Olin College of Engineering
ENGR2410 – Signals and Systems

Reference 4

System response to linear combinations

For any LTI system,

$$e^{j\omega t} \rightarrow \boxed{LTI} \rightarrow H(j\omega)e^{j\omega t}$$

A function with period T such that $v(t+T) = v(t)$ can be represented as

$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$$

In this case,

$$\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} \rightarrow \boxed{LTI} \rightarrow \sum_{n=-\infty}^{\infty} H\left(j\frac{2\pi}{T}n\right) c_n e^{j\frac{2\pi}{T}nt}$$

Orthogonality

Complex exponentials are orthogonal functions since

$$\int_T e^{j\omega t} dt = \begin{cases} T & \omega = 0 \\ 0 & \text{otherwise} \end{cases}$$

Assuming

$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$$

$$v(t) = ... + c_{-1}e^{-j\frac{2\pi}{T}t} + c_0 + c_1 e^{j\frac{2\pi}{T}t} + ... + c_n e^{j\frac{2\pi}{T}nt} + ...$$

Multiply by $e^{-j\frac{2\pi}{T}nt}$ to "isolate" the c_n coefficient

$$v(t)e^{-j\frac{2\pi}{T}nt} = ... + c_{-1}e^{-j\frac{2\pi}{T}(n+1)t} + c_0 e^{-j\frac{2\pi}{T}nt} + c_1 e^{j\frac{2\pi}{T}(1-n)t} + ... + c_n + ...$$

$$\int_T v(t)e^{-j\frac{2\pi}{T}nt} dt = ... + \cancel{\int_T c_{-1}e^{-j\frac{2\pi}{T}(n+1)t}}dt + \cancel{\int_T c_0 e^{-j\frac{2\pi}{T}nt}}dt + \cancel{\int_T c_1 e^{j\frac{2\pi}{T}(1-n)t}}dt + ...$$

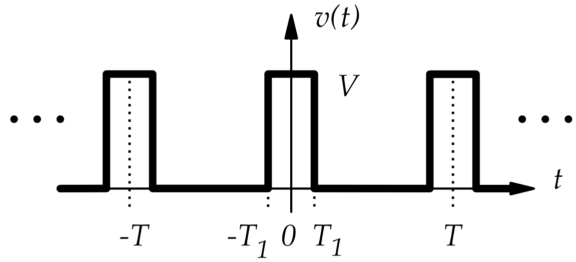
$$... + \int_T c_n dt + ...$$

$$\int_T v(t)e^{-j\frac{2\pi}{T}nt} dt = c_n T$$

$$\boxed{c_n = \frac{1}{T}\int_T v(t)e^{-j\frac{2\pi}{T}nt} dt}$$

For example, if

$$v(t) = \begin{cases} V & -T_1 + nT < t < T_1 + nT, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

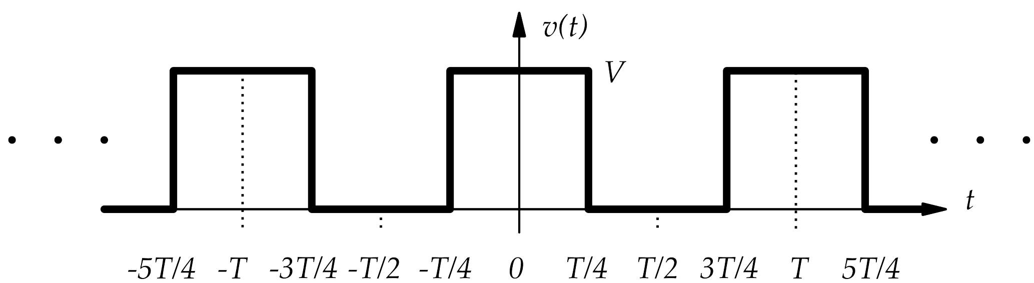


then

$$c_n = \frac{1}{T}\int_{-T_1}^{T_1} V e^{-j\frac{2\pi}{T}nt}dt = \frac{V}{T}\frac{-T}{j2\pi n}e^{-j\frac{2\pi}{T}nt}\Big]_{-T_1}^{T_1} = \frac{V}{\pi n}\left[\frac{e^{j2\pi n\frac{T_1}{T}} - e^{-j2\pi n\frac{T_1}{T}}}{2j}\right]$$

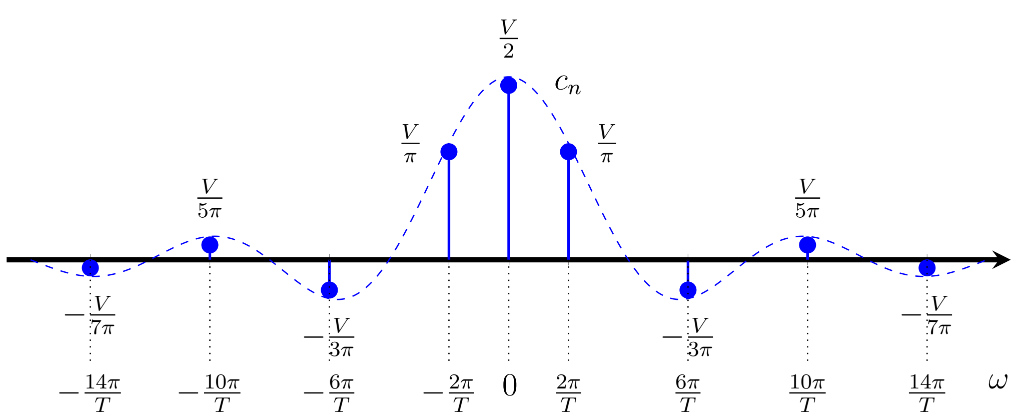
$$c_n = 2V\frac{T_1}{T}\frac{\sin\left(2\pi n\frac{T_1}{T}\right)}{2\pi n\frac{T_1}{T}} = 2V\frac{T_1}{T}\text{sinc}\left(2\pi n\frac{T_1}{T}\right)$$

where $\text{sinc}(x) = \sin(x)/x$. If $T_1 = T/4$, the square wave has a 50% duty cycle,



and the coefficients become

$$c_n = \frac{V}{2}\text{sinc}\left(\frac{\pi}{2}n\right) \qquad \Rightarrow \qquad c_0 = V/2, c_1 = V/\pi$$



We can approximate the square wave using a sinusoid with an offset by combining c_{-1}, c_1 and c_0,

$$v(t) = \frac{V}{2} + \frac{2V}{\pi}\cos\left(\frac{2\pi}{T}t\right) + ... = \frac{V}{2}\left[1 + \underbrace{\frac{4}{\pi}}_{1.273...}\cos\left(\frac{2\pi}{T}t\right) + ...\right]$$

Visit http://falstad.com/fourier for a cool visualization!

Fourier transform

Orthogonality allowed us to represent *periodic* functions using a summation of exponentials. In order to eliminate this restriction, we can take the limit as T goes to infinity, but that turns the summation into an integral of exponentials.

$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} \qquad \Leftrightarrow \qquad c_n = \frac{1}{T}\int_T v(t)e^{-j\frac{2\pi}{T}nt}dt$$

Substitute the equation for the coefficients inside the summation,

$$v(t) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{T}\int_T v(t)e^{-j\frac{2\pi}{T}nt}dt\right)e^{j\frac{2\pi}{T}nt}$$

Define $\Delta\omega = 2\pi/T$, $\omega = \Delta\omega n = 2\pi n/T$. This turns the summation into a Riemann sum. The inner integral is simply a function of ω, and this times $e^{j\omega t}/2\pi$ is the "height" of the rectangle in the Riemann sum. The remaining $\Delta\omega$ is the "width" of the rectangle, and will become the infinitesimal $d\omega$ of the integration as $T \to \infty$.

$$v(t) = \sum_{\omega=-\infty}^{\infty}\underbrace{\left(\int_T v(t)e^{-j\omega t}dt\right)}_{V(j\omega)}e^{j\omega t}\cdot\underbrace{\frac{\Delta\omega}{2\pi}}_{1/T}$$

If we integrate a period that includes $\omega = 0$, the limits of the inner integral will extend to $t = \pm\infty$ as $T \to \infty$, and the sum over ω will converge to the integral

$$v(t) = \int_{\omega=-\infty}^{\infty}\underbrace{\left(\int_{t=-\infty}^{\infty} v(t)e^{-j\omega t}dt\right)}_{V(j\omega)}e^{j\omega t}\cdot\frac{d\omega}{2\pi}, \qquad d\omega = \lim_{T\to\infty}\Delta\omega$$

Separating the integrals,

$$v(t) = \int_{\omega=-\infty}^{\infty} V(j\omega)e^{j\omega t}\cdot\frac{d\omega}{2\pi} = \mathscr{F}^{-1}\{v(t)\}$$

$$V(j\omega) = \int_{t=-\infty}^{\infty} v(t)e^{-j\omega t}dt = \mathscr{F}\{V(j\omega)\}$$

The function $V(j\omega)$ is the *Fourier transform* of $v(t)$. Similarly, $v(t)$ is the *inverse Fourier transform* of $V(j\omega)$.