Chapter 1
The Foundations : Logic and Proofs
Kenneth H. Rosen 7^{th} edition

Section 1.3 : Propositional Equivalences

Tautology, Contradiction & Contingency

- **Tautology:**
  - A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.
- **Contradiction:**
  - A compound proposition that is always false is called a contradiction.
- **Contingency:**
  - A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Tautology, Contradiction & Contingency(Contd.)

Example 1:

- **Tautology:**
  - Consider the truth table of $p \lor \neg p$. Because $p \lor \neg p$ is always true, it is a tautology.
- **Contradiction:**
  - Consider the truth table of $p \land \neg p$. Because $p \land \neg p$ is always false, it is a contradiction.

p	$\neg p$	$p \lor \neg p$	$p \land \neg p$
F	T	T	F
T	F	T	F

Logical Equivalences

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
  - The compound propositions $\boldsymbol{p}$ and $\boldsymbol{q}$ are called logically equivalent if $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$ is a tautology.
  - The notation $\boldsymbol{p} \equiv \boldsymbol{q}$ denotes that $\boldsymbol{p}$ and $\boldsymbol{q}$ are logically equivalent.

Logical Equivalences(Contd.)

- **Example 1:**
  - Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.
- **Solution:**
  - We can verify from the following truth table that, $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
F	F	T	T	F	T	T
F	T	T	F	T	F	F
T	F	F	T	T	F	F
T	T	F	F	T	F	F

Logical Equivalences(Contd.)

- **Exercises:**
  - Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.
  - Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

Logical Equivalence Rules

- **Important rules**

Equivalences	***Name***	***Equivalences***	***Name***
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
$\neg(\neg p) \equiv p$	Double Negation Law	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

Logical Equivalence Rules(Contd.)

- **<u>Important Rules Regarding Conditionals</u>**

• $p \to q \equiv \neg p \vee q$ • $p \to q \equiv \neg q \to \neg p$ • $p \vee q \equiv \neg p \to q$ • $p \wedge q \equiv \neg(p \to \neg q)$ • $\neg(p \to q) \equiv p \wedge \neg q$ • $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ • $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$ • $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$ • $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$	• $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ • $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ • $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ • $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

- **Example 1:**
  - Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

- **Solution:**

$$\begin{aligned}
\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by Rule} \\
&\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan Law} \\
&\equiv p \wedge \neg q && \text{by the Double Negation Law}
\end{aligned}$$

Constructing New Logical Equivalences(Contd.)

- **Example 2:**
  - Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

- **Solution:**

$\neg(p \vee (\neg p \wedge q))$	$\equiv$	$\neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
	$\equiv$	$\neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
	$\equiv$	$\neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv$	$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
	$\equiv$	$F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
	$\equiv$	$(\neg p \wedge \neg q) \vee F$	by the commutative law of disjunction
	$\equiv$	$\neg p \wedge \neg q$	by the identity law for F

Constructing New Logical Equivalences(Contd.)

- **Example 3:**
  - Show that $(p \land q) \rightarrow (p \lor q)$ is a tautology.

- **Solution:**

$(p \land q) \rightarrow (p \lor q)$	$\equiv$	$\neg(p \land q) \lor (p \lor q)$	by law of conditional
	$\equiv$	$(\neg p \lor \neg q) \lor (p \lor q)$	by the first De Morgan law
	$\equiv$	$(\neg p \lor p) \lor (\neg q \lor q)$	by the associative and commutative laws of disjunction (or simply rearranging the terms)
	$\equiv$	$T \lor T$	by negation law and the commutative law of disjunction
	$\equiv$	T	by the domination law

Propositional Satisfiability

- A compound proposition is $\boldsymbol{satisfiable}$ if there is an assignment of truth values to its variables that makes it $\boldsymbol{true}$.
- When no such assignments exists, that is, when the compound proposition is $\boldsymbol{false}$ for all assignments of truth values to its variables, the compound proposition is $\boldsymbol{unsatisfiable}$.
- To show that a compound proposition is $\boldsymbol{unsatifiable}$, we need to show that every assignment of truth values to its variables makes it $\boldsymbol{false}$.
- We can logically reason with the values of each variable. But in our case, we will use the truth table.

Propositional Satisfiability(Contd.)

- **Example 1:**
  - Determine the satisfiability of the compound proposition
    - $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$

Propositional Satisfiability(Contd.)

- **Solution:**
  - Let $s = (p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \vee \neg q$	$q \vee \neg r$	$r \vee \neg p$	s
F	F	F	T	T	T	T	T	T	T
F	F	T	T	T	F	T	F	T	F
F	T	F	T	F	T	F	T	T	F
F	T	T	T	F	F	F	T	T	F
T	F	F	F	T	T	T	T	F	F
T	F	T	F	T	F	T	F	T	F
T	T	F	F	F	T	T	T	F	F
T	T	T	F	F	F	T	T	T	T

Propositional Satisfiability(Contd.)

Since there is at least one combination of input for the variables $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ of the compound proposition, which gives a true value for the compound proposition $\boldsymbol{s}$, we can say that the $\boldsymbol{s}$ is satisfiable.

Propositional Satisfiability(Contd.)

- **Exercises:**
  - Determine the satisfiability of each of the compound propositions
    - $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$
    - $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$

THE END