# Finding Area by the Limit Definition

The next theorem shows that the equivalence of the limits (as $n \to$ ) of the upper and lower sums is not mere coincidence. It is true for all functions that are continuous and nonnegative on the closed interval $[a, b]$.

**THEOREM 4.3 Limits of the Lower and Upper Sums**

Let $f$ be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}
\lim_{n\to\infty} s(n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\,\Delta x \\
&= \lim_{n\to\infty} S(n)
\end{aligned}$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of $f$ on the $i$th subinterval.

# Finding Area by the Limit Definition

This means that you are free to choose an *arbitrary* $x$-value in the $i$th subinterval, as shown in the *definition of the area of a region in the plane*.

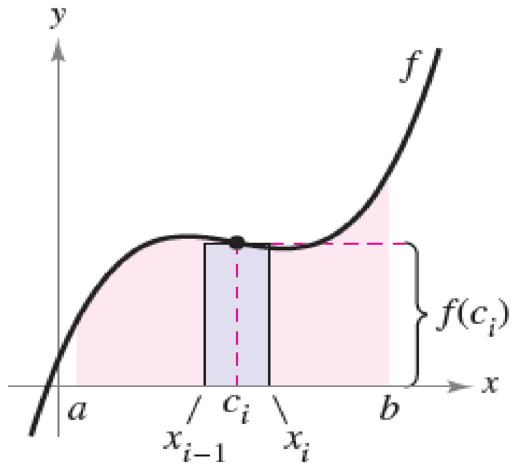
**Definition of the Area of a Region in the Plane**

Let $f$ be continuous and nonnegative on the interval $[a, b]$. (See Figure 4.13.) The area of the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

where $x_{i-1} \le c_i \le x_i$ and

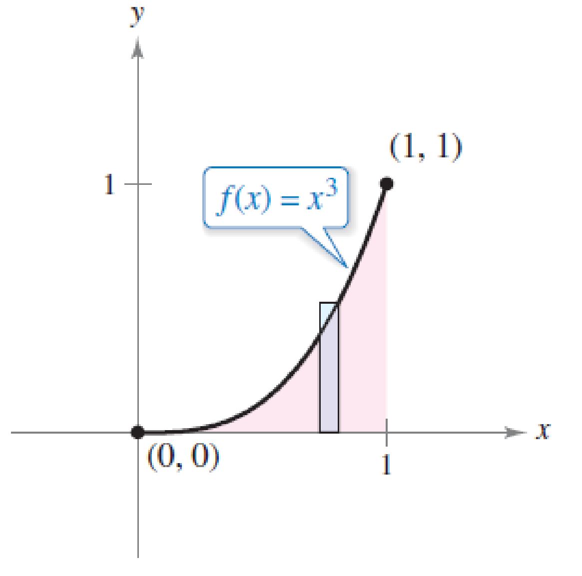
$$\Delta x = \frac{b-a}{n}.$$




The width of the $i$th subinterval is $\Delta x = x_i - x_{i-1}$.
**Figure 4.13**

# Example 5 – *Finding Area by the Limit Definition*

Find the area of the region bounded by the graph of $f(x) = x^3$, the $x$-axis, and the vertical lines $x = 0$ and $x = 1$, as shown in Figure 4.14.




The area of the region bounded by the graph of $f$, the $x$-axis, $x = 0$, and $x = 1$ is $\frac{1}{4}$.

**Figure 4.14**

# Example 5 – *Solution*

Begin by noting that $f$ is continuous and nonnegative on the interval [0, 1]. Next, partition the interval [0, 1] into $n$ subintervals, each of width $\Delta x = 1/n$.

According to the definition of area, you can choose any $x$-value in the $i$th subinterval.

For this example, the right endpoints $c_i = i/n$ are convenient.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right) \qquad \text{Right endpoints: } c_i = \frac{i}{n}$$

# Example 5 – *Solution*

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right)$$ Right endpoints: $c_i = \dfrac{i}{n}$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) \quad = \frac{1}{4}$$

The area of the region is $\frac{1}{4}$

# Finding Area by the Limit Definition

In general, a good value to choose is the midpoint of the interval, $c_i = (x_{i-1} + x_i) / 2$, and apply the **Midpoint Rule.**

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x. \qquad \text{Midpoint Rule}$$

# Riemann Sums

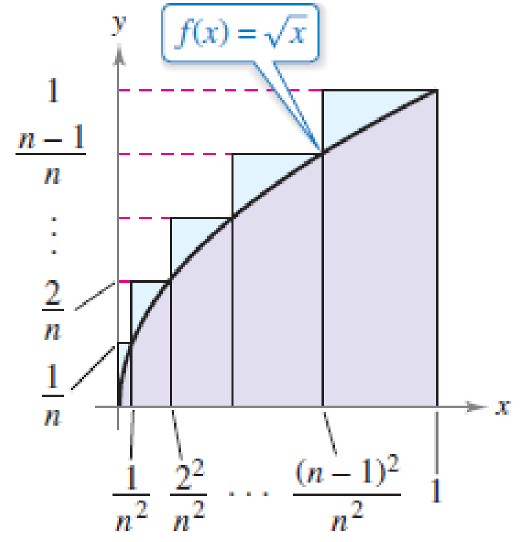
# Example 1 – *A Partition with Subintervals of Unequal Widths*

Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the $x$-axis for $0 \le x \le 1$, as shown in Figure 4.18. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

where $c_i$ is the right endpoint of the partition given by

$c_i = i^2/n^2$ and $\Delta x_i$ is the width of the $i^{\text{th}}$ interval.




The subintervals do not have equal widths.

**Figure 4.18**

# Example 1 – *Solution*

The width of the *i* th interval is

$$\Delta x_i = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2}$$

$$= \frac{i^2 - i^2 + 2i - 1}{n^2}$$

$$= \frac{2i-1}{n^2}.$$

So, the limit is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}} \left(\frac{2i-1}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - i)$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[2\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{n(n+1)}{2}\right]$$

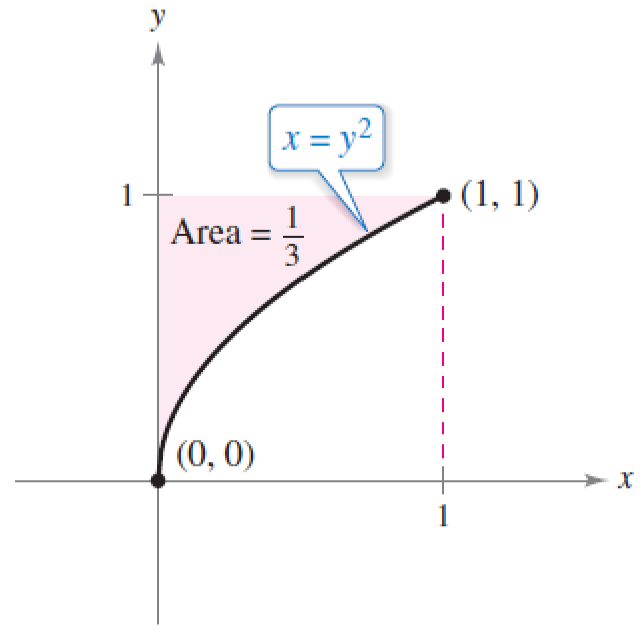
$$= \lim_{n\to\infty} \frac{4n^3 + 3n^2 - n}{6n^3}$$

$$= \lim_{n\to\infty} \left(\frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}\right)$$

$$= \frac{2}{3}.$$

# Riemann Sums

We know that the region shown in Figure 4.19 has an area of $\frac{1}{3}$.




The area of the region bounded by the graph of $x = y^2$ and the $y$-axis for $0 \leq y \leq 1$ is $\frac{1}{3}$.

**Figure 4.19**

# Riemann Sums

Because the square bounded by $0 \le x \le 1$ and $0 \le y \le 1$ has an area of 1, you can conclude that the area of the region shown in Figure 4.18 has an area of .

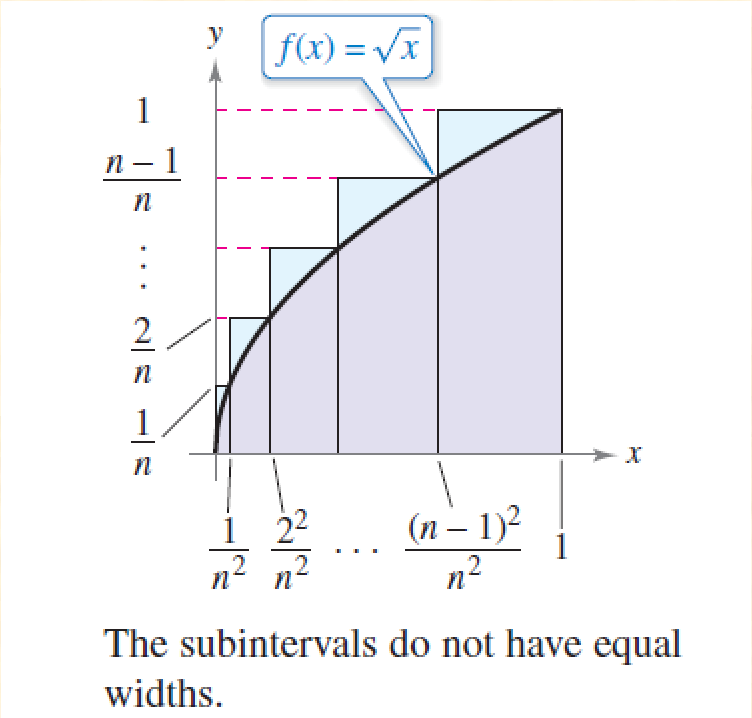



**Figure 4.18**

# Riemann Sums

This agrees with the limit found in Example 1, even though that example used a partition having subintervals of unequal widths.

The reason this particular partition gave the proper area is that as $n$ increases, the *width of the largest subinterval approaches zero*.

This is a key feature of the development of definite integrals.

# Riemann Sums

In the definition of a Riemann sum below, note that the function $f$ has no restrictions other than being defined on the interval $[a, b]$.

**Definition of Riemann Sum**

Let $f$ be defined on the closed interval $[a, b]$, and let $\Delta$ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where $\Delta x_i$ is the width of the $i$th subinterval

$$[x_{i-1}, x_i]. \qquad i\text{th subinterval}$$

If $c_i$ is *any* point in the $i$th subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of $f$ for the partition $\Delta$.

# Riemann Sums

The width of the largest subinterval of a partition $\Delta$ is the **norm** of the partition and is denoted by $||\Delta||$. If every subinterval is of equal width, the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}. \qquad \text{Regular partition}$$

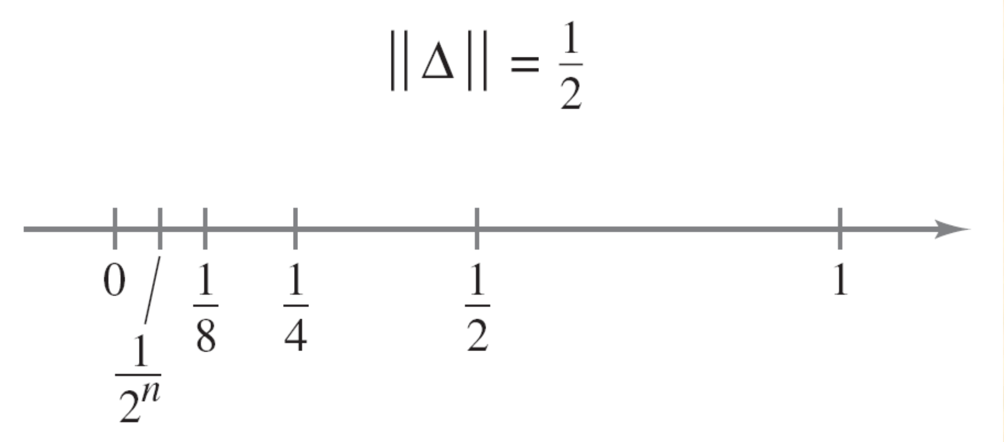
For a **general partition**, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b-a}{\|\Delta\|} \leq n \qquad \text{General partition}$$

# Riemann Sums

As shown in Figure 4.20, for any positive value of $n$, the norm of the partition $\Delta_n$ is $\frac{1}{2}$.

So, letting $n$ approach infinity does not force $||\Delta||$ to approach 0. In a regular partition, however, the statements $||\Delta|| \to 0$ and $n \to \infty$ are equivalent.




$n \to \infty$ does not imply that $\|\Delta\| \to 0$.

# Definite Integrals

**Definition of Definite Integral**

If $f$ is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions $\Delta$

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

exists (as described above), then $f$ is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i = \int_a^b f(x)\, dx.$$

The limit is called the **definite integral** of $f$ from $a$ to $b$. The number $a$ is the **lower limit** of integration, and the number $b$ is the **upper limit** of integration.

# Example 2 – Evaluating a Definite Integral as a Limit

Evaluate the definite integral

$$\int_{-2}^{1} 2x\, dx.$$

Solution:

The function $f(x) = 2x$ is integrable on the interval $[-2, 1]$ because it is continuous on $[-2, 1]$.

Moreover, the definition of integrability implies that any partition whose norm approaches 0 can be used to determine the limit.

# Example 2 – Solution

For computational convenience, define $\Delta$ by subdividing [–2, 1] into $n$ subintervals of equal width

$$\Delta x_i = \Delta x = \frac{b - a}{n} = \frac{3}{n}.$$

Choosing $c_i$ as the right endpoint of each subinterval produces

$$c_i = a + i(\Delta x) = -2 + \frac{3i}{n}.$$

# Example 2 – Solution

So, the definite integral is

$$\int_{-2}^{1} 2x\,dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left(-2 + \frac{3i}{n}\right)$$

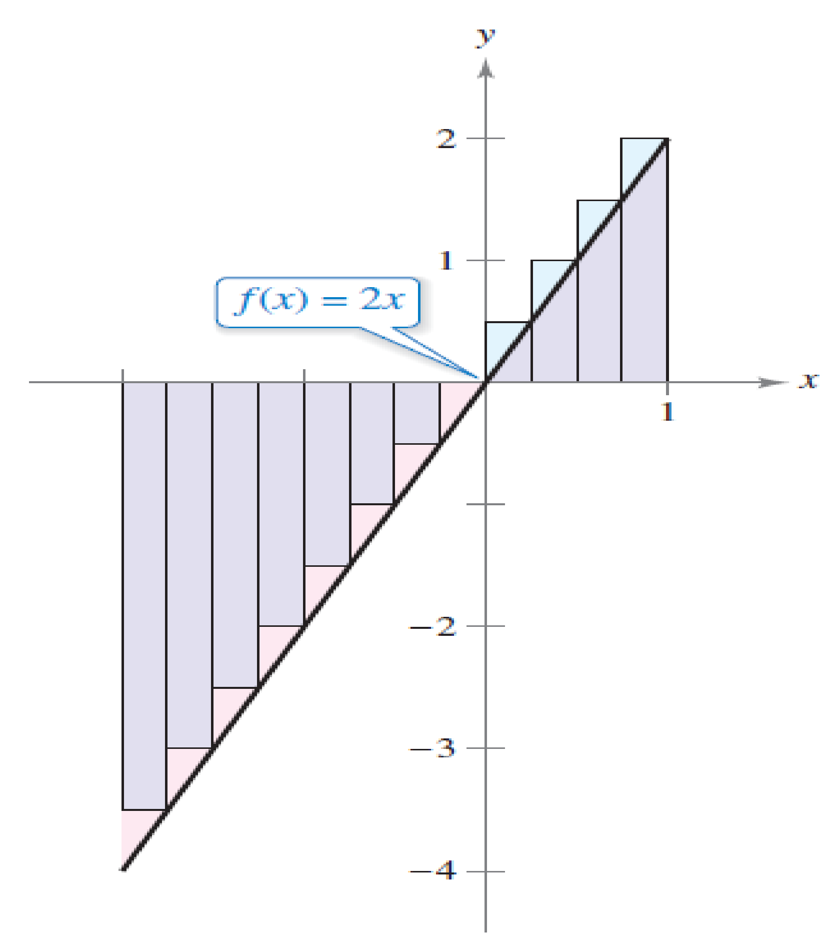
$$= \lim_{n\to\infty} \frac{6}{n} \left(-2\sum_{i=1}^{n} 1 + \frac{3}{n}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty} \frac{6}{n} \left\{-2n + \frac{3}{n}\left[\frac{n(n+1)}{2}\right]\right\}$$

$$= \lim_{n\to\infty} \left(-12 + 9 + \frac{9}{n}\right)$$

$$= -3.$$

# Example 2 – Solution




Because the definite integral is negative, it does not represent the area of the region.

**Figure 4.21**

# Properties of Definite Integrals

**Definitions of Two Special Definite Integrals**

1. If $f$ is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$.
2. If $f$ is integrable on $[a, b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

**THEOREM 4.6 Additive Interval Property**

If $f$ is integrable on the three closed intervals determined by $a$, $b$, and $c$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

See Figure 4.25.

# Properties of Definite Integrals

**THEOREM 4.7 Properties of Definite Integrals**

If $f$ and $g$ are integrable on $[a, b]$ and $k$ is a constant, then the functions $kf$ and $f \pm g$ are integrable on $[a, b]$, and

**1.** $\displaystyle \int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

**2.** $\displaystyle \int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

# Properties of Definite Integrals

**THEOREM 4.8 Preservation of Inequality**

1. If $f$ is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)\,dx.$$

2. If $f$ and $g$ are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every $x$ in $[a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

# The Fundamental Theorem of Calculus

**THEOREM 4.9 The Fundamental Theorem of Calculus**

If a function $f$ is continuous on the closed interval $[a, b]$ and $F$ is an antiderivative of $f$ on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

# The Mean Value Theorem for Integrals

**THEOREM 4.10 Mean Value Theorem for Integrals**

If $f$ is continuous on the closed interval $[a, b]$, then there exists a number $c$ in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

# Average Value of a Function

**Definition of the Average Value of a Function on an Interval**

If $f$ is integrable on the closed interval $[a, b]$, then the **average value** of $f$ on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

See Figure 4.32.

# Example 4 – Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

Solution:

The average value is

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 (3x^2 - 2x)\,dx$$

$$= \frac{1}{3}\Big[x^3 - x^2\Big]_1^4$$

$$= \frac{1}{3}[64 - 16 - (1 - 1)]$$

$$= \frac{48}{3} \quad = 16.$$

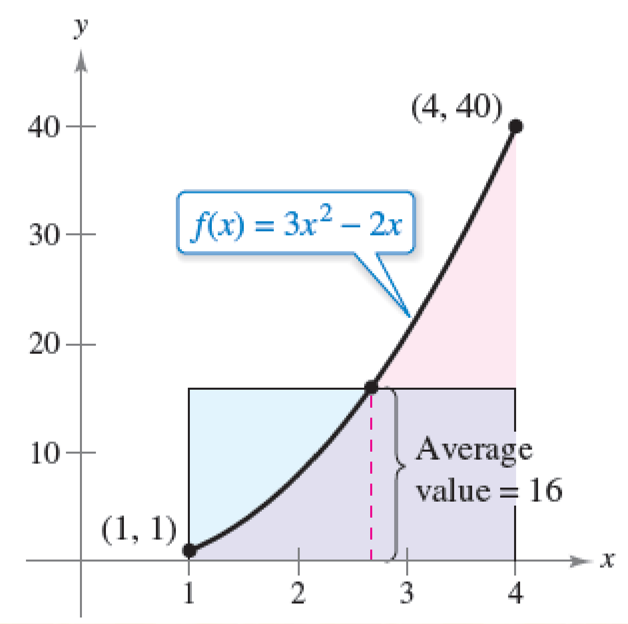



Figure 4.33

# The Second Fundamental Theorem of Calculus

**The Definite Integral as a Number**

Constant

$$\int_a^b f(x)\,dx$$

Constant

$f$ is a function of $x$.

**The Definite Integral as a Function of $x$**

$F$ is a function of $x$.

$$F(x) = \int_a^x f(t)\,dt$$

Constant

$f$ is a function of $t$.

# Example 6 – The Definite Integral as a Function

Evaluate the function

$$F(x) = \int_0^x \cos t \, dt$$

at $x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3},$ and $\frac{\pi}{2}$.

Solution:

You could evaluate five different definite integrals, one for each of the given upper limits.

However, it is much simpler to fix *x* (as a constant) temporarily to obtain

$$\int_0^x \cos t \, dt = \sin t \Big]_0^x = \sin x - \sin 0 = \sin x.$$

# The Second Fundamental Theorem of Calculus

**THEOREM 4.11 The Second Fundamental Theorem of Calculus**

If $f$ is continuous on an open interval $I$ containing $a$, then, for every $x$ in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

# Example 7 – The Second Fundamental Theorem of Calculus

Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2 + 1}\, dt\right].$

Solution:

Note that $f(t) = \sqrt{t^2 + 1}$ is continuous on the entire real number line.

So, using the Second Fundamental Theorem of Calculus, you can write

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2 + 1}\, dt\right] = \sqrt{x^2 + 1}.$$

# Thanks a lot …