Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 1

Problem 1: *(2 points)*

A. Find explicit algebraic expressions for A and θ in terms of a and b such that

$$Ae^{j\theta} = a + jb$$

where $j^2 = -1$ and $e^{j\theta} = \cos\theta + j\sin\theta$. Use the complex plane to illustrate.

B. Invert the transformation of part A and find explicit algebraic expressions for a and b in terms of A and θ. Again, use the complex plane to illustrate.

C. Find explicit algebraic expressions for A and θ such that

$$Ae^{j\theta} = \frac{jb_1}{a_2 + jb_2} \frac{a_3 + jb_3}{-a_4}$$

Problem 2: *(4 points)* This problems uses a *similarity transformation* to solve a system of coupled, first-order differential equations.

A. Solve the "easy" problem: show that the solution for the system of equations

$$\begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

B. Solve the "hard" problem: show that the solution for the system of equations

$$\begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}$$

Hint: Start by showing that

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

(Note that the columns of the second matrix are the eigenvectors of the first matrix.) Express the solution vector as a scaled sum of these eigenvectors and show that this transforms the original problem into the "easy" problem from Part A.

C. Generalize. Follow the same steps to show that the solution to

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x} = Ve^{\Lambda t}V^{-1}\mathbf{x}_0$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}, \quad AV = V\Lambda, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

and V is an invertible matrix.

$A = V\Lambda V^{-1}$ is a similarity transformation *such that A and Λ are* similar *matrices. Moreover, since Λ is a diagonal matrix and V is invertible, A is a* diagonalizable *matrix. Check out* http://en.wikipedia.org/wiki/Diagonalizable_matrix *and* http://planetmath.org/diagonalization*.*

Problem 3: *(4 points)*

A. Show that the general solution for the underdamped second order differential equation

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0, \qquad \omega_0 > \alpha$$

is

$$x = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}), \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Hint: Guess Ae^{st} as the homogeneous solution and substitute.

B. The specific solution with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ is

$$x = x_0\sqrt{1 + (\alpha/\omega_d)^2}e^{-\alpha t}\cos(\omega_d t - \tan^{-1}\alpha/\omega_d)$$

Assume $x_0 = \alpha = 1$. Use a computer to graph this solution for $\omega_d = 1$ and $\omega_d = 10$. For each value of ω_d, verify the initial conditions graphically and zoom out to show the asymptotic behavior.

C. Find the step response of the circuit shown below. Assume that $\frac{R}{2} < \sqrt{\frac{L}{C}}$ and let $v_{in} = Vu(t)$. Find an expression for v_{out} and graph it. *Hint 1: Since the circuit was at rest for* $t < 0$, *you can assume* $v_{out}(0) = 0$ *and* $\dot{v}_{out}(0) = 0$. *Hint 2: Use a linear transformation of the solution from the previous part to avoid deriving the solution again.*