An alternative way of evaluating Fabius function at dyadic rationals

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Jeroen Fabius introduced the Fabius function [Fab66] as an example of an infinitely differentiable function that is nowhere analytic. Although according to [dR17], the function has been rediscovered many times in different contexts. One of the methods for evaluating the function at dyadic rationals has been presented in [Hau20]. This paper presents an alternative method for evaluating the Fabius function at dyadic rationals, summarised in Theorem 4.

The definition of the Fabius function follows directly from the definitions of the original paper [Fab66]. Assume u_i are independent and identically distributed samples from the uniform distribution on the unit interval (i.e., $u_k \sim U[0,1]\,\forall\, k \in \mathbb{N}$). Define a random variable $Y = \sum_{k=1}^{n} u_k/2^k$ which follows distribution $\mathcal{F}_n$ (i.e. $Y \sim \mathcal{F}_n$) with probability density function (PDF) p_n and cumulative distribution function (CDF) F_n. The Fabius distribution $\mathcal{F}$ with PDF p and CDF F is obtained by computing the limit of $\mathcal{F}_n$ as $n \to \infty$. It has been shown [Fab66] that for the Fabius distribution $p(y) = 2F(2y)$ if $0 \le y \le 1/2$. This relation will be used to evaluate the Fabius function in Theorem 4 when the input is a dyadic rational.

Define

$$Q(x,k,m) = \sum_{j=0}^{k-1} (-1)^{s_2(j)} \left(x-j\right)^m \tag{1}$$

$$G(r,k,t) = Q(r2^k, 2^k, k+t) \tag{2}$$

$$= \sum_{j=0}^{2^k-1} (-1)^{s_2(j)} \left(r2^k - j\right)^{k+t} \tag{3}$$

where $s_2(j)$ is a number of ones in the binary expansion of j.

Lemma 1. *If $x \in \mathbb{R}$ and $k, m \in \mathbb{N}$ then $Q(x, 2^k, k) = k!2^{\binom{k}{2}}$ and $Q(x, 2^k, m) = 0$ if $m < k$*

Proof. The lemma can be proved by induction. It is easy to show that $Q(x, 2^k, 0) = 0$ if $k > 0$ since the sum contains an equal number of even and odd powers of -1. Next, assume $Q(x, 2^k, m) = 0$ holds for all m if $m < k$. Then

$$\begin{aligned}
Q(x, 2^k, m+1) &= \sum_{j=0}^{2^k-1} (-1)^{s_2(j)} (x-j)^{m+1} \\
&= Q(x, 2^{k-1}, m+1) - \sum_{j=0}^{2^{k-1}-1} (-1)^{s_2(j)} (x-j-2^{k-1})^{m+1} \\
&= Q(x, 2^{k-1}, m+1) - \sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^i 2^{i(k-1)} Q(x, 2^{k-1}, m-i+1) \\
&= \sum_{i=1}^{m+1} \binom{m+1}{i} (-1)^{i+1} 2^{i(k-1)} Q(x, 2^{k-1}, m-i+1) \\
&= (m+1) 2^{k-1} Q(x, 2^{k-1}, m)
\end{aligned}$$

where in the second equality, we used the fact that $s_2(j+2^k) = s_2(j)+1$ if $j < 2^k$, and in the last step, we used induction to replace all but one term of the sum with zero. Solving the recursion, we get

$$Q(x, 2^k, m+1) = (m+1)! 2^{\binom{k}{2} - \binom{k-m}{2}} Q(x, 2^{k-m-1}, 0)$$

If $k = m+1$ then $Q(x,1,0) = 1$ and if $k > m+1$ then $Q(x, 2^{k-m-1}, 0) = 0$. Substituting both cases into the recursion finishes the proof. ■

Theorem 1. *If $n \in \mathbb{N}$ and p_n is a PDF of $\mathcal{F}_n$ then for $m \in \{0, 1, ..., 2^n - 2\}$ and $y \in \left[\frac{m}{2^n}, \frac{m+1}{2^n}\right]$*

$$p_n(y) = \frac{2^{\binom{n+1}{2}}}{(n-1)!} \sum_{k=0}^{m} (-1)^{s_2(k)} \left(y - \frac{k}{2^n}\right)^{n-1} \tag{4}$$

Proof. Define $x_{n+1} = x_n + u_{n+1}/2^{n+1}$ where $x_{n+1} \sim \mathcal{F}_{n+1}$, $x_n \sim \mathcal{F}_n$, $u_{n+1} \sim U[0,1]$ and $x_0 = 0$. Applying the convolution of two probability distributions, we get

$$\begin{aligned}
p_{n+1}(y) &= \int_{-\infty}^{\infty} p_n(x) q_{n+1}(y-x) dx \\
&= 2^{n+1} \int_{-\infty}^{\infty} p_n(x) \left[0 \le y - x \le 1/2^{n+1}\right] dx \\
&= 2^{n+1} \int_{y-1/2^{n+1}}^{y} p_n(x) dx
\end{aligned}$$

where [...] is a notation for Iverson bracket which is equal to 1 if the boolean statement inside the brackets is true and 0 if it's false.

We finish the proof with an inductive argument. If $n = 1$ then from (4) we get $p_1(y) = 2\left[0 \le y \le 1/2\right]$ which is a PDF of the uniform distribution $U[0, 1/2]$. Assume that (4) holds for p_n. It is a piece-wise polynomial, which means that in order to complete the induction proof, we must consider four separate cases:

Case 1: $y \in \left[0, \frac{1}{2^{n+1}}\right]$

$$p_{n+1}(y) = 2^{n+1} \int_0^y p_n(x) dx = 2^{n+1} \int_0^y \frac{2^{\binom{n+1}{2}}}{(n-1)!} x^{n-1} dx = \frac{2^{\binom{n+2}{2}}}{n!} y^n$$

Case 2: $y \in \left[\frac{2m}{2^{n+1}}, \frac{2m+1}{2^{n+1}}\right]$ where $m \in \{1, ..., 2^n - 2\}$

$$
\begin{aligned}
p_{n+1}(y) &= 2^{n+1}\left(\int_{y-1/2^{n+1}}^{m/2^n} p_n(x)dx + \int_{m/2^n}^{y} p_n(x)dx\right) \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\left(-\sum_{k=0}^{m-1}(-1)^{s_2(k)}\left(y - \frac{1}{2^{n+1}} - \frac{k}{2^n}\right)^n + \sum_{k=0}^{m}(-1)^{s_2(k)}\left(y - \frac{k}{2^n}\right)^n\right) \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\left(\sum_{k=0}^{m-1}(-1)^{s_2(2k+1)}\left(y - \frac{2k+1}{2^{n+1}}\right)^n + \sum_{k=0}^{m}(-1)^{s_2(2k)}\left(y - \frac{2k}{2^{n+1}}\right)^n\right) \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\sum_{k=0}^{2m}(-1)^{s_2(k)}\left(y - \frac{k}{2^{n+1}}\right)^n
\end{aligned}
$$

where the third equality uses the fact that $s_2(2k) = s_2(k)$ and $s_2(2k+1) = s_2(k) + 1$

Case 3: $y \in \left[\frac{2m+1}{2^{n+1}}, \frac{2m+2}{2^{n+1}}\right]$ where $m \in \{0, ..., 2^n - 2\}$

$$
\begin{aligned}
p_{n+1}(y) &= 2^{n+1}\int_{y-1/2^{n+1}}^{y} p_n(x)dx \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\left(\sum_{k=0}^{m}(-1)^{s_2(k)}\left(y - \frac{2k}{2^{n+1}}\right)^n + \sum_{k=0}^{m}(-1)^{s_2(k)+1}\left(y - \frac{2k+1}{2^{n+1}}\right)^n\right) \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\sum_{k=0}^{2m+1}(-1)^{s_2(k)}\left(y - \frac{k}{2^{n+1}}\right)^n
\end{aligned}
$$

Case 4: $y \in \left[\frac{2^{n+1}-2}{2^{n+1}}, \frac{2^{n+1}-1}{2^{n+1}}\right]$

$$
\begin{aligned}
p_{n+1}(y) &= 2^{n+1}\int_{y-1/2^{n+1}}^{1-1/2^n} p_n(x)dx \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\left(\sum_{k=0}^{2^n-2}(-1)^{s_2(k)}\left(1 - \frac{1}{2^n} - \frac{k}{2^n}\right)^n + \sum_{k=0}^{2^n-2}(-1)^{s_2(k)+1}\left(y - \frac{2k+1}{2^{n+1}}\right)^n\right) \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\left(\frac{1}{2^{n^2}}\sum_{k=0}^{2^n-1}(-1)^{s_2(k)}\left(\left(1 - \frac{1}{2^n}\right)2^n - k\right)^n + \sum_{k=0}^{2^n-2}(-1)^{s_2(2k+1)}\left(y - \frac{2k+1}{2^{n+1}}\right)^n\right) \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\left(\frac{1}{2^{n^2}}G\left(1 - \frac{1}{2^n}, n, 0\right) + \sum_{k=0}^{2^n-2}(-1)^{s_2(2k+1)}\left(y - \frac{2k+1}{2^{n+1}}\right)^n\right) \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\left(\frac{1}{2^{n^2}}G(y, n, 0) + \sum_{k=0}^{2^n-2}(-1)^{s_2(2k+1)}\left(y - \frac{2k+1}{2^{n+1}}\right)^n\right) \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\left(\sum_{k=0}^{2^n-1}(-1)^{s_2(2k)}\left(y - \frac{2k}{2^{n+1}}\right)^n + \sum_{k=0}^{2^n-2}(-1)^{s_2(2k+1)}\left(y - \frac{2k+1}{2^{n+1}}\right)^n\right) \\
&= \frac{2^{\binom{n+2}{2}}}{n!}\sum_{k=0}^{2^{n+1}-2}(-1)^{s_2(k)}\left(y - \frac{k}{2^{n+1}}\right)^n
\end{aligned}
$$

where fifth equality uses the fact that $G(1 - 1/2^n, n, 0) = G(y, n, 0)$ which holds according to Lemma 1 and definition (2).

Four cases cover all possible values of $y \in [0, 1 - 1/2^{n+1}]$, which completes the proof by induction. ■

We must rely on coefficients with double and triple indices in the upcoming lemmas and theorems. For simplicity, coefficients with triple indices will be specified without a space or commas. For example, variable α_{tij} points to a coefficient associated with a tuple (t, i, j). In addition, brackets will be used to separate indices when addition or subtraction is used. For example, $\alpha_{(t-i)(i-j)j}$ is associated with a tuple $(t-i, i-j, j)$.

Lemma 2. *If $r \in \mathbb{R}$, $k, t \in \mathbb{N}$ and G is defined by equation (2) then*

$$G(r,k,t) = (k+t)!2^{\binom{k}{2}} \sum_{i=0}^{t} 2^{ik} \sum_{j=0}^{i} \alpha_{tij} r^j \tag{5}$$

$$a_{tij} = (-1)^{i-j} \binom{i}{j} \left(\frac{1}{i!(t-i)!} - q_{ti} \right) + f_{tij} \tag{6}$$

$$q_{ti} = \sum_{a=i}^{t-1} \sum_{b=i}^{a} \sum_{c=i}^{b} \binom{c}{i} \frac{\alpha_{abc}}{(t-a+1)!(2^{t+b-a-i}-1)} \tag{7}$$

$$f_{tij} = \sum_{a=j}^{i-1} \sum_{b=j}^{a} \sum_{c=j}^{b} (-1)^{c-j} \binom{b}{c} \binom{c}{j} \frac{\alpha_{(t-i+a)ab}}{(i-a+1)!(2^{i-c}-1)} \tag{8}$$

Proof. We start the proof by defining the recursion on G

$$\begin{aligned}
G(r,k,t) &= \sum_{i=0}^{2^k-1} (-1)^{s_2(i)} \left(r2^k - i\right)^{k+t} \\
&= \sum_{i=0}^{2^{k-1}-1} (-1)^{s_2(i)} \left(r2^k - i\right)^{k+t} - \sum_{i=0}^{2^{k-1}-1} (-1)^{s_2(i)} \left(r2^k - 2^{k-1} - i\right)^{k+t} \\
&= \sum_{i=0}^{2^{k-1}-1} (-1)^{s_2(i)} 2^{k-1} \sum_{j=0}^{k+t-1} (r2^k - i)^j ((2r-1)2^{k-1} - i)^{k+t-j-1} \\
&= \sum_{i=0}^{2^{k-1}-1} (-1)^{s_2(i)} 2^{k-1} \sum_{j=0}^{k+t-1} \sum_{s=0}^{j} \binom{j}{s} 2^{(k-1)s} ((2r-1)2^{k-1} - i)^{k+t-s-1} \\
&= \sum_{i=0}^{2^{k-1}-1} (-1)^{s_2(i)} 2^{k-1} \sum_{s=0}^{k+t-1} \binom{k+t}{s+1} 2^{(k-1)s} ((2r-1)2^{k-1} - i)^{k+t-s-1} \\
&= \sum_{s=0}^{k+t-1} \binom{k+t}{s+1} 2^{(k-1)(s+1)} \sum_{i=0}^{2^{k-1}-1} (-1)^{s_2(i)} ((2r-1)2^{k-1} - i)^{k+t-s-1} \\
&= \sum_{s=0}^{t} \binom{k+t}{s+1} 2^{(k-1)(s+1)} G(2r-1, k-1, t-s)
\end{aligned}$$

The second equality is obtained by splitting the sum into two sums and using the fact that $s_2(j+2^k) = s_2(j)+1$ if $j \in \{0, 1, ..., 2^k-1\}$. The fifth equality is obtained by interchanging the sums and using the fact that $\sum_{j=s}^{m} \binom{j}{s} = \binom{m+1}{s+1}$. The final equality uses the definition of G and Lemma 1 to eliminate the terms for which $t < s$.

The rest of the proof can be finished by induction. We can see the equation (5) holds for $G(r,0,0)$ for all r, since $G(r,0,0) = \alpha_{000} = 1$. Next we assume by strong induction that $G(r,k',t')$ holds for all r when $k' < k$, $t' < t$. Finally, we use previously derived recursion to show that the formula holds for larger k and t.

With induction, we can represent previously derived recursion in the form $x_k = a_k x_{k-1} + b_k$ where

$$
\begin{aligned}
x_m &= G((r-1)2^{k-m}+1, m, t) \\
a_m &= (m+t)2^{m-1} \\
b_m &= \sum_{s=1}^{t} \binom{m+t}{s+1} 2^{(m-1)(s+1)} G((r-1)2^{k-m+1}+1, m-1, t-s)
\end{aligned}
$$

We can solve recursion for x_k and get a solution in the following form

$$
\begin{aligned}
x_k &= x_0 \prod_{s=1}^{k} a_s + \sum_{m=1}^{k} b_m \prod_{s=m+1}^{k} a_s \\
x_0 &= ((r-1)2^k+1)^t
\end{aligned}
$$

where $\prod_{s=k+1}^{k} a_s = 1$.

For b_m, we can expand G terms by induction and get the following formula

$$
\begin{aligned}
b_m &= \sum_{s=1}^{t} \binom{m+t}{s+1} 2^{(m-1)(s+1)} G((r-1)2^{k-m+1}+1, m-1, t-s) \\
&= 2^{\binom{m}{2}} \sum_{s=1}^{t} \frac{(m+t)!}{(s+1)!} 2^{(m-1)s} \sum_{a=0}^{t-s} 2^{a(m-1)} \sum_{b=0}^{a} \alpha_{(t-s)ab} ((r-1)2^{k-m+1}+1)^b \\
&= (m+t)! 2^{\binom{m}{2}} \sum_{s=1}^{t} \sum_{a=0}^{t-s} \sum_{b=0}^{a} \frac{\alpha_{(t-s)ab} 2^{(a+s)(m-1)}}{(s+1)!} ((r-1)2^{k-m+1}+1)^b \\
&= (m+t)! 2^{\binom{m}{2}} \sum_{s=1}^{t} \sum_{a=0}^{t-s} \sum_{b=0}^{a} \sum_{c=0}^{b} \alpha_{(t-s)ab} \binom{b}{c} \frac{2^{(a+s-c)(m-1)+ck}}{(s+1)!} (r-1)^c
\end{aligned}
$$

next, we can find expressions for sums and products required for the x_k formula

$$
\begin{aligned}
\prod_{s=m+1}^{k} a_s &= \frac{(k+t)!}{(m+t)!} 2^{\binom{k}{2}-\binom{m}{2}} \\
x_0 \prod_{s=1}^{k} a_s &= \frac{(k+t)!}{t!} 2^{\binom{k}{2}} ((r-1)2^k+1)^t \\
&= \frac{(k+t)!}{t!} 2^{\binom{k}{2}} \sum_{i=0}^{t} \binom{t}{i} 2^{ik} (r-1)^i \\
&= (k+t)! 2^{\binom{k}{2}} \sum_{i=0}^{t} 2^{ik} \sum_{j=0}^{i} \frac{(-1)^{i-j}}{t!} \binom{t}{i} \binom{i}{j} r^j \\
\sum_{m=1}^{k} b_m \prod_{s=m+1}^{k} a_s &= (k+t)! 2^{\binom{k}{2}} \sum_{m=0}^{k-1} \sum_{s=1}^{t} \sum_{a=0}^{t-s} \sum_{b=0}^{a} \sum_{c=0}^{b} \alpha_{(t-s)ab} \binom{b}{c} \frac{2^{(a+s-c)m+ck}}{(s+1)!} (r-1)^c \\
&= (k+t)! 2^{\binom{k}{2}} \sum_{s=1}^{t} \sum_{a=0}^{t-s} \sum_{b=0}^{a} \sum_{c=0}^{b} \alpha_{(t-s)ab} \binom{b}{c} \frac{1}{(s+1)!} \frac{2^{(a+s)k} - 2^{ck}}{2^{a+s-c}-1} (r-1)^c
\end{aligned}
$$

In the last formula, we need to separate nested sums into two parts to finish the proof. The first sum is over 2^{ck} terms, and the second one is over $2^{(a+s)k}$. Next, we re-index the sum in order to obtain 2^{ik}

terms, which gives us

$$\sum_{m=1}^{k} b_m \prod_{s=m+1}^{k} a_s = (k+t)!2^{\binom{k}{2}} \left(\sum_{i=0}^{t-1} 2^{ik} \sum_{j=0}^{i} (-1)^{i-j+1} \binom{i}{j} q_{ti} r^j + \sum_{i=1}^{t} 2^{ik} \sum_{j=0}^{i-1} f_{tij} r^j \right)$$

$$q_{ti} = \sum_{a=i}^{t-1} \sum_{b=i}^{a} \sum_{c=i}^{b} \binom{c}{i} \frac{\alpha_{abc}}{(t-a+1)!(2^{t+b-a-i}-1)}$$

$$f_{tij} = \sum_{a=j}^{i-1} \sum_{b=j}^{a} \sum_{c=j}^{b} (-1)^{c-j} \binom{b}{c} \binom{c}{j} \frac{\alpha_{(t-i+a)ab}}{(i-a+1)!(2^{i-c}-1)}$$

Finally, we get the answer by plugging everything into the recursion formula using the fact that $q_{tt} = f_{tii} = 0$, which gives us the desired formula

$$G(r,k,t) = (k+t)!2^{\binom{k}{2}} \sum_{i=0}^{t} 2^{ik} \sum_{j=0}^{i} \alpha_{tij} r^j$$

$$a_{tij} = (-1)^{i-j} \binom{i}{j} \left(\frac{1}{i!(t-i)!} - q_{ti} \right) + f_{tij}$$

■

Theorem 2. *If $0 \leq j \leq i \leq t$ and α_{tij} is defined by equation (6) then*

$$\alpha_{tij} = \alpha_{(t-i)00}\, \alpha_{(i-j)(i-j)0}\, \alpha_{jjj} \tag{9}$$

Proof. With equation (6), we can simplify each of the decomposition terms

$$\alpha_{jjj} = \frac{1}{j!} - q_{jj} + f_{jjj} = \frac{1}{j!} \tag{10}$$

$$\alpha_{(i-j)(i-j)0} = \frac{(-1)^{i-j}}{(i-j)!} + f_{(i-j)(i-j)0} \tag{11}$$

$$\alpha_{(t-i)00} = \frac{1}{(t-i)!} - q_{(t-i)0} \tag{12}$$

where q_{ti} is define in equation (7) and f_{tij} in equation (8). In the above equalities, we used the identity $q_{tt} = f_{tii} = 0$ to make simplifications. The main result can be proved by induction. Multiplying the terms above gives us

$$\begin{aligned}
\alpha_{(t-i)00}\, \alpha_{(i-j)(i-j)0}\, \alpha_{jjj} &= \left(\frac{1}{(t-i)!} - q_{(t-i)0} \right) \left(\frac{(-1)^{i-j}}{(i-j)!} + f_{(i-j)(i-j)0} \right) \frac{1}{j!} \\
&= (-1)^{i-j} \binom{i}{j} \left(\frac{1}{i!(t-i)!} - \frac{1}{i!} q_{(t-i)0} \right) + f_{(i-j)(i-j)0} \frac{1}{j!} \left(\frac{1}{(t-i)!} - q_{(t-i)0} \right)
\end{aligned}$$

The RHS equals α_{tij} if the two identities below hold

$$\begin{aligned}
q_{ti} &= \frac{1}{i!} q_{(t-i)0} \\
f_{tij} &= \frac{1}{j!} f_{(i-j)(i-j)0} \left(\frac{1}{(t-i)!} - q_{(t-i)0} \right) \\
&= \frac{1}{j!} f_{(i-j)(i-j)0} \alpha_{(t-i)00}
\end{aligned}$$

The validity of the equalities above can be proved by induction. We start the induction by noticing that the identity holds trivially if $t, i, j \in \{0, 1\}$ and $j \leq i \leq t$. Next, assuming that induction holds, we can get the identity

$$
\begin{aligned}
\alpha_{(a+i)(b+i)(c+i)} &= \alpha_{(a-b)00}\alpha_{(b-c)(b-c)0}\alpha_{(c+i)(c+i)(c+i)} \\
&= \alpha_{(a-b)00}\alpha_{(b-c)(b-c)0}\frac{1}{(c+i)!} \\
&= \alpha_{(a-b)00}\alpha_{(b-c)(b-c)0}\alpha_{ccc}\frac{c!}{(c+i)!} \\
&= \alpha_{abc}\frac{c!}{(c+i)!}
\end{aligned}
$$

From equation (7) we see that q_{ti} depends only on α_{abc} terms for which $a < t$. By strong induction in combination with equation (7) we get

$$
\begin{aligned}
q_{ti} &= \sum_{a=i}^{t-1}\sum_{b=i}^{a}\sum_{c=i}^{b}\binom{c}{i}\frac{\alpha_{abc}}{(t-a+1)!(2^{t+b-a-i}-1)} \\
&= \sum_{a=0}^{t-i-1}\sum_{b=0}^{a}\sum_{c=0}^{b}\binom{c+i}{i}\frac{\alpha_{(a+i)(b+i)(c+i)}}{(t-a-i+1)!(2^{t+b-a-i}-1)} \\
&= \sum_{a=0}^{t-i-1}\sum_{b=0}^{a}\sum_{c=0}^{b}\binom{c+i}{i}\frac{c!}{(c+i)!}\frac{\alpha_{abc}}{(t-a-i+1)!(2^{t+b-a-i}-1)} \\
&= \frac{1}{i!}\sum_{a=0}^{t-i-1}\sum_{b=0}^{a}\sum_{c=0}^{b}\frac{\alpha_{abc}}{(t-i-a+1)!(2^{t-i+b-a}-1)} \\
&= \frac{1}{i!}q_{(t-i)0}
\end{aligned}
$$

Similarly, proof of the f_{tij} identity requires a different α_{tij} identity, which we assume holds by induction

$$
\begin{aligned}
\alpha_{(a+c+j)(a+j)(b+j)} &= \alpha_{c00}\alpha_{(a-b)(a-b)0}\alpha_{(b+j)(b+j)(b+j)} \\
&= \alpha_{c00}\alpha_{(a-b)(a-b)0}\frac{1}{(b+j)!} \\
&= \alpha_{c00}\alpha_{(a-b)(a-b)0}\alpha_{bbb}\frac{b!}{(b+j)!} \\
&= \alpha_{c00}\alpha_{aab}\frac{b!}{(b+j)!}
\end{aligned}
$$

From equation (8) we see that f_{tij} depends only on α_{abc} terms for which $a < t$. By strong induction, we get

$$
\begin{aligned}
f_{tij} &= \sum_{a=j}^{i-1}\sum_{b=j}^{a}\sum_{c=j}^{b}(-1)^{c-j}\binom{b}{c}\binom{c}{j}\frac{\alpha_{(t-i+a)ab}}{(i-a+1)!(2^{i-c}-1)} \\
&= \sum_{a=0}^{i-j-1}\sum_{b=0}^{a}\sum_{c=0}^{b}(-1)^{c}\binom{b+j}{c+j}\binom{c+j}{j}\frac{\alpha_{(t-i+a+j)(a+j)(b+j)}}{(i-j-a+1)!(2^{i-j-c}-1)} \\
&= \alpha_{(t-i)00}\sum_{a=0}^{i-j-1}\sum_{b=0}^{a}\sum_{c=0}^{b}(-1)^{c}\binom{b+j}{c+j}\binom{c+j}{j}\frac{b!}{(b+j)!}\frac{\alpha_{aab}}{(i-j-a+1)!(2^{i-j-c}-1)} \\
&= \frac{1}{j!}\alpha_{(t-i)00}\sum_{a=0}^{i-j-1}\sum_{b=0}^{a}\sum_{c=0}^{b}(-1)^{c}\binom{b}{c}\frac{\alpha_{aab}}{(i-j-a+1)!(2^{i-j-c}-1)} \\
&= \frac{1}{j!}\alpha_{(t-i)00}f_{(i-j)(i-j)0}
\end{aligned}
$$

This completes the proof. ∎

Define $\beta_i = \alpha_{ii0}$ and $\gamma_i = \alpha_{i00}$. Theorem 2 allows us to define α_{tij} in terms of β and γ

$$\alpha_{tij} = \frac{\beta_{i-j}\,\gamma_{t-i}}{j!} \tag{13}$$

The following lemma shows an essential relation between β_i and γ_j

Theorem 3. *Define $g(z) = \beta_0 + \beta_1 z + \beta_2 z^2 + \ldots$ and $f(z) = \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \ldots$. Then*

$$g(z)f(z) = 1 \tag{14}$$

$$g(z)f(2z) = B^+(z) = \frac{z}{1-e^{-z}} \tag{15}$$

$$g(-z)f(z) = e^z \tag{16}$$

where $B^+(z)$ is a generating function for the Bernoulli numbers with $B_1^+ = 1/2$.

Proof. From definition for $G(r,k,t)$ in equation (2), we can easily show that $G(r,0,t) = r^t$. Similarly, we can set $k = 0$ in equation (5) to get the following relation

$$G(r,0,t) = t!\sum_{i=0}^{t}\sum_{j=0}^{i}\alpha_{tij}r^j = t!\sum_{j=0}^{t}\sum_{i=j}^{t}\alpha_{tij}r^j = r^t + t!\sum_{j=0}^{t-1}\sum_{i=j}^{t}\alpha_{tij}r^j$$

where in the last equality we used the fact that $\alpha_{ttt} = 1/t!$. If $t > 0$, then we can interpret G as a polynomial in r, which means that the above equation's RHS must equal r^t for all r. It implies that if $0 \le j < t$, then

$$\sum_{i=j}^{t}\alpha_{tij} = 0$$

Re-indexing the sum, replacing $t-j$ with t and using decomposition of α_{tij} from equation (13) gives us

$$\sum_{i=0}^{t}\gamma_{t-i}\beta_i = 0 \tag{17}$$

For the $t = 0$ case, we can easily show that $\beta_0\gamma_0 = 1$ since $\beta_0 = \gamma_0 = 1$. This proves the identity in equation (14), namely $g(z)f(z) = 1$.

Similarly, to prove equation (15), we can consider the case where $r = 0$ and $k = 1$. If we plug these values into equations (5) and (2), use the identity $\alpha_{ti0} = \beta_i\gamma_{t-i}$ then we get

$$(t+1)!\sum_{i=0}^{t}2^i\beta_i\gamma_{t-i} = (-1)^t \tag{18}$$

From this, the formula for $g(2z)f(z) = (1-e^{-z})/z$ immediately follows. Rearranging the terms with the help of the equation (14) gives us the desired result in equation (15).

The $zg(2z) = g(z)(1-e^{-z})$ identity also gives us a recursive definition of β_t, which we will use to prove the final identity in equation (16)

$$\beta_t = \frac{1}{2^t-1}\sum_{i=0}^{t-1}\frac{(-1)^{t-i}\beta_i}{(t-i+1)!} \tag{19}$$

To prove the equation (16), it is sufficient to show that the following identity holds

$$\sum_{i=0}^{t}\frac{\beta_i}{(t-i)!} = (-1)^t\beta_t \tag{20}$$

To do that we use definition of β_t, namely $\beta_t = \alpha_{tt0} = (-1)^t/t! + f_{tt0}$ where f_{tt0} is defined in equation (8). First, we rewrite f_{tt0} by cancelling some terms

$$f_{tt0} = \sum_{a=0}^{t-1} \sum_{b=0}^{a} \sum_{c=0}^{b} \frac{(-1)^c \beta_{a-b}}{(t-a+1)!(2^{t-c}-1)c!(b-c)!} \tag{21}$$

Next, we apply the change of variables in summation

$$\begin{cases} 0 \le a \le t-1 \\ 0 \le b \le a \\ 0 \le c \le b \end{cases} \iff \begin{cases} 0 \le c \le t-1 \\ c \le a \le t-1 \\ c \le b \le a \end{cases} \iff \begin{cases} 0 \le t-c-1 \le t-1 \\ 0 \le a-c \le t-c-1 \\ 0 \le a-b \le a-c \end{cases} \iff \begin{cases} 0 \le k \le t-1 \\ 0 \le j \le k \\ 0 \le i \le j \end{cases}$$

where we used the following substitution $k = t - c - 1$, $j = a - c$ and $i = a - b$. We can plug new variables into the equation (21) and get

$$f_{tt0} = \sum_{k=0}^{t-1} \sum_{j=0}^{k} \sum_{i=0}^{j} \frac{(-1)^{t-k-1} \beta_i}{(k-j+2)!(2^{k+1}-1)(t-k-1)!(j-i)!}$$

Next, we can simplify the equation above by using an inductive argument on equation (20). The identity holds trivially for $t = 0$. Next, by strong induction, we assume that it holds for all $j < t$, which gives us

$$\begin{aligned} f_{tt0} &= \sum_{k=0}^{t-1} \sum_{j=0}^{k} \sum_{i=0}^{j} \frac{(-1)^{t-k-1} \beta_i}{(k-j+2)!(2^{k+1}-1)(t-k-1)!(j-i)!} \\ &= \sum_{k=0}^{t-1} \sum_{j=0}^{k} \frac{(-1)^{t-k-1}}{(k-j+2)!(2^{k+1}-1)(t-k-1)!} \sum_{i=0}^{j} \frac{\beta_i}{(j-i)!} \\ &= \sum_{k=0}^{t-1} \frac{(-1)^t}{(t-k-1)!(2^{k+1}-1)} \sum_{j=0}^{k} \frac{(-1)^{k-j+1} \beta_j}{(k-j+2)!} \\ &= (-1)^t \sum_{k=1}^{t} \frac{\beta_k}{(t-k)!} \end{aligned}$$

where in the third equality we applied induction to get $(-1)^j \beta_j$ where $j < t$ and in the fourth equality we used recursive definition of β_{k+1} from equation (19) to make further simplifications.

And finally

$$(-1)^t \beta_t = \frac{1}{t!} + (-1)^t f_{tt0} = \frac{1}{t!} + \sum_{k=1}^{t} \frac{\beta_k}{(t-k)!} = \sum_{k=0}^{t} \frac{\beta_k}{(t-k)!}$$

which gives us $g(-z) = g(z)e^z$ relation and completes the proof ∎

Theorem 4. *If F is a CDF of the Fabius distribution, $k, m \in \mathbb{N}$ such that $0 \le k/2^m \le 1$ then*

$$F\left(\frac{k}{2^m}\right) = \sum_{i=0}^{m} F\left(\frac{1}{2^i}\right) \frac{(-1)^i}{(m-i)!} \frac{2^{\binom{i}{2}}}{2^{\binom{m}{2}}} \sum_{j=0}^{k-1} (-1)^{s_2(j)} (k-j)^{m-i} \tag{22}$$

$$F\left(\frac{1}{2^m}\right) = \frac{1}{2^{\binom{m}{2}}(2^m-1)} \sum_{i=0}^{m-1} \frac{2^{\binom{i}{2}}}{(m-i+1)!} F\left(\frac{1}{2^i}\right) \tag{23}$$

Proof. We assume $n \geq m$. Then from Theorem 1 we get

$$\begin{aligned}
p_n\left(\frac{k}{2^m}\right) &= \frac{2^{\binom{n+1}{2}}}{(n-1)!}\sum_{j=0}^{k2^{n-m}-1}(-1)^{s_2(j)}\left(\frac{k}{2^m}-\frac{j}{2^n}\right)^{n-1} \\
&= \frac{2^{\binom{n+1}{2}}}{2^{n(n-1)}(n-1)!}\sum_{i=0}^{k-1}\sum_{j=0}^{2^{n-m}-1}(-1)^{s_2(i)+s_2(j)}\left((k-i)2^{n-m}-j\right)^{n-1} \\
&= \frac{2^{\binom{n+1}{2}}}{2^{n(n-1)}(n-1)!}\sum_{r=0}^{k-1}(-1)^{s_2(r)}G(k-r,n-m,m-1) \\
&= \frac{2^{\binom{n+1}{2}+\binom{n-m}{2}}}{2^{n(n-1)}}\sum_{r=0}^{k-1}(-1)^{s_2(r)}\sum_{i=0}^{m-1}2^{i(n-m)}\sum_{j=0}^{i}\alpha_{(m-1)ij}(k-r)^j \\
&= 2^{\binom{m+1}{2}}\sum_{i=0}^{m-1}\frac{1}{2^{n(m-i-1)+im}}\sum_{j=0}^{i}\alpha_{(m-1)ij}Q(k,k,j)
\end{aligned}$$

where the second equality uses the fact that $s_2(i2^k+j) = s_2(i)+s_2(j)$ if $j < 2^k$ and the fourth equality follows from equation (5). Combining the equation above with the fact that $F(y) = p(y/2)/2$ we get

$$\begin{aligned}
F\left(\frac{k}{2^m}\right) &= \frac{1}{2}\,p\left(\frac{k}{2^{m+1}}\right) \\
&= \lim_{n\to\infty}\frac{1}{2}\,p_n\left(\frac{k}{2^{m+1}}\right) \\
&= \lim_{n\to\infty}2^{\binom{m+2}{2}}\sum_{i=0}^{m}\frac{1}{2^{n(m-i)+i(m+1)+1}}\sum_{j=0}^{i}\alpha_{mij}Q(k,k,j) \\
&= \frac{2^{\binom{m+2}{2}}}{2^{m^2+m+1}}\sum_{j=0}^{m}\alpha_{mmj}Q(k,k,j) \\
&= \frac{1}{2^{\binom{m}{2}}}\sum_{j=0}^{m}\beta_{m-j}\frac{Q(k,k,j)}{j!} \\
&= \frac{1}{2^{\binom{m}{2}}}\sum_{i=0}^{m}\beta_i\frac{Q(k,k,m-i)}{(m-i)!}
\end{aligned}$$

Next, consider the case where $F(1/2^m)$. Then from formula for $F(k/2^m)$ and equation (20), which follows from Theorem 3, we get

$$F\left(\frac{1}{2^m}\right) = \frac{1}{2^{\binom{m}{2}}}\sum_{i=0}^{m}\frac{\beta_i}{(m-i)!} = \frac{(-1)^m}{2^{\binom{m}{2}}}\beta_m$$

Rearranging the terms gives us the formula for the β_m terms in terms of the Fabius function

$$\beta_m = (-1)^m 2^{\binom{m}{2}}F\left(\frac{1}{2^m}\right) \tag{24}$$

Substituting the equation (24) into the equation (19) and formula for $F(k/2^m)$ finishes the proof. ■

It is important to note that β_m in equation (24) has quite a similar relation to the Fabius function as a previously discovered constant d_n in [dR17] and w_n in [BR23].

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