Olin College of Engineering
ENGR2410 – Signals and Systems

Assignment 3

Problem 1

A. Find the equivalent impedance of a series RLC combination.

Solution:

$$Z = R + Ls + \frac{1}{Cs}$$
$$Z = \frac{s^2 + \frac{1}{L/R}s + \frac{1}{LC}}{s/L}$$
$$Z = \frac{1}{Cs}\left(\frac{s^2 + \frac{1}{L/R}s + \frac{1}{LC}}{\frac{1}{LC}}\right)$$
$$Z = \frac{L\left(s^2 + \frac{1}{L/R}s + \frac{1}{LC}\right)}{s}$$

B. Find an expression for the quality factor Q of the series RLC in terms of the characteristic impedance $Z_0 = \sqrt{L/C}$. Recall that Q is defined as $\frac{\omega_0}{2\alpha}$.

Solution:

$$Q = \frac{1}{\sqrt{LC}}L/R = \frac{\sqrt{L/C}}{R}$$

C. Find a condition for R such that $Q \gg 1$. In the limit, is R acting more like a short or an open circuit?

Solution:

$$R \ll \sqrt{L/C}$$

R is approaching a short.

D. Repeat the previous three parts for the parallel RLC combination.

Solution:

$$Z = R||Ls||\frac{1}{Cs} = R||\frac{Ls}{LCs^2+1} = \frac{s/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$Z = \frac{1}{C} \cdot \frac{s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

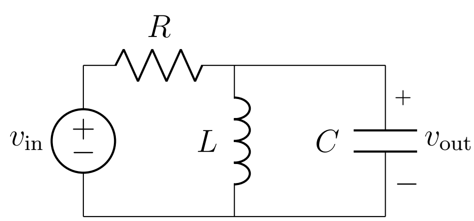
$$Q = \frac{1}{\sqrt{LC}}RC = \frac{R}{\sqrt{L/C}}$$

$$R \gg \sqrt{L/C}$$

R is approaching an open.

Problem 2

A. Find the transfer function for the parallel RLC circuit using impedances.



Solution:

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{Ls||\frac{1}{Cs}}{R + Ls||\frac{1}{Cs}}$$

$$H(s) = \frac{\frac{Ls}{LCs^2+1}}{R + \frac{Ls}{LCs^2+1}}$$

$$H(s) = \frac{Ls}{RLCs^2 + Ls + R}$$

$$H(s) = \frac{\frac{1}{RC}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

B. Show that at low frequencies a capacitor may be replaced with an open circuit and an inductor may be replaced with a short circuit. Show that the inverse is true at high frequencies.

Solution:
The complex impedance of a capacitor is $\frac{1}{Cj\omega}$. The impedance of the capacitor is related inversely to frequency. At low frequencies $\omega \to 0$, the impedance of the capacitor is really large and the capacitor acts like an open circuit. At high frequencies $\omega \to \infty$, the impedance of the capacitor is really small and the capacitor acts like a short circuit.

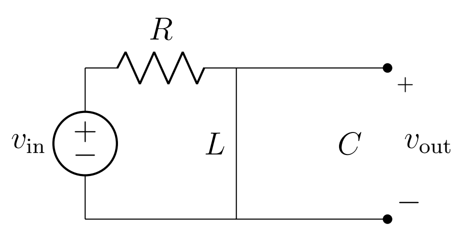
$$\begin{aligned}\lim_{\omega \to 0} \frac{1}{Cj\omega} &= \infty \\ \lim_{\omega \to \infty} \frac{1}{Cj\omega} &= 0\end{aligned}$$

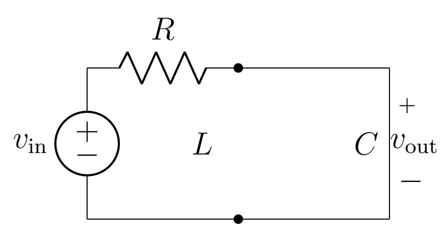
The complex impedance of an inductor is $Lj\omega$. The impedance of the inductor is related directly to frequency. At low frequencies $\omega \to 0$, the impedance of the inductor is really small and the inductor acts like a short circuit. At high frequencies $\omega \to \infty$, the impedance of the inductor is really large and the inductor acts like an open circuit.

$$\begin{aligned}\lim_{\omega \to 0} Lj\omega &= 0 \\ \lim_{\omega \to \infty} Lj\omega &= \infty\end{aligned}$$

C. Draw equivalent circuits for low frequencies and high frequencies. Use them to verify the extremes of the Bode plot from last week.

Solution:
The left circuit depicts how the original circuit behaves at low frequencies when the capacitor acts like an open circuit and the inductor acts like a short. The right circuit depicts the original circuit at high frequencies when the capacitor acts like a short and the inductor acts like an open circuit. The result in either of these cases is that $v_{out} = 0$.





D. What is the equivalent impedance of the parallel LC combination at resonance? What can you replace it with?

Solution:

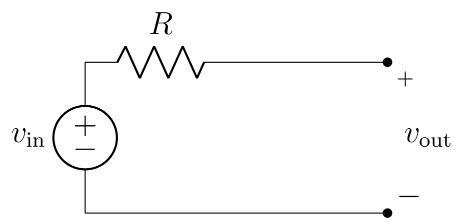
$$Z = Ls || \frac{1}{Cs} = \frac{1}{C} \cdot \frac{s}{s^2 + \frac{1}{LC}}$$

$$Z(j\omega_0) = Z\left(j\frac{1}{\sqrt{LC}}\right) = Ls || \frac{1}{Cs} = \frac{1}{C} \cdot \frac{j\frac{1}{\sqrt{LC}}}{-\frac{1}{LC} + \frac{1}{LC}} = \frac{1}{C} \cdot \frac{j\frac{1}{\sqrt{LC}}}{0}$$

A division by zero indicates that the current will be zero for any voltage across the combination, therefore, the combination acts like an open circuit at resonance.

E. Draw an equivalent circuit at resonance. Does it correspond to your Bode plot?

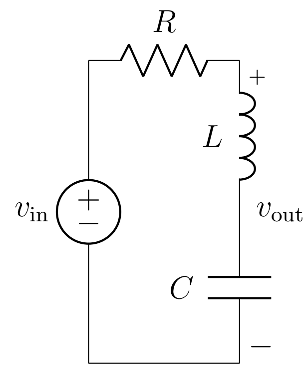
Solution:



At resonance, $v_{out} = v_{in}$, or $H(s) = 1$ which corresponds to the Bode plot. Note that it would be wrong to draw the L and the C open; neither is open, the parallel combination acts like an open.

Problem 3

A. Find the transfer function for the circuit shown below using impedances.



Solution:

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{Ls + \frac{1}{Cs}}{R + Ls + \frac{1}{Cs}}$$

$$H(s) = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{1}{L/R}s + \frac{1}{LC}}$$

$$\omega_0^2 = \frac{1}{LC} \qquad \alpha = \frac{1}{2L/R}$$

$$H(j\omega) = \frac{\omega_0^2 - \omega^2}{\omega_0^2 - \omega^2 + j2\alpha\omega}$$

B. Sketch the Bode plot of the magnitude of H(s).

Solution:

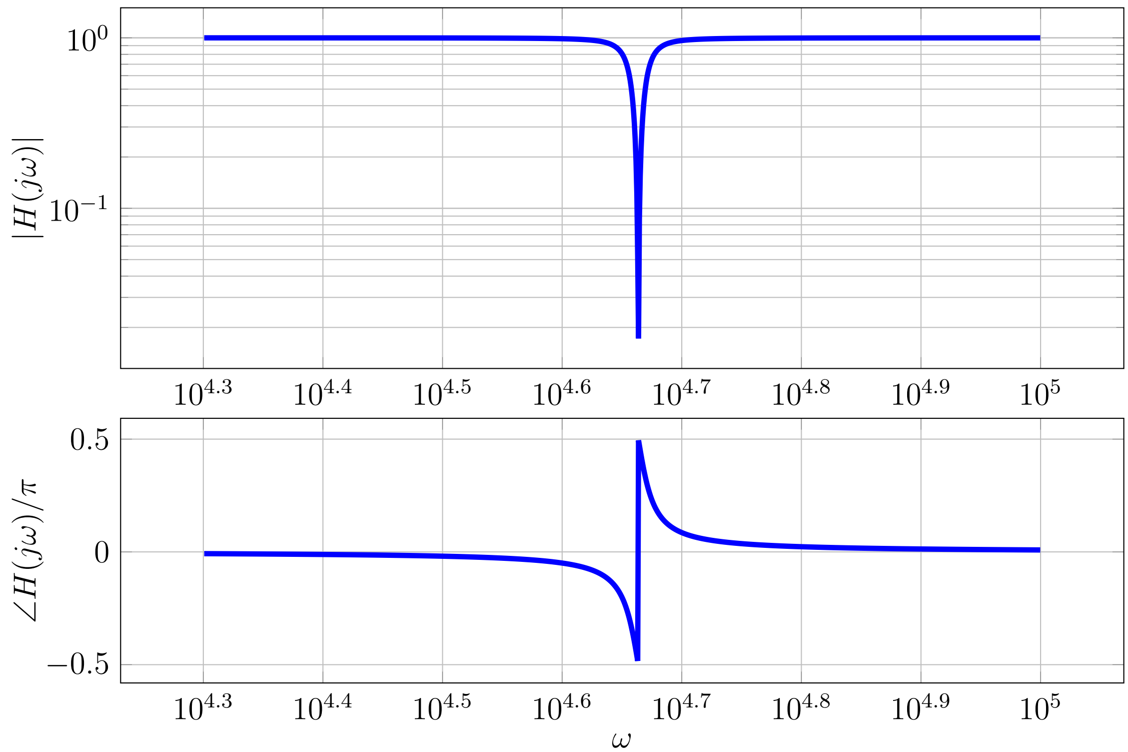
$$\omega \to 0 \qquad H(j\omega) \approx 1$$

$$\omega \to \infty \qquad H(j\omega) \approx 1$$

At resonance, $\omega = \omega_0$, and the series LC acts like a short.

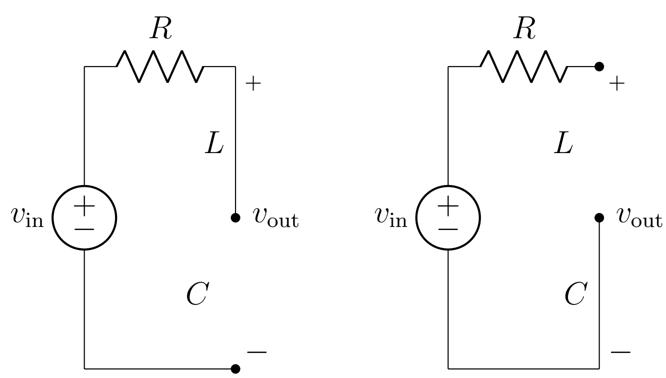
$$\omega = \omega_0 \qquad H(j\omega) = 0$$

The angle is significantly harder in this case and there is no easy way to get the behavior at resonance. I have included the graph of the angle below. Try to see if you can relate it to the transfer function!



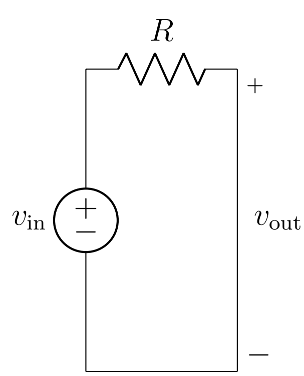
C. Recall that at low frequencies a capacitor may be replaced with an open circuit and an inductor may be replaced with a short circuit and the inverse is true at high frequencies. Draw equivalent circuits for low frequencies and high frequencies. Use them to verify the extremes of the Bode plot.

Solution:
The left circuit depicts how the original circuit behaves at low frequencies when the capacitor acts like an open circuit and the inductor acts like a short. The right circuit depicts the original circuit at high frequencies when the capacitor acts like a short and the inductor acts like an open circuit. The result in either of these cases is that $v_{out} = v_{in}$ as expected, since current flows through R.



D. Draw an equivalent circuit at resonance.

Solution:
At resonance, $v_{out} = 0$, or $H(s) = 0$.



Course feedback

Feel free to send any additional feedback directly to us.

Name (optional):

A. End time: How long did the assignment take you?

B. Are the lectures understandable and engaging?

C. Was the assignment effective in helping you learn the material?

D. Are you getting enough support from the teaching team?

E. Are the connections between lecture and assignment clear?

F. Are the objectives of the course clear? Do you feel you are making progress towards those objectives?

G. Anything else?