# CSE 4203
# DISCRETE MATHEMATICS

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# Introduction

- Contents
  - Discrete Mathematics and its applications- 7th edition – Kenneth H. Rosen
  - References and slides as provided in the class.
- Quizzes and Assignments
  - 4 quizzes ( 2 before mid + 2 after mid ). Best 3 will be counted.
  - There will be surprise quizzes as well.
  - 2 assignments will be given.

# Chapter 1
# The Foundations : Logic and Proofs
# Kenneth H. Rosen 7th edition

Section 1.1 : Propositional Logic

# Propositios

- A declarative sentence, i.e. declares a fact
  - Either true
  - Or false
  - But not both

| Propositions | Not propositions |
|---|---|
| 2+2=4 | What time is it? |
| 2+2=3 | x+1=3 |
| Dhaka is the capital of Bangladesh | x + y = z |
| Toronto is the capital of USA | Read this carefully |

# Propositional variables

- Variables used to represent propositions
- **Example:**
  - Donald is a good guy.
  - Rebecca is a talented girl.
- Assigning variables to propositions:
  - p = "Donald is a good guy."
  - q = "Rebecca is a talented girl."
- Most commonly used variables are p, q, r, s…
- The truth values of a proposition is denoted by
  - T, if the proposition is true.
  - F, if the proposition is false.
  - We can also use the following convention
    - 1, if the proposition is true.
    - 0, if the proposition is false.

# Propositional variables (contd.)

- Converting sentences into propositions:
  - Consider the sentence, X+Y=Z
  - This is not a proposition.
  - The sentence contains variables only.
  - To convert this sentence into a proposition, we have to assign values to the variables.
  - Let's say, X=2, Y=4, Z=6
    - Thus, X+Y=Z or 2+4=6 becomes a proposition as the outcome is true.
  - Again let's say X=2,Y=4,Z=3
    - Again, X+Y=Z or 2+4=3 becomes a proposition as the outcome is false.

# Propositional calculus/logic

- Area of logic that deals with propositions, is called propositional calculus or propositional logic.

# Compound Propositions

- New propositions are formed by combining one or more propositions.
- These are formed from existing propositions using logical operators
- The newly formed propositions are termed as COMPOUND PROPOSITIONS.

# Logical Operators

**Logical Operators**

- Negation ( ¬ )
- Exclusive Disjunction/ XOR ( ⊕ )
- Conjunction / AND ( ∧ )
- Disjunction / OR ( ∨ )
- Conditionals / Implications ( → )
- Bi-conditionals ( ↔ )

# Negation Operator ($\neg$)

- **Definition:**
  - Let $p$ be a proposition.
  - The negation of $p$, denoted by $\neg p$ (also denoted by $\overline{p}$), is the proposition, "It is not the case that $p$."
  - The proposition $\neg p$ or $\overline{p}$ is read "**not p**."
  - The truth value of $\neg p$, is the opposite of that of $p$.

| $p$ | $\overline{p}$ |
| --- | --- |
| T | F |
| F | T |

# Negation Operator (¬) (contd.)

- **Examples:**
  - $p$ : "Adam has a high resolution PC."
  - $\neg p$ :
    - "Adam does not have a high resolution PC." **Or**,
    - "It is not the case that Adam has a high resolution PC."

  - $p$ : "Dean's smartphone has at least 32GB of memory."
  - $\neg p$ :
    - "Dean's smartphone does not have at least 32GB of memory." **Or**,
    - "It is not the case that Dean's smartphone has at least 32GB of memory." **Or**,
    - "Dean's smartphone has less than 32 GB of memory."

# Conjunction / AND operator ( $\wedge$ )

- **<u>Definition:</u>**
  - Let $p$ and $q$ be propositions.
  - The conjunction of $p$ and $q$, denoted by $p \wedge q$, is the proposition ,"$p$ *and* $q$"
  - The conjunction $p \wedge q$ is true when both $p$ and $q$ are true and is false otherwise.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| F | F | F |
| F | T | F |
| T | F | F |
| T | T | T |

## Conjunction / AND operator ( $\wedge$ )(Contd.)

- **Examples:**
  - $p$ : "Dean's PC has more than 32GB of free hard disk space."
  - $q$ : "The processor in Dean's PC runs faster than 1GHz."
  - $p \wedge q$ :
    - "Dean's PC has more than 32GB of free hard disk space and the processor in Dean's PC runs faster than 1GHz." **Or**,
    - "Dean's PC has more than 32GB of free hard disk space and its processor runs faster than 1GHz."

# Disjunction / OR operator ( ∨ )

- **Definition:**
  - Let $p$ and $q$ be propositions.
  - The disjunction of $p$ and $q$, denoted by $p \lor q$, is the proposition "$p$ *or* $q$."
  - The disjunction $p \lor q$ is false when both $p$ and $q$ are false and is true otherwise.

| $p$ | $q$ | $p \lor q$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | T |

# Exclusive OR/ XOR operator ( $\oplus$ )

- **Definition:**
  - Let $p$ and $q$ be propositions.
  - The exclusive or of $p$ and $q$, denoted by $p \oplus q$, is the proposition "***p or q but not both***".
  - The exclusive or $p \oplus q$ is true when **EXACTLY ONE** of $p$ and $q$ is true and is false otherwise.

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | F |

# Conditionals/Implications

- **Definition:**
  - Let $p$ and $q$ be propositions.
  - The conditional statement $p \rightarrow q$ is the proposition "$if\ p, then\ q$."
  - The conditional statement $p \rightarrow q$ is false when $p$ is true and $q$ is false, and true otherwise.
  - In the conditional statement $p \rightarrow q$,
    - $p$ is called the hypothesis (or antecedent or premise)
    - $q$ is called the conclusion (or consequence).

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

# Conditionals/Implications(Contd.)

- Terminologies to express implication:
  - "if p, then q"
  - "p implies q"
  - "if p, q"
  - "p only if q"
  - "p is sufficient for q"
  - "a sufficient condition for q is p"
  - "q if p"
  - "q whenever p"
  - "q when p"
  - "q is necessary for p"
  - "a necessary condition for p is q"
  - "q follows from p"
  - "q unless ¬p"

# Conditionals/Implications(Contd.)

- A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract.
- Confusions:
  - Of the various ways to express the conditional statement $p \rightarrow q$, the two that seem to cause the most confusion are
    - "$p$ ***only if*** $q$"
    - "$q$ ***unless*** $\neg p$"

# Conditionals/Implications(Contd.)

- "$p$ *only if* $q$"
  - Expresses the same thing as "*if* $p$, *then* $q$."
  - It means, $p$ cannot be true when $q$ is not true.
  - i.e. the statement is false if $p$ is true, but $q$ is false.
  - When $p$ is false, $q$ may be either true or false, because the statement says nothing about the truth value of $q$.

| $p$ | $q$ | $p \rightarrow q$ [ p only if q ] |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

# Conditionals/Implications(Contd.)

- **"$q$ *unless* $\neg p$"**
  - Expresses the same thing as "*if* $p$, *then* $q$,"
  - It means, if $\neg p$ is false, then $q$ must be true.
  - i.e. the statement "$q$ *unless* $\neg p$" is false when $p$ is true but $q$ is false, but it is true otherwise.
  - Consequently, "$q$ *unless* $\neg p$" and $p \rightarrow q$ always have the same truth value.

| $p$ | $q$ | $p \rightarrow q$ [ q unless ¬p ] |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

# Conditionals/Implications(Contd.)

- **Example 1:**
  - p :“Maria learns discrete mathematics”
  - q : “Maria will find a good job.”
- Express the statement $p \rightarrow q$ as a statement in English.

# Conditionals/Implications(Contd.)

- **Solution:**
  - "***If*** Maria learns discrete mathematics, ***then*** she will find a good job." **Or**,
  - "Maria will find a good job ***when*** she learns discrete mathematics." **Or**,
  - "***For*** Maria to get a good job, ***it is sufficient*** for her to learn discrete mathematics." **Or**,
  - "Maria will find a good job ***unless*** she does ***not*** learn discrete mathematics."

# Converse, Contrapositive and Inverse

- Converse
  - The proposition $q \rightarrow p$ is called the converse of $p \rightarrow q$.
- Contrapositive
  - The proposition $\neg q \rightarrow \neg p$ is called the contrapositive of $p \rightarrow q$.
- Inverse
  - The proposition $\neg p \rightarrow \neg q$ is called the inverse of $p \rightarrow q$.

# Converse, Contrapositive and Inverse(Contd.)

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \rightarrow q$ **(Original)** | $\neg q \rightarrow \neg p$ **(Contrapositive)** | $q \rightarrow p$ **(Converse)** | $\neg p \rightarrow \neg q$ **(Inverse)** |
|---|---|---|---|---|---|---|---|
| F | F | T | T | T | T | T | T |
| F | T | T | F | T | T | F | F |
| T | F | F | T | F | F | T | T |
| T | T | F | F | T | T | T | T |

# Converse, Contrapositive and Inverse(Contd.)

- Equivalence:
  - When two compound propositions always have the same truth value we call them **equivalent.**
- The original conditional statement and its contrapositive are equivalent.
- The converse and the inverse of a conditional statement are also equivalent, but neither is equivalent to the original conditional statement.

# Converse, Contrapositive and Inverse(Contd.)

- **<u>Example 1:</u>**
  - Find the converse, contrapositive and inverse of the following statement.
    - “The home team wins whenever it is raining?”

# Converse, Contrapositive and Inverse(Contd.)

- **Solution:**
  - **Step 1:** Convert the original statement into "If hypothesis, then conclusion" structure.
    - "If it is raining, then the home team wins."
  - **Step 2:** Assign variables to the hypothesis and conclusion
    - $\boldsymbol{p}$ : "It is raining."
    - $\boldsymbol{q}$ : "The home team wins."
    - Thus, the statement becomes of the form "$\boldsymbol{If\ p, then\ q}$".
  - **Step 3:** Find logical expressions
    - Converse : $\boldsymbol{q} \rightarrow \boldsymbol{p}$
    - Contrapositive : $\neg \boldsymbol{q} \rightarrow \neg \boldsymbol{p}$
    - Inverse : $\neg \boldsymbol{p} \rightarrow \neg \boldsymbol{q}$

# Converse, Contrapositive and Inverse(Contd.)

- Solution(Contd.):
  - **Step 4:** Assign values to the variables.
    - **Converse:**
      - "If the home team wins, then it is raining."
    - **Contrapositive:**
      - "If the home team does not win, then it is not raining."
    - **Inverse:**
      - "If it is not raining, then the home team does not win."

# Bi-conditionals

- **Definition:**
  - Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be propositions.
  - The bi-conditional statement $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$ is the proposition "$\boldsymbol{p}$ ***if and only if*** $\boldsymbol{q}$."
  - The bi-conditional statement $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$ is true when $\boldsymbol{p}$ and $\boldsymbol{q}$ have the same truth values, and is false otherwise.
  - Bi-conditional statements are also called bi-implications.

| $\boldsymbol{p}$ | $\boldsymbol{q}$ | $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$ |
|---|---|---|
| F | F | T |
| F | T | F |
| T | F | F |
| T | T | T |

# Bi-conditionals(Contd.)

- Terminologies to express bi-conditionals:
  - “p if and only if q.”
  - “p is necessary and sufficient for q”
  - “if p then q, and conversely”
  - “p iff q.”

# Bi-conditionals(Contd.)

- **<u>Example 1:</u>**
  - $p$ :"You can take the flight,"
  - $q$ :"You buy a ticket."
  - Then $p \leftrightarrow q$ means,
    - "You can take the flight if and only if you buy a ticket."
- Implicit use:
  - Bi-conditionals are not always explicit in natural language
  - "if and only if" is rarely used in common language.
  - Often expressed using an "if, then" or an "only if" construction.
  - The other part of the "if and only if" is implicit.
  - That is, the converse is implied, but not stated.
  - We need to make an assumption whether a conditional statement in natural language implicitly includes its converse

# Bi-conditionals(Contd.)

- Example of Implicit Use:
  - "If you finish your meal, then you can have dessert."
- What it really means:
  - "You can have dessert if and only if you finish your meal."
- This last statement is logically equivalent to the two statements
  - "If you finish your meal, then you can have dessert"
    **And**,
  - "You can have dessert only if you finish your meal."

# Truth table of compound propositions

- Construct the truth table part by part in terms of logical expressions.
- **Example 1:**
  - Find the truth table of the following compound expression
  - $(\mathbf{p} \lor \neg \mathbf{q}) \rightarrow (\mathbf{p} \land \mathbf{q})$
- **Solution:**

| $p$ | $q$ | $\neg q$ | $p \lor \neg q$ | $p \land q$ | $(\mathbf{p} \lor \neg \mathbf{q}) \rightarrow (\mathbf{p} \land \mathbf{q})$ |
|---|---|---|---|---|---|
| F | F | T | T | F | F |
| F | T | F | F | F | T |
| T | F | T | T | F | F |
| T | T | F | T | T | T |

# Precedence of Logical Operators

- The operators with lower precedence values have higher priorities.
- That means, the operation of those operators must be performed before that of the low priority operators.

| Operators | Precedence |
| --- | --- |
| ¬ (Negation) | 1 |
| ∧ (Conjunction) | 2 |
| ∨ (Disjunction) | 3 |
| → (Implication) | 4 |
| ↔ (Bi-conditionals) | 5 |

# Precedence of Logical Operators(Contd.)

- **Example:**
  - "$\neg p \land q$" is the conjunction of $\neg p$ and $q$, namely, "$(\neg p) \land q$", **NOT** the negation of the conjunction of $p$ and $q$, namely "$\neg(p \land q)$".
  - "$p \land q \lor r$" means "$(p \land q) \lor r$" and **NOT** "$p \land (q \lor r)$".
  - "$p \lor q \to r$" is the same as "$(p \lor q) \to r$".

# Logic and Bit Operations

- A bit is a symbol with two possible values,
  - 0 (zero)
  - 1 (one)
- A bit can be used to represent a truth value, because there are two truth values,
  - True (T)
  - False (F)
- 1 (one) represents T (true).
- 0 (zero) represents F (false).

# Logic and Bit Operations(Contd.)

- Logical operations using 1s (Ones) and 0s (Zeros)

| $p$ | $q$ | $p \vee q$ | $p \wedge q$ | $p \oplus q$ | $p \rightarrow q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 1 |

# Logic and Bit Operations(Contd.)

- Bit Strings
  - A bit string is a sequence of zero or more bits.
  - The length of this string is the number of bits in the string.
  - 101010011 is a bit string of length nine.
- We can extend bit operations to bit strings.
- We define the bitwise OR, bitwise AND, and bitwise XOR of two strings of the same length to be the strings that have as their bits, the OR, AND, and XOR of the corresponding bits in the two strings, respectively.
- We will split bit strings into blocks of four bits to make them easier to read.

# Logic and Bit Operations(Contd.)

- **Example 1:**
  - Find the bitwise OR, bitwise AND, and bitwise XOR of the bit strings 01 1011 0110 and11 0001 1101.
- **Solution:**

| | |
|---|---|
| 01 1011 0110 | String 1 |
| 11 0001 1101 | String 2 |
| 11 1011 1111 | Bitwise OR |
| 01 0001 0100 | Bitwise AND |
| 10 1010 1011 | Bitwise XOR |

THE END