Olin College of Engineering
ENGR2410 – Signals and Systems

Quiz 1

Name:____________________________________

You have 90 uninterrupted minutes to complete this quiz. You may use one page of notes in addition to the overview and tables document in the website. Use alternate methods to confirm or check your answers. Awareness of errors and/or the right answer is worth quite a bit of partial credit! Good luck!

Problem 1 Find the Fourier series coefficients c_n for the functions shown below such that

$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$$

as well as the average power $< p >= \frac{1}{T}\int_{t_0}^{t_0+T} |v(t)|^2 dt$.

A. $v(t) = V + \frac{V}{2}\cos\left(\frac{2\pi}{T}t\right)$

Solution:
The fastest solution is to recognize the coefficients directly by rewriting as a sum of complex exponentials. Average power is then the sum of the squared magnitude of all the coefficients.

$$v(t) = V + \frac{V}{2}\left[\frac{e^{j\frac{2\pi}{T}t} + e^{-j\frac{2\pi}{T}t}}{2}\right] = \underbrace{V}_{c_0} + \underbrace{\frac{V}{4}}_{c_1} e^{j\frac{2\pi}{T}t} + \underbrace{\frac{V}{4}}_{c_{-1}} e^{-j\frac{2\pi}{T}t}$$

$$< p > = \underbrace{V^2}_{|c_0|^2} + \underbrace{\frac{1}{16}V^2}_{|c_1|^2} + \underbrace{\frac{1}{16}V^2}_{|c_{-1}|^2} = \frac{18}{16}V^2 = \frac{9}{8}V^2$$

An alternate way to find the coefficients is to use $c_n = \frac{1}{T}\int_{t_0}^{t_0+T} v(t)e^{-j\frac{2\pi}{T}nt}dt$. In this case, the easiest way to find a closed form solution is to integrate from $-\frac{T}{2}$ to $\frac{T}{2}$.

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\underbrace{V}_{c_0} + \underbrace{\frac{V}{4}}_{c_1} e^{j\frac{2\pi}{T}t} + \underbrace{\frac{V}{4}}_{c_{-1}} e^{-j\frac{2\pi}{T}t}\right] e^{-j\frac{2\pi}{T}nt}dt \\
&= \frac{1}{T}\underbrace{V}_{c_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\frac{2\pi}{T}nt}dt + \frac{1}{T}\underbrace{\frac{V}{4}}_{c_1}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j\frac{2\pi}{T}t}e^{-j\frac{2\pi}{T}nt}dt + \ldots \\
&\qquad \frac{1}{T}\underbrace{\frac{V}{4}}_{c_{-1}}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\frac{2\pi}{T}t}e^{-j\frac{2\pi}{T}nt}dt \\
&= \frac{1}{T}\underbrace{V}_{c_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\frac{2\pi}{T}nt}dt + \frac{1}{T}\underbrace{\frac{V}{4}}_{c_1}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j\frac{2\pi}{T}(1-n)t}dt + \frac{1}{T}\underbrace{\frac{V}{4}}_{c_{-1}}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\frac{2\pi}{T}(1+n)t}dt \\
&= \frac{1}{\cancel{T}}\underbrace{V}_{c_0}\frac{\cancel{T}}{-j2\pi n}\left[e^{-j\frac{2\pi}{T}nt}\right]_{-\frac{T}{2}}^{\frac{T}{2}} + \frac{1}{\cancel{T}}\underbrace{\frac{V}{4}}_{c_1}\frac{\cancel{T}}{j2\pi(1-n)}\left[e^{j\frac{2\pi}{T}(1-n)t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} + \ldots \\
&\qquad \frac{1}{\cancel{T}}\underbrace{\frac{V}{4}}_{c_{-1}}\frac{\cancel{T}}{-j2\pi(1+n)}\left[e^{-j\frac{2\pi}{T}(1+n)t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} \\
&= \underbrace{V}_{c_0}\frac{1}{-j2\pi n}\left[e^{-j\pi n} - e^{j\pi n}\right] + \underbrace{\frac{V}{4}}_{c_1}\frac{1}{j2\pi(1-n)}\left[e^{j\pi(1-n)} - e^{-j\pi(1-n)}\right] + \ldots \\
&\qquad \underbrace{\frac{V}{4}}_{c_{-1}}\frac{1}{-j2\pi(1+n)}\left[e^{-j\pi(1+n)} - e^{j\pi(1+n)}\right]
\end{aligned}
$$

Use the negative in the denominators to write as $\sin(\theta) = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$,

$$
\begin{aligned}
&= \underbrace{V}_{c_0}\frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j}\right] + \underbrace{\frac{V}{4}}_{c_1}\frac{1}{\pi(1-n)}\left[\frac{e^{j\pi(1-n)} - e^{-j\pi(1-n)}}{2j}\right] + \ldots \\
&\qquad \underbrace{\frac{V}{4}}_{c_{-1}}\frac{1}{\pi(1+n)}\left[\frac{e^{j\pi(1+n)} - e^{-j\pi(1+n)}}{2j}\right]
\end{aligned}
$$

$$c_n = \underbrace{V}_{c_0} \underbrace{\frac{\sin(\pi n)}{\pi n}}_{1 \text{ if } n=0} + \underbrace{\frac{V}{4}}_{c_1} \underbrace{\frac{\sin(\pi(1-n)}{\pi(1-n)}}_{1 \text{ if } n=1} + \underbrace{\frac{V}{4}}_{c_{-1}} \underbrace{\frac{\sin(\pi(1+n)}{\pi(1+n)}}_{1 \text{ if } n=-1}; \text{ (otherwise 0)}$$

$$c_n = \begin{cases} V & n = 0 \\ \frac{V}{4} & n = 1, -1 \\ 0 & \text{otherwise} \end{cases}$$

B. $v(t) = V\sin\left(\frac{2\pi}{T}t\right) + \frac{V}{2}\cos\left(\frac{2\pi}{T}t\right)$

Solution:

$$\begin{aligned}
v(t) &= V\left[\frac{e^{j\frac{2\pi}{T}t} - e^{-j\frac{2\pi}{T}t}}{2j}\right] + \frac{V}{2}\left[\frac{e^{j\frac{2\pi}{T}t} + e^{-j\frac{2\pi}{T}t}}{2}\right] \\
&= \underbrace{\frac{V}{2j} + \frac{V}{4}}_{c_1} e^{j\frac{2\pi}{T}t} + \underbrace{-\frac{V}{2j} + \frac{V}{4}}_{c_{-1}} e^{-j\frac{2\pi}{T}t} = \underbrace{-j\frac{V}{2} + \frac{V}{4}}_{c_1} e^{j\frac{2\pi}{T}t} + \underbrace{j\frac{V}{2} + \frac{V}{4}}_{c_{-1}} e^{-j\frac{2\pi}{T}t} \\
< p > &= \underbrace{\left[\frac{1}{4} + \frac{1}{16}\right]V^2}_{|c_1|^2} + \underbrace{\left[\frac{1}{4} + \frac{1}{16}\right]V^2}_{|c_{-1}|^2} = \frac{10}{16}V^2 = \frac{5}{8}V^2
\end{aligned}$$

C. $v(t) = V\sin\left(\frac{2\pi}{T}t\right) + \frac{V}{2}\cos\left(2\frac{2\pi}{T}t\right)$

Solution:

$$\begin{aligned}
&c_1 = V/(2j), c_{-1} = -V/(2j), c_{-2} = V/4, c_2 = V/4 \\
&< p >= |c_{-1}|^2 + |c_1|^2 + |c_{-2}|^2 + |c_2|^2 = (1/4 + 1/4 + 1/16 + 1/16)V^2 \\
&= (1/2 + 1/8)V^2 = 5V^2/8
\end{aligned}$$

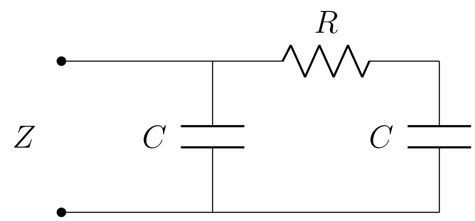
$$\begin{aligned}
v(t) &= V\left[\frac{e^{j\frac{2\pi}{T}t} - e^{-j\frac{2\pi}{T}t}}{2j}\right] + \frac{V}{2}\left[\frac{e^{j2\frac{2\pi}{T}t} + e^{-j2\frac{2\pi}{T}t}}{2}\right] \\
&= \underbrace{\frac{V}{2j}}_{c_1} e^{j\frac{2\pi}{T}t} + \underbrace{-\frac{V}{2j}}_{c_{-1}} e^{-j\frac{2\pi}{T}t} + \underbrace{\frac{V}{4}}_{c_2} e^{j2\frac{2\pi}{T}t} + \underbrace{\frac{V}{4}}_{c_{-2}} e^{-j2\frac{2\pi}{T}t} \\
< p > &= \underbrace{\frac{1}{4}V^2}_{|c_1|^2} + \underbrace{\frac{1}{4}V^2}_{|c_{-1}|^2} + \underbrace{\frac{1}{16}V^2}_{|c_2|^2} + \underbrace{\frac{1}{16}V^2}_{|c_{-2}|^2} = \frac{10}{16}V^2 = \frac{5}{8}V^2
\end{aligned}$$

Problem 2

A. Show that the impedance Z in the circuit below is

$$Z = \frac{1}{Cs} || \left(R + \frac{1}{Cs} \right) = \frac{1}{Cs} \cdot \frac{s + \frac{1}{RC}}{s + \frac{2}{RC}}$$

where $||$ is the parallel operator, $R_1 || R_2 = \frac{R_1 R_2}{R_1 + R_2}$.



Solution:

$$Z = \frac{1}{Cs} || \left(R + \frac{1}{Cs} \right)$$

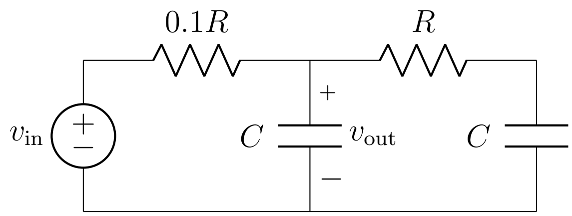
$$Z = \frac{1}{Cs} || \left(\frac{RCs + 1}{Cs} \right)$$

$$Z = \frac{\frac{RCs+1}{(Cs)^2}}{\frac{RCs+2}{Cs}}$$

$$Z = \frac{1}{Cs} \cdot \frac{RCs + 1}{RCs + 2}$$

$$Z = \frac{1}{Cs} \cdot \frac{s + \frac{1}{RC}}{s + \frac{2}{RC}}$$

B. Find the transfer function V_{out}/V_{in} in the circuit below.



Solution:

$$H(s) = \frac{Z}{0.1R + Z}$$

$$H(s) = \frac{\frac{1}{Cs} \cdot \frac{RCs+1}{RCs+2}}{0.1R + \frac{1}{Cs} \cdot \frac{RCs+1}{RCs+2}}$$

$$H(s) = \frac{RCs + 1}{0.1RCs(RCs + 2) + RCs + 1}$$

$$H(s) = \frac{RCs + 1}{0.1(RC)^2 s^2 + 1.2RCs + 1}$$

$$H(s) = \frac{1}{0.1RC} \frac{s + \frac{1}{RC}}{\left[s^2 + \frac{12}{RC}s + \frac{10}{(RC)^2}\right]}$$

Alternative forms:

$$H(s) = \frac{10}{RC} \frac{s + \frac{1}{RC}}{\left[s^2 + \frac{12}{RC}s + \frac{10}{(RC)^2}\right]}$$

$$H(j\omega) = \frac{10}{RC} \frac{j\omega + \frac{1}{RC}}{\left[-w^2 + \frac{12}{RC}j\omega + \frac{10}{(RC)^2}\right]}$$

$$H(s) = \frac{\frac{10}{RC}s + \frac{10}{(RC)^2}}{s^2 + \frac{12}{RC}s + \frac{10}{(RC)^2}}$$

$$H(s) = \frac{10RCs + 10}{(RC)^2 s^2 + 12RCs + 10}$$

$$H(s) = \frac{RCs + 1}{0.1(RC)^2 s^2 + 1.2RCs + 1}$$

C. Find the natural frequencies of the system. Is the system underdamped ($\omega_0 \gg \alpha$), or overdamped? Make reasonable approximations (less than 10% error) when appropriate. *Hint: You may make the approximation* $\sqrt{26} \approx 5$.

Solution:

$$s^2 + \frac{12}{RC}s + \frac{10}{(RC)^2} = 0$$

$$\alpha = \frac{6}{RC} \qquad \omega_0 = \frac{\sqrt{10}}{RC}, \quad \alpha > \omega_0 \text{ (overdamped)}$$

$$s = -\frac{6 \pm \sqrt{6^2 - 10}}{RC} = \frac{6 \pm \sqrt{26}}{RC} \approx -\frac{6 \pm 5}{RC}$$

$$s_1 = -\frac{11}{RC}, \quad s_2 = -\frac{1}{RC}$$

D. Sketch the Bode plot using asymptotic approximations (i.e., let $\omega \to \infty$ and $\omega \to 0$). Again, make reasonable approximations. *Hint:* $\left|\frac{1+10j}{-9+12j}\right| = 0.67, \angle\left(\frac{1+10j}{-9+12j}\right) = -\pi/4.2$

Solution:

$$s \to 0, H(s) \approx 1 \qquad s \to \infty, H(s) \approx \frac{10}{RCs}$$

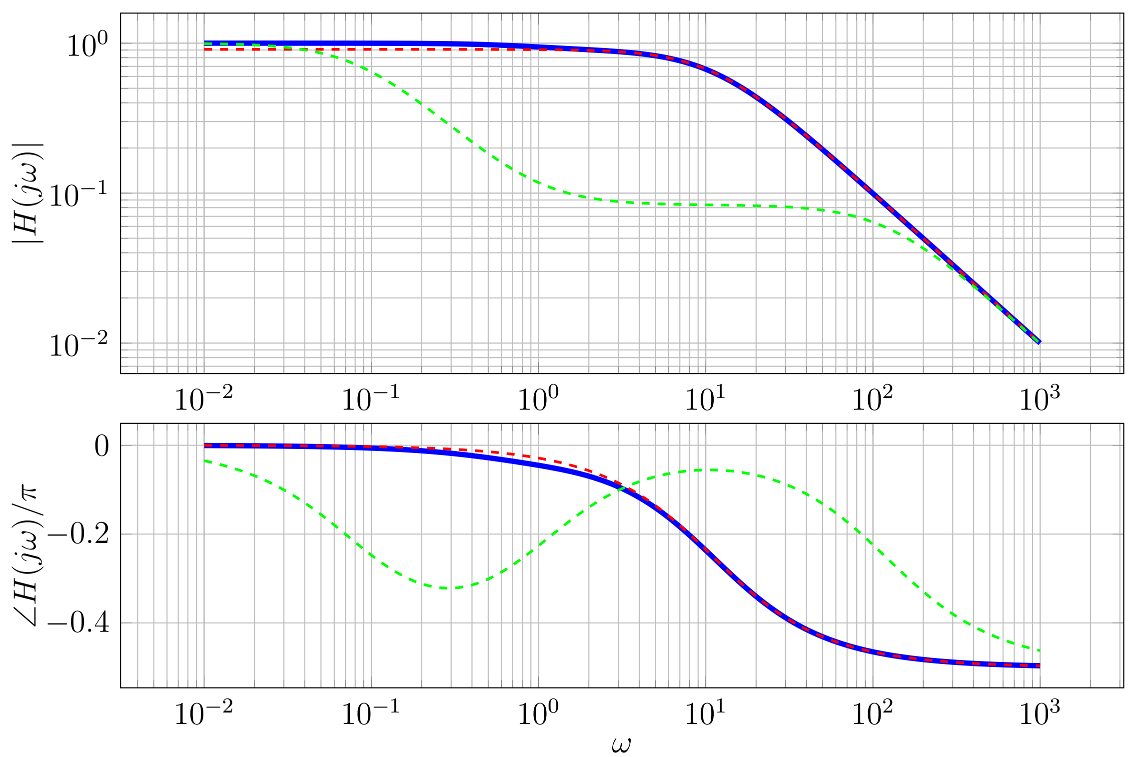
Intersection at $\frac{10}{RC\omega} = 1, \omega = 10/RC$. The value at the intersection is

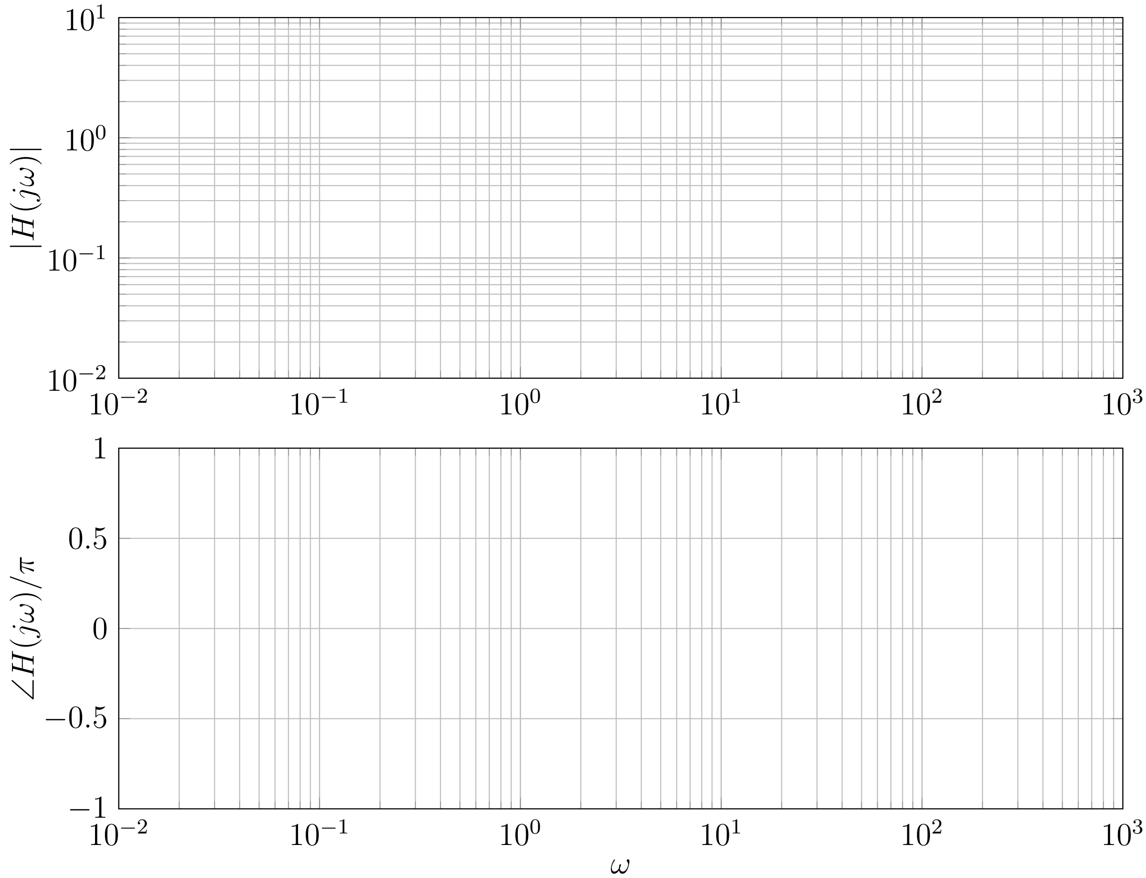
$$H\left(j\frac{10}{RC}\right) = \frac{1+10j}{-9+12j} \approx \frac{10j}{10(-1+j)}; \left|\frac{10j}{10(-1+j)}\right| = 0.71; \angle\frac{10j}{10(-1+j)} = -\frac{\pi}{4}$$

The plots assume $RC = 1$. *Note than even though there are two roots, they are close enough that they look like a first order system! In particular,*

$$H(s) \approx \frac{1}{0.1RC}\frac{\cancel{s+\frac{1}{RC}}}{\left[\cancel{s+\frac{1}{RC}}\right]\left[s+\frac{11}{RC}\right]} \approx \frac{1}{0.1RC\left[s+\frac{11}{RC}\right]}$$

The exact plot below is blue, and the approximation is red. Note the small effect of the second root. The green shows the two roots clearly by increasing α by a factor of ten.




10^{1}
10^{0}
10^{-1}
10^{-2}
$|H(j\omega)|$
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3}
1
0.5
0
-0.5
-1
$\angle H(j\omega)/\pi$
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3}
ω

E. Find $v_{out}(t)$ if $v_{in}(t)$ is a 1 Volt step at $t = 0$ (i.e., $v_{in}(t) = 1 \text{ V} \cdot u(t)$).

(i) Find the particular, or steady-state, solution when all transients have died.

Solution:

$$v_{out,particular}(t) = 1 \text{ V}$$

(ii) Write the general solution, which includes the particular and the homogeneous solution before specifying the initial conditions.

Solution:

$$v_{out,general}(t) = 1 \text{ V} + A_1 e^{-\frac{1}{RC}t} + A_2 e^{-\frac{11}{RC}t}$$

(iii) The capacitor is initially discharged ($v_{out}(0) = 0$). The initial current through the capacitor is $i_C(0) = 10\text{ V}/R$, which means that the initial slope at $t = 0$ is $\dot{v}_{out}(0) = i_C(0)/C = 10\text{ V}/(RC)$. Use these initial conditions to find an algebraic expression for $v_{out}(t)$.

Solution:

$$v_{out}(t) = 1\text{ V} + A_1 e^{-\frac{1}{RC}t} + A_2 e^{-\frac{11}{RC}t}$$

$$v_{out}(0) = 1\text{ V} + A_1 + A_2 = 0 \Rightarrow A_2 = -1\text{ V} - A_1$$

$$\dot{v}_{out}(t) = -\frac{1}{RC}A_1 e^{-\frac{1}{RC}t} - \frac{11}{RC}A_2 e^{-\frac{11}{RC}t}$$

$$\dot{v}_{out}(0) = -\frac{1}{RC}A_1 - \frac{11}{RC}A_2 = \frac{10\text{ V}}{RC}$$

$$-A_1 - 11(-1\text{ V} - A_1) = 10\text{ V}$$

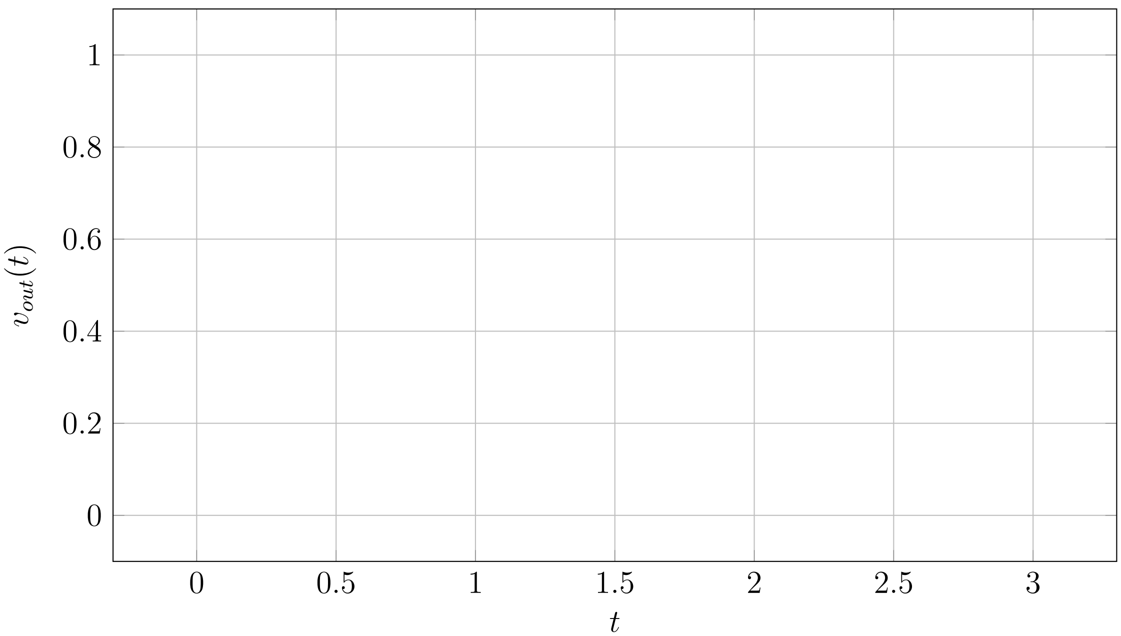
$$-A_1 + 11\text{ V} + 11A_1) = 10\text{ V}$$

$$10A_1 = -1\text{ V} \Rightarrow A_1 = -\frac{1}{10}\text{ V}$$

$$A_2 = -1\text{ V} + \frac{1}{10}\text{ V} = -\frac{9}{10}\text{ V}$$

$$v_{out}(t) = 1\text{ V} - \frac{1}{10}\text{ V} \cdot e^{-\frac{1}{RC}t} - \frac{9}{10}\text{ V} \cdot e^{-\frac{11}{RC}t}$$

(iv) Sketch $v_{out}(t)$.



Solution:
Assuming $RC = 1$. The red trace is the fast exponent that dies out quickly. This corresponds to the first capacitor charging quickly through the $0.1R$ resistor. The green trace is the slow response when the second capacitor charges slowly once the first capacitor is essentially fully charged.