Beta and Gamma Function

- Beta and gamma functions are popular functions in mathematics. Gamma is a single variable function while beta is a dual variable function.
- The Gamma function and Beta functions belong to the category of the special transcendental functions and are defined in terms of improper definite integrals.
- These functions are very useful in many areas like asymptotic series, Riemann-zeta function, number theory, etc. and also have many applications in engineering and physics.
- The Gamma function was first introduced by Swiss mathematician Leonhard Euler(1707-1783).

Gamma Function

Let n be any positive number. Then the definite integral $\int_0^\infty x^{n-1}\, e^{-x}\, dx$, for $n > 0$ is called gamma function of n which is denoted by Γn and it is defined as

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx, \text{ for } n > 0$$

Example 1-*Gamma Function*

Prove that $\Gamma(1)=1$

Proof: We know by definition of Gamma function

$$\Gamma(n)=\int_0^{\infty} x^{n-1}\, e^{-x}\, dx$$

$$\therefore \Gamma(1)=\int_0^{\infty} x^{1-1}\, e^{-x}\, dx$$

$$=\int_0^{\infty} x^{0}\, e^{-x}\, dx \qquad \left[\because e^{-\infty}=\frac{1}{e^{\infty}}=\frac{1}{\infty}=0\right]$$

$$=\left[-e^{-x}\right]_0^{\infty}=-\{0-1\}=1$$

Example 2-*Gamma Function*

Prove that $\Gamma(n+1) = n!$

Proof: We know by definition of Gamma function

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$$

$$\therefore \Gamma(n+1) = \int_0^\infty x^n\, e^{-x}\, dx$$

$$= \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty -e^{-x} n x^{n-1}\, dx$$

$$= 0 + \int_0^\infty e^{-x} n x^{n-1}\, dx \quad = n\int_0^\infty x^{n-1} e^{-x}\, dx$$

$$= n\Gamma(n)$$

Example 2-*Gamma Function*

$$\therefore \Gamma(n+1) = n\Gamma(n)$$

Replacing n by n-1, n-2, n-3, …….3, 2, 1 we get

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$

$$\Gamma(n-2) = (n-3)\Gamma(n-3)$$

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$$\Gamma 4 = 3\Gamma(3) \qquad \Gamma 3 = 2\Gamma(2) \qquad \Gamma 2 = 1\Gamma(1) = 1 \qquad \because \Gamma(1) = 1$$

$$\therefore \Gamma(n+1) = n(n-1)(n-2)(n-3)\ldots 3.2.1 = n!$$

Example 3-*Gamma Function*

Prove that $\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{2}$

Proof: We know by definition of Gamma function

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\,dx = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\,dx$$

$$= \int_0^\infty e^{-t^2} t^{-1}.2t\,dt;\ \text{by puting } x = t^2$$

$$= 2\int_0^\infty e^{-t^2}\,dt \qquad\qquad \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$= 2.\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

Beta Function

The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx, \text{ for } m>0,\ n>0$

is called beta function of m, n. We write as

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx, \text{ for } m>0,\ n>0$$

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Example 4-*Beta Function*

Prove that $\beta(m,n)=\beta(n,m)$

Proof: We know by definition of Beta function

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$=\int_0^1 (1-x)^{m-1}(1-(1-x))^{n-1}dx \qquad \because \int_0^a f(x)\,dx=\int_0^a f(a-x)dx$$

$$=\int_0^1 (1-x)^{m-1}x^{n-1}dx$$

$$=\int_0^1 x^{n-1}(1-x)^{m-1}dx$$

$$=\beta(n,m)$$

Example 5-*Beta Function*

Show that $\beta(m,n)=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$

Proof: We know by definition of Beta function

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$=\int_\infty^0\left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left\{-\frac{1}{(1+y)^2}dy\right\}$$

$$=-\int_\infty^0\frac{1}{(1+y)^{m-1}}\cdot\frac{y^{n-1}}{(1+y)^{n-1}}\cdot\frac{1}{(1+y)^2}dy$$

$$=\int_0^\infty\frac{y^{n-1}}{(1+y)^{m+n}}dy$$

Let, $x=\frac{1}{1+y}$

$\therefore dx=-\frac{1}{(1+y)^2}dy$

and, $1-x=1-\frac{1}{1+y}=\frac{y}{1+y}$

x	1	0
y	0	∞

Example 6-*Beta Function*

Prove that $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof: We know by definition of Gamma function

$$\Gamma(n)=\int_0^\infty x^{n-1}\,e^{-x}\,dx$$

$$=\int_0^\infty (yz)^{n-1}\,e^{-yz}\,zdy$$

$$=\int_0^\infty y^{n-1}\,z^n\,e^{-yz}\,dy$$

Let, $x = zy$

$\therefore\ dx = zdy$

Multiplying both sides by $z^{m-1}e^{-z}$ **and then integrating w.r.to z from 0 to ∞**

$$\Gamma(n)\int_0^\infty z^{m-1}e^{-z}dz=\int_0^\infty\left[z^{m-1}e^{-z}\int_0^\infty y^{n-1}\,z^n\,e^{-yz}\,dy\right]dz$$

Example 7-*Beta Function*

$$\Gamma(n)\int_0^\infty z^{m-1}e^{-z}dz=\int_0^\infty\left[z^{m-1}e^{-z}\int_0^\infty y^{n-1}\,z^n\,e^{-yz}\,dy\right]dz$$

$$\Gamma(n)\Gamma(m)=\int_0^\infty\left[\int_0^\infty z^{m+n-1}e^{-z(1+y)}dz\right]y^{n-1}dy$$

$$\Gamma(n)\Gamma(m)=\int_0^\infty\frac{1}{(1+y)^{m+n}}\Gamma(m+n)\,y^{n-1}dy$$

$$\frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}=\int_0^\infty\frac{1}{(1+y)^{m+n}}y^{n-1}dy$$

$$\frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)}=\beta(m,n)$$

Let, $z(1+y)=u$

$$dz=\frac{1}{1+y}du$$

$$\int_0^\infty z^{m+n-1}e^{-z(1+y)}dz$$

$$=\int_0^\infty\left(\frac{u}{1+y}\right)^{m+n-1}e^{-u}\,\frac{1}{1+y}du$$

$$=\frac{1}{(1+y)^{m+n}}\int_0^\infty u^{(m+n)-1}e^{-u}du$$

$$=\frac{1}{(1+y)^{m+n}}\Gamma(m+n)$$

Example 8-*Beta Function*

Evaluate $\int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx$

Solution:

$$\int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx \qquad \left[\because \beta(m,n)=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy\right]$$

$$=\int_0^\infty \frac{x^{9-1}}{(1+x)^{15+9}}\,dx$$

$$=\beta(15,9) \qquad \beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$=\frac{\Gamma(9)\Gamma(15)}{\Gamma(9+15)}$$

$$=\frac{\Gamma(9)\Gamma(15)}{\Gamma(24)}=\frac{8!\,14!}{23!}$$

Example 9-*Beta Function*

Evaluate $$\int_0^\infty \frac{x^8\left(1-x^6\right)}{\left(1+x\right)^{24}}\,dx$$

Solution:
$$\int_0^\infty \frac{x^8\left(1-x^6\right)}{\left(1+x\right)^{24}}\,dx = \int_0^\infty \frac{x^8-x^{14}}{\left(1+x\right)^{24}}\,dx = \int_0^\infty \frac{x^8}{\left(1+x\right)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{\left(1+x\right)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{\left(1+x\right)^{15+9}}\,dx - \int_0^\infty \frac{x^{15-1}}{\left(1+x\right)^{9+15}}\,dx$$

$$= \beta(15,9) - \beta(9,15) \qquad \left[\because\ \beta(m,n) = \int_0^\infty \frac{y^{n-1}}{\left(1+y\right)^{m+n}}\,dy\right]$$

$$= \beta(9,15) - \beta(9,15) \qquad \left[\because\ \beta(m,n) = \beta(n,m)\right]$$

$$= 0$$

Example 11-*Gamma Function*

Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Proof: we Know that

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}=\int_0^{\frac{\pi}{2}}(\sin\theta)^p(\cos\theta)^q\ d\theta$$

$$\frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{0+0+2}{2}\right)}=\int_0^{\frac{\pi}{2}}(\sin\theta)^0(\cos\theta)^0\ d\theta$$

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2}{2}\right)}=\int_0^{\frac{\pi}{2}}d\theta \quad \Rightarrow\left(\Gamma\left(\frac{1}{2}\right)\right)^2=2.[\theta]_0^{\frac{\pi}{2}}=2.\frac{\pi}{2}=\pi$$

$$\therefore\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

Example 12-*Gamma Function*

Evaluate: $\Gamma\left(\frac{7}{2}\right)$

we Know that $\Gamma(n)=(n-1)\Gamma(n-1)$

$$\Gamma\left(\frac{7}{2}\right)=\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$$

$$\Gamma\left(\frac{7}{2}\right)=\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)$$

$$\Gamma\left(\frac{7}{2}\right)=\left(\frac{5}{2}\right)\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)$$

$$\Gamma\left(\frac{7}{2}\right)=\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)$$

$$\Gamma\left(\frac{7}{2}\right)=\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$$

$$\Gamma\left(\frac{7}{2}\right)=\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}$$

Example 13-*Gamma Function*

Evaluate: $\Gamma(9)$

we Know that

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(9) = (9-1)\Gamma(9-1)$$

$$\Gamma(9) = (8)\Gamma(8)$$

$$\Gamma(9) = (8)(8-1)\Gamma(8-1)$$

$$\Gamma(9) = (8)(7)\Gamma(7)$$

$$\Gamma(9) = 8.7.6.5.4.3.2.1.\Gamma(1)$$

$$\Gamma(9) = 8.7.6.5.4.3.2.1 = 8!$$

Example 14-*Gamma Function*

Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^7 x \; dx$

Sol^n: we Know that $\int_0^{\frac{\pi}{2}} \left(\sin^p \theta\right)\left(\cos^q \theta\right) \; d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^6 x \cos^7 x \; dx = \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{2\Gamma\left(\frac{6+7+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{8}{2}\right)}{2\Gamma\left(\frac{15}{2}\right)} = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma(4)}{2.\frac{13}{2}.\frac{11}{2}.\frac{9}{2}.\frac{7}{2}\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{48}{9009}$$

Try Yourself

$$(1)\int_0^a x^4\left(a^2-x^2\right)^{1/2}\,dx \quad \left[\text{Hints: } x=a\sin\theta\right]$$

$$(2)\int_0^\infty \frac{x^4}{\left(1+x^2\right)^4}\,dx \quad \left[\text{Hints: } x=\tan\theta\right]$$

$$(3)\int_0^\infty \frac{x^2}{\left(1+x^4\right)}\,dx \quad \left[\text{Hints: } x^2=\tan\theta\right]$$

$$(4)\int_0^1 \frac{x^7}{\left(1-x^4\right)^{1/2}}\,dx \quad \left[\text{Hints: } x^2=\sin\theta\right]$$

$$(5)\int_0^1 \frac{x}{\left(1-x^5\right)^{1/2}}\,dx \quad \left[\text{Hints: } x^5=t\right]$$

$$(6)\int_0^2 x\,\sqrt[3]{\left(8-x^3\right)}\,dx \quad \left[\text{Hints: } x^3=8t\right]$$

$$(7)\int_0^4 x^{3/2}\,\sqrt{\left(16-x^2\right)}\,dx \quad \left[\text{Hints: } x^2=16t\right]$$

$$(8)\int_{-\infty}^{\infty} e^{-x^2}\,dx \quad \left[\text{Hints: } x^2=\mathrm{t}\right]$$

Thanks a lot …