Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 6

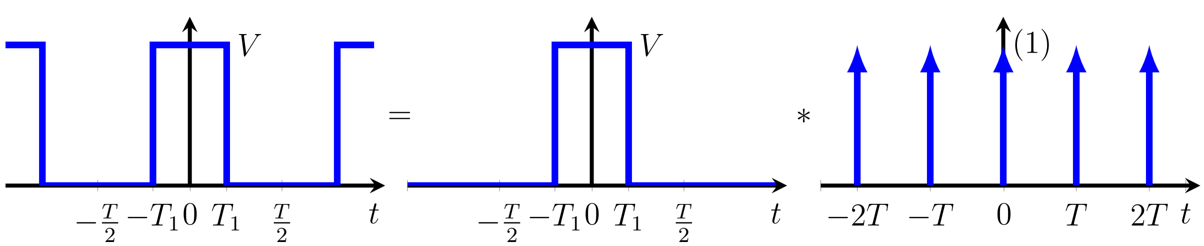
Problem 1 You have shown that a *periodic* function can be expressed using an infinite, but *discrete* set of complex exponentials. We found that in order to express a *non-periodic* function, we need an infinitely dense (i.e., *continuous*) set of complex exponentials. In this problem, you will find the continuous time Fourier transform of periodic functions in order to close the loop and verify that the integral simplifies to a summation!

A. Find an expression for the continuous time Fourier transform of a pulse train, $\mathscr{F}\{v(t)\}$, where

$$v(t) = \begin{cases} V & -T_1 + nT < t < T_1 + nT, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

This is the same function we explored in Reference 4. *Hint: In Reference 6, we showed that convolution with a shifted impulse creates a copy of the original signal shifted by that amount. Use this to express $v(t)$ as the convolution of a pulse and an impulse train, as shown below.*

$$v(t) = \left\{\begin{matrix} V & -T_1 < t < T_1 \\ 0 & \text{otherwise} \end{matrix}\right\} * \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$



Solution:

In Reference 6, we showed that

$$\mathscr{F}\{x(t) * y(t)\} = X(j\omega)Y(j\omega)$$

Therefore,

$$\mathscr{F}\{v(t)\} = \mathscr{F}\left\{\begin{matrix} V & -T_1 < t < T_1 \\ 0 & \text{otherwise} \end{matrix}\right\} \cdot \mathscr{F}\left\{\sum_{k=-\infty}^{+\infty} \delta(t - kT)\right\}$$

In Reference 5, we showed that

$$v(t) = \begin{cases} V & -T_1 < t < T_1 \\ 0 & \text{otherwise} \end{cases} \qquad \overset{\mathscr{F}}{\Longleftrightarrow} \qquad 2VT_1 \operatorname{sinc}(\omega T_1)$$

In Reference 6, we showed that the Fourier transform of the impulse train $x(t)$ of period T and unit area is another impulse train of both period and area $2\pi/T$,

$$\sum_{k=-\infty}^{+\infty} \delta(t-kT) \qquad \overset{\mathscr{F}}{\Longleftrightarrow} \qquad \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T}n\right)$$

Therefore,

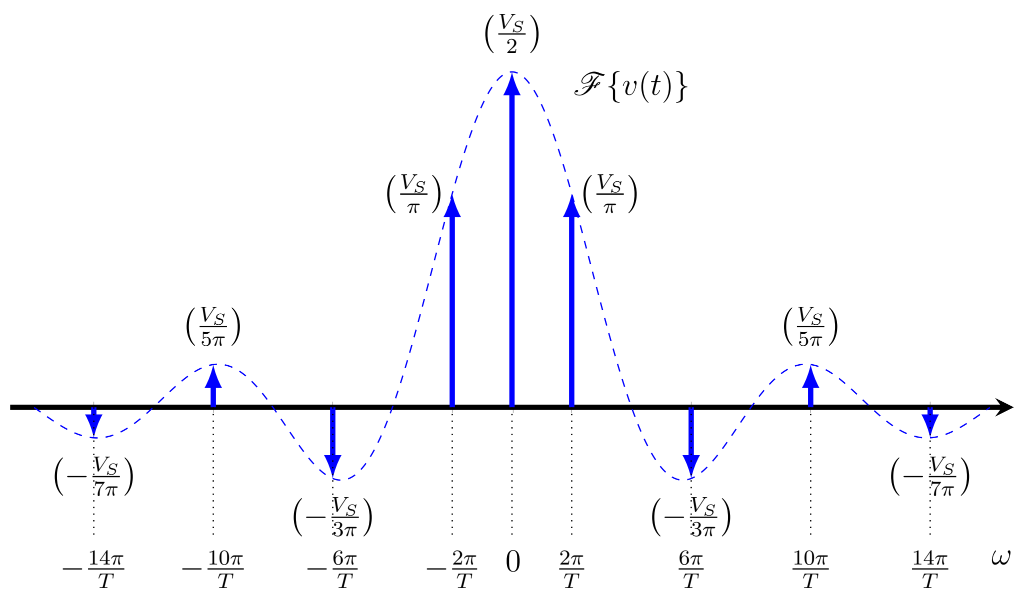
$$\mathscr{F}\{v(t)\} = 2VT_1 \operatorname{sinc}(\omega T_1) \cdot \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{T}n\right)$$

$$\mathscr{F}\{v(t)\} = 2VT_1 \cdot \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty} \operatorname{sinc}\left(\frac{2\pi}{T}nT_1\right)\delta\left(\omega - \frac{2\pi}{T}n\right)$$

$$\mathscr{F}\{v(t)\} = 2\pi \cdot 2V\frac{T_1}{T}\sum_{n=-\infty}^{+\infty} \operatorname{sinc}\left(2\pi n\frac{T_1}{T}\right)\delta\left(\omega - 2\pi n\frac{1}{T}\right)$$

B. Graph $\mathscr{F}\{v(t)\}$ for the case $T_1 = T/4$ and $V = \frac{V_S}{2\pi}$ and compare it to the coefficients of the square wave of period T.

Solution:



C. Show that the Fourier transform you found is equivalent to the coefficients for the even pulse train with period T and pulse width $2T_1$,

$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}, \qquad c_n = 2V\frac{T_1}{T}\mathrm{sinc}\left(2\pi n\frac{T_1}{T}\right)$$

Solution:

$$v(t) = \int_{-\infty}^{\infty} \mathscr{F}\{v(t)\}e^{j\omega t}\frac{d\omega}{2\pi}$$

$$v(t) = \int_{-\infty}^{\infty} \cancel{2\pi} \cdot 2V\frac{T_1}{T}\sum_{n=-\infty}^{+\infty} \mathrm{sinc}\left(2\pi n\frac{T_1}{T}\right)\delta\left(\omega - 2\pi n\frac{1}{T}\right)e^{j\omega t}\frac{d\omega}{\cancel{2\pi}}$$

$$v(t) = \sum_{n=-\infty}^{+\infty} 2V\frac{T_1}{T}\mathrm{sinc}\left(2\pi n\frac{T_1}{T}\right)\int_{-\infty}^{\infty}\delta\left(\omega - 2\pi n\frac{1}{T}\right)e^{j\omega t}d\omega$$

$$v(t) = \sum_{n=-\infty}^{+\infty} \underbrace{2V\frac{T_1}{T}\mathrm{sinc}\left(2\pi n\frac{T_1}{T}\right)}_{c_n}e^{j\frac{2\pi}{T}nt}$$

D. Previously, you found that the frequency content of a period T function only exists in the harmonics of $2\pi/T$ (i.e., multiples of $\frac{2\pi}{T}n$, where n is an integer) and must be zero elsewhere. Can you generalize the transform of a periodic function $x(t) = x(t+T)$? For consistency of notation, define $x_T(t)$ as a single period of $x(t)$ and $X_T(j\omega) = \mathscr{F}\{x_T(t)\}$.

Solution:

$$\mathscr{F}\{x(t)\} = \mathscr{F}\{x_T(t)\} \cdot \mathscr{F}\left\{\sum_{k=-\infty}^{+\infty}\delta(t-kT)\right\}$$

$$\mathscr{F}\{x(t)\} = X_T(j\omega)\cdot\frac{2\pi}{T}\sum_{n=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi}{T}n\right)$$

$$\mathscr{F}\{x(t)\} = \sum_{n=-\infty}^{+\infty}\frac{2\pi}{T}\cdot X_T(j\omega)\delta\left(\omega - \frac{2\pi}{T}n\right)$$

$$\mathscr{F}\{x(t)\} = \sum_{n=-\infty}^{+\infty}\frac{2\pi}{T}\cdot X_T\left(j\frac{2\pi}{T}n\right)\delta\left(\omega - \frac{2\pi}{T}n\right)$$

E. Generalize the equivalence to series

$$c_n = \frac{1}{T} \cdot X_T\left(j\frac{2\pi}{T}n\right).$$

Solution:

$$x(t) = \int_{-\infty}^{\infty} \mathscr{F}\{x(t)\} e^{j\omega t} \frac{d\omega}{2\pi}$$

$$x(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{+\infty} \frac{\cancel{2\pi}}{T} \cdot X_T\left(j\frac{2\pi}{T}n\right) \delta\left(\omega - \frac{2\pi}{T}n\right) e^{j\omega t} \frac{d\omega}{\cancel{2\pi}}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T} \cdot X_T\left(j\frac{2\pi}{T}n\right) \int_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T}n\right) e^{j\omega t} d\omega$$

$$x(t) = \sum_{n=-\infty}^{+\infty} \underbrace{\frac{1}{T} \cdot X_T\left(j\frac{2\pi}{T}n\right)}_{c_n} e^{j\frac{2\pi}{T}nt}$$

Course feedback

Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?