

# Olin College of Engineering

## ENGR2410 – Signals and Systems

### Reference 1

#### Change of Basis and Diagonalization

The system

$$\begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} = \begin{bmatrix} 3x_{in} + y_{in} \\ x_{in} + 3y_{in} \end{bmatrix}$$

takes the  $[x_{in} \ y_{in}]^T$  vector<sup>1</sup> as input and outputs the vector  $[x_{out} \ y_{out}]^T$ ,

$$\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}} \rightarrow \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix}.$$

The eigenvectors and eigenvalues for this matrix are

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can express any  $[x_{in} \ y_{in}]^T$  as a linear combination of these eigenvectors so that

$$\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Note that the eigenvectors are the columns of the matrix.

If we express  $[x_{in} \ y_{in}]^T$  as a linear combination of the eigenvectors, finding the output becomes easier since

$$\begin{aligned} \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix} &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2 \right) \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_1 + \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2 \\ &= 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_1 + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2. \end{aligned}$$

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<sup>1</sup> $v^T$  is the transpose of vector  $v$ .

Using more compact matrix notation,

$$\begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

However, in order to use this equation, we have to transform the input from  $[x_{in} \ y_{in}]^T$  into  $[a_1 \ a_2]^T$ ,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}.$$

Combining these last two equations,

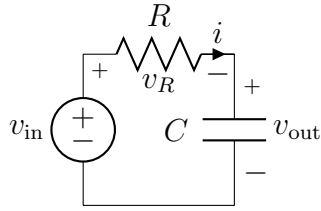
$$\begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}.$$

Using the same system notation as before,

$$\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}_{\text{transform input into eigenvector space}} \rightarrow \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{inverse transform}} \rightarrow \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix}.$$

### RC circuit

We can express the behavior of the RC circuit shown below as a differential equation in terms of  $v_{out}$ :



$$v_{in} = v_R + v_{out}$$

$$v_{in} = Ri + v_{out}$$

$$v_{in} = RC\dot{v}_{out} + v_{out}$$

$$\dot{v}_{out} + \frac{1}{RC}v_{out} = \frac{1}{RC}v_{in} \quad (\text{standard form})$$

Consider an input signal that is constant for  $t > 0$ ,

$$v_{in}(t) = V, t > 0$$

The *particular solution* solves the full differential equation:

$$\dot{v}_{out,p} + \frac{1}{RC}v_{out,p} = \frac{1}{RC}V, t > 0$$

Because  $v_{in}$  is constant for  $t > 0$ , it is reasonable to guess the constant function  $K$  for  $v_{out,p}$

$$v_{out,p}(t) = K$$

$$\dot{v}_{out,p}(t) = 0$$

Substituting,

$$0 + \frac{1}{RC}K = \frac{1}{RC}V$$

$$\Rightarrow K = V$$

$$\Rightarrow v_{out,p}(t) = V$$

This output satisfies the differential equation, but we have to add *homogeneous solutions* that solve the homogeneous equation:

$$\dot{v}_{out,h} + \frac{1}{RC}v_{out,h} = 0$$

If these solutions exist, then  $v_{out} = v_{out,h} + v_{out,p}$  is the *general solution* for the original equation.

Solving the homogeneous equation,

$$\begin{aligned}\frac{dv_{out,h}}{v_{out,h}} &= -\frac{1}{RC}dt \\ \ln v_{out,h} &= -\frac{t}{RC} + A' \\ v_{out,h} &= e^{A'} e^{-t/RC} \\ v_{out,h} &= Ae^{-t/RC}\end{aligned}$$

The general solution is

$$\begin{aligned}v_{out} &= v_{out,p} + v_{out,h} \\ &= V + Ae^{-t/RC}\end{aligned}$$

This is certainly a solution to the original equation, but it still contains the arbitrary constant  $A$ . We must also consider the initial conditions.

If we assume that the system was at rest for  $t < 0$ , then  $v_{out}(0) = 0$ . In this case,

$$v_{out}(0) = V + A = 0 \Rightarrow A = -V$$

so that the final solution is

$$v_{out}(t) = V(1 - e^{-t/RC}), t > 0$$

In system notation,

$$Vu(t) \rightarrow \boxed{\text{RC circuit}} \rightarrow V(1 - e^{-t/RC})u(t)$$

where  $u(t)$  is the unit step function:

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

The output is the *step response* of the system.