1 Matematica numerica

1.1 Quick Start

1.2 Table of Theorems

1.3 Basic Theorem Environments

Let's start with the most fundamental definition.

Definizione 1.3.1

A natural number is called a *prime number* if it is greater than 1 and cannot be written as the product of two smaller natural numbers.

Esempio. The numbers 2, 3, and 17 are prime. As proven in Corollario 1.3.2.1, this list is far from complete! See Teorema 1.3.2 for the full proof.

Teorema 1.3.2 (Euclid's Theorem)

There are infinitely many prime numbers.

Dimostrazione. By contradiction: Suppose $p_1, p_2, ..., p_n$ is a finite enumeration of all primes. Let $P = p_1 p_2 ... p_n$. Since $P + 1$ is not in our list, it cannot be prime. Thus, some prime p_j divides $P + 1$. Since p_j also divides P, it must divide their difference $(P + 1) - P = 1$, a contradiction. □

Corollario 1.3.2.1

There is no largest prime number.

Lemma 1.3.3

There are infinitely many composite numbers.

1.4 Functions and Continuity

Teorema 1.4.1 (Continuity Theorem)

If a function f is differentiable at every point, then f is continuous.

Consiglio

Teorema 1.4.1 tells us that differentiability implies continuity, but not vice versa. For example, $f(x) = |x|$ is continuous but not differentiable at $x = 0$. For a deeper understanding of continuous functions, see Teorema A.1.1 in the appendix.

1.5 Geometric Theorems

Teorema 1.5.1 (Pythagorean Theorem)

In a right triangle, the square of the hypotenuse equals the sum of squares of the other two sides: $x^2 + y^2 = z^2$

Importante

Teorema 1.5.1 is one of the most fundamental and important theorems in plane geometry, bridging geometry and algebra.

Corollario 1.5.1.1

There exists no right triangle with sides measuring 3cm, 4cm, and 6cm. This directly follows from Teorema 1.5.1.

Lemma 1.5.2

Given two line segments of lengths a and b, there exists a real number r such that $b = ra$.

1.6 Algebraic Structures

Definizione 1.6.1 (Ring)

Let R be a non-empty set with two binary operations $+$ and $\cdot$, satisfying:
1. $(R, +)$ is an abelian group
2. $(R, \cdot)$ is a semigroup
3. The distributive laws hold

Then $(R, +, \cdot)$ is called a ring.

Proposizione 1.6.2

Every field is a ring, but not every ring is a field. This concept builds upon Definizione 1.6.1.

Esempio. Consider Definizione 1.6.1. The ring of integers $\mathbb{Z}$ is not a field, as no elements except ± 1 have multiplicative inverses.

A Theorion Appendices

A.1 Advanced Analysis

Teorema A.1.1 (Maximum Value Theorem)

A continuous function on a closed interval must attain both a maximum and a minimum value.

Avvertenza

Both conditions of this theorem are essential:
- The function must be continuous
- The domain must be a closed interval

A.2 Advanced Algebra Supplements

Assioma A.2.1 (Group Axioms)

A group $(G, \cdot)$ must satisfy:
1. Closure
2. Associativity
3. Identity element exists
4. Inverse elements exist

Postulato A.2.2 (Fundamental Theorem of Algebra)

Every non-zero polynomial with complex coefficients has a complex root.

Osservazione

This theorem is also known as Gauss's theorem, as it was first rigorously proved by Gauss.

A.3 Common Problems and Solutions

Problema. Prove: For any integer $n > 1$, there exists a sequence of n consecutive composite numbers.

Soluzione. Consider the sequence: $n! + 2, n! + 3, ..., n! + n$

For any $2 \leq k \leq n$, $n! + k$ is divisible by k because: $n! + k = k\left(\frac{n!}{k} + 1\right)$

Thus, this forms a sequence of $n - 1$ consecutive composite numbers.

Esercizio.
1. Prove: The twin prime conjecture remains unproven.
2. Try to explain why this problem is so difficult.

Conclusione. Number theory contains many unsolved problems that appear deceptively simple yet are profoundly complex.

A.4 Important Notes

Nota

Remember that mathematical proofs should be both rigorous and clear. Clarity without rigor is insufficient, and rigor without clarity is ineffective.

Attenzione

When dealing with infinite series, always verify convergence before discussing other properties.

> Mathematics is the queen of sciences, and number theory is the queen of mathematics. — Gauss

Chapter Summary:
- We introduced basic number theory concepts
- Proved several important theorems
- Demonstrated different types of mathematical environments

A.5 Restated Theorems

Teorema 1.3.2 (Euclid's Theorem)

There are infinitely many prime numbers.

Teorema 1.4.1 (Continuity Theorem)

If a function f is differentiable at every point, then f is continuous.

Teorema 1.5.1 (Pythagorean Theorem)

In a right triangle, the square of the hypotenuse equals the sum of squares of the other two sides: $x^2 + y^2 = z^2$

Teorema A.1.1 (Maximum Value Theorem)

A continuous function on a closed interval must attain both a maximum and a minimum value.