Antiderivatives

To find a function F whose derivative is $f(x) = 3x^2$, you might use your knowledge of derivatives to conclude that

$$F(x) = x^3 \quad \text{because} \quad \frac{d}{dx}[x^3] = 3x^2.$$

The function F is an *antiderivative* of f.

Definition of Antiderivative

A function F is an **antiderivative** of f on an interval I when $F'(x) = f(x)$ for all x in I.

Antiderivatives

Note that F is called *an* antiderivative of f rather than *the* antiderivative of f. To see why, observe that

$$F_1(x) = x^3, \quad F_2(x) = x^3 - 5, \quad \text{and} \quad F_3(x) = x^3 + 97$$

are all derivatives of $f(x) = 3x^2$. In fact, for any constant C, the function $F(x) = x^3 + C$ is an antiderivative of f.

THEOREM 4.1 Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$ for all x in I, where C is a constant.

Antiderivatives

The family of functions represented by G is the general antiderivative of f, and $G(x) = x^2 + C$ is the general solution of the *differential equation*

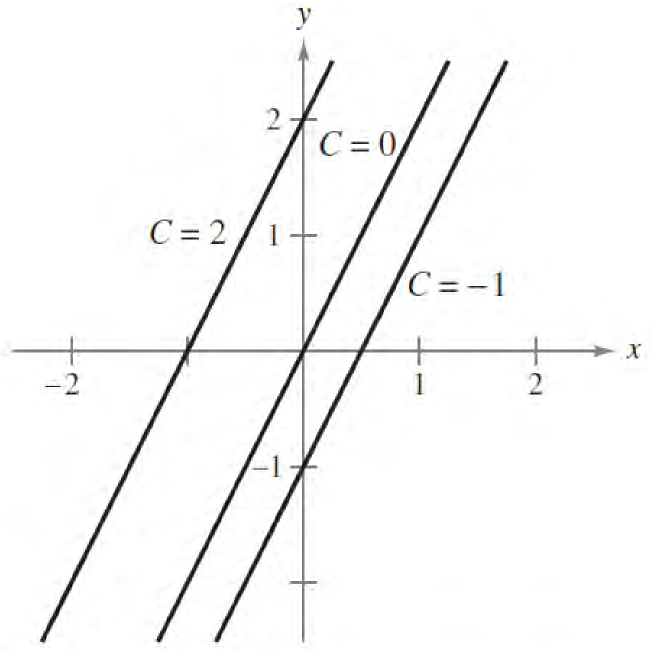
$G'(x) = 2x.$ Differential equation

A differential equation in x and y is an equation that involves x, y, and derivatives of y.

For instance, $y' = 3x$ and $y' = x^2 + 1$ are examples of differential equations.

Antiderivatives

The graphs of several functions of the form $y = 2x + C$ are shown in Figure 4.1.



Functions of the form $y = 2x + C$

Figure 4.1

Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\,dx.$$

The operation of finding all solutions of this equation is called antidifferentiation (or indefinite integration) and is denoted by an integral sign $\int$.

Antiderivatives

The general solution is denoted by

$$y = \int \underbrace{f(x)}_{\text{Integrand}} \, \overbrace{dx}^{\text{Variable of integration}} = \underbrace{F(x)}_{\text{An antiderivative of } f(x)} + \overbrace{C}^{\text{Constant of integration}}.$$

The expression $\int f(x)dx$ is read as the *antiderivative of f with respect to x*. So, the differential *dx* serves to identify *x* as the variable of integration. The term indefinite integral is a synonym for antiderivative.

Basic Integration Rules

Basic Integration Rules

Differentiation Formula	Integration Formula	
$\frac{d}{dx}[C] = 0$	$\int 0\, dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\, dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\, dx = k\int f(x)\, dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule

Basic Integration Rules

Basic Integration Rules

Differentiation Formula	Integration Formula
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$

Initial Conditions and Particular Solutions

- You have already seen that the equation $y = \int f(x)dx$ has many solutions (each differing from the others by a constant).
- This means that the graphs of any two antiderivatives of f are vertical translations of each other.

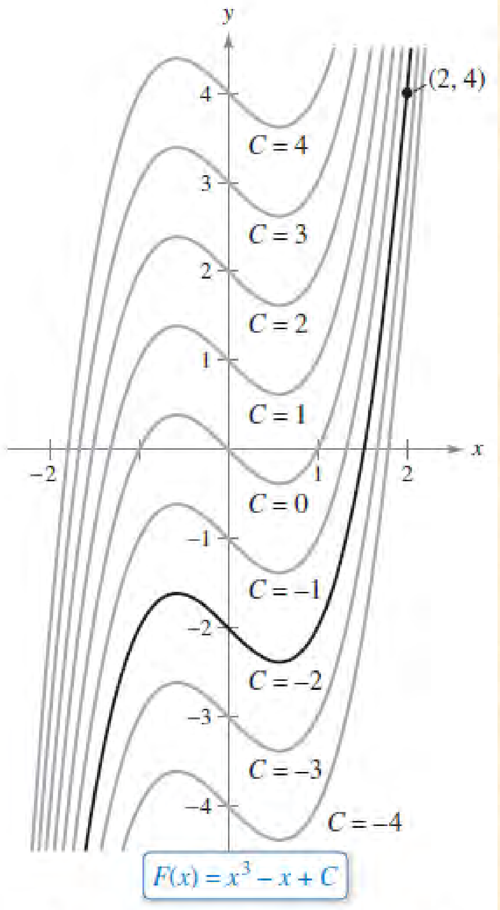
For example, Figure 4.2 shows the graphs of several antiderivatives of the form

$$y = \int (3x^2 - 1)\,dx = x^3 - x + C \qquad \text{General solution}$$

for various integer values of C.

Each of these antiderivatives is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2 - 1.$$



The particular solution that satisfies the initial condition $F(2) = 4$ is $F(x) = x^3 - x - 2$.

Figure 4.2

Initial Conditions and Particular Solutions

In many applications of integration, you are given enough information to determine a particular solution. To do this, you need only know the value of $y = F(x)$ for one value of x. This information is called an initial condition.

For example, in Figure 4.2, only one curve passes through the point (2, 4).

To find this curve, you can use the general solution

$F(x) = x^3 - x + C$ General solution

and the initial condition

$F(2) = 4.$ Initial condition

Initial Conditions and Particular Solutions

By using the initial condition in the general solution, you can determine that

$F(2) = 8 - 2 + C = 4$

which implies that $C = -2$.

So, you obtain

$F(x) = x^3 - x - 2.$ Particular solution

Example 1 – Finding a Particular Solution

Find the general solution of and find the particular solution that satisfies the initial condition $F(1) = 0$.

Solution:

To find the general solution, integrate to obtain

$$F(x) = \int \frac{1}{x^2}\,dx \qquad F(x) = \int F'(x)dx$$

$$= \int x^{-2}\,dx \qquad \text{Rewrite as a power.}$$

$$= \frac{x^{-1}}{-1} + C \qquad \text{Integrate.}$$

Example 1 – Finding a Particular Solution

$$= -\frac{1}{x} + C, \quad x > 0.$$

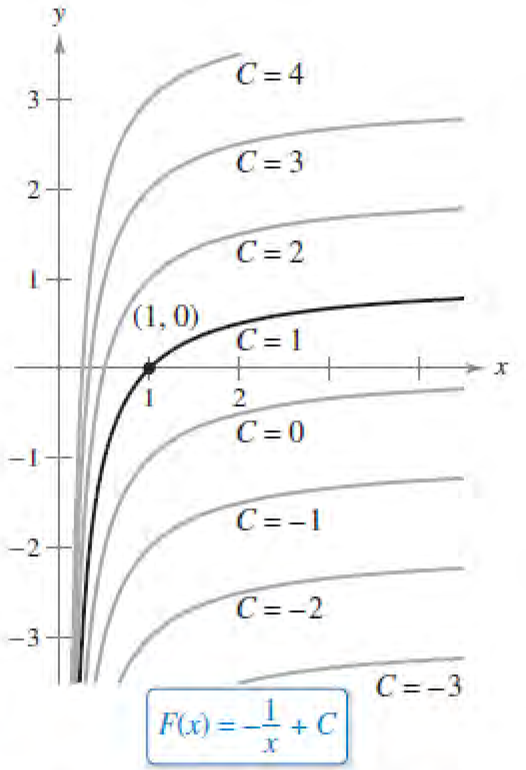
Using the initial condition $F(1) = 0$, you can solve for C as follows.

$$F(1) = -\frac{1}{1} + C = 0 \quad \Rightarrow \quad C = 1$$

So, the particular solution, as shown in Figure 4.3, is

$$F(x) = -\frac{1}{x} + 1, \quad x > 0.$$

Particular solution

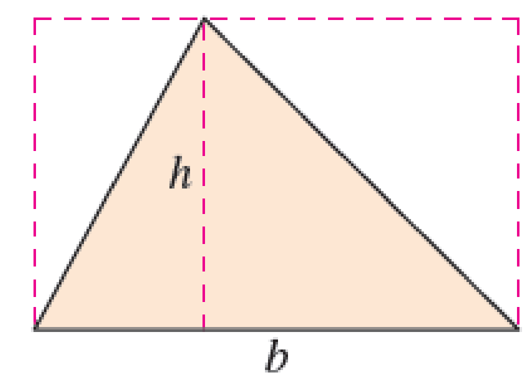


The particular solution that satisfies the initial condition $F(1) = 0$ is $F(x) = -(1/x) + 1, x > 0$.

Figure 4.3

Area

For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle, as shown in Figure 4.5.



Triangle: $A = \frac{1}{2}bh$

Figure 4.5

Area

Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown in Figure 4.6.

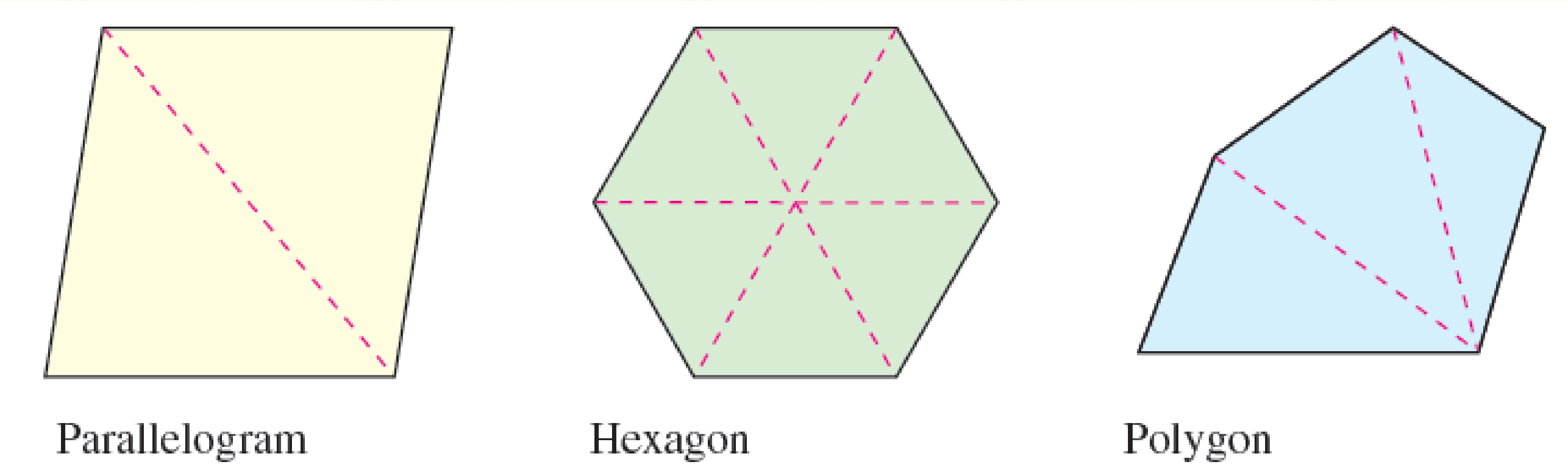
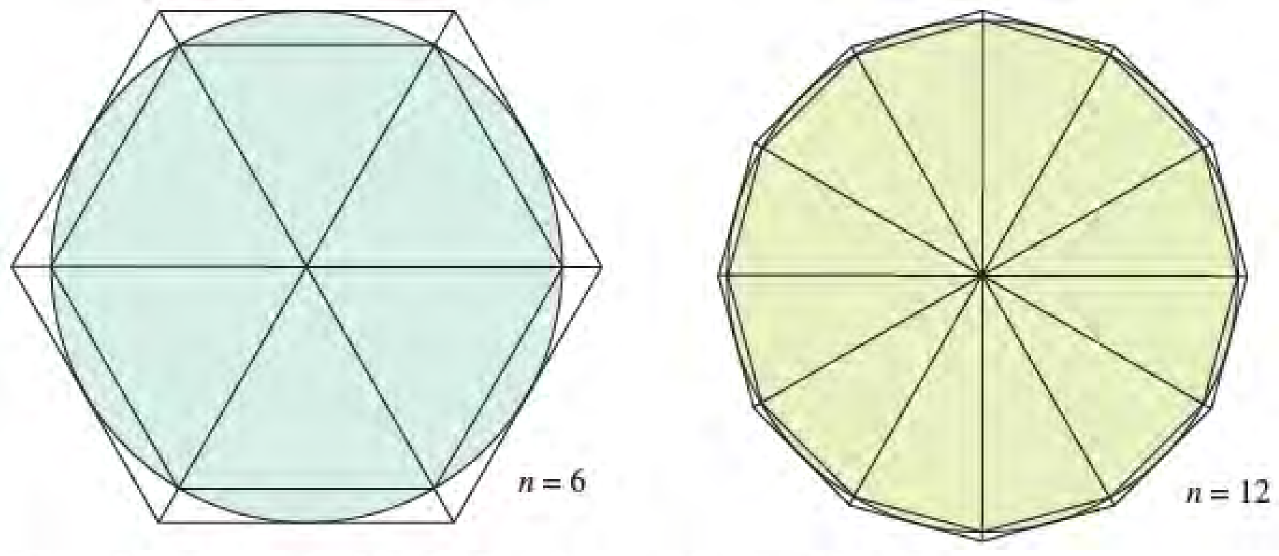


Figure 4.6

Area

For instance, in Figure 4.7, the area of a circular region is approximated by an *n*-sided inscribed polygon and an *n*-sided circumscribed polygon.



The exhaustion method for finding the area of a circular region

Figure 4.7

Example 1 – Approximating the Area of a Plane Region

Use the five rectangles in Figure 4.8(a) and (b) to find *two* approximations of the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the x-axis between $x = 0$ and $x = 2$.

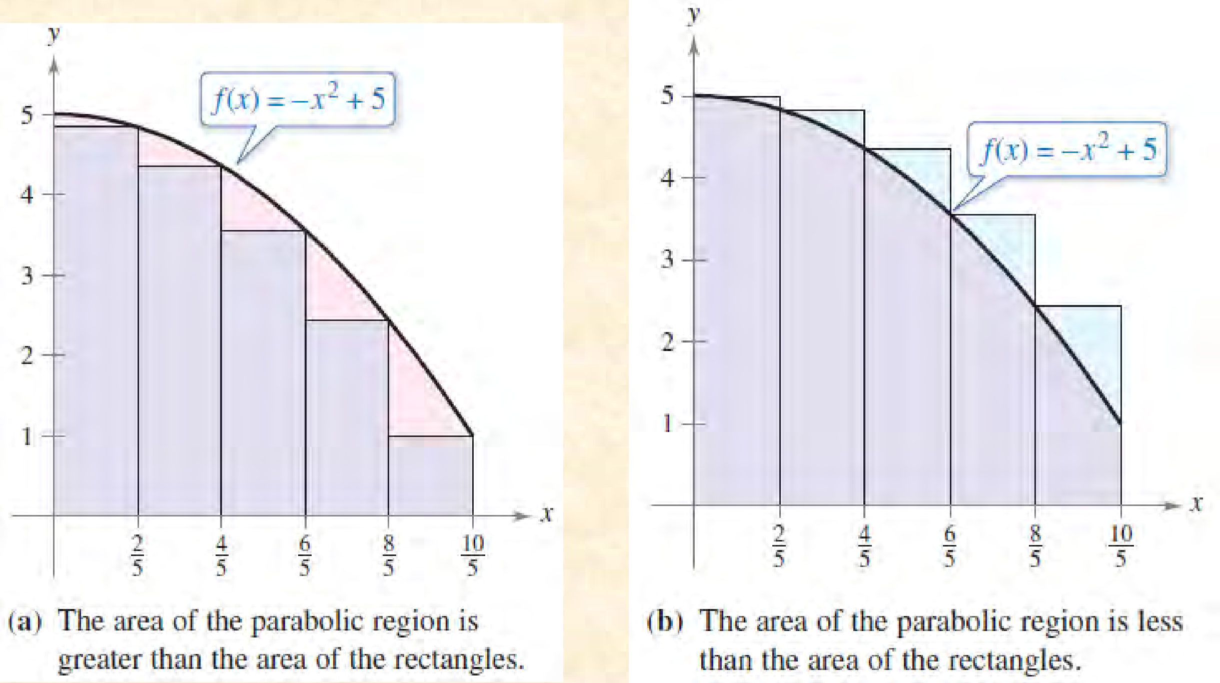


Figure 4.8

Example 3(a) – Solution

The right endpoints of the five intervals are

$$\frac{2}{5}i \qquad \text{Right endpoints}$$

where i = 1, 2, 3, 4, 5.

The width of each rectangle is $\frac{2}{5}$, and the height of each

rectangle can be obtained by evaluating f at the right endpoint of each interval.

$$\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \left[\frac{6}{5}, \frac{8}{5}\right], \left[\frac{8}{5}, \frac{10}{5}\right]$$

Evaluate f at the right endpoints of these intervals.

Example 3(a) – Solution

The sum of the areas of the five rectangles is

$$\sum_{i=1}^{5} \underbrace{f\left(\frac{2i}{5}\right)}_{\text{Height}} \underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} = \sum_{i=1}^{5}\left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48.$$

Because each of the five rectangles lies inside the parabolic region, you can conclude that the area of the parabolic region is greater than 6.48.

Example 3(b) – Solution

The left endpoints of the five intervals are

$$\frac{2}{5}(i-1) \qquad \text{Left endpoints}$$

where i = 1, 2, 3, 4, 5.

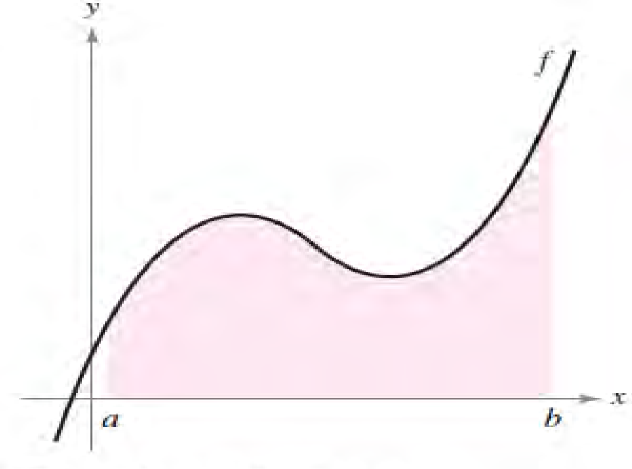
The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the left endpoint of each interval. So, the sum is

$$\sum_{i=1}^{5} \overbrace{f\left(\frac{2i-2}{5}\right)}^{\text{Height}} \overbrace{\left(\frac{2}{5}\right)}^{\text{Width}} = \sum_{i=1}^{5}\left[-\left(\frac{2i-2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08.$$

Finding Area by the Limit Definition

Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$ as shown in Figure 4.9.

The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.



The region under a curve

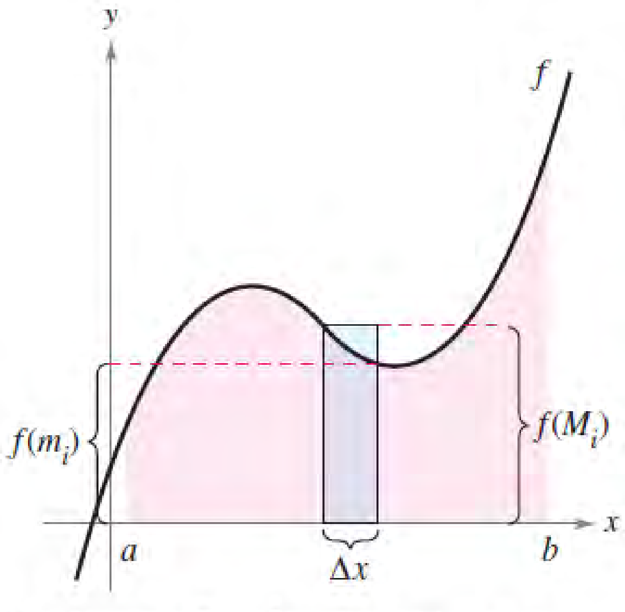
Figure 4.9

Finding Area by the Limit Definition

To approximate the area of the region, begin by subdividing the interval [*a*, *b*] into *n* subintervals, each of width

$$\Delta x = \frac{b - a}{n}$$

as shown in Figure 4.10.



The interval $[a, b]$ is divided into n subintervals of width $\Delta x = \dfrac{b - a}{n}$.

Figure 4.10

Finding Area by the Limit Definition

The endpoints of the intervals are

$$\underbrace{a + 0(\Delta x)}_{a = x_0} < \underbrace{a + 1(\Delta x)}_{x_1} < \underbrace{a + 2(\Delta x)}_{x_2} < \cdots < \underbrace{a + n(\Delta x)}_{x_n = b}.$$

Because *f* is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in *each* subinterval.

$f(m_i)$ = Minimum value of $f(x)$ in ith subinterval

$f(M_i)$ = Maximum value of $f(x)$ in ith subinterval

Finding Area by the Limit Definition

Next, define an inscribed rectangle lying *inside* the *i*th subregion and a circumscribed rectangle extending *outside* the *i*th subregion. The height of the *i*th inscribed rectangle is $f(m_i)$ and the height of the *i*th circumscribed rectangle is $f(M_i)$.

For *each i*, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\begin{pmatrix}\text{Area of inscribed}\\ \text{rectangle}\end{pmatrix} = f(m_i)\,\Delta x \le f(M_i)\,\Delta x = \begin{pmatrix}\text{Area of circumscribed}\\ \text{rectangle}\end{pmatrix}$$

Finding Area by the Limit Definition

The sum of the areas of the inscribed rectangles is called a **lower sum**, and the sum of the areas of the circumscribed rectangles is called an **upper sum**.

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x \qquad \text{Area of inscribed rectangles}$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x \qquad \text{Area of circumscribed rectangles}$$

Finding Area by the Limit Definition

From Figure 4.11, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

$$s(n) \le (\text{Area of region}) \le S(n)$$

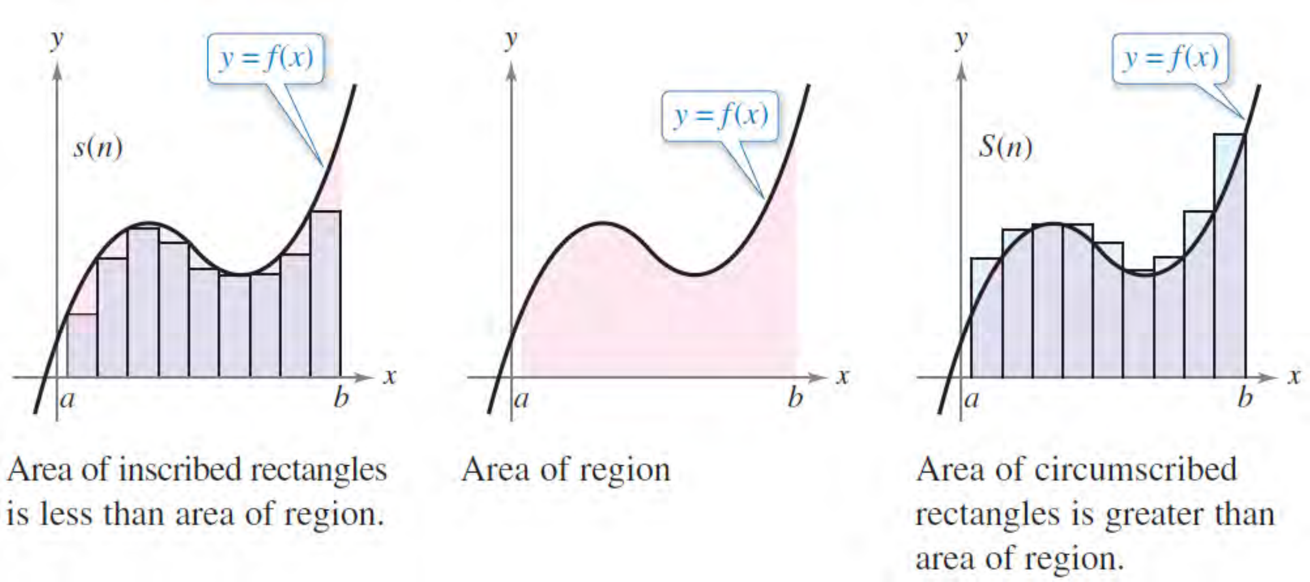


Figure 4.11

Example 3 – Finding Upper and Lower Sums for a Region

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 2$.

Solution:

To begin, partition the interval [0, 2] into n subintervals, each of width

$$\Delta x = \frac{b - a}{n}$$

$$= \frac{2 - 0}{n} \quad = \frac{2}{n}.$$

Example 4 – Solution

Figure 4.12 shows the endpoints of the subintervals and several inscribed and circumscribed rectangles.

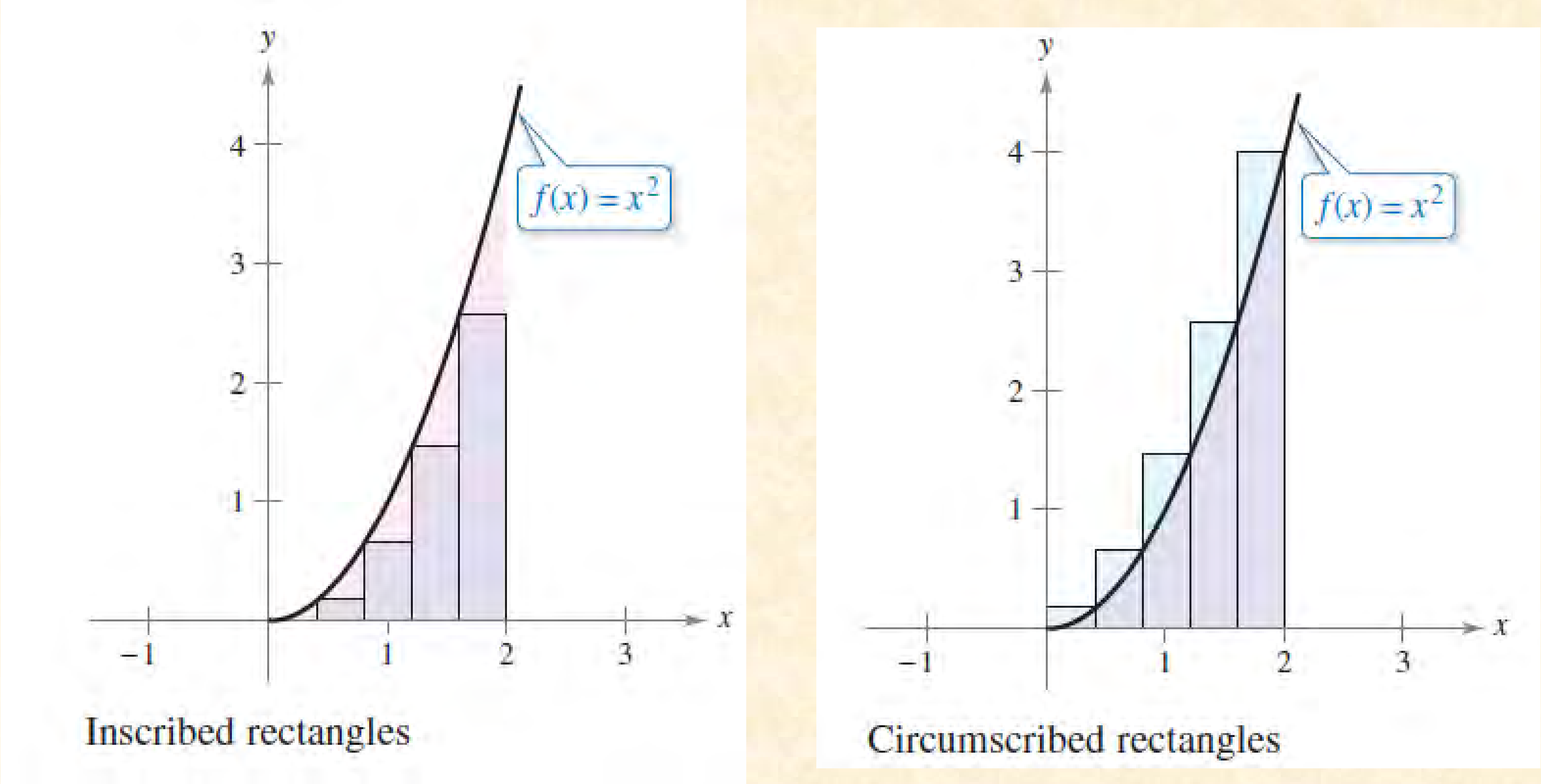


Figure 4.12

Example 4 – Solution

Because f is increasing on the interval [0, 2], the minimum value on each subinterval occurs at the left endpoint, and the maximum value occurs at the right endpoint.

Left Endpoints	**Right Endpoints**
$m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$	$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

Using the left endpoints, the lower sum is

$$s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x = \sum_{i=1}^{n} f\left[\frac{2(i-1)}{n}\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \left[\frac{2(i-1)}{n}\right]^2\left(\frac{2}{n}\right)$$

Example 4 – Solution

$$= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right)(i^2 - 2i + 1)$$

$$= \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)$$

$$= \frac{8}{n^3}\left\{\frac{n(n+1)(2n+1)}{6} - 2\left[\frac{n(n+1)}{2}\right] + n\right\}$$

$$= \frac{4}{3n^3}(2n^3 - 3n^2 + n)$$

$$= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.$$

Lower sum

Example 4 – Solution

Using the right endpoints, the upper sum is

$$S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(\frac{8}{n^3}\right)i^2$$

$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{4}{3n^3}(2n^3 + 3n^2 + n)$$

$$= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}.$$ Upper sum

Finding Area by the Limit Definition

In Theorem 4.3, the same limit is attained for both the minimum value $f(m_i)$ and the maximum value $f(M_i)$.

So, it follows from the Squeeze Theorem (Theorem 1.8) that the choice of x in the ith subinterval does not affect
the limit.

THEOREM 1.8 The Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L.

Thanks a lot …