Net Change Theorem

The Fundamental Theorem of Calculus states that if f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

But because $F'(x) = f(x)$, this statement can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where the quantity $F(b) - F(a)$ represents the *net change of* $F(x)$ on the interval $[a, b]$.

Net Change Theorem

THEOREM 4.12 The Net Change Theorem

If $F'(x)$ is the rate of change of a quantity $F(x)$, then the definite integral of $F'(x)$ from a to b gives the total change, or **net change**, of $F(x)$ on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F(x)$$

Example 9 – *Using the Net Change Theorem*

A chemical flows into a storage tank at a rate of $(180 + 3t)$ liters per minute, where t is the time in minutes and $0 \leq t \leq 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

Solution:

Let $c(t)$ be the amount of the chemical in the tank at time t.

Then $c'(t)$ represents the rate at which the chemical flows into the tank at time t.

Example 9 – *Using the Net Change Theorem*

During the first 20 minutes, the amount that flows into the tank is

$$\int_0^{20} c'(t)\,dt = \int_0^{20} (180 + 3t)\,dt$$

$$= \left[180t + \frac{3}{2}t^2\right]_0^{20}$$

$$= 3600 + 600 = 4200.$$

So, the amount of the chemical that flows into the tank during the first 20 minutes is 4200 liters.

Integration by Substitution

Pattern Recognition

Recall that for the differentiable functions

$y = F(u)$ and $u = g(x)$

the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

These results are summarized in the following theorem.

Example 3 – *Multiplying and Dividing by a Constant*

Find the indefinite integral.

$$\int x(x^2 + 1)^2 \, dx.$$

Solution:

This is similar to the integral given in Example 1, except that the integrand is missing a factor of 2.

Recognizing that $2x$ is the derivative of $x^2 + 1$, you can let

$$g(x) = x^2 + 1$$

and supply the $2x$ as shown.

$$\int x(x^2 + 1)^2 \, dx = \int (x^2 + 1)^2 \left(\frac{1}{2}\right)(2x) \, dx \qquad \text{Multiply and divide by 2.}$$

Example 3 – *Solution*

$$= \frac{1}{2}\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\, \overbrace{(2x)}^{g'(x)}\, dx$$ Constant Multiple Rule

$$= \frac{1}{2}\left[\frac{(x^2 + 1)^3}{3}\right] + C$$ Integrate.

$$= \frac{1}{6}(x^2 + 1)^3 + C$$ Simplify.

Change of Variables for Indefinite Integrals

The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)dx$, and the integral in Theorem 4.13 takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

Example 4 – *Change of Variables*

Find $\int \sqrt{2x - 1}\, dx.$

Solution:

First, let u be the inner function, $u = 2x - 1$.

Then calculate the differential du to be $du = 2dx$.

Now, using $\sqrt{2x - 1} = \sqrt{u}$ and $dx = du/2,$ substitute to obtain

$$\int \sqrt{2x - 1}\, dx = \int \sqrt{u}\left(\frac{du}{2}\right) \qquad \text{Integral in terms of } u$$

Example 4 – *Solution*

$$= \frac{1}{2}\int u^{1/2}\,du$$ Constant Multiple Rule

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C$$ Antiderivative in terms of u

$$= \frac{1}{3}u^{3/2} + C$$ Simplify.

$$= \frac{1}{3}(2x - 1)^{3/2} + C.$$ Antiderivative in terms of x

The General Power Rule for Integration

THEOREM 4.14 The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Change of Variables for Definite Integrals

THEOREM 4.15 Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Change of Variables for Definite Integrals

Evaluate $\int_0^1 x(x^2 + 1)^3 \, dx.$

Solution:

To evaluate this integral, let $u = x^2 + 1$.

Then, you obtain $du = 2x\,dx.$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	**Upper Limit**
When $x = 0,\ u = 0^2 + 1 = 1.$	When $x = 1,\ u = 1^2 + 1 = 2.$

Example 8 – *Solution*

Now, you can substitute to obtain

$$\int_0^1 x(x^2+1)^3\,dx = \frac{1}{2}\int_0^1 (x^2+1)^3(2x)\,dx \qquad \text{Integration limits for } x$$

$$= \frac{1}{2}\int_1^2 u^3\,du \qquad \text{Integration limits for } u$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}.$$

Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult.

Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the *y*-axis or about the origin by recognizing the integrand to be an even or odd function (see Figure 4.40).

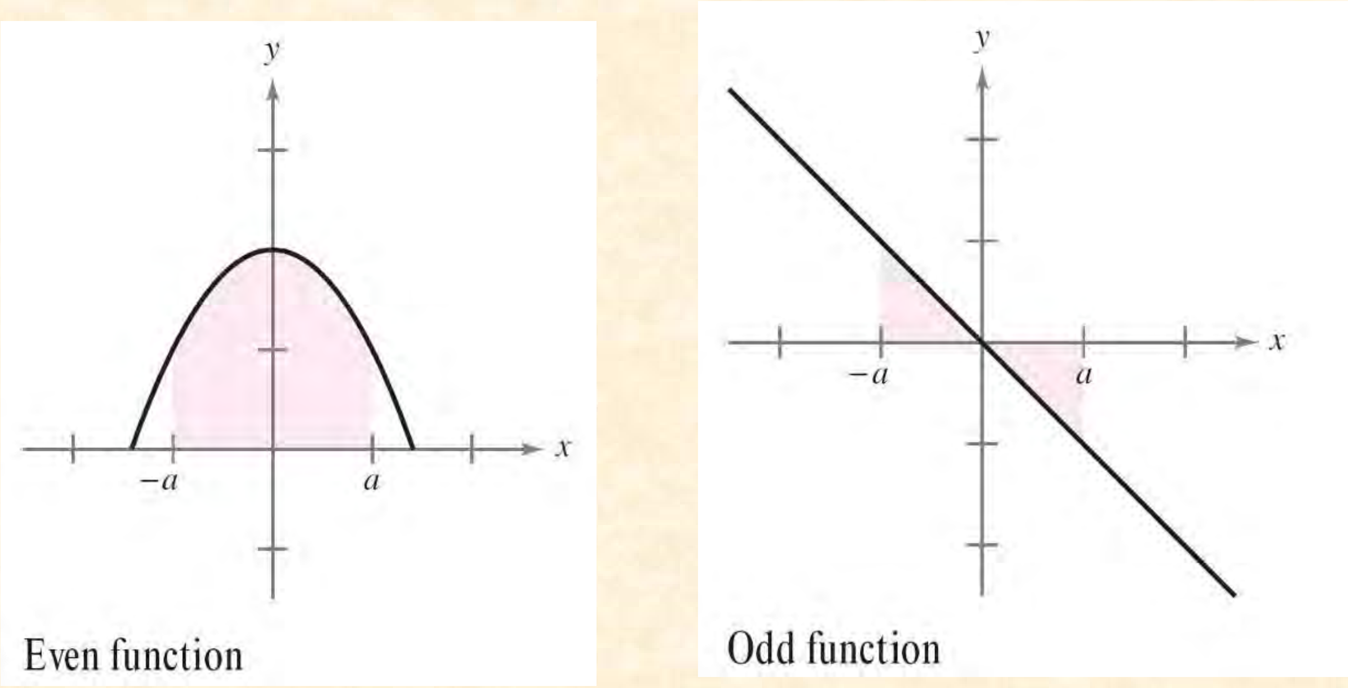


Figure 4.40

Integration of Even and Odd Functions

THEOREM 4.16 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 10 – *Integration of an Odd Function*

Evaluate the definite integral.

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\, dx.$$

Solution:

Letting $f(x) = \sin^3 x \cos x + \sin x \cos x$ produces

$$f(-x) = \sin^3(-x) \cos(-x) + \sin(-x) \cos(-x)$$

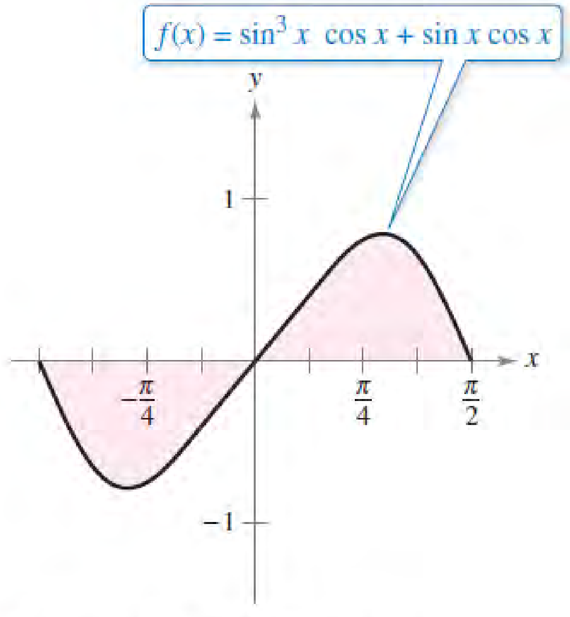
$$= -\sin^3 x \cos x - \sin x \cos x$$

$$= -f(x).$$

Example 10 – *Solution*

So, f is an odd function, and because f is symmetric about the origin over $[-\pi/2, \pi/2]$, you can apply Theorem 4.16 to conclude that

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\, dx = 0.$$



Because f is an odd function,

$$\int_{-\pi/2}^{\pi/2} f(x)\, dx = 0.$$

Figure 4.41

From Figure 4.41, you can

See that the two regions on

either side of the y-axis have the same area.

However, because one lies below the x-axis

and one lies above it, integration produces a cancellation effect.

8 Integration Techniques and Improper Integrals

Fitting Integrands to Basic Integration Rules

REVIEW OF BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\arctan \dfrac{u}{a} + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\operatorname{arcsec} \dfrac{|u|}{a} + C$

Example 1 – *A Comparison of Three Similar Integrals*

Find each integral.

a. $\displaystyle\int \frac{4}{x^2 + 9}\,dx$

b. $\displaystyle\int \frac{4x}{x^2 + 9}\,dx$

c. $\displaystyle\int \frac{4x^2}{x^2 + 9}\,dx$

Example 1(a) – *Solution*

Use the Arctangent Rule and let $u = x$ and $a = 3$.

$$\int \frac{4}{x^2 + 9}\,dx = 4\int \frac{1}{x^2 + 3^2}\,dx \qquad \text{Constant Multiple Rule}$$

$$= 4\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C \qquad \text{Arctangent Rule}$$

$$= \frac{4}{3}\arctan\frac{x}{3} + C \qquad \text{Simplify.}$$

Example 1(b) – *Solution*

The Arctangent Rule does not apply because the numerator contains a factor of x.

Consider the Log Rule and let $u = x^2 + 9$. Then $du = 2x\,dx$, and you have

$$\int \frac{4x}{x^2+9}\,dx = 2\int \frac{2x\,dx}{x^2+9} \qquad \text{Constant Multiple Rule}$$

$$= 2\int \frac{du}{u} \qquad \text{Substitute: } u = x^2 + 9.$$

$$= 2\ln|u| + C \qquad \text{Log Rule}$$

$$= 2\ln(x^2+9) + C. \qquad \text{Rewrite as a function of } x.$$

Example 1(c) – *Solution*

Because the degree of the numerator is equal to the degree of the denominator, you should first use division to rewrite the improper rational function as the sum of a polynomial and a proper rational function.

$$\int \frac{4x^2}{x^2+9}\,dx = \int \left(4 + \frac{-36}{x^2+9}\right) dx \qquad \text{Rewrite using long division.}$$

$$= \int 4\,dx - 36\int \frac{1}{x^2+9}\,dx \qquad \text{Rewrite as two integrals.}$$

$$= 4x - 36\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C \qquad \text{Integrate.}$$

$$= 4x - 12\arctan\frac{x}{3} + C \qquad \text{Simplify.}$$

Fitting Integrands to Basic Integration Rules

PROCEDURES FOR FITTING INTEGRANDS TO BASIC INTEGRATION RULES

Technique	Example
Expand (numerator).	$(1+e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\dfrac{1+x}{x^2+1} = \dfrac{1}{x^2+1} + \dfrac{x}{x^2+1}$
Complete the square.	$\dfrac{1}{\sqrt{2x-x^2}} = \dfrac{1}{\sqrt{1-(x-1)^2}}$
Divide improper rational function.	$\dfrac{x^2}{x^2+1} = 1 - \dfrac{1}{x^2+1}$
Add and subtract terms in numerator.	$\dfrac{2x}{x^2+2x+1} = \dfrac{2x+2-2}{x^2+2x+1} = \dfrac{2x+2}{x^2+2x+1} - \dfrac{2}{(x+1)^2}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate.	$\dfrac{1}{1+\sin x} = \left(\dfrac{1}{1+\sin x}\right)\left(\dfrac{1-\sin x}{1-\sin x}\right) = \dfrac{1-\sin x}{1-\sin^2 x} = \dfrac{1-\sin x}{\cos^2 x} = \sec^2 x - \dfrac{\sin x}{\cos^2 x}$

Integration by Parts

THEOREM 8.1 Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du.$$

Example 1 – *Integration by Parts*

Find $\int xe^x \, dx.$

Solution:

To apply integration by parts, you need to write the integral in the form

There are several ways to do this.

$$\int \underbrace{(x)}_{u}\underbrace{(e^x\,dx)}_{dv}, \quad \int \underbrace{(e^x)}_{u}\underbrace{(x\,dx)}_{dv}, \quad \int \underbrace{(1)}_{u}\underbrace{(xe^x\,dx)}_{dv}, \quad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

The guidelines for integration by parts in earlier slide suggest the first option because the derivative of $u = x$ is simpler than x, and $dv = e^x\,dx$ is the most complicated portion of the integrand that fits a basic integration formula.

Example 1 – *Solution*

$$dv = e^x\,dx \Rightarrow v = \int dv = \int e^x\,dx = e^x$$

$$u = x \Rightarrow du = dx$$

Now, integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx \qquad \text{Substitute.}$$

$$= xe^x - e^x + C. \qquad \text{Integrate.}$$

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

Example 7 – *Using the Tabular Method*

Find $\int x^2 \sin 4x \, dx.$

Solution:

Begin as usual by letting $u = x^2$ and $dv = v' \, dx = \sin 4x \, dx$. Next, create a table consisting of three columns, as shown.

Alternate Signs	u and Its Derivatives	v' and Its Antiderivatives
$+$	x^2	$\sin 4x$
$-$	$2x$	$-\frac{1}{4}\cos 4x$
$+$	2	$-\frac{1}{16}\sin 4x$
$-$	0	$\frac{1}{64}\cos 4x$

Differentiate until you obtain 0 as a derivative.

Example 7 – *Solution*

The solution is obtained by adding the signed products of the diagonal entries:

$$\int x^2 \sin 4x \, dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32} \cos 4x + C$$

Integrals Involving Powers of Sine and Cosine

In this section, you will study techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x \, dx \qquad \text{and} \qquad \int \sec^m x \tan^n x \, dx$$

where either *m* or *n* is a positive integer.

To find antiderivatives for these forms, try to break them into combinations of trigonometric integrals to which you can apply the Power Rule.

Integrals Involving Powers of Sine and Cosine

For instance, you can find $\int \sin^5 x \cos x\, dx$ with the Power Rule by letting $u = \sin x$. Then, $du = \cos x\, dx$ and you have

$$\int \sin^5 x \cos x\, dx = \int u^5\, du = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C.$$

To break up $\int \sin^m x \cos^n x\, dx$ into forms to which you can apply the Power Rule, use the following relationships.

$\sin^2 x + \cos^2 x = 1$	Pythagorean identity
$\sin^2 x = \frac{1 - \cos 2x}{2}$	Power-reducing formula for $\sin^2 x$
$\cos^2 x = \frac{1 + \cos 2x}{2}$	Power-reducing formula for $\cos^2 x$

Integrals Involving Powers of Sine and Cosine

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SINE AND COSINE

1. When the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then expand and integrate.

$$\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x\, dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x\, dx}^{\text{Save for } du} = \int (1 - \cos^2 x)^k \cos^n x \sin x\, dx$$

2. When the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then expand and integrate.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x\, dx = \int (\sin^m x)\overbrace{(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x\, dx}^{\text{Save for } du} = \int (\sin^m x)(1 - \sin^2 x)^k \cos x\, dx$$

3. When the powers of both the sine and cosine are even and nonnegative, make repeated use of the formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in the second guideline.

Example 1 – *Power of Sine Is Odd and Positive*

Find $\int \sin^3 x \cos^4 x \, dx.$

Solution:

Because you expect to use the Power Rule with $u = \cos x$, *save one sine factor* to form *du* and convert the remaining sine factors to cosines.

$$\int \sin^3 x \cos^4 x \, dx = \int \sin^2 x \cos^4 x (\sin x) \, dx = \int (1 - \cos^2 x) \cos^4 x \sin x \, dx$$ Trigonometric identity

$$= \int (\cos^4 x - \cos^6 x) \sin x \, dx = \int \cos^4 x \sin x \, dx - \int \cos^6 x \sin x \, dx$$ Rewrite.

$$= -\int \cos^4 x(-\sin x) \, dx + \int \cos^6 x(-\sin x) \, dx = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$ Integrate.

Integrals Involving Powers of Secant and Tangent

The guidelines below can help you find integrals of the form

$$\int \sec^m x \tan^n x \, dx.$$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SECANT AND TANGENT

1. When the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{Even}}} x \tan^n x \, dx = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x \, dx}^{\text{Save for } du} = \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x \, dx$$

2. When the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x \, dx = \int (\sec^{m-1} x)\overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x \, dx}^{\text{Save for } du} = \int (\sec^{m-1} x)(\sec^2 x - 1)^k \sec x \tan x \, dx$$

Integrals Involving Powers of Secant and Tangent

3. When there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x\,dx = \int (\tan^{n-2} x)\overbrace{(\tan^2 x)}^{\text{Convert to secants}}\,dx = \int (\tan^{n-2} x)(\sec^2 x - 1)\,dx$$

4. When the integral is of the form

$$\int \sec^m x\,dx$$

where m is odd and positive, use integration by parts.

5. When the first four guidelines do not apply, try converting to sines and cosines.

Example 4 – *Power of Tangent Is Odd and Positive*

Find $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx.$

Solution:

Because you expect to use the Power Rule with *u* = sec *x*, *save a factor of* (*sec x tan x*) to form *du* and convert the remaining tangent factors to secants.

$$\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx = \int (\sec x)^{-1/2}\tan^3 x\,dx = \int (\sec x)^{-3/2}(\tan^2 x)(\sec x \tan x)\,dx$$ Rewrite.

$$= \int (\sec x)^{-3/2}(\sec^2 x - 1)(\sec x \tan x)\,dx$$ Trigonometric identity

$$= \int [(\sec x)^{1/2} - (\sec x)^{-3/2}](\sec x \tan x)\,dx \quad = \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C$$ Integrate.

Integrals Involving Sine-Cosine Products

Integrals involving the products of sines and cosines of two angles occur in many applications.

In such instances, you can use the following product-to-sum formulas.

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$$

Example 8 – *Using a Product-to-Sum Formula*

Find $\int \sin 5x \cos 4x \, dx.$

Solution:

Considering the second product-to-sum formula, you can write

$$\int \sin 5x \cos 4x \, dx = \frac{1}{2}\int (\sin x + \sin 9x) \, dx$$

$$= \frac{1}{2}\left(-\cos x - \frac{\cos 9x}{9}\right) + C$$

$$= -\frac{\cos x}{2} - \frac{\cos 9x}{18} + C.$$

Trigonometric Substitution

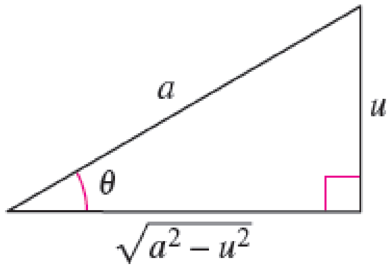
Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a \sin \theta.$$

Then $\sqrt{a^2 - u^2} = a \cos \theta$, where

$$-\pi/2 \le \theta \le \pi/2.$$

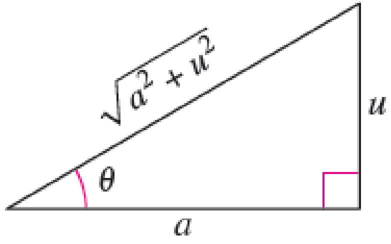


2. For integrals involving $\sqrt{a^2 + u^2}$, let

$$u = a \tan \theta.$$

Then $\sqrt{a^2 + u^2} = a \sec \theta$, where

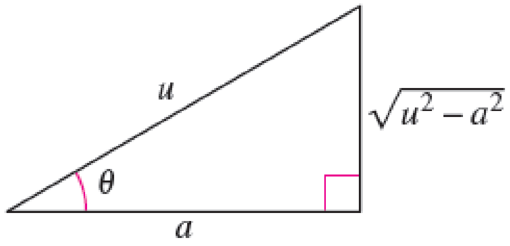
$$-\pi/2 < \theta < \pi/2.$$



3. For integrals involving $\sqrt{u^2 - a^2}$, let

$$u = a \sec \theta.$$

Then



$$\sqrt{u^2 - a^2} = \begin{cases} a \tan \theta \text{ for } u > a, \text{ where } 0 \le \theta < \pi/2 \\ -a \tan \theta \text{ for } u < -a, \text{ where } \pi/2 < \theta \le \pi. \end{cases}$$

Example 1 – *Trigonometric Substitution: u = a sin θ*

Find $\displaystyle\int \frac{dx}{x^2\sqrt{9 - x^2}}.$

Solution:

First, note that the basic integration rules do not apply.

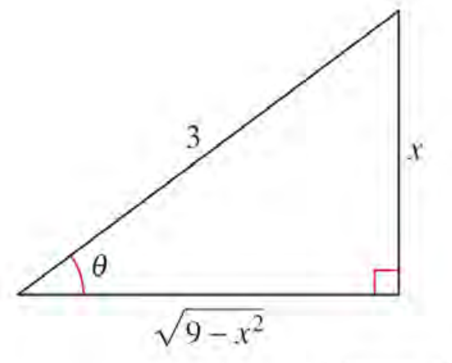
To use trigonometric substitution, you should observe that $\sqrt{9 - x^2}$ is of the form $\sqrt{a^2 - u^2}.$

So, you can use the substitution $x = a \sin \theta = 3 \sin \theta.$

Example 1 – *Solution*

Using differentiation and the triangle shown in Figure 8.6, you obtain

$$dx = 3\cos\theta\, d\theta, \qquad \sqrt{9 - x^2} = 3\cos\theta, \qquad \text{and} \qquad x^2 = 9\sin^2\theta.$$



$\sin\theta = \dfrac{x}{3}$, $\cot\theta = \dfrac{\sqrt{9 - x^2}}{x}$

Figure 8.6

So, trigonometric substitution yields

$$\int \frac{dx}{x^2\sqrt{9 - x^2}} = \int \frac{3\cos\theta\, d\theta}{(9\sin^2\theta)(3\cos\theta)} \qquad \text{Substitute.}$$

$$= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta} \qquad \text{Simplify.}$$

$$= \frac{1}{9}\int \csc^2\theta\, d\theta \qquad \text{Trigonometric identity}$$

$$= -\frac{1}{9}\cot\theta + C \qquad \text{Apply Cosecant Rule.}$$

$$= -\frac{1}{9}\left(\frac{\sqrt{9 - x^2}}{x}\right) + C \qquad \text{Substitute for } \cot\theta.$$

$$= -\frac{\sqrt{9 - x^2}}{9x} + C.$$

Trigonometric Substitution

THEOREM 8.2 Special Integration Formulas ($a > 0$)

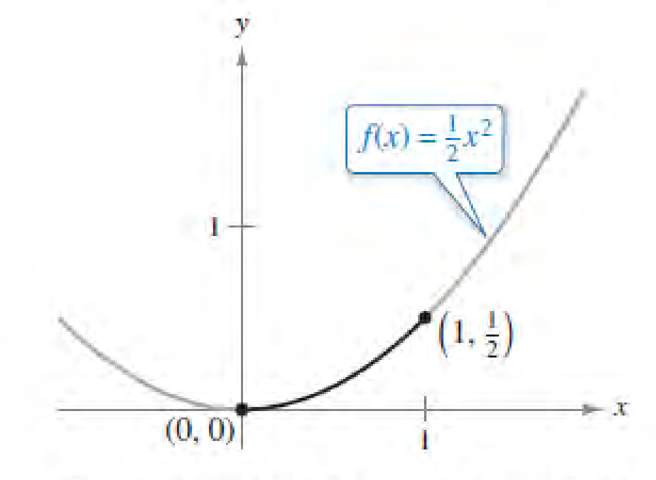
1. $\displaystyle\int \sqrt{a^2 - u^2}\, du = \frac{1}{2}\left(u\sqrt{a^2 - u^2} + a^2 \arcsin \frac{u}{a}\right) + C$

2. $\displaystyle\int \sqrt{u^2 - a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln\left|u + \sqrt{u^2 - a^2}\right|\right) + C, \quad u > a$

3. $\displaystyle\int \sqrt{u^2 + a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 + a^2} + a^2 \ln\left|u + \sqrt{u^2 + a^2}\right|\right) + C$

Example 5 – *Finding Arc Length*

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$ (see Figure 8.10).



The arc length of the curve from (0, 0) to $\left(1, \frac{1}{2}\right)$

Figure 8.10

Example 5 – *Solution*

Refer to the arc length formula.

$$s = \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx \qquad \text{Formula for arc length}$$

$$= \int_0^1 \sqrt{1 + x^2}\, dx \qquad f'(x) = x$$

$$= \int_0^{\pi/4} \sec^3 \theta \, d\theta \qquad \text{Let } a = 1 \text{ and } x = \tan \theta.$$

$$= \frac{1}{2}\left[\sec \theta \tan \theta + \ln|\sec \theta + \tan \theta|\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right] \qquad \approx 1.148$$

Practice

All Examples and related Problems from Exercise of the Textbook (Larson Calculus)

Partial Fractions

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide when improper:** When $N(x)/D(x)$ is an improper fraction (that is, when the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Example 1 – *Distinct Linear Factors*

Write the partial fraction decomposition for $\dfrac{1}{x^2 - 5x + 6}$.

Solution:

Because $x^2 - 5x + 6 = (x - 3)(x - 2)$, you should include one partial fraction for each factor and write

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x-3} + \frac{B}{x-2}$$

where A and B are to be determined.

Multiplying this equation by the least common denominator $(x - 3)(x - 2)$ yields the basic equation

$1 = A(x - 2) + B(x - 3)$. Basic equation.

Example 1 – *Solution*

Because this equation is to be true for all *x*, you can substitute any *convenient* values for *x* to obtain equations in *A* and *B*.

The most convenient values are the ones that make particular factors equal to 0.

To solve for *A*, let $x = 3$.

$1 = A(3 – 2) + B(3 – 3)$ Let $x = 3$ in basic equation.

$1 = A(1) + B(0)$

$1 = A$

To solve for *B*, let $x = 2$.

$1 = A(2 – 2) + B(2 – 3)$ Let $x = 2$ in basic equation.

$1 = A(0) + B(–1)$

$–1 = B$

So, the decomposition is

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}$$

as shown at the beginning of this section.

Example 3 – *Distinct Linear and Quadratic Factors*

Find $\displaystyle\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}\,dx.$

Solution:

Because $(x^2 - x)(x^2 + 4) = x(x - 1)(x^2 + 4)$ you should include one partial fraction for each factor and write

$$\frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 4}.$$

Multiplying by the least common denominator $x(x - 1)(x^2 + 4)$ yields the *basic equation*

$2x^3 - 4x - 8 = A(x - 1)(x^2 + 4) + Bx(x^2 + 4) + (Cx + D)(x)(x - 1).$

Example 3 – *Solution*

To solve for A, let $x = 0$ and obtain

$$-8 = A(-1)(4) + 0 + 0$$

$$2 = A.$$

To solve for B, let $x = 1$ and obtain

$$-10 = 0 + B(5) + 0$$

$$-2 = B.$$

At this point, C and D are yet to be determined. You can find these remaining constants by choosing two other values for x and solving the resulting system of linear equations.

Using $x = -1$, $A = 2$, and $B = -2$, you can write

$$-6 = (2)(-2)(5) + (-2)(-1)(5) + (-C + D)(-1)(-2)$$

$$2 = -C + D.$$

Example 3 – *Solution*

For $x = 2$, you have

$$0 = (2)(1)(8) + (-2)(2)(8) + (2C + D)(2)(1)$$

$$8 = 2C + D.$$

Solving the linear system by subtracting the first equation from the second

$$-C + D = 2$$

$$2C + D = 8$$

yields $C = 2$.

Consequently, $D = 4$, and it follows that

$$\int \frac{2x^3 - 4x - 8}{x(x-1)(x^2+4)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right) dx$$

$$= 2\ln|x| - 2\ln|x-1| + \ln(x^2+4) + 2\arctan\frac{x}{2} + C.$$

Thanks a lot …