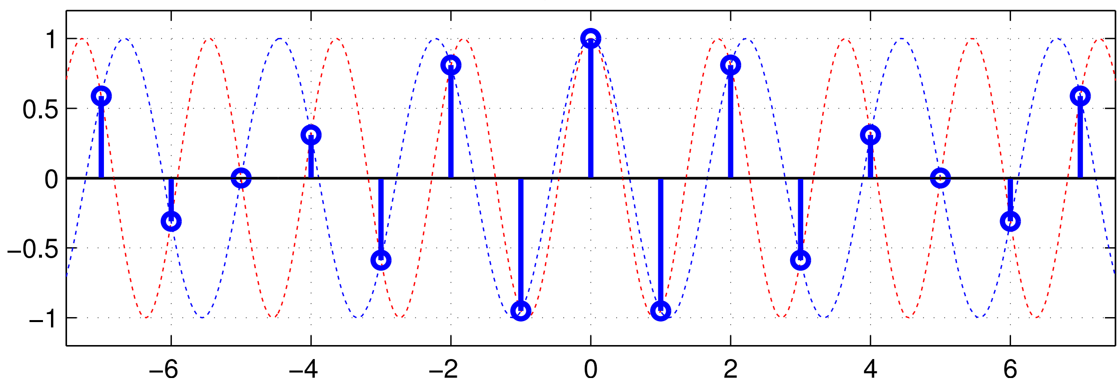
Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 8

Problem 1 *(2 points)*

A. Verify that $\cos(1.1\pi n)$ aliases to $\cos(0.9\pi n)$ by creating a plot of $\cos(1.1\pi t)$ and $\cos(0.9\pi t)$ for $-7 \leq t \leq 7$ and then plotting $\cos(1.1\pi n)$ for $n \in \{-7, -6, ...7\}$ on the same axes. Explain clearly this plot.

Solution:
Since $\cos(1.1\pi n)$ aliases to $\cos(0.9\pi n)$, the underlying graphs of $\cos(1.1\pi t)$ and $\cos(0.9\pi t)$ intersect whenever t is equal to an integer n.

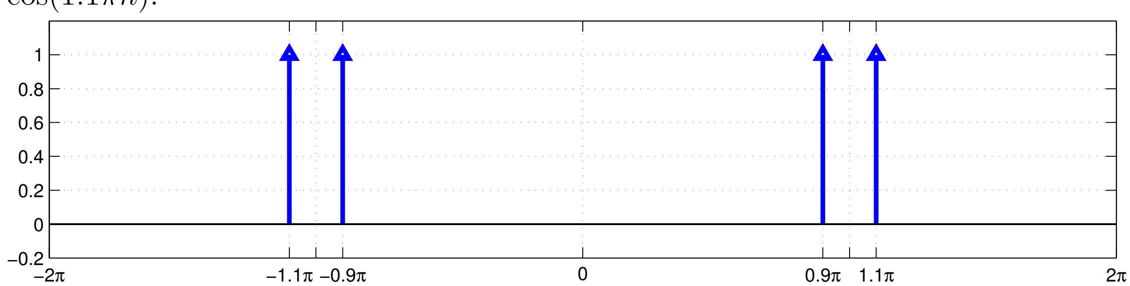


B. Find the transform of $\cos(\Omega_0 n)$.

Solution:
$\cos(\Omega_0 n) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \sum_k \pi\delta(\Omega - \Omega_0 - 2\pi k) + \pi\delta(\Omega + \Omega_0 - 2\pi k), \quad k \in \mathbb{Z}$

C. Sketch the transforms of $\cos(0.9\pi n)$ and $\cos(1.1\pi n)$ from -2π to 2π. Explain clearly.

Solution:
Both transforms look identical, since the impulses at $\pm 0.9\pi$ show up at $\pm 1.1\pi$ in the case of $\cos(0.9\pi n)$, and the impulses at $\pm 1.1\pi$ alias down $\pm 0.9\pi$ in the case of $\cos(1.1\pi n)$.

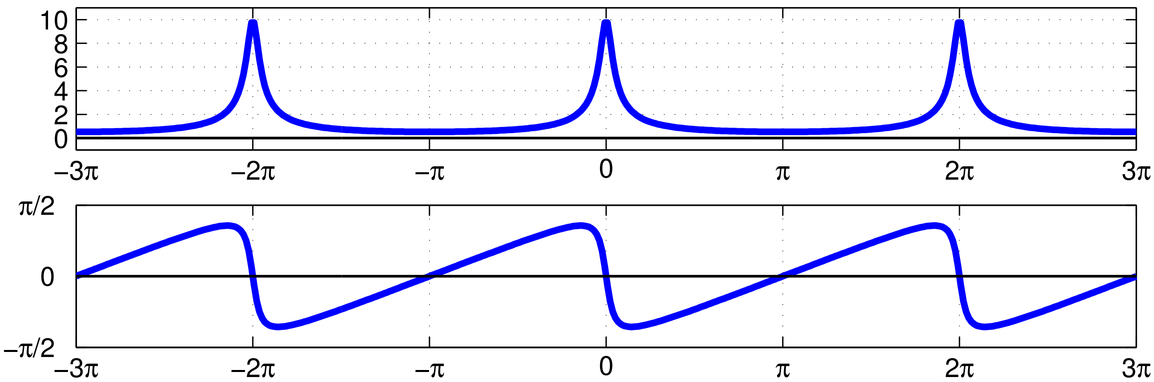


Problem 2 *(3 points)* The transfer function of the first-order difference equation

$$y[n] - ay[n-1] = x[n] \qquad \text{is} \qquad H(\Omega) = \frac{1}{1 - ae^{-j\Omega}}.$$

A. Plot the magnitude and phase of $H(\Omega)$ from -3π to 3π when $a = 0.9$.

Solution:

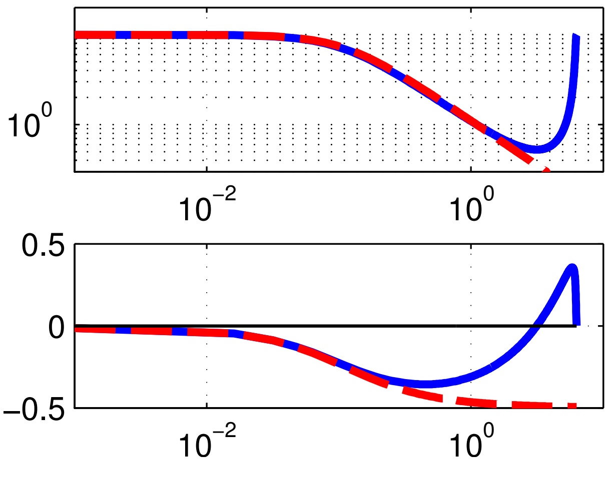


B. Since $e^x \approx 1 + x$ when $x \ll 1$,

$$H_{approx}(\Omega) = \frac{\frac{1}{a}}{\frac{1-a}{a} + j\Omega} \approx H(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

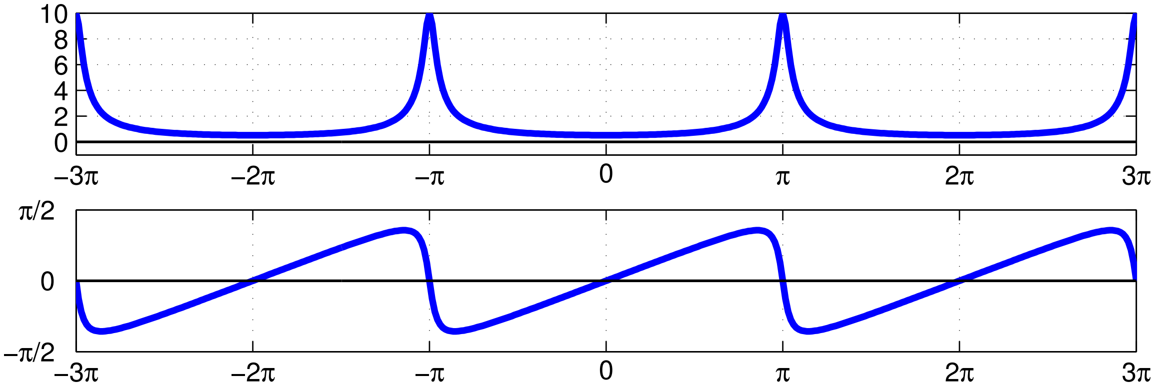
when $\Omega \approx 2\pi n$. Make a Bode plot of both $H(\Omega)$ and $H_{approx}(\Omega)$ when $a = 0.9$ and $10^{-3} < \Omega < 2\pi$. What kind of filter is this?

Solution:
The filter passes frequencies around 0, so it is a low pass filter.



C. Redo the part A when $a = -0.9$. What kind of filter is this? Explain clearly.

Solution:
The filter passes frequencies around π. This is the highest frequency a discrete system can have (any higher frequencies will be aliased down, as shown in Problem 1), so it is a high pass filter.



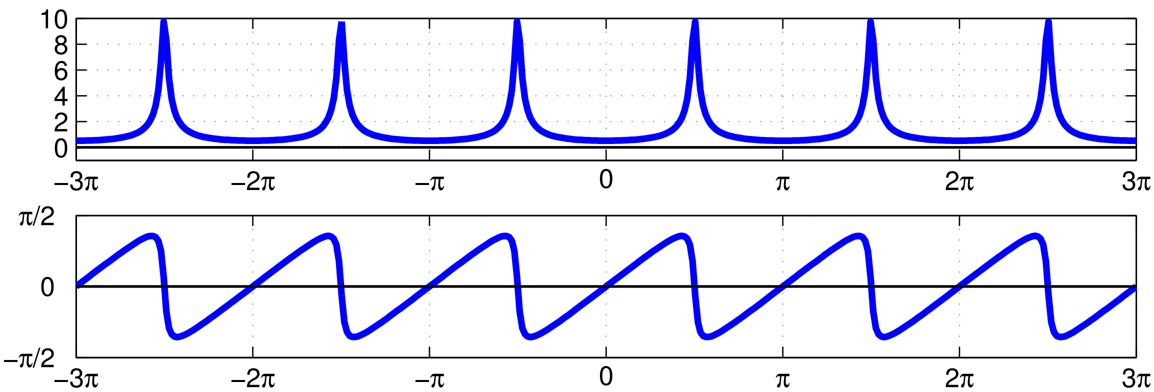
D. Find the transfer function for the difference equation below and plot it as in part A. What kind of filter is this? Explain clearly.

$$y[n] + 0.9y[n-2] = x[n]$$

Solution:

$$H(\Omega) = \frac{1}{1 + 0.9e^{-2j\Omega}}$$

The filter passes frequencies around $\pi/2$ and rejects both frequencies around 0 and π. $\pi/2$ is the center frequency between the lowest discrete frequency (0) and the highest (π), so it is a band pass filter.

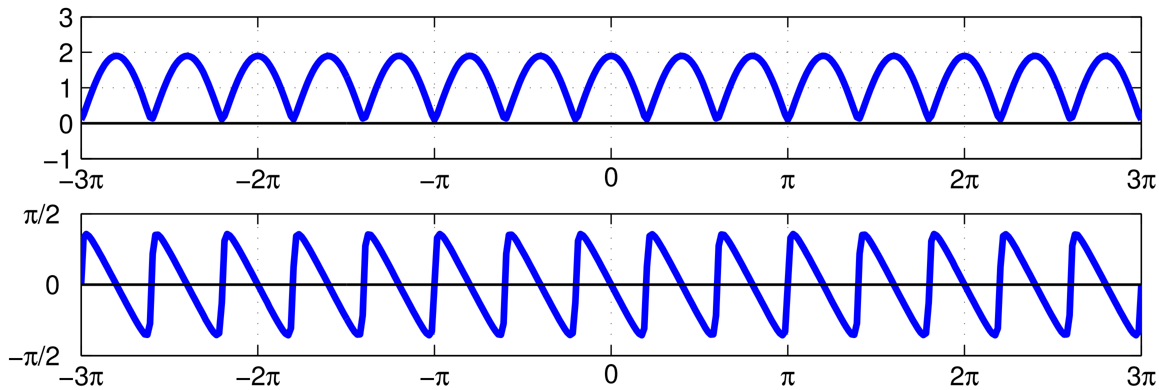


E. Find the transfer function for the difference equation below and plot it as in part A. This is called a *comb filter*.

$$y[n] = x[n] - 0.9x[n-5]$$

Solution:

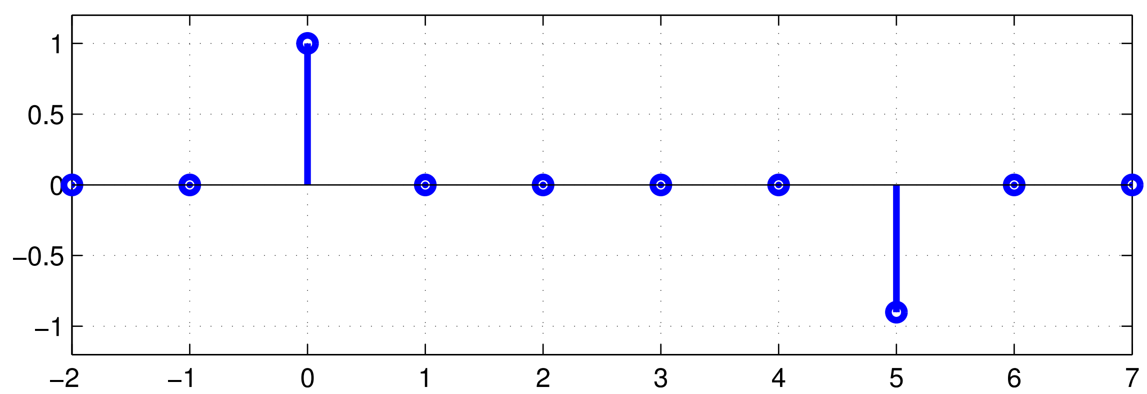
$$H(\Omega) = 1 - 0.9e^{-5j\Omega}$$



F. Find and sketch the impulse response for the comb filter of part E. *This type of filter is a* finite impulse response *(FIR) filter. The first order difference equation of part A is an* infinite impulse response *(IIR) filter.*

Solution:

$$h[n] = \mathscr{F}^{-1}\{H(\Omega)\} = \mathscr{F}^{-1}\{1 - 0.9e^{-5j\Omega}\} = \delta[n] - 0.9\delta[n-5]$$

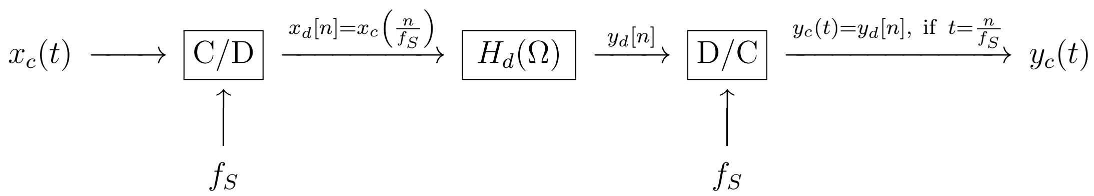


Problem 3 *(5 points)*

A. Find and sketch the transfer function $H_c(j\omega)$ such that

$$y_c(t) = x_c\left(t - \frac{1}{3f_S}\right)$$

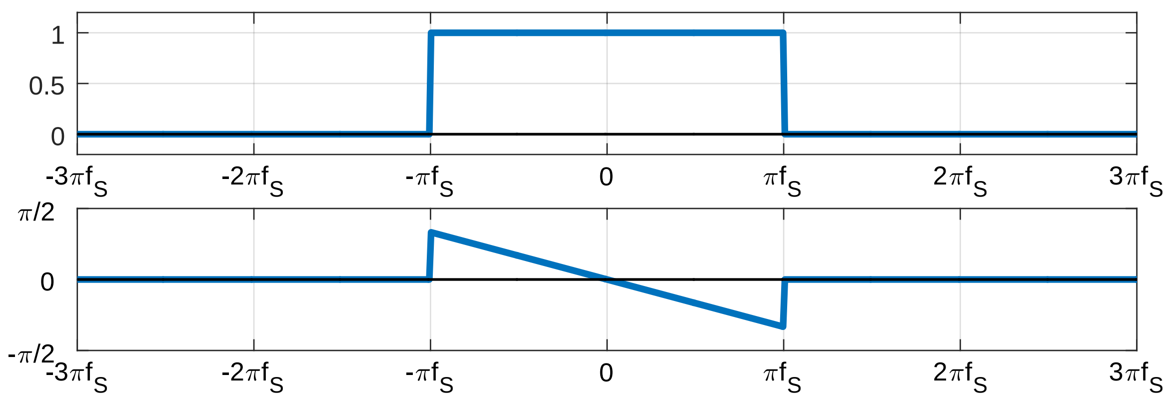
in the system shown below, assuming $x_c(t)$ is bandlimited by f_{max} such that the sampling frequency $f_S > 2f_{max}$.



Solution:

$$Y_c(j\omega) = X_c(j\omega)e^{-j\frac{\omega}{3f_S}}$$

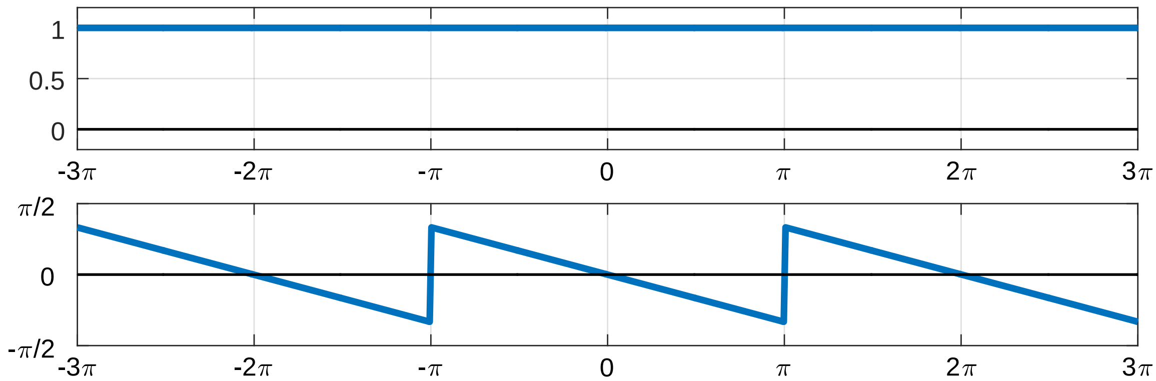
$$H_c(j\omega) = \begin{cases} e^{-j\frac{\omega}{3f_S}} & -2\pi\frac{f_S}{2} \leq \omega \leq 2\pi\frac{f_S}{2} \\ 0 & \text{otherwise} \end{cases}$$



B. Find and sketch $H_d(\Omega)$.

Solution:

$$H_d(\Omega) = e^{-j\frac{\Omega - 2\pi k}{3}}, \quad -\pi + 2\pi k \le \Omega \le \pi + 2\pi k, \quad k \in \mathbb{Z}$$



C. Find the naive expression for $y_d[n]$ in terms of $x_d[n]$ by transforming $H_d(\Omega)$. Note that while your result is technically true, it cannot be applied literally! The next two parts give us the actual answer.

Solution:

$$Y_d(\Omega) = X_d(\Omega)e^{-j\frac{\Omega}{3}}$$

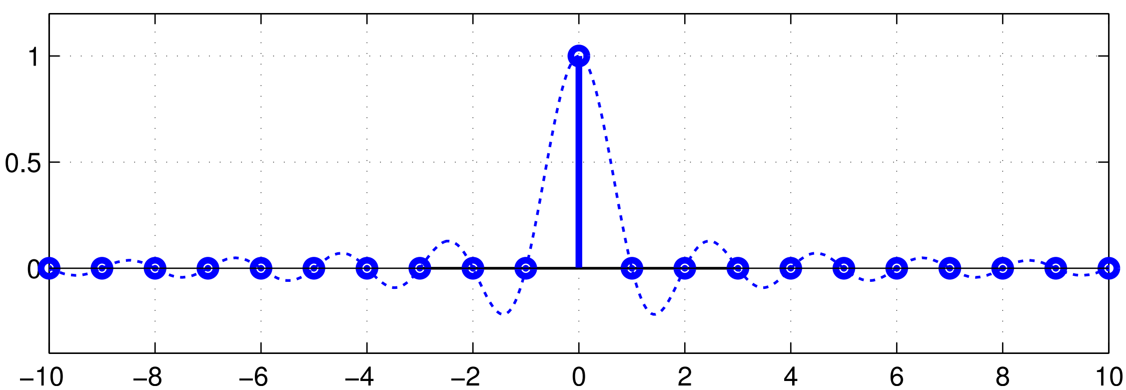
$$y_d[n] = x_d[n - 1/3]$$

Non-sensical, since n must be an integer!

D. Assume $x_c(t) = \text{sinc}(\pi f_s t)$. Verify that $x_d[n] = \delta[n]$. Combine both plots in the same set of axes.

Solution:

$$x_d[n] = x_c(n/f_S) = \text{sinc}(\pi f_S n/f_S) = \text{sinc}(\pi n) = \delta[n]$$



E. Find $y_c(t)$ and $y_d[n]$. Explain why $y_d[n] = h_d[n]$. Combine both plots in the same set of axes.

Solution:
Since $H_c(j\omega)$ is a delay of $\frac{1}{3f_S}$, then $y_c(t) = \mathrm{sinc}\left[\pi f_S \left(t - \frac{1}{3f_S}\right)\right]$.
We can invoke again that $y_d[n] = y_c(n/f_S)$ to find $y_d[n]$:

$$y_d[n] = \mathrm{sinc}\left[\pi\left(n - 1/3\right)\right]$$

Regardless of the definition of $x_c(t)$, since $x_d[n] = \delta[n]$, y_d must be the impulse response $h_d[n]$.