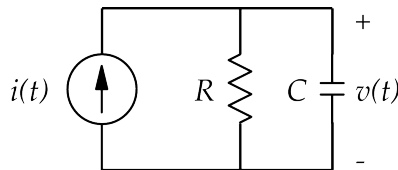


Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 5

Problem 1 (6 points) In lecture, we showed that if the input current $i(t) = Q_0\delta(t)$ in the circuit below, then the output voltage $v(t) = \frac{Q_0}{C}e^{-t/\tau}$, $\tau = RC$. Since $i(t)$ is an impulse input, $v(t)$ is the impulse response of the system.



- A. Find the impedance of the circuit $Z(j\omega) = V(j\omega)/I(j\omega)$.

Solution:

$$Z(j\omega) = \frac{1}{C} \frac{1}{(j\omega + \frac{1}{RC})}$$

- B. Find the frequency content of the impulse response $v(t)$, $V(j\omega)$, by multiplying the Fourier transform of the input $i(t)$ with the impedance $Z(j\omega)$.

Solution:

$$I(j\omega) = Q_0$$

$$V(j\omega) = \frac{Q_0}{C} \frac{1}{(j\omega + \frac{1}{RC})}$$

- C. Take the Fourier transform of the impulse response $v(t)$ using the integral definition of the Fourier transform directly. Note that you have just found another important Fourier transform pair!

Solution:

$$\begin{aligned} V(j\omega) &= \mathcal{F}\{v(t)\} = \frac{Q_0}{C} \mathcal{F}\{e^{-t/\tau}\} = \frac{Q_0}{C} \int_{t=0}^{\infty} e^{-t/\tau} e^{-j\omega t} dt = \frac{Q_0}{C} \int_{t=0}^{\infty} e^{-(1/\tau + j\omega)t} dt \\ &= \frac{Q_0}{C} \frac{-1}{(1/\tau + j\omega)} e^{-(1/\tau + j\omega)t} \Big|_{t=0}^{\infty} = \frac{Q_0}{C} \frac{-1}{(1/\tau + j\omega)} [0 - 1] = \frac{Q_0}{C} \frac{1}{(1/\tau + j\omega)}, \tau = RC \end{aligned}$$

Note that we have just found that:

$$\mathcal{F}\{e^{-t/\tau}\} = \frac{1}{1/\tau + j\omega}$$

This implies that:

$$e^{-t/\tau} = \mathcal{F}^{-1}\left\{\frac{1}{1/\tau + j\omega}\right\}$$

More generally, we write:

$$e^{-t/\tau} \xleftrightarrow{\mathcal{F}} \frac{1}{1/\tau + j\omega}$$

This transform will be useful in part A of problem 2.

- D. Compare the answers to the previous two parts. Explain the relationship between them clearly.

Solution:

We have arrived at the Fourier transform of the impulse response by taking the Fourier transform directly and indirectly by multiplying the transfer function of the system times the transform of the impulse. In system notation,

$$\begin{array}{ccccc} Q_0\delta(t) & \longrightarrow & \boxed{\text{RC ckt}} & \longrightarrow & \frac{Q_0}{C}e^{-t/\tau} \\ \mathcal{F} \downarrow & & & & \downarrow \mathcal{F} \\ Q_0 & \longrightarrow & \boxed{\frac{1}{C} \frac{1}{(j\omega + \frac{1}{RC})}} & \longrightarrow & \frac{Q_0}{C} \frac{1}{(j\omega + \frac{1}{RC})} \end{array}$$

The remaining parts should convince you that since the impulse is the derivative of the step, the impulse response must be the derivative of the step response.

- E. Find $v(t)$ if $i(t) = I_0u(t)$. Use either a differential equation, or circuit analysis and your knowledge of first-order systems.

Solution:

$$v(t) = I_0R(1 - e^{-t/\tau}) \quad \tau = RC$$

- F. Find the operator that transforms the input from part E into the input from part A. Be careful with any constants needed.

Solution:

$$Q_0\delta(t) = \underbrace{\left[\frac{Q_0}{I_0} \frac{d}{dt}\right]}_{\text{operator}} I_0u(t)$$

- G. Apply the operator you found in the previous part to the step response you found in part E and compare it to the impulse response of the circuit. Explain your result.

Solution:

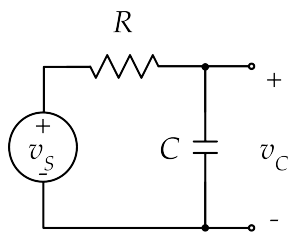
$$\underbrace{\left[\frac{Q_0}{I_0} \frac{d}{dt} \right]}_{\text{operator}} \underbrace{I_0 R (1 - e^{-t/\tau})}_{\text{step response}} = \frac{Q_0 R}{\tau} e^{-t/\tau} = \frac{Q_0 \cancel{R}}{\cancel{R} C} e^{-t/\tau} = \underbrace{\frac{Q_0}{C} e^{-t/\tau}}_{\text{impulse response}}$$

The derivative can be taken before or after finding the response of the system. In either case, we find the impulse response. In system notation,

$$\begin{array}{ccccc} I_0 u(t) & \longrightarrow & \boxed{\text{RC ckt}} & \longrightarrow & I_0 R (1 - e^{-t/\tau}) \\ \left[\frac{Q_0}{I_0} \frac{d}{dt} \right] \downarrow & & & & \downarrow \left[\frac{Q_0}{I_0} \frac{d}{dt} \right] \\ Q_0 \delta(t) & \longrightarrow & \boxed{\text{RC ckt}} & \longrightarrow & \frac{Q_0}{C} e^{-t/\tau} \end{array}$$

Problem 2 (4 points) This problem emphasizes that distributions like the impulse are mathematical idealizations. Thus, the details of the distribution are not important as long as the effect is the same. In other words, if it behaves like an impulse, it is an impulse.

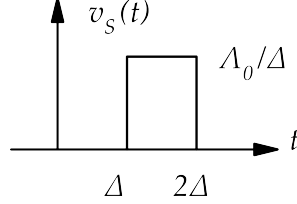
- A. Find an algebraic expression for the impulse response $v_C(t)$ if $v_S(t) = \Lambda_0\delta(t)$ using frequency analysis. Λ_0 is the product of voltage and time, V·s, and has those units (which turn out to be the units of magnetic flux for a really good reason if you're curious!)



Solution:

$$\begin{aligned}
 v_S(t) &= \Lambda_0\delta(t) \\
 v_s(j\omega) &= \mathcal{F}\{v_S(t)\} = \Lambda_0 \\
 H(j\omega) &= \frac{\frac{1}{\tau}}{\frac{1}{\tau} + j\omega}, \quad \tau = RC \\
 v_C(j\omega) &= H(j\omega)v_s(j\omega) = \frac{\frac{\Lambda_0}{\tau}}{\frac{1}{\tau} + j\omega} \\
 v_C(t) &= \mathcal{F}^{-1}\left\{\frac{\frac{\Lambda_0}{\tau}}{\frac{1}{\tau} + j\omega}\right\} = \frac{\Lambda_0}{\tau}e^{-\frac{t}{\tau}}u(t)
 \end{aligned}$$

- B. For the same circuit as in part A, assume that $v_S(t)$ is as shown below and find an algebraic expression for $v_C(t)$. Let $\Delta \rightarrow 0$. Show that $v_C(t)$ approaches the impulse response you found in the previous part. Note that $v_S(0) = 0$ for all values of Δ . Think about how to define the function it approaches. *Hint: Note that in the limit only one part of the solution matters. Also, you will find it useful to remember that $e^x \approx 1 + x$ if $x \ll 1$.*



Solution:

$$v_C(t) = \begin{cases} 0 & t < 0 \\ \frac{\Lambda_0}{\Delta}(1 - e^{-(t-\Delta)/\tau}) & \Delta < t < 2\Delta, \quad \tau = RC \\ \frac{\Lambda_0}{\Delta}(1 - e^{-\Delta/\tau})e^{-(t-2\Delta)/\tau} & t > 2\Delta \end{cases}$$

Letting $\Delta \rightarrow 0$ we can show that $v_C(t)$ approaches the impulse response we found in the previous part.

$$\lim_{\Delta \rightarrow 0} v_C(t) = \lim_{\Delta \rightarrow 0} \frac{\Lambda_0}{\Delta}(1 - e^{-\Delta/\tau})e^{-(t-2\Delta)/\tau}, \quad t > 2\Delta$$

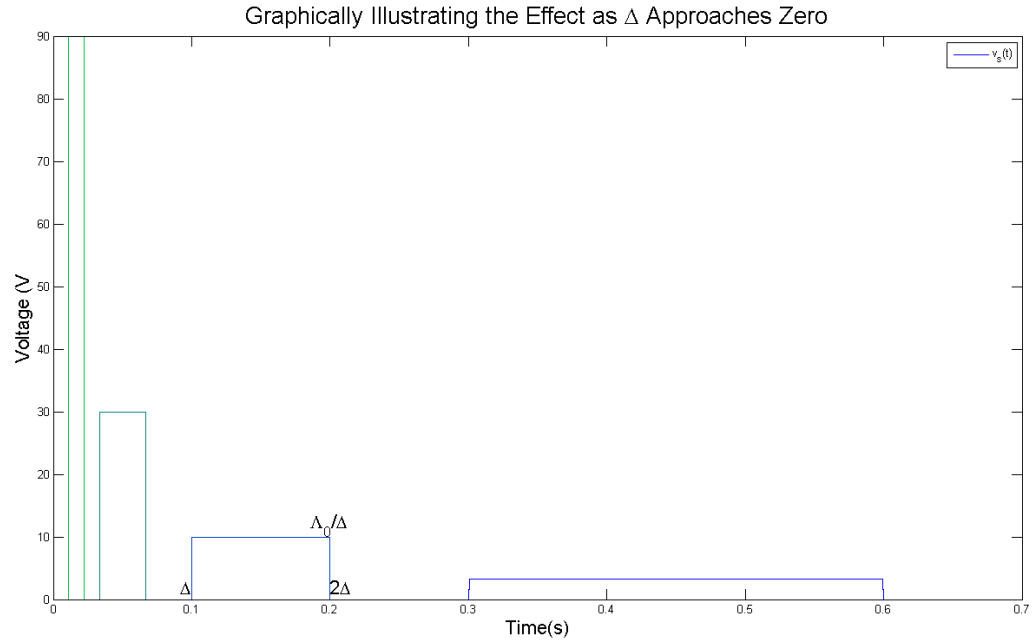
$$\lim_{\Delta \rightarrow 0} v_C(t) = \lim_{\Delta \rightarrow 0} \frac{\Lambda_0}{\Delta}(e^{2\Delta/\tau} - e^{\Delta/\tau})e^{-t/\tau}, \quad t > 2\Delta$$

$$\lim_{\Delta \rightarrow 0} v_C(t) = \lim_{\Delta \rightarrow 0} \frac{\Lambda_0}{\Delta} \left(1 + \frac{2\Delta}{\tau} - 1 - \frac{\Delta}{\tau} \right) e^{-t/\tau}, \quad t > 2\Delta$$

$$\lim_{\Delta \rightarrow 0} v_C(t) = \lim_{\Delta \rightarrow 0} \frac{\Lambda_0}{\cancel{\Delta}} \left(\frac{\cancel{\Delta}}{\tau} \right) e^{-t/\tau}, \quad t > 2\Delta$$

$$\lim_{\Delta \rightarrow 0} v_C(t) = \frac{\Lambda_0}{\tau} e^{-t/\tau}, \quad t > 0$$

We can graphically illustrate the effect as $\Delta \rightarrow 0$. The rectangle pulse becomes taller and thinner. Eventually, it will have a height of ∞ and a width of 0, like an impulse. Also, similar to an impulse, the area of the rectangle is constant, even when $\Delta \rightarrow 0$: $Area = \frac{\Lambda_0}{\Delta} \cdot \Delta = \Lambda_0$. Thus, as $\Delta \rightarrow 0$, $v_S(t)$ becomes an impulse with area Λ_0 .



Problem 3 (0 points) This *optional* problem illustrates the inverse frequency-time relationship of the Fourier transform. This relationship is reflected in quantum physics as the *Heisenberg Uncertainty Principle* because the momentum and position of a particle are Fourier transforms of each other.

A. Using the fact that

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

show that the Fourier transform of the Gaussian e^{-t^2} is the Gaussian $\sqrt{\pi}e^{-\frac{\omega^2}{4}}$. *Hint: Complete the square of the exponential and make a substitution.*

Solution:

$$\begin{aligned} \mathcal{F} \{e^{-t^2}\} &= \int_t e^{-t^2} e^{-j\omega t} dt \\ &= \int_t e^{-t^2 - j\omega t} dt \\ &= \int_t e^{-t^2 - j\omega t + \frac{\omega^2}{4}} e^{-\frac{\omega^2}{4}} dt \\ &= \int_t e^{-(t - \frac{j\omega}{2})^2} e^{-\frac{\omega^2}{4}} dt \end{aligned}$$

$$\text{Let } u = t - \frac{j\omega}{2}, \quad du = dt$$

$$\begin{aligned} &= e^{-\frac{\omega^2}{4}} \int_t e^{-u^2} du \\ &= \sqrt{\pi} e^{-\frac{\omega^2}{4}} \end{aligned}$$

- B. Show the scaling property of the Fourier Transform, $\mathcal{F}\{x(at)\} = \frac{1}{|a|}X(\frac{j\omega}{a})$.

Solution:

$$\begin{aligned}
 & \text{if } a > 0, \\
 & \mathcal{F}^{-1}\left\{\frac{1}{|a|}X\left(\frac{j\omega}{a}\right)\right\} = \frac{1}{|a|} \int_{\omega} X\left(\frac{j\omega}{a}\right) e^{j\omega t} d\omega \\
 & \text{Let } u = \frac{\omega}{a}, \quad d\omega = a du \\
 & \mathcal{F}^{-1}\left\{\frac{1}{|a|}X\left(\frac{j\omega}{a}\right)\right\} = \frac{a}{|a|} \int_u X(ju) e^{ju(at)} du \\
 & \quad \quad \quad = x(at) \\
 & \text{if } a < 0, \\
 & \mathcal{F}^{-1}\left\{\frac{1}{|a|}X\left(\frac{j\omega}{a}\right)\right\} = \frac{-|a|}{|a|} \int_{-\infty}^{\infty} X(ju) e^{ju(at)} du \\
 & \quad \quad \quad = \frac{|a|}{|a|} \int_{\infty}^{-\infty} X(ju) e^{ju(at)} du \\
 & \quad \quad \quad = x(at)
 \end{aligned}$$

- C. Use the scaling property to find the Fourier Transform of $e^{-\frac{t^2}{\sigma^2}}$.

Solution:

$$\begin{aligned}
 \mathcal{F}\{x(at)\} &= \frac{1}{|a|}X\left(\frac{j\omega}{a}\right) \\
 \mathcal{F}\{e^{-t^2}\} &= \sqrt{\pi}e^{-\frac{\omega^2}{4}} \\
 \mathcal{F}\left\{e^{-\frac{t^2}{\sigma^2}}\right\} &= |\sigma|\sqrt{\pi}e^{-\frac{\omega^2\sigma^2}{4}}
 \end{aligned}$$

- D. Define the width of the Gaussian as the time or frequency where the function reaches $\frac{1}{e}$ of its peak value. Show that the width of the frequency Gaussian is inversely proportional to the width of the time Gaussian.

Solution:

The peak value of the function, $e^{-\frac{t^2}{\sigma^2}}$, occurs at $t = 0$.

$$\begin{aligned}
 e^{-\frac{t_w^2}{\sigma^2}} &= \frac{1}{e} \\
 -\frac{t_w^2}{\sigma^2} &= -1 \\
 t_w^2 &= \sigma^2 \\
 t_w &= \pm\sigma
 \end{aligned}$$

The peak value of the function, $|\sigma|\sqrt{\pi}e^{-\frac{\omega^2\sigma^2}{4}}$, occurs at $\omega = 0$.

$$\begin{aligned} |\sigma|\sqrt{\pi}e^{-\frac{\omega_w^2\sigma^2}{4}} &= |\sigma|\sqrt{\pi} \cdot \frac{1}{e} \\ e^{-\frac{\omega_w^2\sigma^2}{4}} &= \frac{1}{e} \\ \frac{\omega_w^2\sigma^2}{4} &= 1 \\ \omega_w^2 &= \frac{4}{\sigma^2} \\ \omega_w &= \pm \frac{2}{\sigma} \end{aligned}$$

Therefore, $|t_w| \cdot |\omega_w| = 2$. Thus, the width of the frequency Gaussian is inversely proportional to the width of the time Gaussian.

Cultural note: In quantum physics, quantities such as position and momentum are described by random variables. The quantity σ^2 in this problem is equivalent to the variance of one these random variables, say position. Since the momentum is the Fourier transform of the position, the variance of the momentum is proportional to $1/\sigma^2$. Using any other probability distribution instead of Gaussians increases the product of the variances. That is, the uncertainty product is minimized when the particle position and momentum are described by Gaussians.