Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 10

Problem 1 *(2 points)* Use the Laplace transform to verify that the step response of the system $\dot{y} + y = x$ is $y(t) = (1 - e^{-t})u(t)$. Be sure to indicate the regions of convergence of any functions in the s-plane. You will have to refresh your partial fraction expansions.

Solution:

$$\dot{y} + y = x \quad \overset{\mathscr{L}}{\Longleftrightarrow} \quad sY(s) + Y(s) = X(s)$$

$$H(s) = \frac{1}{s+1}, \quad \text{Re}\{s\} > -1$$

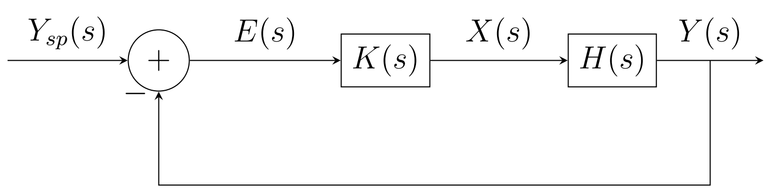
Since $\mathscr{L}\{u(t)\} = 1/s, \quad \text{Re}\{s\} > 0,$

$$Y(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}, \quad \text{Re}\{s\} > 0$$

$$y(t) = u(t) - e^{-t}u(t) = (1 - e^{-t})u(t)$$

Problem 2 *(2 points)* In this problem, you will explore the properties of integral control, as compared to proportional control. Recall that with proportional control, the DC gain depends on the amount of feedback, K_p. This is known as *offsett error*. Integral control eliminates this error at the cost of introducing oscillations and possibly making the overall system unstable.

A. Find the DC gain of the system $Y(s)/Y_{sp}(s)$ below if you use an integral controller $K(s) = K_I/s$ for any $H(s)$ and verify that it is independent of the value of K_I.



Solution:

$$\frac{Y}{Y_{sp}} = \frac{\frac{K_I}{s}H(s)}{1 + \frac{K_I}{s}H(s)} = \frac{K_I H(s)}{s + K_I H(s)} \Rightarrow \lim_{s \to 0} \frac{Y}{Y_{sp}} = 1$$

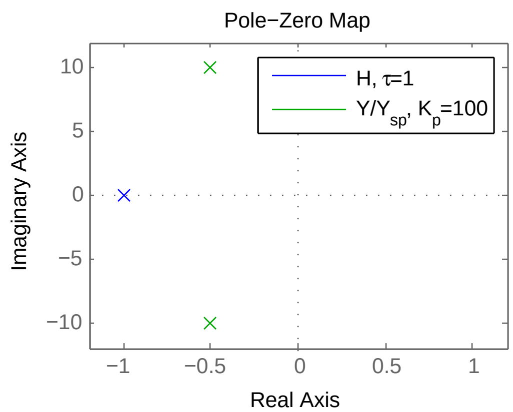
Regardless of the value of K_I.

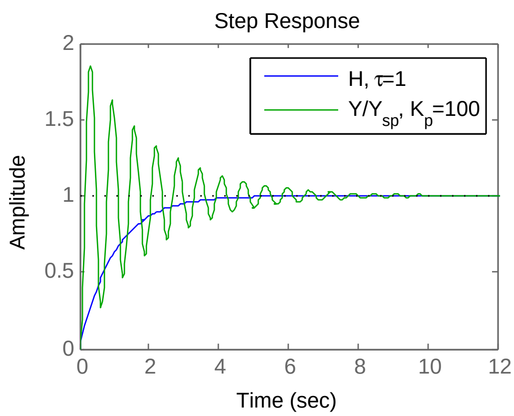
B. Assume $H(s) = \frac{1/\tau}{s+1/\tau}$. Find $Y(s)/Y_{sp}(s)$. Find the pole(s) of the system assuming $K_I \gg 1/\tau$. Compare the pole-zero diagram and step response of $H(s)$ and $Y(s)/Y_{sp}(s)$ in this case.

Solution:

$$\frac{Y}{Y_{sp}} = \frac{K_I/\tau}{s^2 + s/\tau + K_I/\tau}$$

$$s \approx -1/2\tau \pm j\sqrt{K_I/\tau}$$





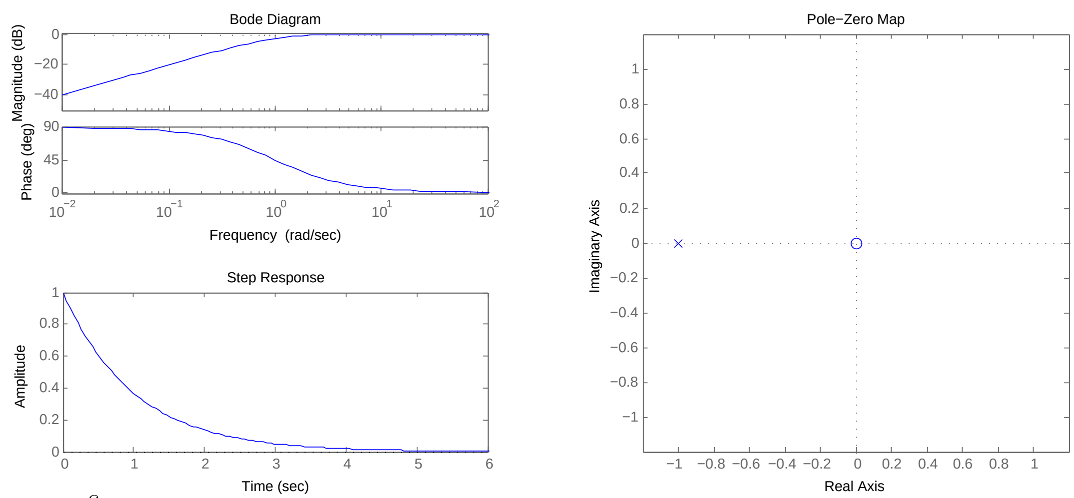
Problem 3 *(2 points)* Use Matlab to analyze the behavior of the systems listed. Feel free to use the code shown below.

```
s=tf('s');h=(s^2+1)/(s^2+3*s+1)
subplot 311;bode(h)
subplot 312;pzmap(h)
subplot 313;step(h)
```

For each system, note the relationship between all three plots: order of the system, number of poles and zeros, real or complex poles, oscillations and so forth. Hand in a couple of sentences for each system describing its behavior and any notable characteristics concisely.

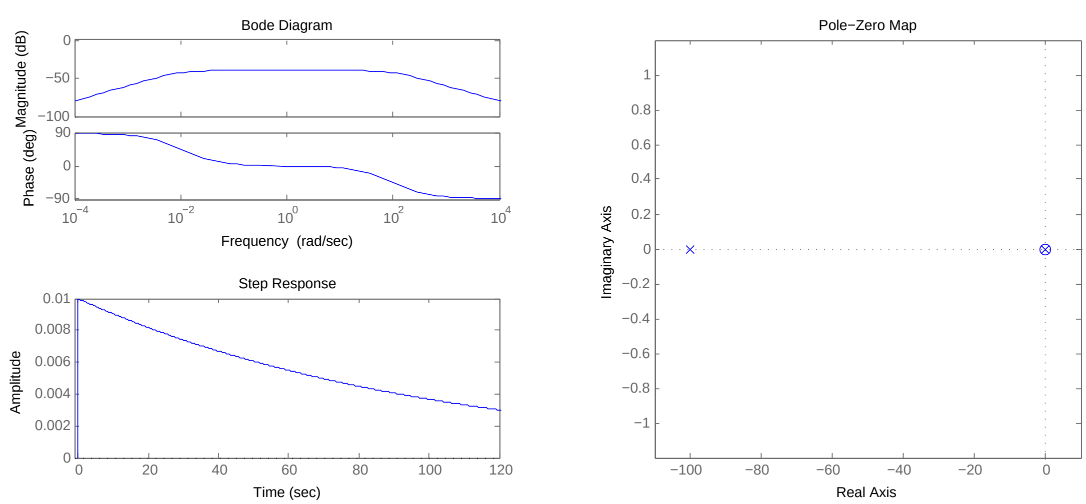
$A.\ \dfrac{s}{s+1}$

Solution:



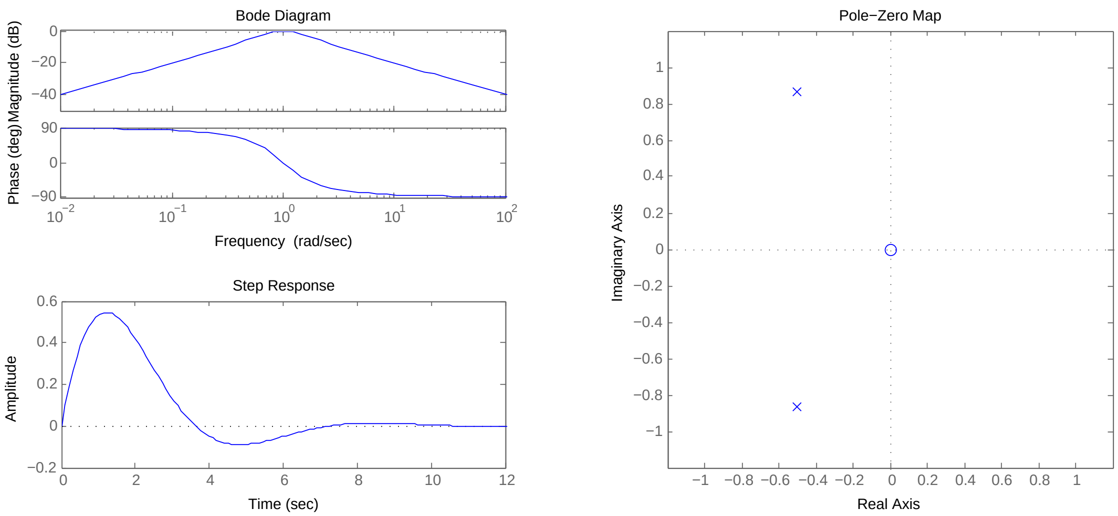
$B.\ \dfrac{s}{s^2+100s+1}$

Solution:



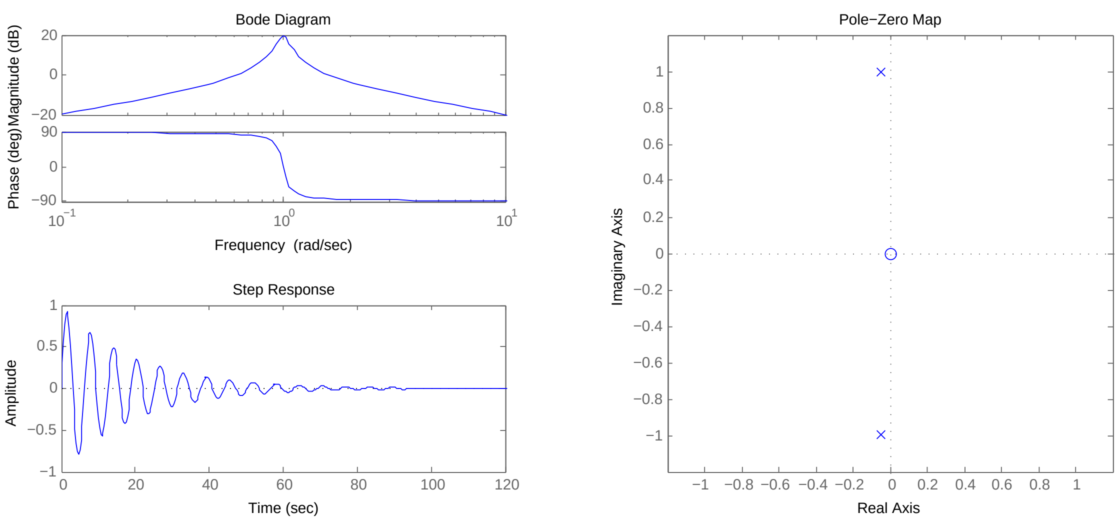
$$C.\ \frac{s}{s^2 + s + 1}$$

Solution:



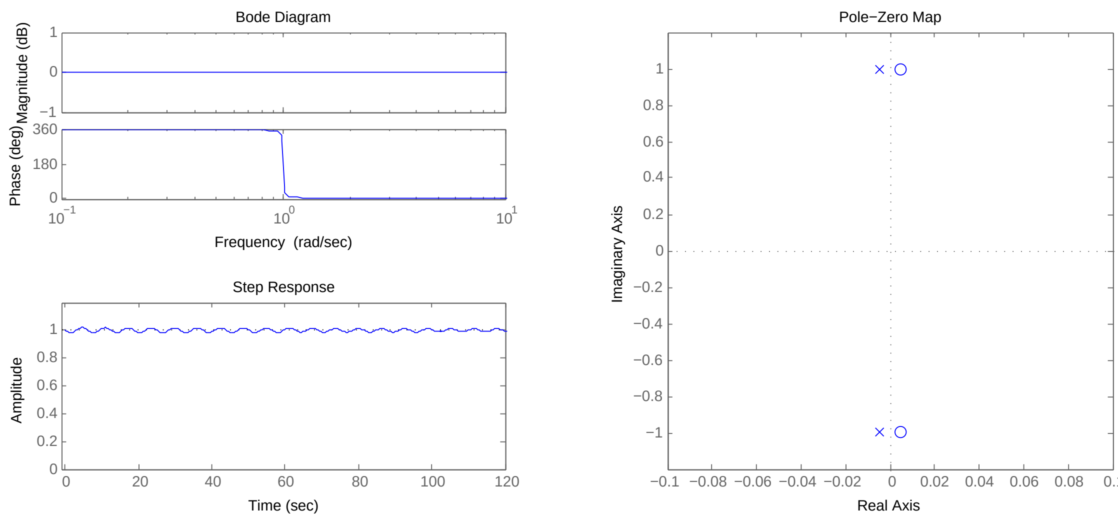
$$D.\ \frac{s}{s^2 + 0.1s + 1}$$

Solution:



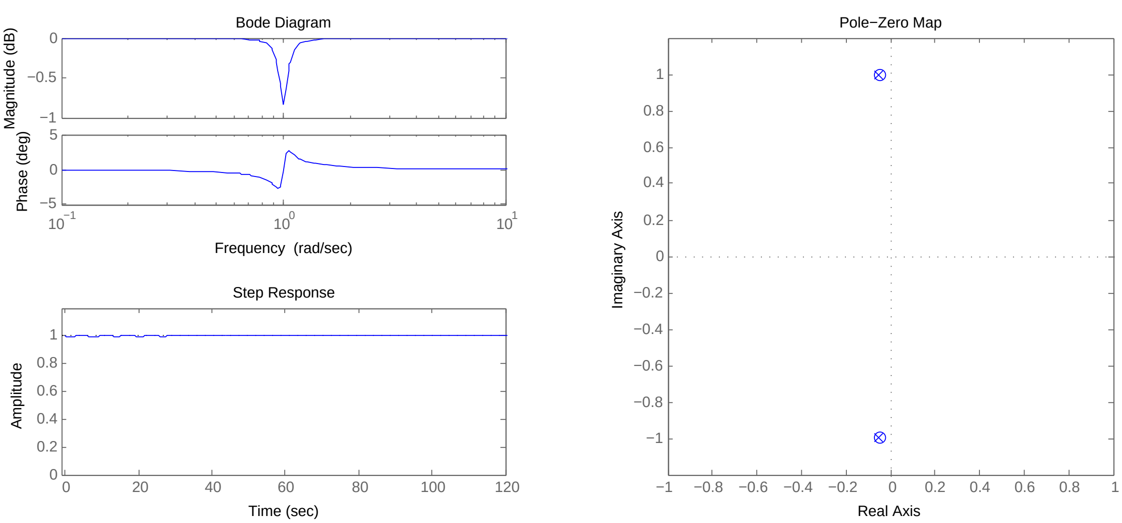
$E.\ \dfrac{s^2 - 0.01s + 1}{s^2 + 0.01s + 1}$

Solution:



$F.\ \dfrac{s^2 + 0.1s + 1}{s^2 + 0.11s + 1}$

Solution:



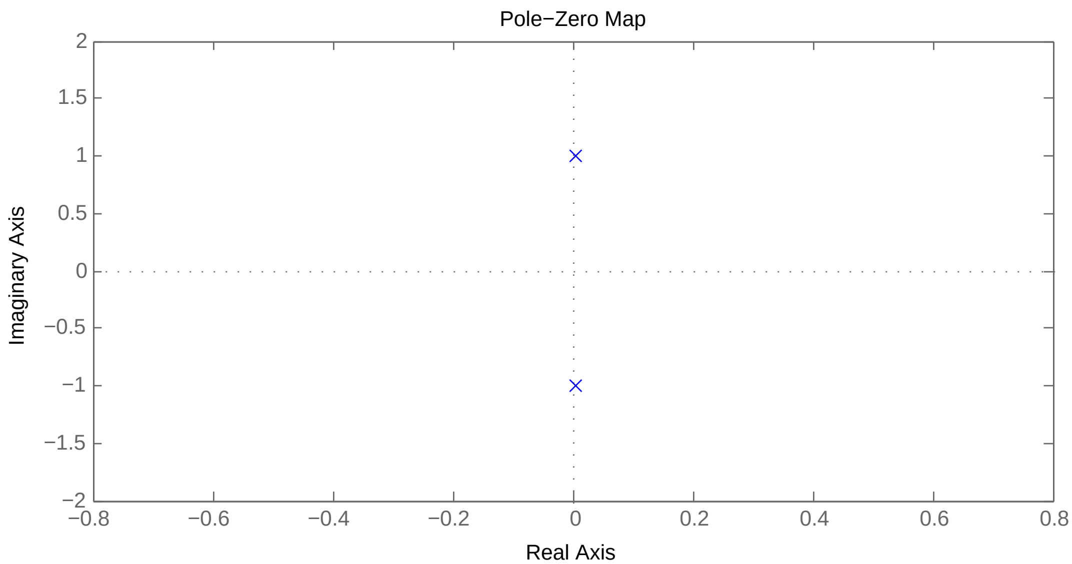
Problem 4 *(4 points)* You are asked to stabilize the system

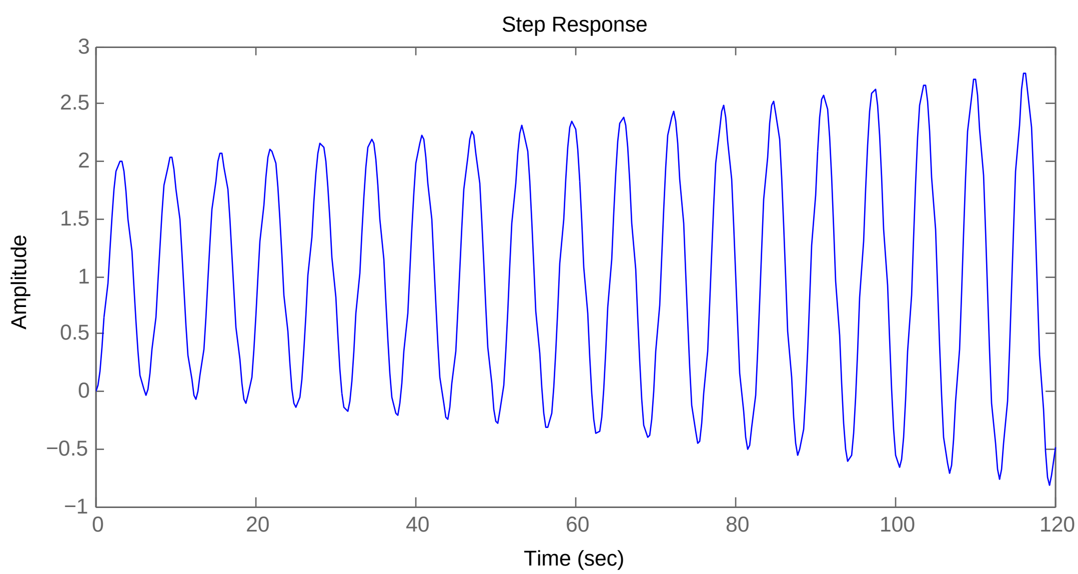
$$H(s) = \frac{1}{s^2 - 0.01s + 1}$$

Do the algebra by hand in this problem. Matlab will introduce numerical errors that will give you the wrong answer!

A. Plot the step response and pole-zero map of this system using Matlab.

Solution:

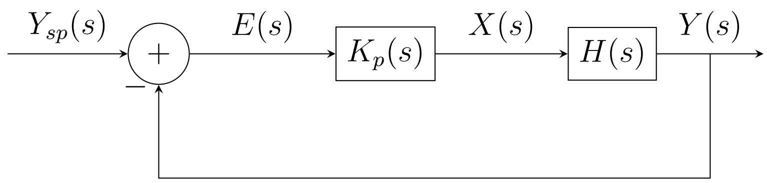




Note that the poles in the system lie in the right-hand plane of the pole-zero map–hence the system's instability.

B. Use the pole-zero map to show the effect of using proportional control on this system. Show the step response of at least two feedback gains to illustrate. Can you stabilize the system?

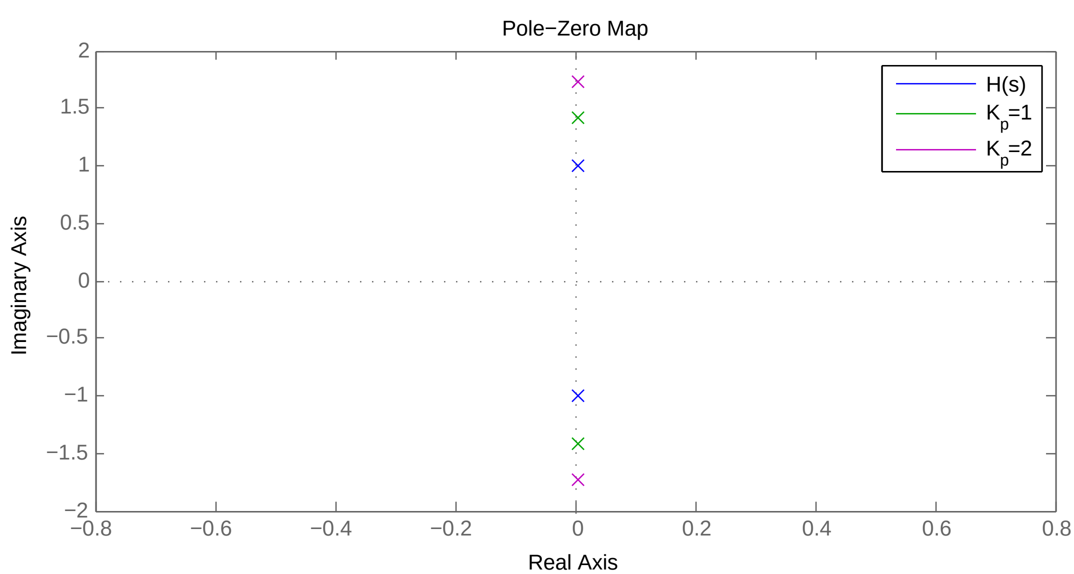
Solution:

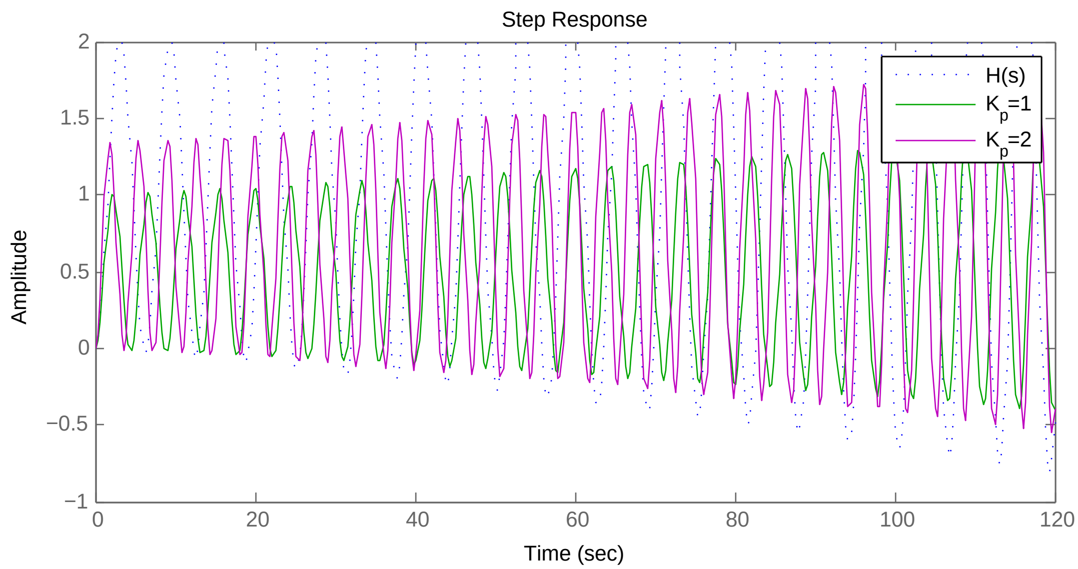


To find the transfer function of this system, we use Black's equation:

$$\begin{aligned}
Y/Y_{sp} &= \frac{KH}{1+KH} \\
K &= K_p \\
Y/Y_{sp} &= \frac{K_pH}{K_pH+1} \\
&= \frac{\frac{K_p}{s^2-0.01s+1}}{1+\frac{K_p}{s^2-0.01s+1}} \\
&= \frac{K_p}{s^2-0.01s+1+K_p}
\end{aligned}$$

Using Matlab, let's simulate a couple of values of K_p.





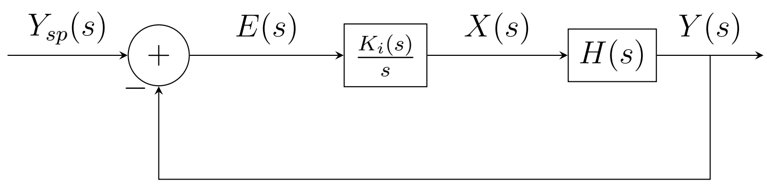
These are some samples of some possible plots. What's important to note here is that no matter how we try, we can't get those poles to go to the left of the y-axis, and therefore can never achieve stability. In fact, if we just go through with the quadratic equation, we can find the poles:

$$\begin{aligned} s_p &= \frac{0.01}{2} \pm \sqrt{\frac{0.01^2}{4} - (1+K_p)^2} \\ &\approx 0.005 \pm j|1+K_p| \end{aligned}$$

Given $K_p > 0$, $\mathrm{Re}\{s_p\} = 0.005$ And no matter what, $0.005 > 0$.

C. Repeat part B using integral control.

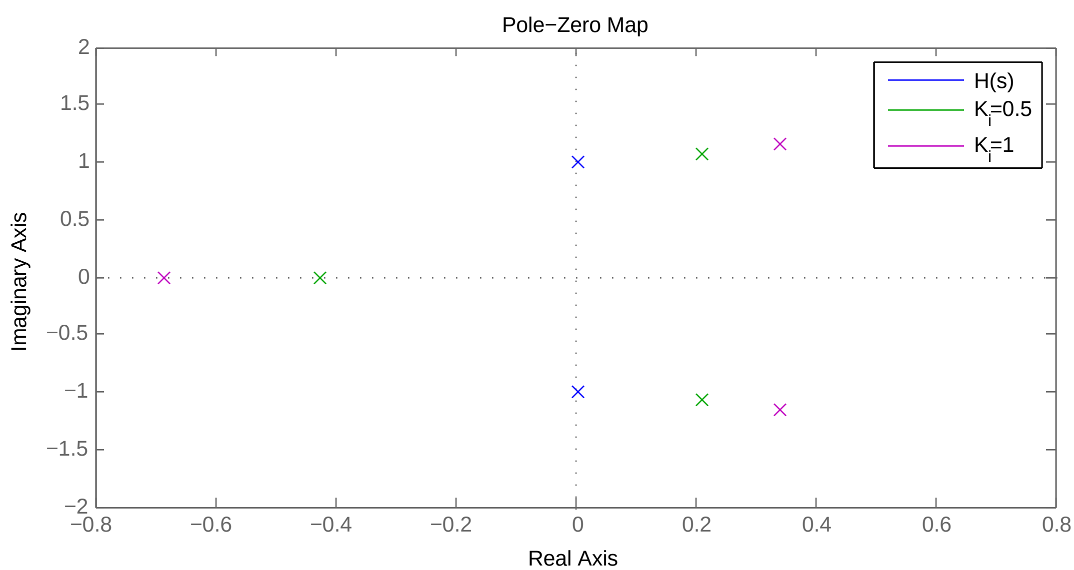
Solution:
Now, we're looking at this system:

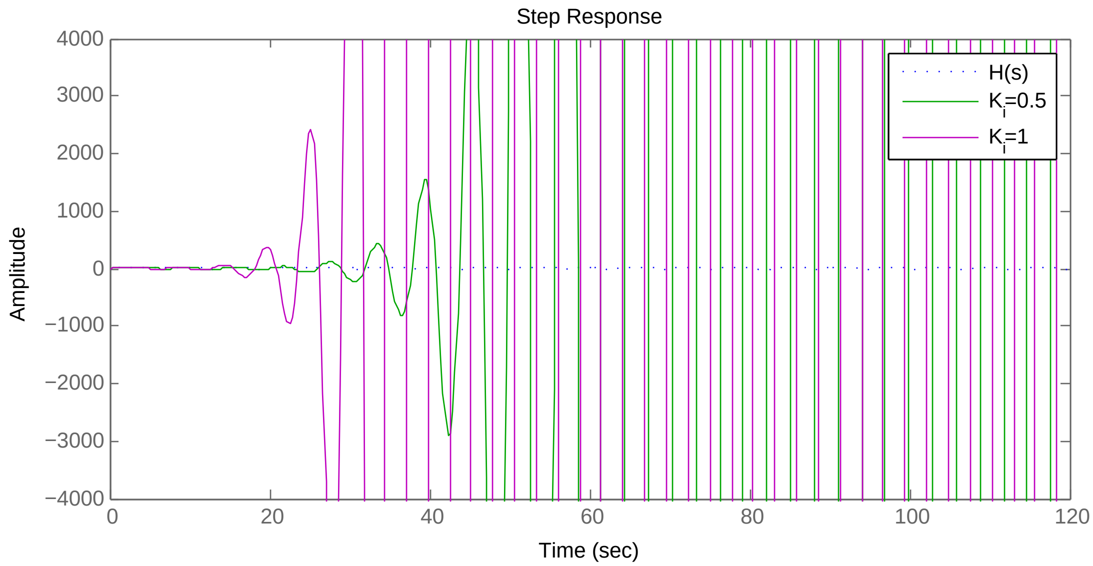


and we again find the transfer function using Black's Formula:

$$
\begin{aligned}
Y/Y_{sp} &= \frac{KH}{1+KH} \\
K &= \frac{K_i}{s} \\
Y/Y_{sp} &= \frac{\frac{HK_i}{s}}{\frac{K_iH}{s}+1} \\
&= \frac{\frac{K_i}{s(s^2-0.01s+1)}}{\frac{K_i}{s(s^2-0.01s+1)}+1} \\
&= \frac{K_i}{s^3-0.01s^2+s+K_i}
\end{aligned}
$$

Let's plot some Matlab:

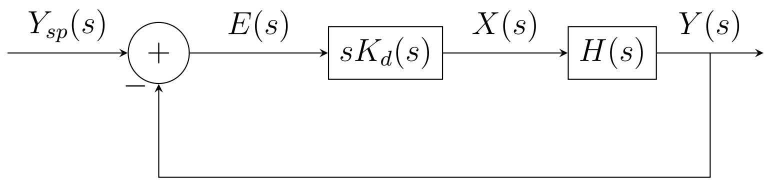




There really doesn't seem to be a way to stabilize the system, and it is much worse in this case! If you keep trying different values, you'll again notice that the poles never move to the left of the y-axis, so again, no chance for stability.

D. Repeat part B using differential (or derivative) control.

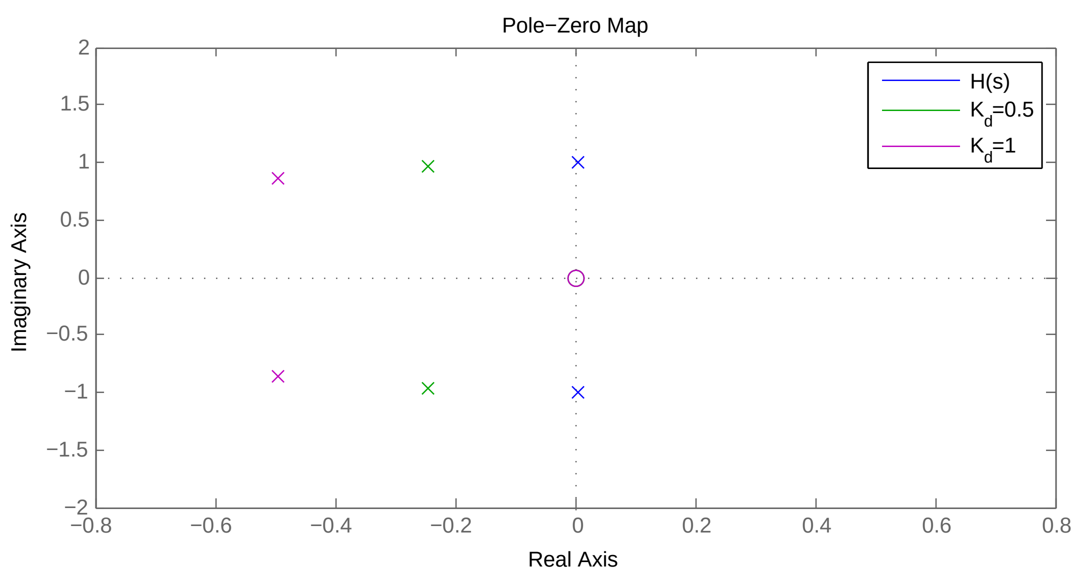
Solution:
New block diagram:

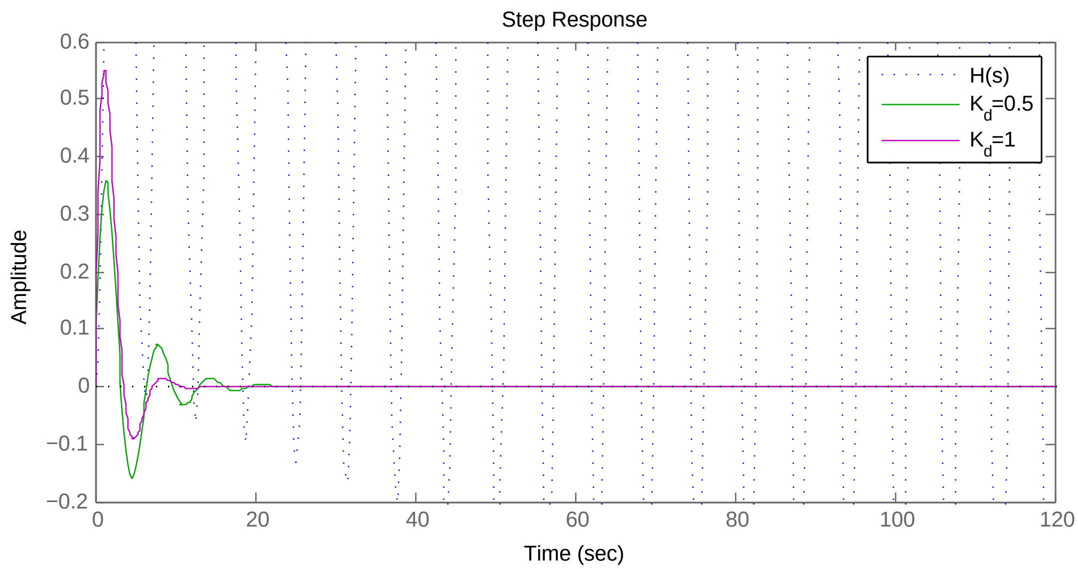


Transfer function via Black's:

$$
\begin{aligned}
Y/Y_{sp} &= \frac{KH}{1+KH} \\
K &= sK_d \\
Y/Y_{sp} &= \frac{sK_dH}{sK_dH+1} \\
&= \frac{\frac{sK_d}{s^2-0.01s+1}}{\frac{sK_d}{s^2-0.01s+1}+1} \\
&= \frac{sK_d}{s^2-0.01s+1+sK_d} \\
&= \frac{sK_d}{s^2+(K_d-0.01)s+1}
\end{aligned}
$$

Matlab fun:

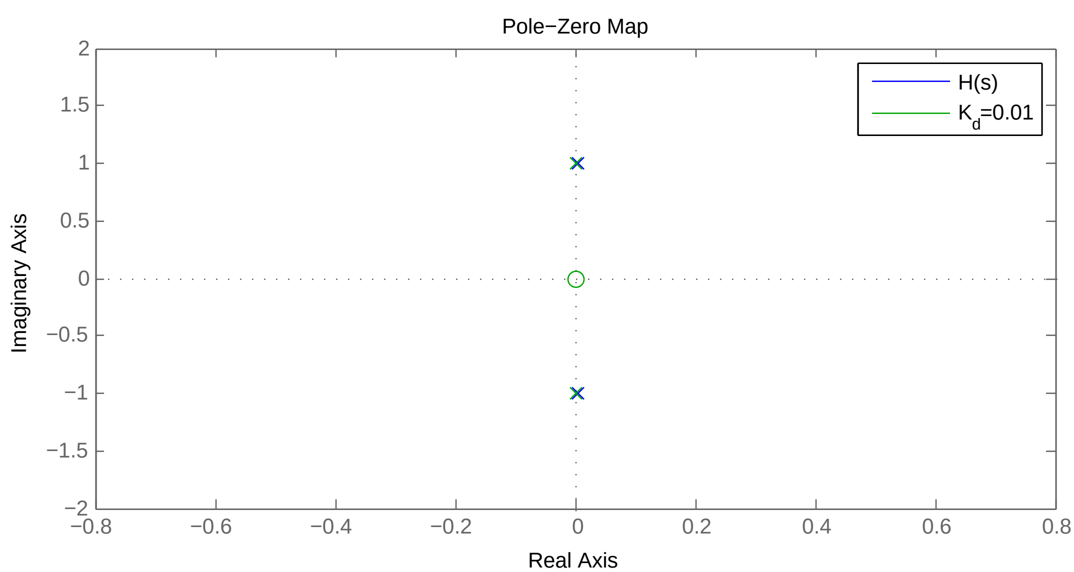


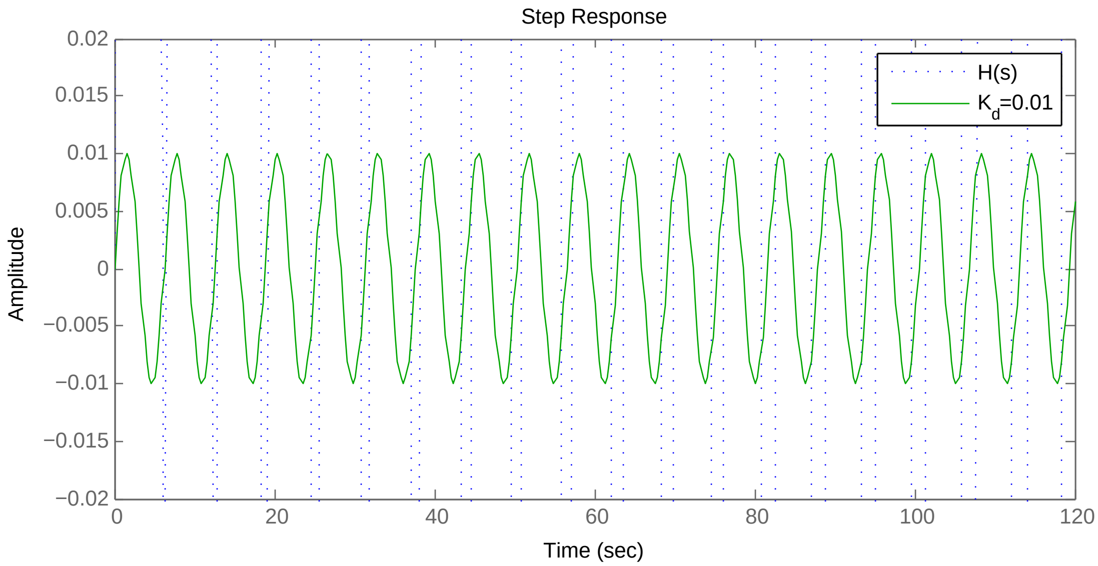


Hey, cool! In this case, we can stabilize the system! Let's do some quick quadratic fun to find the poles:

$$\begin{aligned} s_p &= \frac{0.01 - K_d}{2} \pm \sqrt{\frac{(K_d - 0.01)^2}{4} - 1^2} \\ &= \frac{0.01 - K_d}{2} \pm \sqrt{\frac{(K_d - 0.01)^2}{4} - 1} \end{aligned}$$

For $K_d > 0.01$, $\mathrm{Re}\{s_p\} < 0$, finally allowing some stability! Let's actually look at the special case where $K_d = 0.01$:





Interesting! When the poles are exactly on the y-axis, the system neither converges nor diverges! How cool!