Olin College of Engineering
ENGR2410 – Signals and Systems

Quiz 5

Instructions

A. Collaboration is not allowed on quizzes.

B. Students may only use a page of notes and the tables from the website during the quizzes.

C. Time is limited to one continuous hour.

D. Quizzes are due at the beginning of lecture on Thursday.

E. Late or missed quizzes will be given a score of zero. Any excuses must come directly from the Office of Student Life.

F. The two lowest quiz scores will be eliminated to allow for unforeseeable circumstances.

G. In case of doubt, students are expected to base their behavior on the values expressed in the Honor Code.

Name:

Start time:

Problem 1 *(10 points)*

A. Show the time shift property of the Fourier transform, $\mathscr{F}\{x(t+T)\} = X(j\omega)e^{j\omega T}$ where $\mathscr{F}\{x(t)\} = X(j\omega)$ denotes that $X(j\omega)$ is the Fourier transform of $x(t)$. *Hint: Use the substitution* $t' = t + T$.

Solution:
We know $\mathscr{F}\{x(t)\} = X(j\omega) = \int_{t=-\infty}^{t=\infty} x(t)e^{-j\omega t}dt$. So let's try to write $\mathscr{F}\{x(t+T)\}$ in a form we recognize.

$$\begin{aligned} \mathscr{F}\{x(t+T)\} &= \int_{t=-\infty}^{t=\infty} x(t+T)e^{-j\omega t}dt \\ \text{Let } t' &= t+T \\ t &= t'-T, \qquad dt = dt' + 0 = dt' \end{aligned}$$

If t goes from $-\infty$ to ∞ in the integral before the substitution then $t' - T$ must also go from $-\infty$ to ∞.

$$\begin{aligned} \mathscr{F}\{x(t+T)\} &= \int_{t'=-\infty}^{t'=\infty} x(t')e^{-j\omega(t'-T)}dt' \\ &= \int_{-\infty}^{\infty} x(t')e^{-j\omega t'}e^{j\omega T}dt' \end{aligned}$$

Since the term $e^{j\omega T}$ is independent of t' we can bring it outside the integral.

$$\begin{aligned} \mathscr{F}\{x(t+T)\} &= e^{j\omega T}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j\omega t'}dt'}_{X(j\omega)} \\ &= e^{j\omega T}X(j\omega) \end{aligned}$$

B. A periodic function $x(t)$ has the property $x(t) = x(t+T)$. Show that the Fourier transform of this equation implies that the Fourier transform of $x(t)$ can only have non-zero frequency components at $\omega = \frac{2\pi k}{T}$, where k is any integer. *This closes the loop back to the Fourier series by showing that the frequency content of a period T function only exists in the harmonics of $2\pi/T$.*

Solution:

$$\begin{aligned} x(t) &= x(t+T) \\ \mathscr{F}\{x(t)\} &= \mathscr{F}\{x(t+T)\} \\ X(j\omega) &= X(j\omega)\underbrace{e^{j\omega T}}_{\text{delay}} \\ 1 &= e^{j\omega T} \\ \therefore \omega T &= 2\pi k;\ k \in \mathbb{Z} \\ \omega &= \frac{2\pi k}{T} \end{aligned}$$

Thus, $X(j\omega)$ can only be non-zero when ω is a multiple of $\frac{2\pi}{T}$.

C. Show that

$$\cos(\omega_0 t) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad 2\pi\left[\frac{1}{2}\delta(\omega-\omega_0)+\frac{1}{2}\delta(\omega+\omega_0)\right]$$

Hint: Recall that $\cos(\omega t) = \frac{e^{j\omega t}+e^{-j\omega t}}{2}$ *and the "picking" property of the impulse,* $\int_{-\infty}^{\infty}\delta(x-x_0)f(x)dx = f(x_0)$.

Solution:
Since the Fourier transform is unique, we only need to show it in one direction. In particular,

$$\mathscr{F}\{\cos(\omega_0 t)\} = \int_{t=-\infty}^{t=\infty}\cos(\omega_0 t)e^{-j\omega t}dt$$

requires integration by parts. Rewriting this integral as

$$\mathscr{F}\{\cos(\omega_0 t)\} = \frac{1}{2}\int_{t=-\infty}^{t=\infty}e^{j\omega_0 t}e^{-j\omega t}dt + \frac{1}{2}\int_{t=-\infty}^{t=\infty}e^{-j\omega_0 t}e^{-j\omega t}dt$$

makes it easier. However, computing the inverse Fourier transform is simpler:

$$\begin{aligned}
&\mathscr{F}^{-1}\left\{2\pi\left[\frac{1}{2}\delta(\omega-\omega_0)+\frac{1}{2}\delta(\omega+\omega_0)\right]\right\}\\
&= \cancel{2\pi}\frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}\frac{d\omega}{\cancel{2\pi}} + \cancel{2\pi}\frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega+\omega_0)e^{j\omega t}\frac{d\omega}{\cancel{2\pi}}\\
&= \frac{1}{2}e^{j\omega_0 t}+\frac{1}{2}e^{-j\omega_0 t}\\
&= \cos(\omega_0 t)
\end{aligned}$$

Course feedback
Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the quiz take you?

B. Was the quiz a fair measure of your understanding?

C. Was the assignment effective preparation for the quiz?

D. Is the Monday session effective?

E. Are the connections between lecture, assignment and quiz clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?

Assignment grades

Date:

Assignment number:

Group member 1:

Grade:

Group member 2:

Grade:

Group member 3:

Grade: