Chapter 1
The Foundations : Logic and Proofs
Kenneth H. Rosen 7th edition

Section 1.7 :Introduction to Proofs

Some Terminologies

- **Theorem:**
  - A statement that can be shown to be true.
  - Usually reserved for a statement that is considered at least somewhat important.
  - Less important theorems sometimes are called **Propositions.**

- **Proof:**
  - Used to demonstrate that a theorem is true.
  - A valid argument that establishes the truth of a theorem.

Some Terminologies(Contd.)

- **Axioms:**
  - Statements that are assumed to be true.
  - May be stated using primitive terms that do not require definition.

- **Lemma:**
  - A less important theorem that is helpful in the proof of other results .

- **Corollary:**
  - A theorem that can be established directly from a theorem that has been proved.

Some Terminologies(Contd.)

- **<u>Conjecture:</u>**
  - A statement that is being proposed to be a true statement, usually on the basis of
    - Some partial evidence Or,
    - A heuristic argument Or,
    - The intuition of an expert
  - When a proof of a conjecture is found, the conjecture becomes a theorem.
  - Many times conjectures are shown to be false, so they are not theorems.

Methods of Proving Theorems

- Direct Proofs
- Proof by Contraposition
- Vacuous and Trivial Proofs
- Proof by Contradiction

Direct Proof

- **Example 1:**
- Give a direct proof of the theorem
  "$If\ n\ is\ an\ odd\ integer, then\ n^2\ is\ odd.$"

- **Solution:**

  This theorem states, $\forall n P\ ((n)\ \rightarrow\ Q(n))$, where

  $P(n)$ is "$n\ is\ an\ odd\ integer.$"

  $Q(n)$ is "$n^2\ is\ odd.$"

  Let, n is an odd integer.

  Thus, by the definition of an odd number, $n = 2k + 1$, where k is some integer.

Direct Proof(Contd.)

Thus, we can write,

$$\begin{array}{rcl|l} n & = & 2k+1 & By\ definition \\ n^2 & = & (2k+1)^2 & Squaring\ both\ sides \\ & = & 4k^2+4k+1 & \\ & = & 2(2k^2+2k)+1 & \\ & = & 2N+1 & N=(2k^2+2k) \end{array}$$

Thus from the above deduction, we can see that n^2 is also an odd number by the definition of odd number.

Thus, we can say that, "$If\ n\ is\ an\ odd\ integer, then\ n^2\ is\ odd.$"

Direct Proof(Contd.)

- **Example 2:**
- Give a direct proof that
  $$"If\ m\ and\ n\ are\ both\ perfect\ squares, then\ mn\ is\ also\ a\ perfect\ square."$$
- **Solution:**

  Let us assume that both m and n are perfect squares.

  Let us consider two integers s and t. Whose squares are respectively s^2 and t^2.

  Thus, by definition, $m = s^2$ and $n = t^2$.

Direct Proof(Contd.)

Thus,

$$
\begin{aligned}
mn &= s^2t^2 \\
&= (ss)(tt) \\
&= (st)(st) \\
&= (st)^2
\end{aligned}
$$

Thus, by definition, mn is also a perfect square.

Thus, we can say that,

"$If\ m\ and\ n\ are\ both\ perfect\ squares, then\ mn$
$is\ also\ a\ perfect\ square.$"

Direct Proof(Contd.)

- **<u>Example 3:</u>**
- Prove that,
  "$The\ sum\ of\ two\ rational\ numbers\ is\ rational.$"

- **<u>Solution:</u>**

  Suppose that r and s are rational numbers. From the definition of a rational number, it follows that there are,

  Integers p and q, with $q \neq 0$, such that $r = \frac{p}{q}$, p, q are co-primes and

  integers t and u, with $u \neq 0$, such that $s = \frac{t}{u}$, s, t are co-primes.

  Therefore,

$$r + s \quad = \frac{p}{q} + \frac{t}{u} \quad = \frac{pu + tq}{qu}$$

Direct Proof(Contd.)

Because $q \neq 0$ and $u \neq 0$, it follows that, $qu \neq 0$

Consequently, we have expressed $r + s$ as the ratio of two integers, $pu + qt$ and qu, where $qu \neq 0$. This means that $r + s$ is rational.

Thus, we can say that,
"$The\ sum\ of\ two\ rational\ numbers\ is\ rational.$"

Proof by Contraposition

- **Example 1:**
- Prove that, "$if\ n\ is\ an\ integer\ and\ 3n\ +\ 2\ is\ odd, then\ n\ is\ odd.$"

- **Solution:**

Let, us assume that the conclusion of the conditional statement "$If\ 3n\ +\ 2\ is\ odd, then\ n\ is\ odd$" is $false$; namely, we assume that n is even.

Then, by definition, $n = 2k$. Substituting $2k$ for n we see that,

$$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1) \end{aligned}$$

Proof by Contraposition(Contd.)

This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd.

This is the negation of the premise of the theorem.

Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded;

We have proved the theorem "$If\ 3n + 2\ is\ odd, then\ n\ is\ odd$."

Proof by Contraposition(Contd.)

- **Example 2:**
- Prove that.
  "$If\ n\ is\ an\ integer\ and\ n^2\ is\ odd, then\ n\ is\ odd.$"

- **Solution:**

Let us assume that n is not odd. This means that n is even. This implies that there exists an integer k such that $n = 2k$.

Thus,

$$
\begin{aligned}
n^2 &= (2k)^2 && \text{Squaring both sides} \\
&= 4k^2 && \\
&= 2(2k^2) && \\
&= 2t && t = 2k^2
\end{aligned}
$$

Proof by Contraposition(Contd.)

Which implies that n^2 is also even.

We have proved that, "$If\ n\ is\ an\ integer\ and\ n^2\ is\ odd, then\ n$ $is\ odd.$" by contraposition.

Proof by Contradiction

- **Example 1:**
- Prove that "$\sqrt{2}$ *is irrational*".
- **Solution:**

Let, $s =$ "$\sqrt{2}$ *is irrational*"

$\therefore \neg s =$ "$\sqrt{2}$ *is rational.*"

If $\neg s$ is true then,

$$\begin{array}{rcl|l} \sqrt{2} & = & \dfrac{p}{q} & p, q \in N, q \neq 0 \text{ and } p, q \text{ are co} - \text{primes} \\ 2 & = & \dfrac{p^2}{q^2} & \text{Squaring both sides} \\ 2q^2 & = & p^2 & \end{array}$$

Proof by Contradiction(Contd.)

By the definition of an even integer it follows that p^2 is even. We next use the fact that if p^2 is even, p must also be even.

Now, as p is even, $p = 2c$ by definition of an even number for an integer c

Therefore,

$$\begin{array}{rcl|l} 2q^2 & = & (2c)^2 & \textit{Substituting } 2c \textit{ for } p \\ 2q^2 & = & 4c^2 & \\ q^2 & = & 2c^2 & \textit{Dividing both sides by } 2 \end{array}$$

Thus, we can see that, q^2 is an even number, thus, q is also even.

Now, the assumption $\neg s$ leads to $\sqrt{2} = \frac{p}{q}$, where p, q are co-primes i.e. they don't have any common factor other than 1.

Proof by Contradiction(Contd.)

But, we can see that, p, q both are even and both are divisible by 2.

Thus, we can see that, our assumption $\neg s$ leads to a contradiction.
Therefore, $\neg s$ must be $false,$ that is s is $true$.

Therefore, "$\sqrt{2}\ is\ irrational.$"

Proof by Contradiction(Contd.)

- **Example 2:**
- Prove that, "$If\ n\ is\ an\ integer\ and\ 3n\ +\ 2\ is\ odd, then\ n\ is\ odd.$"

- **Solution:**

  Let,

  $p = $ "$3n\ +\ 2\ is\ odd$"

  $q = $ "$n\ is\ odd.$"

  To construct a proof by contradiction, let both p and $\neg q$ are true. That is, assume that "$3n\ +\ 2\ is\ odd$" and that "$n\ is\ not\ odd$".

Proof by Contradiction(Contd.)

Since n is not odd, that is n is even.

Then, by definition, $n = 2k$. Substituting $2k$ for n we see that,

$$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1) \end{aligned}$$

Thus, we can see that $3n + 2$ is even. Which is the same as the proposition $\neg p$.

Now, we can see that we assumed p and $\neg q$ to be $true$. Based on this assumption we derived that "$If\ \neg q\ is\ true, \neg p\ is\ also\ true$."

Thus, we see that assuming $\neg q$ as true, leads to both p and $\neg p$ to be true. Thus we have a contradiction.

Thus, we can say that, "$If\ n\ is\ an\ integer\ and\ 3n + 2\ is\ odd, then\ n\ is\ odd$."

Mistakes in Proofs

- **Example 1:**
  - What is wrong with this "proof" that $1 = 2$?

"Proof:" We use these steps, where a and b are two equal positive integers.

Step	***Reason***
1. $a = b$	*Given*
2. $a^2 = ab$	*Multiply both sides of* (1) *by* a
3. $a^2 - b^2 = ab - b^2$	*Subtract* b^2 *from both sides of* (2)
4. $(a - b)(a + b) = b(a - b)$	*Factor both sides of* (3)
5. $(a + b) = b$	*Divide both sides of* (4) *by* $a - b$
6. $2b = b$	*Replace* a *by* b *in* (5) *because* $a = b$ *and simplify*
7. $2 = 1$	*Divide both sides of* (6) *by* b

Mistakes in Proofs(Contd.)

- **Solution:**

  Every step is valid except for one, $step\ 5$ where we divided both sides by $(a - b)$. The error is that $(a - b)$ equals zero; division of both sides of an equation by the same quantity is valid as long as this quantity is not zero.

Mistakes in Proofs(Contd.)

- **Example 2:**
- What is wrong with this "proof?"

"Theorem:" $If\ n^2\ is\ positive, then\ n\ is\ positive.$

"Proof:" Suppose that n^2 is positive. Because the conditional statement"$If\ n\ is\ positive, then\ n^2$ $is\ \ positive$" is true, we can conclude that n is positive.

Mistakes in Proofs(Contd.)

- **Solution:**

Let

$P(n) =$ "n *is positive*."

$Q(n) =$ "n^2 *is positive*".

Then our hypothesis is $Q(n)$.

The statement "*If* n *is positive, then* n^2 *is positive*" is the statement $\forall n(P(n) \rightarrow Q(n))$.

From the hypothesis $Q(n)$ and the statement $\forall n(P(n) \rightarrow Q(n))$ we cannot conclude $P(n)$, because we are not using a valid rule of inference.

A counterexample is supplied by $n = -1$ for which $n^2 = 1$ is positive, but n is negative.

Mistakes in Proofs(Contd.)

- **Example 3:**
- Is the following argument correct? It supposedly shows that, "n *is an even integer whenever* n^2 *is an even integer*."

  Suppose that n^2 is even. Then $n^2 = 2k$ for some integer k.

  Let, $n = 2l$ for some integer l. This shows that n is even.

- **Solution:**

  This argument is incorrect. The statement "$let, n = 2l$ *for some integer* l" occurs in the proof.

  No argument has been given to show that n can be written as $2l$ for some integer l.

  This is circular reasoning because this statement is equivalent to the statement being proved, namely, "n *is even*."

  Of course, the result itself is correct; only the method of proof is wrong

THE END