

Chapter 1
The Foundations : Logic and Proofs
Kenneth H. Rosen 7th edition

Section 1.3 : Propositional Equivalences

Tautology, Contradiction & Contingency

▶ **Tautology:**

- ▶ A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.

▶ **Contradiction:**

- ▶ A compound proposition that is always false is called a contradiction.

▶ **Contingency:**

- ▶ A compound proposition that is neither a tautology nor a contradiction is called a contingency.



Tautology, Contradiction & Contingency(Contd.)

Example I:

▶ **Tautology:**

- ▶ Consider the truth table of $p \vee \neg p$. Because $p \vee \neg p$ is always true, it is a tautology.

▶ **Contradiction:**

- ▶ Consider the truth table of $p \wedge \neg p$. Because $p \wedge \neg p$ is always false, it is a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
F	T	T	F
T	F	T	F



Logical Equivalences

- ▶ Compound propositions that have the same truth values in all possible cases are called logically equivalent.
 - ▶ The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.
 - ▶ The notation $p \equiv q$ denotes that p and q are logically equivalent.



Logical Equivalences(Contd.)

► **Example I:**

- Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

► **Solution:**

- We can verify from the following truth table that, $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
F	F	T	T	F	T	T
F	T	T	F	T	F	F
T	F	F	T	T	F	F
T	T	F	F	T	F	F



Logical Equivalences(Contd.)

► **Exercises:**

- Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.
- Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.



Logical Equivalence Rules

► Important rules

<i>Equivalences</i>	<i>Name</i>	<i>Equivalences</i>	<i>Name</i>
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
$\neg(\neg p) \equiv p$	Double Negation Law	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws



Logical Equivalence Rules(Contd.)

► Important Rules Regarding Conditionals

- $p \rightarrow q \equiv \neg p \vee q$
- $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- $p \vee q \equiv \neg p \rightarrow q$
- $p \wedge q \equiv \neg(p \rightarrow \neg q)$
- $\neg(p \rightarrow q) \equiv p \wedge \neg q$
- $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
- $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
- $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
- $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
- $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
- $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$



Constructing New Logical Equivalences

▶ **Example I:**

- ▶ Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

▶ **Solution:**

$$\begin{aligned}\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by Rule} \\ &\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan Law} \\ &\equiv p \wedge \neg q && \text{by the Double Negation Law}\end{aligned}$$



Constructing New Logical Equivalences(Contd.)

► **Example 2:**

- Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

► **Solution:**

$\neg(p \vee (\neg p \wedge q))$	\equiv	$\neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
	\equiv	$\neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
	\equiv	$\neg p \wedge (p \vee \neg q)$	by the double negation law
	\equiv	$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
	\equiv	$F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
	\equiv	$(\neg p \wedge \neg q) \vee F$	by the commutative law of disjunction
	\equiv	$\neg p \wedge \neg q$	by the identity law for F



Constructing New Logical Equivalences(Contd.)

► **Example 3:**

► Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

► **Solution:**

$$\begin{aligned}(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by law of conditional} \\ &\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative} \\ &&& \text{laws of disjunction (or simply} \\ &&& \text{rearranging the terms)} \\ &\equiv T \vee T && \text{by negation law and the commutative} \\ &&& \text{law of disjunction} \\ &\equiv T && \text{by the domination law}\end{aligned}$$



Propositional Satisfiability

- ▶ A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that makes it *true*.
- ▶ When no such assignments exists, that is, when the compound proposition is *false* for all assignments of truth values to its variables, the compound proposition is *unsatisfiable*.
- ▶ To show that a compound proposition is *unsatisfiable*, we need to show that every assignment of truth values to its variables makes it *false*.
- ▶ We can logically reason with the values of each variable. But in our case, we will use the truth table.



Propositional Satisfiability(Contd.)

- ▶ **Example 1:**

- ▶ Determine the satisfiability of the compound proposition
 - ▶ $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$



Propositional Satisfiability(Contd.)

► **Solution:**

► Let $s = (p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \vee \neg q$	$q \vee \neg r$	$r \vee \neg p$	s
F	F	F	T	T	T	T	T	T	T
F	F	T	T	T	F	T	F	T	F
F	T	F	T	F	T	F	T	T	F
F	T	T	T	F	F	F	T	T	F
T	F	F	F	T	T	T	T	F	F
T	F	T	F	T	F	T	F	T	F
T	T	F	F	F	T	T	T	F	F
T	T	T	F	F	F	T	T	T	T

Propositional Satisfiability(Contd.)

Since there is at least one combination of input for the variables p, q, r of the compound proposition, which gives a true value for the compound proposition s , we can say that the s is satisfiable.



Propositional Satisfiability(Contd.)

► **Exercises:**

- Determine the satisfiability of each of the compound propositions

- $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

- $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$



THE END

