# Olin College of Engineering
# ENGR2410 – Signals and Systems

## Assignment 7

**Problem 1** In this problem, you will consolidate most of the transforms you already know. The purpose is two-fold: first, while these tables exist everywhere, you should be able to decompose problems into pieces that will have transforms. More importantly, a lot of the transforms have been derived as needed in a particular context. In this exercise, focus on the relationships between different transforms. *For every subproblem, find both an algebraic expression and graph the function carefully.*

A. You might have noticed that both the Fourier and inverse Fourier integrals are essentially the same with some sign and factor differences. This means that transforms come in pairs. Show the *duality* property of Fourier transforms,

   If $x(t) \overset{\mathscr{F}}{\Longleftrightarrow} X(\omega)$ then $X(t) \overset{\mathscr{F}}{\Longleftrightarrow} 2\pi x(-\omega)$. (*We have replaced $X(\omega)$ for $X(j\omega)$ to avoid confusion.*)

   **Solution:**

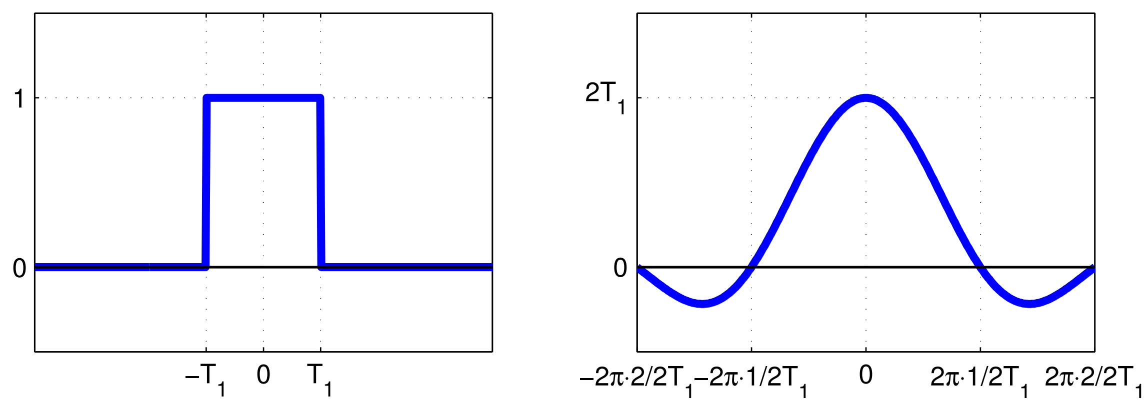
$$\begin{aligned}
x(t) &= \int_{\omega=-\infty}^{\infty} X(\omega) e^{j\omega t} \frac{d\omega}{2\pi} && \text{By definition} \\
2\pi x(-t) &= \int_{\omega=-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega \\
2\pi x(-\omega) &= \int_{t=-\infty}^{\infty} X(t) e^{-j\omega t} dt && \text{Switch } \omega \text{ and } t \\
2\pi x(-\omega) &= \mathscr{F}\{X(t)\}
\end{aligned}$$

B. Recall and graph the transform for a box in the time domain of unit height, from $-T_1$ to $T_1$. Find an expression and graph the inverse transform of a box of unit height in the frequency domain from $-\omega_0$ to $\omega_0$. *This is known as a low-pass "brick" filter. Note that the impulse response starts at* $-\infty$. *Typical systems in the real world are* causal, *which means their response cannot extend to* $-\infty$.

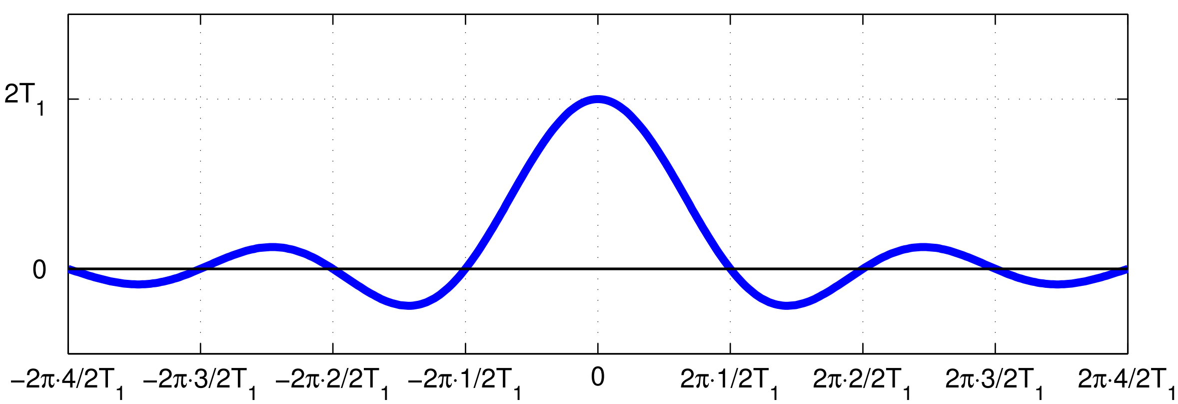
**Solution:**

**sinc in frequency**

$$\mathscr{F}\left\{\Pi(\mathrm{t}/\mathrm{T}_1)\right\} = 2T_1\frac{\sin(\omega T_1)}{\omega T_1} \text{ where } \Pi(\mathrm{x}) = \begin{cases} 1 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$




**Exploded view of sinc in frequency**

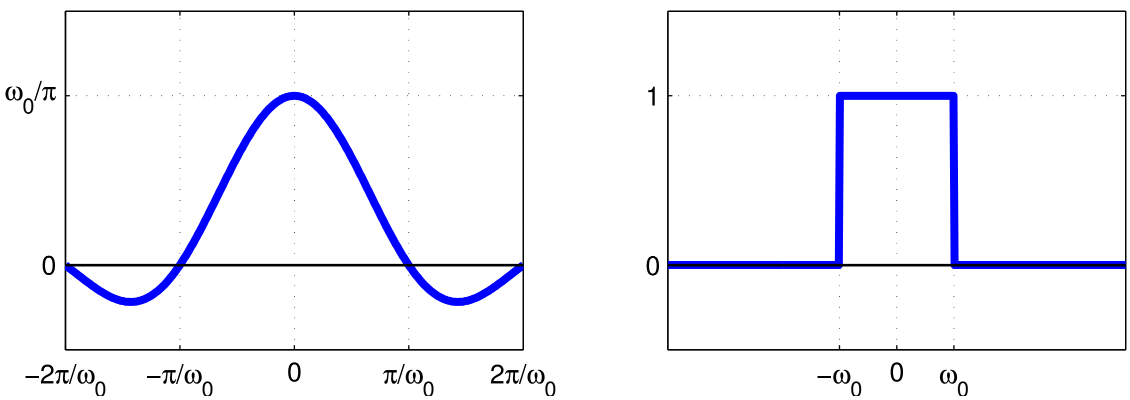


**sinc in time**

Start from sinc in frequency:

$$\begin{aligned}
2T_1\frac{\sin(\omega T_1)}{\omega T_1} &= \int_{t=-T_1}^{T_1} e^{-j\omega t}dt \\
2\omega_0\frac{\sin(\omega_0 t)}{\omega_0 t} &= \int_{\omega=-\omega_0}^{\omega_0} e^{j\omega t}d\omega \qquad \text{Substitute: } \omega=-t,\ t=\omega,\ T_1=\omega_0 \\
\frac{\omega_0}{\pi}\frac{\sin(\omega_0 t)}{\omega_0 t} &= \int_{\omega=-\omega_0}^{\omega_0} e^{j\omega t}\frac{d\omega}{2\pi}
\end{aligned}$$

$$\mathscr{F}\left\{\frac{\omega_0}{\pi}\cdot\frac{\sin(\omega_0 t)}{\omega_0 t}\right\} = \Pi(\omega/\omega_0) \text{ where } \Pi(\text{x}) = \begin{cases}1 & -1\le x\le 1\\ 0 & \text{otherwise}\end{cases}$$



C. Scale the unit boxes in both time and frequency such that the area inside the box is constant. Take the limits as both $T_1 \to 0$ and $\omega_0 \to 0$ such that the boxes approach impulses. Recall or rederive the transforms for impulses in both time and frequency domains and show that you approach this result in both limits. Use sketches as necessary to make your explanation clear.

**Solution:**

$$\Pi(\mathrm{t}/\mathrm{T}_1) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad 2\mathrm{T}_1\frac{\sin(\omega\mathrm{T}_1)}{\omega\mathrm{T}_1} \qquad \text{unit box in time}$$

$$\frac{1}{2T_1}\Pi(\mathrm{t}/\mathrm{T}_1) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \frac{\sin(\omega\mathrm{T}_1)}{\omega\mathrm{T}_1} \qquad \text{scale the box}$$

$$\delta(t) \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad 1 \qquad \text{take } T_1 \to 0$$

$$\frac{\omega_0}{\pi}\cdot\frac{\sin(\omega_0 t)}{\omega_0 t} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \Pi(\omega/\omega_0) \qquad \text{unit box in frequency}$$

$$\frac{1}{2\pi}\cdot\frac{\sin(\omega_0 t)}{\omega_0 t} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \frac{1}{2\omega_0}\Pi(\omega/\omega_0) \qquad \text{scale the box}$$

$$\frac{1}{2\pi} \quad \overset{\mathscr{F}}{\Longleftrightarrow} \quad \delta(\omega) \qquad \text{take } \omega_0 \to 0$$

D. Recall the time shift property of the Fourier transform, where you expressed the transform of $x(t+T)$ in term of the transform of $x(t)$. Find the dual property, the frequency shift property where you express the transform of $X(j\omega - j\omega_0)$ in terms of $X(j\omega)$.

**Solution:**

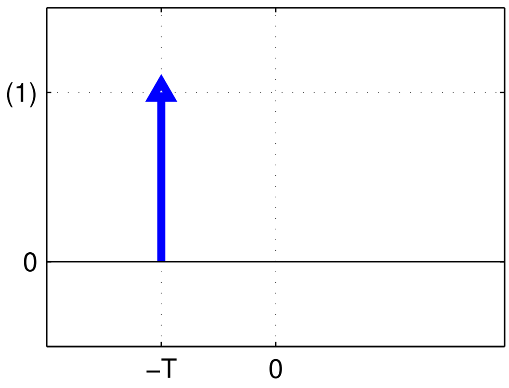
$$\mathscr{F}\{x(t+T)\} = X(j\omega)e^{j\omega T}$$
$$\mathscr{F}\{x(t)e^{j\omega_0 t}\} = X(\omega - \omega_0)$$

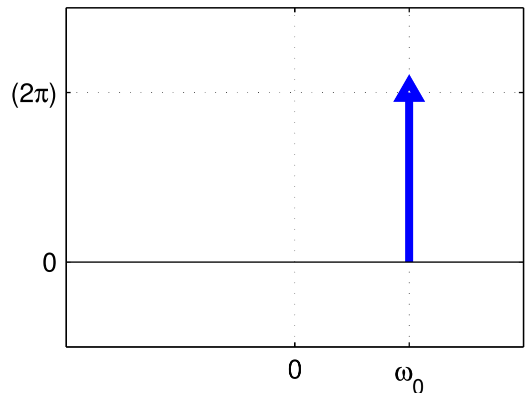
E. Rederive the dual transforms for complex exponentials in both time and frequency by using the shift properties for the case where $x(t) = \delta(t)$ and $X(j\omega) = \delta(\omega)$ as appropriate. Graph the functions that involve impulses.

**Solution:**

$$\mathscr{F}\{\delta(t)\} = 1$$
$$\mathscr{F}\{\delta(t+T)\} = e^{j\omega T}$$

$$\mathscr{F}\{1\} = 2\pi\delta(\omega)$$
$$\mathscr{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$$

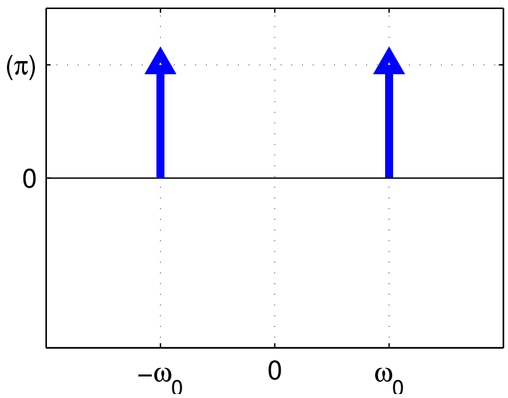


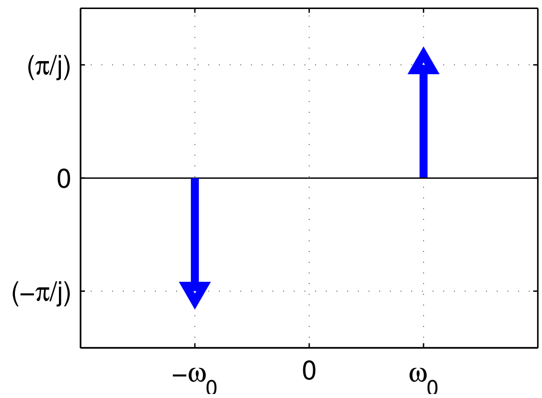




F. Rederive the transform of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ and graph them.

**Solution:**

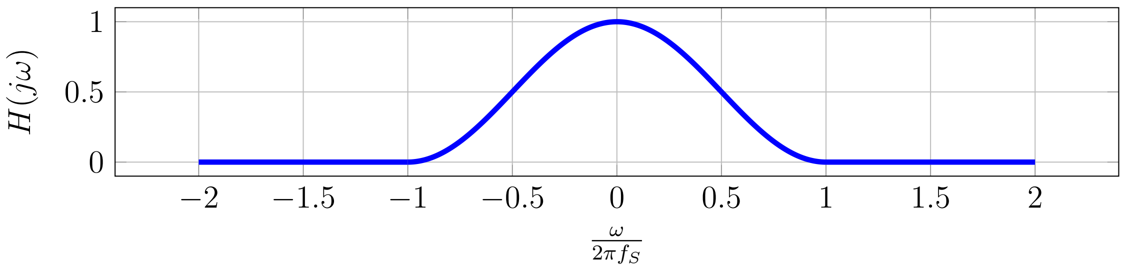
$$\mathscr{F}\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$
$$\mathscr{F}\{\sin(\omega_0 t)\} = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$



**Problem 2** The filter $H(j\omega)$ is known as a *raised-cosine filter*. Find an algebraic expression for the impulse response of the filter $h(t)$ and graph it. *The zero crossings of the impulse response are used to avoid* Inter-Symbol Interference (ISI) *in digital communication systems (for example, see* [http://www.analog.com/media/en/technical-documentation/application-notes/AN-922.pdf](http://www.analog.com/media/en/technical-documentation/application-notes/AN-922.pdf)*).*

$$H(j\omega) = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\omega}{2f_S}\right)\right] & -2\pi f_S \le w \le 2\pi f_S \\ 0 & \text{otherwise} \end{cases}$$

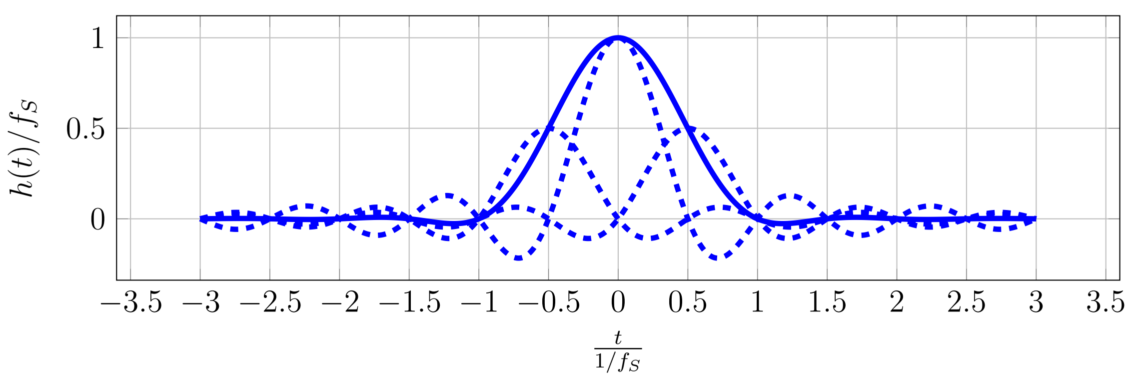



**Solution:**
The original equation can be decomposed into the product

$$H(j\omega) = \frac{1}{2}\left[1+\cos\left(\frac{\omega}{2f_S}\right)\right] \times \begin{cases} 1 & -2\pi f_S \le \omega \le 2\pi f_S \\ 0 & \text{otherwise} \end{cases}$$

The inverse Fourier transform is then

$$\begin{aligned} h(t) =& \frac{1}{2}\mathscr{F}^{-1}\left\{1+\cos\left(\frac{\omega}{2f_S}\right)\right\} * \mathscr{F}^{-1}\left\{\begin{cases} 1 & -2\pi f_S \le \omega \le 2\pi f_S \\ 0 & \text{otherwise} \end{cases}\right\} \\ =& \frac{1}{2}\left[\delta\left(t\right) + \frac{1}{2}\delta\left(t+\frac{1}{2f_s}\right) + \frac{1}{2}\delta\left(t+\frac{1}{2f_s}\right)\right] * 2f_s\text{sinc}(2\pi f_s t) \\ =& f_s\text{sinc}(2\pi f_s t) + \frac{1}{2}f_s\text{sinc}(2\pi f_s t - \pi) + \frac{1}{2}f_s\text{sinc}(2\pi f_s t + \pi) \end{aligned}$$



**Problem 3** One of the most evident effects of sampling is *aliasing*, where frequencies above one half of the sampling frequency appear as lower frequencies added to the original signal. In this problem, you will calculate the frequency of the aliased signal in the case of a sinusoid. For consistency throughout this problem, sketch the all Fourier transforms from -30 kHz to 30 kHz.

*A fascinating debate is whether the human brain samples what we see, and if so, what is the sampling frequency. Check out all the sampling language in* http://www.sciencedirect.com/science/article/pii/S0042698905003883*. However, newer evidence like* http://www.ncbi.nlm.nih.gov/pmc/articles/PMC2856842/ *suggests that sampling may not be enough to explain our perception.*

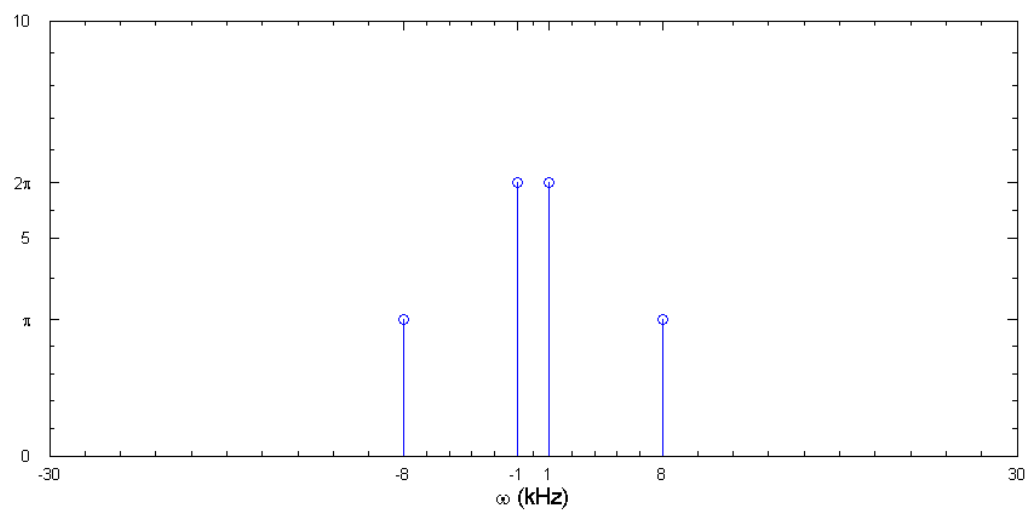
A. Graph the Fourier transform of an input signal

$$x(t) = 2\cos(2\pi \cdot 1 \text{ kHz} \cdot t) + \cos(2\pi \cdot 8 \text{ kHz} \cdot t).$$

**Solution:**
The Fourier transform of $\cos(\omega_0 t)$ is

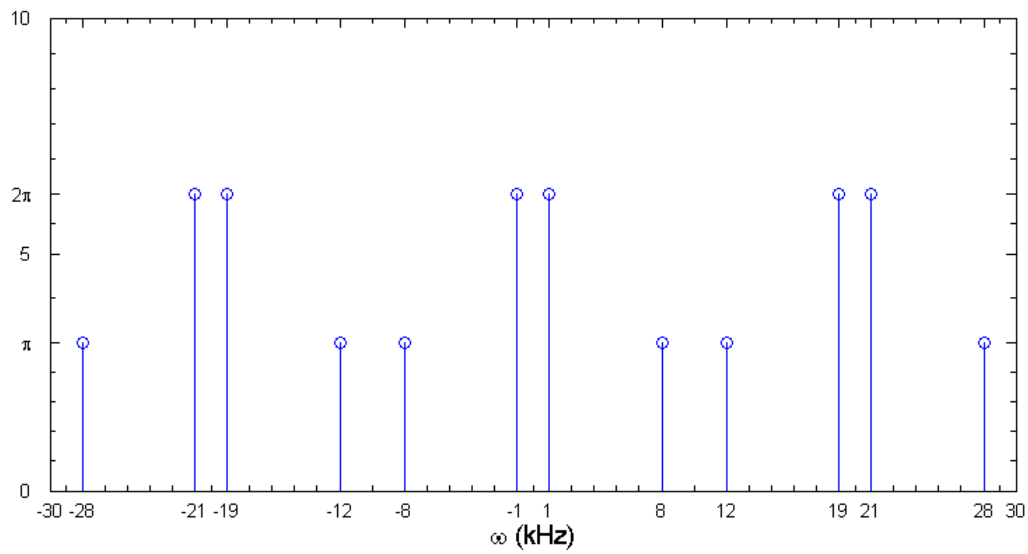
$$\mathscr{F}\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0).$$

The complete graph using superposition is shown below.



B. The input signal $x(t)$ is sampled at 20 kHz. Sketch the Fourier transform of the sampled function, $x_{S1}(t)$.

**Solution:**
The frequency content of the sampled function is periodic with a period of 20 kHz (i.e. the solution repeats every 20 kHz), as shown below.



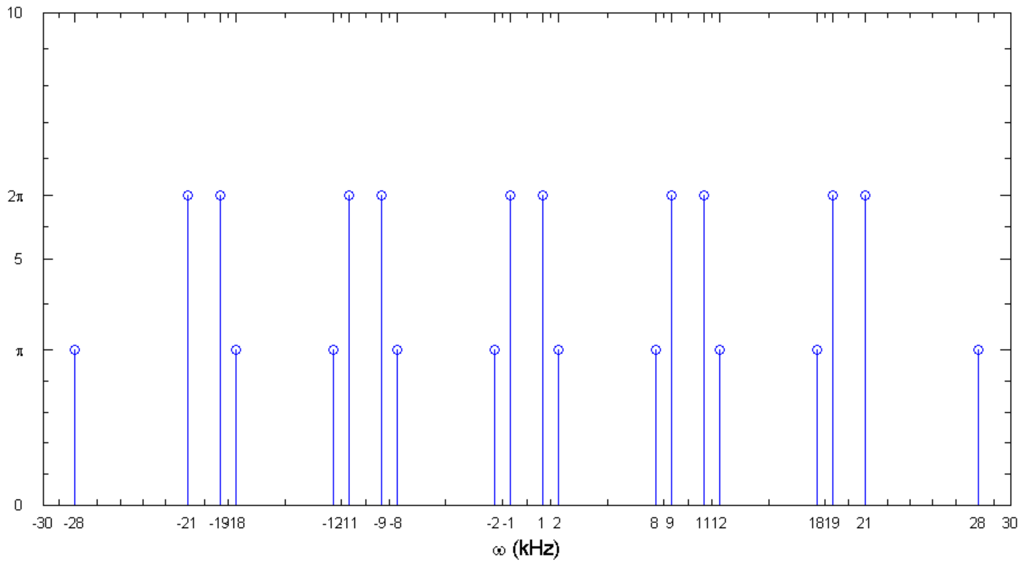

C. The sampled function $x_{S1}(t)$ is passed through an ideal low-pass filter from -10 kHz to 10 kHz. Write an expression for the filtered output $y_1(t)$.

**Solution:**
All of the frequencies above 10 kHz in the output vanish. The resulting frequency content is identical to that of the original function.

$$y_1(t) = 2\cos(2\pi \cdot 1 \text{ kHz} \cdot t) + \cos(2\pi \cdot 8 \text{ kHz} \cdot t)$$

D. The input function $x(t)$ is now sampled at 10 kHz. Graph the Fourier transform of the resulting function, $x_{S2}(t)$.

**Solution:**
The frequency content of the sampled function is periodic with a period of 10 kHz (i.e. the solution repeats every 10 kHz), as shown below. Notice the overlapping frequencies that will lead to aliasing.



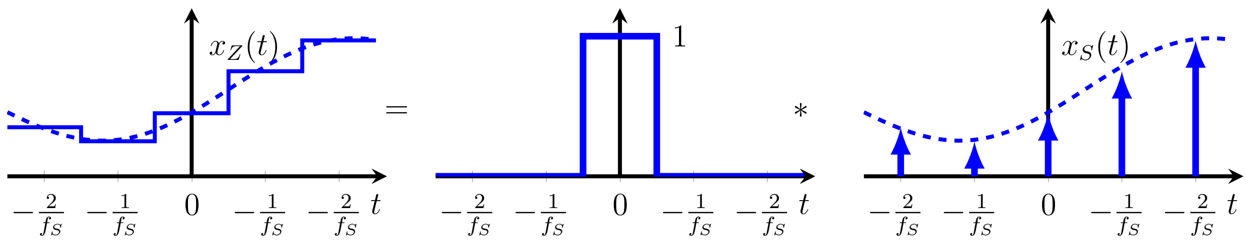

E. The sampled function $x_{S2}(t)$ is passed through an ideal low-pass filter from -10 kHz to 10 kHz. Write an expression for the filtered output $y_2(t)$.

**Solution:**
The overlapping frequencies introduce two additional cosines at 2 kHz and 9 kHz.

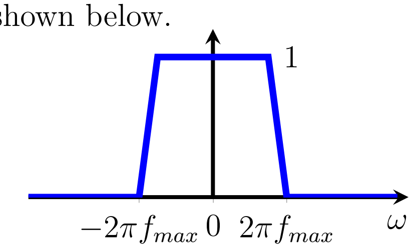
$$y_2(t) = 2\cos(2\pi \cdot 1 \text{ kHz} \cdot t) + \cos(2\pi \cdot 2 \text{ kHz} \cdot t) + \cos(2\pi \cdot 8 \text{ kHz} \cdot t) + 2\cos(2\pi \cdot 9 \text{ kHz} \cdot t)$$

**Problem 4** Real sampling requires finite width pulses. In fact, the most common sampling method is called *zero-order hold*, where the value of the sampled function is held constant until the next sample is taken. For example, see http://eetimes.com/electronics-news/4197022/Switched-Capacitor-Filters-Beat-Active-Filters-at-Their-Own-Game. In this problem, you will show that the highest possible frequency of the original signal $x(t)$ sampled at frequency $f_S$ is decreased by 64% when reconstructed using zero-order hold.

Remember that an ideal sampling of a function $x(t)$ with maximum frequency $f_{max}$ can be modeled by multiplying the function with a pulse train with period $1/f_S$. Reconstructing the resulting sampled signal $x_s(t)$ with zero-order hold can be modeled by convolving it with a pulse of the same width as the sampling period. Call this reconstructed signal $x_Z(t)$.

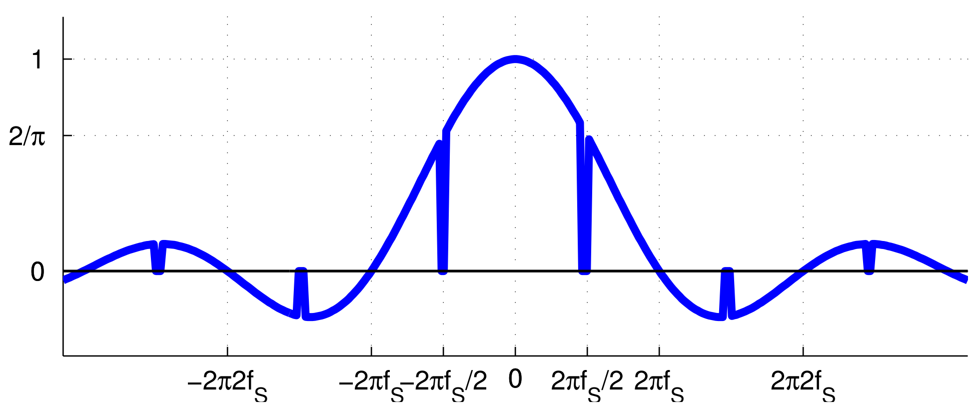



A. Find an expression and sketch the frequency content of $x_Z(t)$. Assume $x(t)$ has frequency content $X(j\omega)$ as shown below.




**Solution:**
In the frequency domain, $X_S(j\omega)$ is multiplied by the transform of a single pulse of width $1/f_S$. The transform of the pulse is $\frac{1}{f_S}\text{sinc}\left(\frac{\omega}{2f_S}\right)$. The resulting transform is

$$X_Z(j\omega) = \frac{1}{\cancel{f_S}}\text{sinc}\left(\frac{\omega}{2f_S}\right)\cancel{f_S}\sum_{n=-\infty}^{+\infty} X[j(\omega - 2\pi f_S n)]$$



B. Assuming no aliasing, what is the attenuation at the highest possible signal frequency?

**Solution:**
If we are sampling at $1/f_S$, the maximum frequency $X(j\omega)$ can have before we see aliasing is $f_S/2$. At this frequency, the gain is

$$\operatorname{sinc}\left(\frac{1}{\cancel{2}\cancel{f_S}}\cancel{2}\pi\frac{\cancel{f_S}}{2}\right) = \frac{2}{\pi} \approx 0.64$$

**Course feedback**
Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?