UNIVERSITY OF AMSTERDAM

ASSIGNMENT 1

Title

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Course:
Stochastic Calculus

Course code:
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Introduction

We are asked to obtain different variables from a US stock on a given day, chosen by us. Our stock of option is Apple "AAPL" on the 7th of March 2022 with price $S_0 = 159.3$ USD. The closing price, on the same day, of a put option that is at or slightly in the money, gives us a p_0 of 3.15 EUR with the strike price K equal to 160.0 EUR. The risk-free US interest rate R (e.g., a Treasury bill rate) with approximately the same time-to-maturity as the option can be found on the website of the Wall Street Journal with an asked annualized yield of 0.674 percent.

We were able to access historical data through yfinance Python library from 12th of December 1980 up until the chosen S_0. To Calculate the historical or realized volatility we took the log daily returns of AAPL the last 6 months.

Question 1

The values obtained from the internet can now be used to calculate the parameters needed to price the option.

(a) To calculate T , we used the $pandas_market_calendars$" Python plug-in which allows precise calculation of the trading days until an option's expiration. This is done by storing calendar information about each stock exchange, which we used to calculate the days until the expiration for our option of choice. The expiration date is given by the 16th of September 2022 about 6 months from now. As, options expire at specific intervals, we picked the closest to 6 months. Since, a trading year consists of 252 trading days and the number of trading days up until the expiring date is calculated to be 135, T is given by $135/252 \approx 0.53571$ years.

(b) From the introduction, R is given by 0.00674. Then we can calculate the continuously compounded interest rate r using $r = \log(1+R) = 0.00672$.

(c) From the historical returns we have estimated the volatility σ using 4 different methods: The formula used in the slides (Handouts Lecture 1), the Parkinson Volatility Model, and lastly using the Rogers and Satchell Volatility Model.

The volatility method in the slides requires the following formula:

$$\hat{\sigma^2} = 252\frac{1}{N-1}\sum_{i=1}^{N}(R_i - \bar{R})^2 \tag{1}$$

where R are daily stock returns.

The Parkinson Historical Volatility squared is calculated by:

$$\hat{\sigma}_p^2 = \frac{1}{N}\sum_{t=1}^{N}\frac{1}{4ln2}(lnH_t - lnL_t)^2 \tag{2}$$

where H_t is the daily high price and L_t is the daily low price.

The Rogers and Satchell Volatility squared Function (1) is calculated by:

$$\hat{\sigma}_{RS}^2(1) = \frac{1}{N}\sum_{t=1}^{N} ln(\frac{H_t}{O_t})[ln(\frac{H_t}{O_t}) - ln(\frac{C_t}{O_t})] + ln(\frac{L_t}{O_t})[ln(\frac{L_t}{O_t}) - ln(\frac{C_t}{O_t})] \tag{3}$$

where $_t$ stands for daily, and H_t stands for the highest price, L_t for the lowest price, O_t for the opening price, C_t for the closing price.

Bellow the volatility of the last 6 months is calculated based on the 3 previously mentioned methods:

$Std[ln(1 + Returns_{Daily})]$	= 0.016331511 EUR
Parkinson Volatility Model	= 0.014818221 EUR
Rogers and Satchell Volatility Model	= 0.014417318 EUR

Including all daily price data points might result to a better comparison between different volatility levels, than including only the Open and Closing prices.

Question 2

We approximated the analytically derived solution of the Black-Scholes Equation using a Binomial tree with the following variables: $\delta t = \frac{T}{N}$, $u = e^{\sigma\sqrt{\delta t}}$, $d = e^{-\sigma\sqrt{\delta t}}$, $p = \frac{e^{r\times\delta t}-d}{u-d}$. Then with these available variables we can use a backwards construction in the binomial tree using the formula:

$$P_0 = e^{-r*\delta t} * (p * u + (1-p) * d) \tag{4}$$

where the payoff of an option is calculated on the last nodes of the tree using $(S-K)_+$ or $(K-S)_+$ for a call or put respectively.

Then for the Black-Schole model we can use the put formula:

$$P_0 = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1) \tag{5}$$

where $d_1 = d_2 + \sigma\sqrt{T} = \frac{log(S_0/K)+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

These results are obtained by writing a function that calculates the binomial tree values of the put option for a given N, number of steps, using the previously mentioned variables. The number of steps and its European put option value is given in table 1.

Option value with 2 steps	= 0.85903 EUR
Option value with 100 steps	= 0.82404 EUR
Option value with 10,000 steps	= 0.82423 EUR
Option value with B-S model	= 0.82422 EUR

Table 1: *Approximation of the binomial tree value vs. the Black-Scholes value.*

From the table it can be seen that the binomial model with 10,000 steps is already equal to the Black-Scholes value to 3 decimal points. The Black-Scholes model is a analytical solution as N→ ∞ and hence therefore has generally speaking a higher accuracy than the binomial tree. The graphical illustration of the convergence of the binomial tree values are given in figure 1 for N in range $[10^1, 10^4]$. The convergence is investigated by writing a script to plot the option's value using the binomial tree function for both European and American options created. The plot also contains the execution time of the code to run, if there are very large values of N to be used the execution time increases exponentially. This is due to the complexity of binomial trees which is $O(n^2)$ which is attributed to looping 2 time over half of the matrix. Hence for a large number of steps the execution time increases exponentially.

To decrease the execution time that it takes to propagate the tree, we used Numba, an advanced Python library which translates python code and excecutes it in optimized C. Allowing in this implementation a 10^4 step binomeal tree to be calculated in about 0.67 sec while in python it took 6.1 secs. Lastly, the figure clearly shows the convergence of the binomial tree value to the expected value from the Black-Scholes equation.

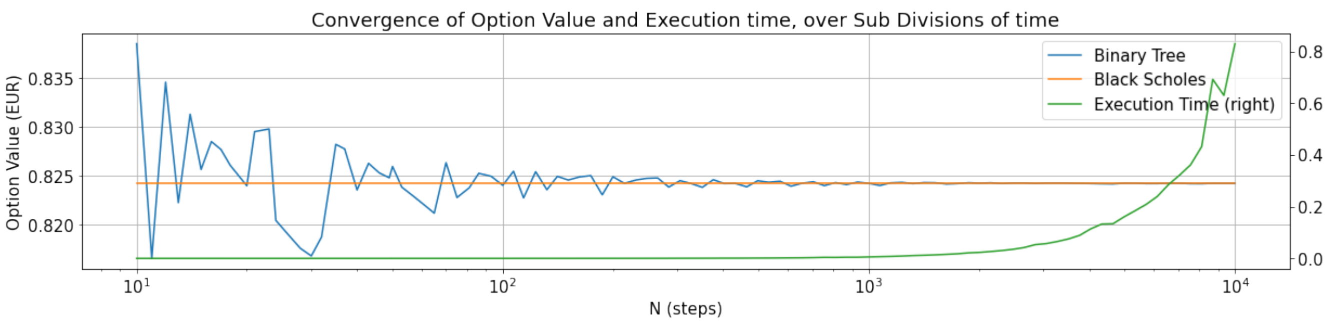

Figure 1: Convergence of the binomial tree approximation of the value of a Put Option, relative to the analytical solution using the Black-Scholes equation. Parameters used $S_0 = 159.3$ USD, $K = 160.0$ EUR, $T = 0.53571$ Years, $\sigma = 0.01633$ EUR, $R = 0.00672$, $N \in [10^1, 10^4]$.

Question 3

An American put option has the opportunity to exercise early. In the case of a put option, the payoff of the option decreases in time due to positive interest rates, hence exercising early could payoff. The binomial tree of a American put and its European counterpart are given in table 2 and 3 respectively. The number of steps, N for both cases, is equal to 3. These are obtained from the binomial tree function with arbitrary N and the parameter set $S_0 = 159.3$ USD, $K = 160.0$ EUR, $T = 0.53571$ Years, $\sigma = 0.01633$, $R = 0.00672\%$.

There is one modification required in the binomial tree, to approximate the value of an American Option. That is checking if exercising at that moment yields a higher payoff.

0.	0.	0.	162.63252
0.	0.	161.51400	160.40318
0.	160.40318	159.30000	158.20440
159.30000	158.20440	157.11634	156.03576
t=0	t=1	t=2	t=3

Table 2: Value of the Stock for N=3

0.	0.	0.	162.63252
0.	0.	161.51400	160.40318
0.	160.40318	159.30000	158.20440
159.30000	158.20440	157.11634	156.03576
t=0	t=1	t=2	t=3

Table 3: Value of the Stock for N=3

0.	0.	0.	0.
0.	0.	0.	0.
0.	0.3077	0.74333	1.79559
0.92329	1.79559	2.88365	3.96423
t=0	t=1	t=2	t=3

Table 4: American put option for N=3

0.	0.	0.	0.
0.	0.	0.	0.
0.	0.30772	0.74333	1.79559
0.82113	1.54880	2.69119	3.96423
t=0	t=1	t=2	t=3

Table 5: European put option for N=3

(a) Using the same parameter set (S_0, K, T, r, σ). We can make use of the created function, with N=10,000 steps, to find the value of both a American and European put option at time 0. For American the put option is worth 0.93201 USD at time 0 . The European put option at time 0 is worth: 0.82320 USD with a difference of 0.10881 USD or 11.67 %. Hence, we indeed see the expected result, that the American put option should be worth at least as much as the European counterpart. From the 2 binomial trees we see that exercising a Put Option early does make sense, as the payoffs close to the root (initial node) are large. This is also reinforced by the figure below, as the American Option's value is larger than the Europeans as the expiration date (T) increases:

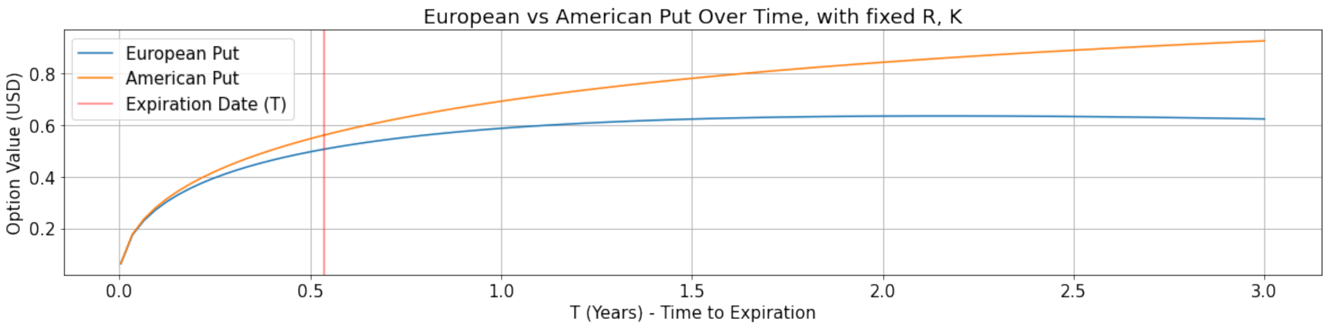

Figure 2: American vs. European put option relative to time, $S_0 = 159.3$ USD, $K = 160.0$ USD, $T = 0.53571$ Years, $\sigma = 0.01633$, $R = 0.00672$, N=10,000.

From figure 2 we can see the values of both options relative to the expiration date T with constant interest R and strike K. The options are computed by the three step binomial tree. It can be seen that the American put option goes up more compared to the European put option.

(b) Does the American put option price seem to converge to a limit as N increases, like the European put option value? Calculate the value of early exercise by subtracting the (binomial) European put option price from the American put option price with the same large N.

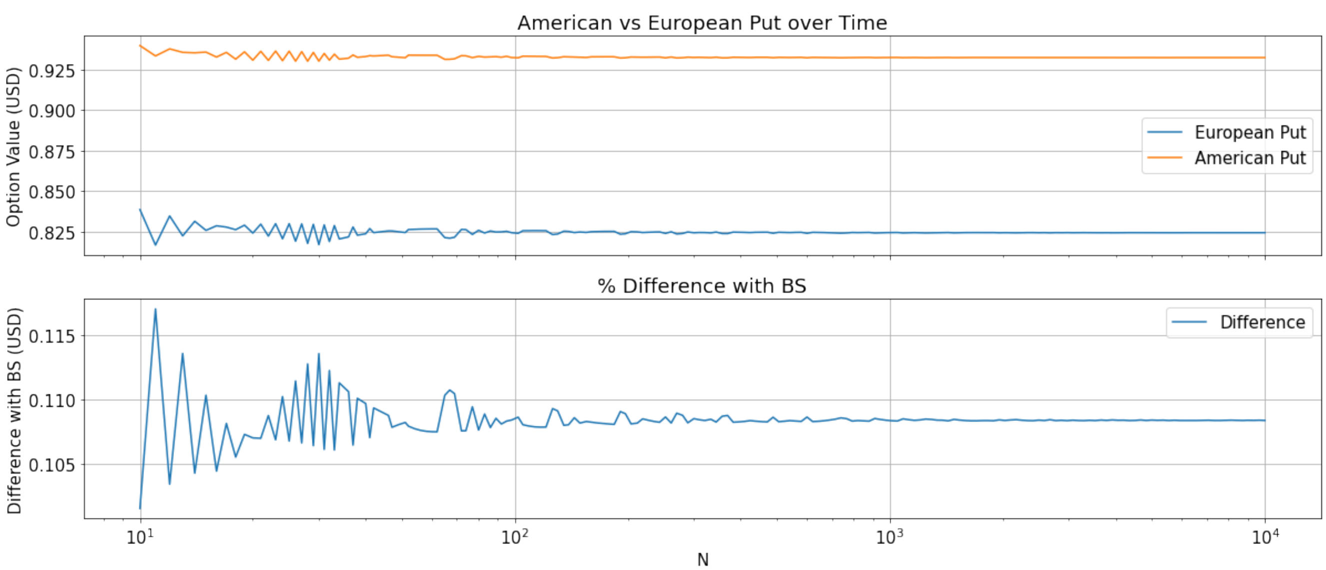

Figure 3: Convergence in the Difference in Value Between the Approximation of an American vs European Put Option using Binomial Trees. $S_0 = 159.3$ USD, $K = 160.0$ USD, $T = 0.53571$ Years, $\sigma = 0.01633$, $R = 0.00672\%$

We can clearly see that the American put option price converges as N increases, just like the European put option. The value of early exercise can be found in the plot of the difference between the American and European put option prices around 0.108

(c) Investigating the effect of the interest rate r by plotting the European and American put options against the continuously compounded r in the range $[-2, 5]$. Both of the options are calculated using N = 100 steps. The plots are given in figure 4. It can be seen that if we set $r = 0$, the value of the American option and its counterpart, the European option are equal to each other. The same can be said for a negative interest rate (r ¡ 0). This is expected due to the fact that exercising early on a American option with a current interest rate being 0 or negative never pays off, and hence the values of both options are similar. When looking at the increase in r the American put option becomes more worth than the European option which again makes sense due to the ability to exercise early in an American option.

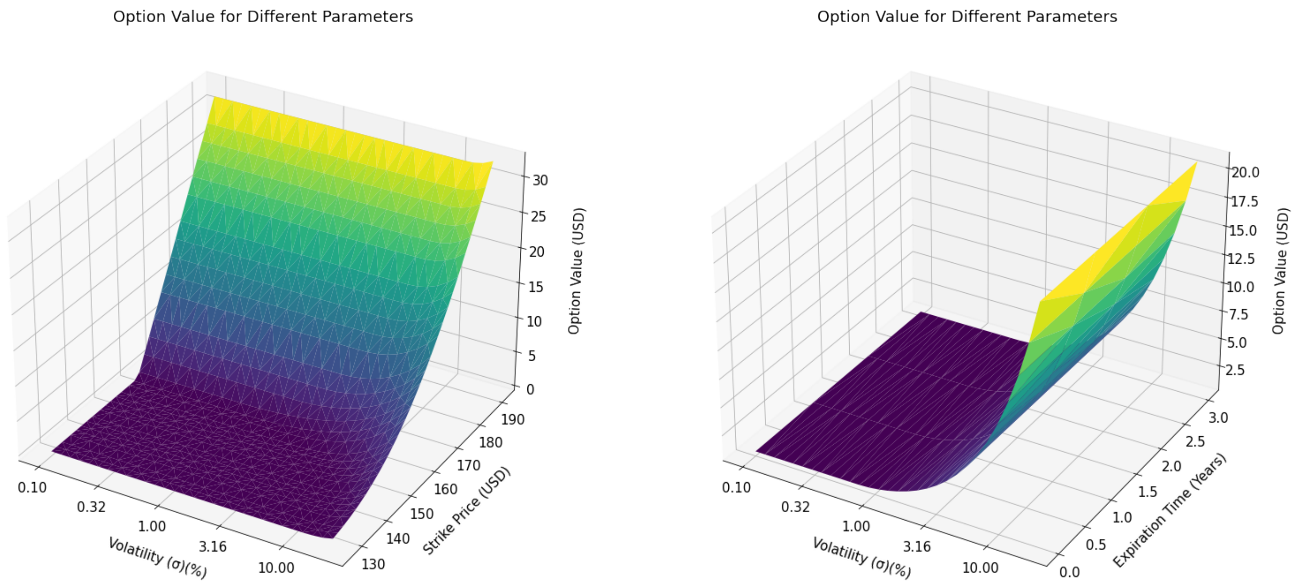

Figure 4: Option Value for Different Parameters. $S_0 = 159.3$ USD, $K = 160.0$ USD, $T = 0.53571$ Years, $\sigma = 0.01633$, $R = 0.00672$

Question 4

The model price may differ substantially from the market value P_0. This could be due to the volatility over period $[0, T]$ might not equal the historical volatility.

Assuming the historical volatility will be equal to the realized volatility over the next 6 months for calculating the option's value could lead to great losses (or profits). As the goal of an option is to reduce the risk and hedge against movements in an underlying stock. Using the historical volatility is not enough.

We developed an algorithm to calculate the Implied Volatility (IV) based on a bump-and-revaluate style algorithm where with very low iterations and high number of binomial tree steps we where able to calculate the IV, fast and accurately.

The first step of the algorithm is to calculate the change in the value of the option with a small change in the sigma ϵ which was kept to 0.0001 to avoid the influence of higher derivatives in the approximation of Vega V which is the rate of change of the option's value with respect to the volatility. Then we approximate the size of the step required to take in the value of the volatility used, to try to match the actual value of the option.

We repeat this process until we find the value of sigma that will result to the actual value of the option, within the error of the binomial tree with the given number of steps we use.

For 5336 steps the approximation error is about 2.50e-05. Based on the difference between the approximation of a European option's value and the analytical solution using the BS equation.

The resulted IV which we found with parameters $S_0 = 159.3$ USD, $K = 160.0$ EUR, $T = 0.53571$ Years, and $R = 0.00672$ is:

Implied Volatility = 0.06529 vs **Historical Volatility = 0.01633** : Difference = 299.795 %

To derive this result it took 4 Iterations of the previously mentioned algorithm and 0.000355 seconds. Also, the way that the algorithm convergence to the optimal value is the following:

Step	$\sigma_{Implied}$
0	0.01633
1	0.07085
2	0.06528
3	0.06529
4	0.06529

References

[1] Chou, R. Y.; Chou,H.; Liu, N. (2009). *Range Volatility Models and Their Applications in Finance.* -

[2] Harbourfront Technologies, (2020). *Parkinson Historical Volatility Calculation - Volatility Analsysi in Python.* URL: https://derivvaluation.medium.com/parkinson-historical-volatility-calculation-volatility-analysis-in-python-8ffee87f1a84