Chapter 1
The Foundations : Logic and Proofs
Kenneth H. Rosen 7th edition

Section 1.4 : Predicates and Quantifiers

Predicates

- Helps in converting a non propositional statement into a proposition.
- What to identify?
  - Variables
  - Predicates
  - Propositional Function

Predicates(Contd.)

- **<u>Example 1:</u>**
  - "Computer x is under attack by an intruder." Convert this sentence into a proposition using predicate logic.

- **<u>Solution:</u>**
  - Let,

    $s =$ "Computer x is under attack by an intruder."

    The variable involved is, x

    The predicate is, "Computer is under attack by an intruder."

    We can denote the statement s by the propositional function $P(x)$, where P denotes the predicate and x is the variable. Once we assign a value to the variable, the statement $P(x)$ is said to have a truth value and thus becomes a proposition.

Predicates(Contd.)

- **Example 2:**
  - Let $A(x)$ = *"$Computer\ x\ is\ under\ attack\ by\ an\ intruder.$"* Suppose that of the computers on campus, only $CS2$ and $MATH1$ are currently under attack by intruders. What are truth values of $A(CS1)$, $A(CS2)$, and $A(MATH1)$?

- **Solution:**

$\boldsymbol{x}$	***Under Attack?***	***Truth Value***
CS1	No	F
CS2	Yes	T
MATH1	Yes	T

Predicates(Contd.)

- Example 3:
  - Consider the statement "$x = y + 3$". Convert this statement into a proposition and find out the truth value for the following pair of inputs as (x, y),
    - $(1,2)$
    - $(3,0)$
- Solution:
  - Let,

    $Q(x, y)$ = "$x = y + 3$"

$(\boldsymbol{x}, \boldsymbol{y})$	$\boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{y})$	$\boldsymbol{Truth\ value}$
$(1,2)$	$1 = 2 + 3$	F
$(3,0)$	$3 = 0 + 3$	T

Predicates(Contd.)

- Pre-Conditions
  - The statements that describe valid input are known as pre-conditions.
- Post-Conditions
  - The conditions that the output should satisfy when the program has run are known as post-conditions.

Predicates(Contd.)

- **Example 3:**
  - The following program, designed to interchange the values of two variables x and y.

$$temp := x$$
$$x := y$$
$$y := temp$$

1. Find predicates that we can use as the precondition and the post-condition to verify the correctness of this program.
2. Then explain how to use them to verify that for all valid input the program does what is intended.

Predicates(Contd.)

- **Solution:**
  - **1.**

x, y must have an initial value before this program can be run.

Thus, for the pre-condition, we can use the predicate $P(x, y)$

Where,

$P(x, y)$ = "$x = a \; and \; y = b$."

Now, we want to verify whether the program is able to swap the values or not.

Thus, for the post-condition, we can use the predicate $Q(x, y)$

Where,

$Q(x, y)$ = "$x = b \; and \; y = a$."

Predicates(Contd.)

- **Solution:**
  - **2.**

To verify that the program executes successfully, we consider that the precondition $P(x, y)$ is true i.e."$x = a \; and \; y = b$" holds.

Now, let us check whether the values of the variables match with those of the post-condition to prove the correctness of the program.

Steps	***Values***	***Decision***
$temp := \; x$	$x = a, temp = a, y = b$	$Precondition \; holds$
$x := y$	$x = b, temp = a, y = b$	$-$
$y := \; temp$	$x = b, temp = a. y = a$	$Postcondition \; holds$

Since, both the pre and post conditions hold, the program is correct.

Quantifier and Classifications

- Quantification
  - Expresses the extent to which a predicate is true over a range of elements.
  - In English, the words $all, some, many, none$ and few are used in quantifications.
- Universal Quantification AND
  - Tells us that a predicate is true for every element under consideration
- Existential quantification OR
  - Tells us that there is one or more element under consideration for which the predicate is true.

Predicate Calculus

- The area of logic that deals with predicates and quantifiers is known as *predicate calculus*

Universal Quantifiers

- Universal Quantification
  - The universal quantification of $P(x)$ is the statement
    "$P(x)$ *for all values of* x *in the domain.*"
  - The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$.
  - Here $\forall$ is called the universal quantifier.
  - An element for which $P(x)$ is false is called a *counterexample* of $\forall x P(x)$.

Universal Quantifiers(Contd.)

- Phrases used to express universal quantifiers
  - “for all”
  - “for every”
  - “all of”
  - “for each”
  - “given any”
  - “for arbitrary”
  - “for each”
  - “for any.”

Existential Quantifiers

- Existential quantification
  - The existential quantification of $P(x)$ is the statement
    "$There\ exists\ an\ element\ x\ in\ the\ domain\ such\ that\ P(x)$."
  - The notation $\exists x P(x)$ denotes the existential quantification of $P(x)$.
  - Here $\exists$ is called the existential quantifier.

Existential Quantifiers(Contd.)

- Phrases used to express existential quantifiers
  - “there exists”
  - “for some”
  - “for at least one”
  - “there is.”

Truth Value of quantifiers

$\textbf{Statement}$	$\textbf{When True}?$	$\textbf{When false}?$
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Examples of Quantifiers

- **Example : I**
  - What is the truth value of the universal quantification for the statement "$x < 2$", where the domain consists of all real numbers?
- **Solution:**
  - Let,

    $Q(x) =$"$x < 2$"

    Here, the universal quantification for $Q(x)$ is $\forall x Q(x)$.

    Since the domain consists of all the real numbers, $\forall x Q(x)$ is $false$ as $Q(x)$ is $false$ for $Q(3)$.

Examples of Quantifiers(Contd.)

- **Example 2:**
  - What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

- **Solution:**

Because the domain is $\{1, 2, 3, 4\}$, the proposition $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

Since the domains contains $positive\ integers \leq 4$ and $P(4)$ is $false$, the quantification $\forall x P(x)$ is $false$

Examples of Quantifiers(Contd.)

- **Example 3:**
  - What does the statement $\forall x N(x)$ mean if $N(x)$ is "$Computer\ x\ is\ connected\ to\ the\ network$" and the domain consists of all computers on campus?

- **Solution:**

The statement $\forall x N(x)$ means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed in English as,

"$Every\ computer\ on\ campus\ is\ connected\ to\ the\ network.$"

Examples of Quantifiers(Contd.)

- **Example 4:**
  - What is the truth value of the existential quantification for the statement "$x > 3$", where the domain consists of all real numbers?

- **Solution:**
  - Let

  $P(x) =$"$x > 3$"

  Here, the existential quantification for $Q(x)$ is $\forall x Q(x)$.

  Since ,"$x > 3$" is sometimes $true$, for instance, when $x = 4$, we can conclude that $\exists x P(x)$, is $true$.

Examples of Quantifiers(Contd.)

- **Example 5:**
  - What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

- **Solution:**

  Because the domain is {1, 2, 3, 4}, the proposition $\exists x P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

  Since the domains contains $positive\ integers \leq 4$ and $P(4)$ is $true$, the quantification $\exists x P(x)$ is $true$

Precedence of quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
- For example, $\forall x P(x) \lor Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$.
- In other words, it means

$$(\forall \boldsymbol{x} \boldsymbol{P}(\boldsymbol{x})) \lor \boldsymbol{Q}(\boldsymbol{x})$$

and not

$$\forall \boldsymbol{x}(\boldsymbol{P}(\boldsymbol{x}) \lor \boldsymbol{Q}(\boldsymbol{x}))$$

Logical Equivalence

- $\neg\forall x P(x) \equiv \exists x \neg P(x)$
  - Consider the statement

"Every student in your class has taken a course in calculus."

Here,
$P(x)$="x *has taken a course in calculus*"
The domain consists of the students of your class.

The statement can be represented using the universal quantifier $\forall x P(x)$
The negation of the statement is

"*It is not the case that every student in your class has taken a course in calculus*."

Logical Equivalence(Contd.)

This negation can be represented by $\neg\forall x P(x)$.

Now, the negation can also be represented as,

"*There is a student in your class who has not taken a course in calculus*"

Which is similar to the existential quantification of the negation of the original proposition and is represented by $\exists x \neg P(x)$.

Thus, we can see that,

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

Logical Equivalences(Contd.)

- $\neg\exists x P(x) \equiv \forall x\, \neg P(x)$
  - Consider the statement,

    "*There is a student in this class who has taken a course in calculus*"

Here,

$P(x)$="*x has taken a course in calculus*"

The domain consists of the students of your class.

The statement can be represented using the existential quantifier $\exists x P(x)$

Logical Equivalences(Contd.)

The negation of the statement is

“$\textit{It is not the case that there is a student in this class who has taken a course in calculus}$.”

This negation can be represented by $\neg\exists \boldsymbol{x} \boldsymbol{P}(\boldsymbol{x})$.

Which is similar to saying,

“$\textit{Every student in this class has not taken calculus}$”

Which is the universal quantification of the negation of the original propositional function, $\forall \boldsymbol{x} \neg \boldsymbol{P}(\boldsymbol{x})$

Thus, we can see that,

$$\neg\exists \boldsymbol{x} \boldsymbol{P}(\boldsymbol{x}) \equiv \forall \boldsymbol{x} \neg \boldsymbol{P}(\boldsymbol{x})$$

Logical equivalences(Contd.)

- The rules for negations for quantifiers are called $De\ Morgan's\ laws\ for\ quantifiers$

$\boldsymbol{Negation}$	$\boldsymbol{Equivalent\ Statement}$	$\boldsymbol{When\ True?}$	$\boldsymbol{When\ False?}$
$\neg\exists x P(x)$	$\forall x \neg P(x)$	$For\ every\ x,$ $P(x)\ is\ false.$	$There\ is\ an\ x\ for\ which$ $P(x)\ is\ true.$
$\neg\forall x P(x)$	$\exists x \neg P(x)$	$There\ is\ an\ x\ for$ $which\ P(x)\ is\ false.$	$P(x)\ is\ true\ for\ every\ x.$

Logical Equivalences(Contd.)

- **Example 1:**
  - What are the negations of the statement? Express using Quantifiers.
    - “There is an honest politician”
    - “All Americans eat cheeseburgers”?

Logical Equivalences(Contd.)

- **Solution 1.1:**
  - Let,

  $H(x) = "x\ is\ honest."$

  Then, the statement "$There\ is\ an\ honest\ politician.$" is represented by $\exists x H(x)$ where the domain consists of all the politicians.

  The negation of this can be represented by $\neg \exists x H(x)$, which is equivalent to $\forall x \neg H(x)$. This is the same as saying,

$$"Every\ politician\ is\ dishonest."$$

  I

Logical Equivalences(Contd.)

- **Solution 1.2:**
  - Let,

    $C(x) = "x\ eats\ cheeseburgers."$

    Then, the statement "$All\ Americans\ eat\ cheeseburgers.$" is represented by $\forall x C(x)$, where the domain consists of all Americans.

    The negation of this can be represented by $\neg\forall x C(x)$, which is equivalent to $\exists x \neg C(x)$. This is the same as saying,

    "$There\ is\ an\ American\ who\ does\ not\ eat\ cheeseburgers.$"

Logical Equivalences(Contd.)

- **<u>Example 2:</u>**
  - Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x[P(x) \wedge \neg Q(x)]$ are logically equivalent.

- **<u>Solution:</u>**

$$
\begin{aligned}
\neg\forall x(P(x) \rightarrow Q(x)) &\equiv \exists x\neg[P(x) \rightarrow Q(x)] \\
&\equiv \exists x\neg[\neg P(x) \vee Q(x)] \\
&\equiv \exists x[\neg\left(\neg P(x)\right) \wedge \neg Q(x)] \\
&\equiv \exists x[P(x) \wedge \neg Q(x)]
\end{aligned}
$$

Logical Expression using Quantifiers

- **Example 1:**
  - Convert the following Sentence into logical expression using predicates and Quantifiers.

$$\text{“}For\ every\ person, if\ he\ is\ a\ student\ of\ this\ class\ then\ he\ has\ studied\ calculus.\text{“}$$

- **Solution:**
  - Let,

    $S(x) = \text{"x is a student of this class"}$

    $Q(x, calculus) = \text{"x has studied subject calculus."}$

    Thus, here the domain consists of every one in the class, and the statement is a conditional which states that. $S(x)$ is the hypothesis and $Q(x, calculus)$ is the conclusion. Thus, the required logical expression becomes,

$$\forall x[(S(x) \rightarrow Q(x, calculus)]$$

Logical Expression using Quantifiers(Contd.)

- **Example 2:**
  - Express the statements

1. *"Some student in this class has visited Mexico"*
2. *"Every student in this class has visited either Canada or Mexico"*

using predicates and quantifiers when the domain consists of students of the class and also when the domain contains all people.

- **Solution 2.1:**
  - Let,

$$M(x) = \text{“}There\ is\ a\ student\ x\ in\ this\ class\ with\ the\ property\ that\ the\ student\ x\ has\ visited\ Mexico\text{”}$$

When the domain consists of all the students in the class, we can quantify this statement using the existential quantifier as $\exists x M(x)$

Logical Expression using Quantifiers(Contd.)

Now, when the domain consists of all people, we look at the problem a little bit differently,

Let,

$S(x) =$ "x is a $student$ in $this$ $class$."

Thus the statement can now be quantified existentially as

$$\exists x(S(x) \land M(x)$$

[GUESS!! Why can't we express this as, $\exists x(S(x) \rightarrow M(x))$?]

Logical Expression using Quantifiers(Contd.)

The reason is that, the statement is existentially quantified, that means, not all the students have visited Mexico and therefore when we look at the domain containing all the people, the statement cannot be expressed as $\exists x\big(S(x) \rightarrow M(x)\big)$, which is true when there is someone not in the class because, in that case, for such a person $x, S(x) \rightarrow M(x)$ becomes either $F \rightarrow T$ or $F \rightarrow F$, both of which are true.

Logical Expression using Quantifiers(Contd.)

- **Solution 2.2:**

Let,

$M(x) =$ "x *has visited Mexico.*"

$C(x) =$ "x *has visited Canada.*"

When the domain consists of the students in the class the statement can be represented as

$$\forall x\big(C(x) \vee M(x)\big)$$

Now, when the domain contains all the people,

Let,

$S(x) =$ "x *is a student in this class.*"

Logical Expression using Quantifiers(Contd.)

With all the people in the domain the statement can be represented as,

$$"For\ every\ person\ x, if\ \ x\ is\ a\ student\ of\ this\ class\ then\ x\ has\ visited\ either\ Canada\ or\ Mexico."$$

This, can be represented using,

$$\forall x(S(x) \rightarrow (C(x) \vee M(x)))$$

Logical Expression using Quantifiers(Contd.)

So from the previous discussions on having a different domain of discourse, for

- **Existentially Quantified Statements**
  - **Local domain** → predicate represented as stated.
  - **Universal domain** → predicate represented in conjunction with the domain.
- **Universally Quantified Statements**
  - **Local domain** → predicate represented as stated.
  - **Universal domain** → predicate represented as conditional in the form [$\boldsymbol{domain\ (as\ hypothesis)} \rightarrow \boldsymbol{predicate\ (as\ conclusion)}$]

THE END