{t=\infty}e^{j\omega_0 t}e^{-j\omega t}dt + \frac{1}{2}\int_{t=-\infty}^{t=\infty}e^{-j\omega_0 t}e^{-j\omega t}dt$$

makes it easier. However, computing the inverse Fourier transform is simpler:

$$\begin{aligned}
&\mathscr{F}^{-1}\left\{2\pi\left[\frac{1}{2}\delta(\omega-\omega_0)+\frac{1}{2}\delta(\omega+\omega_0)\right]\right\}\\
&= \cancel{2\pi}\frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}\frac{d\omega}{\cancel{2\pi}} + \cancel{2\pi}\frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega+\omega_0)e^{j\omega t}\frac{d\omega}{\cancel{2\pi}}\\
&= \frac{1}{2}e^{j\omega_0 t}+\frac{1}{2}e^{-j\omega_0 t}\\
&= \cos(\omega_0 t)
\end{aligned}$$

Course feedback

Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?