Reduction Formula

- In the solution of many physical or engineering problems, we have to integrate some integrands involving powers or products of trigonometric functions. In this unit we shall develop a quicker method for evaluating these integrals. We shall consider some standard forms of integrands one by one, and derive formulas to integrate them.

- By using the method of integration by parts we shall try to express such an integral in terms of another similar integral with a lower value of the parameter. You will see that by the repeated use of this technique, we shall be able to evaluate the given integral.

Reduction Formula

- A Reduction Formula for an integral is a formula which connects integral linearly with another integral of the same type, but of the lower degree.
- Reduction formula is generally obtained by repeated application of integration by parts.
- Reduction formula are used as a tool to find area and volume.
- A reduction formula simplifies a complex integral into a more manageable form by reducing the degree of the integrand. It often uses recursive relationships.

Example 1

The integrand in $\int x^n e^x dx$ depends on *x* and also on the parameter *n* which is the exponent of x,

Solution:

Let, $I_n = \int x^n e^x dx = x^n e^x - \int n x^{n-1} e^x dx$ [Integration by parts]

$= x^n e^x - n\int x^{n-1} e^x dx = x^n e^x - nI_{n-1}$ [the exponent of *x* is reduced by 1.]

$$\therefore \int x^4 e^x dx = x^4 e^x - 4I_3 = x^4 e^x - 4\left[x^3 e^x - 3I_2\right]$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24I_1 = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24I_0$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C$$

Reduction Formula

A formula of the form

$$\int f(x,n)dx = g(x) - \int f(x,k)dx; k < n$$

Reduction Formula for $\sin^n x$

Let, $I_n = \int \sin^n x dx$

Integrating by parts by taking $\sin^{n-1} x$ as first function and $\sin x$ as second function.

$$I_n = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x.\cos x(-\cos x)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 xdx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\left(1-\sin^2 x\right)dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} xdx - (n-1)\int \sin^n xdx$$

$$= -\sin^{n-1} x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore I_n = \frac{-\sin^{n-1} x\cos x}{n} + \frac{n-1}{n} I_{n-2}$$ is the required reduction formula for $\int \sin^n x dx$

Reduction Formula for $cos^n x$

Let, $I_n = \int \cos^n x dx$

Do yourself

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$$\therefore I_n = \frac{-\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$ is the required reduction formula for $\int \cos^n x dx$

Example 1-Using Reduction Formula

Using Reduction Formula Evaluate the Definite Integral:

$$\int_0^{\pi/2} \sin^5 x dx$$

Solution:

$$\text{We have } \int \sin^n x dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$\therefore \int_0^{\pi/2} \sin^n x dx = \left[\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx ; n \geq 2$$

$$\therefore \int_0^{\pi/2} \sin^5 x dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x dx = \frac{4}{5}.\frac{2}{3}\int_0^{\pi/2} \sin x dx = \frac{8}{15}(-\cos x)_0^{\pi/2} = \frac{8}{15}$$

Example 1-Using Reduction Formula

Evaluate the Definite Integral:

$$\int_0^{\pi/2} \sin^n x dx$$

Solution:

$$\text{We have } \int \sin^n x dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$\therefore \int_0^{\pi/2} \sin^n x dx = \left[\frac{-\sin^{n-1} x \cos x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx; n \geq 2$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2} = \left(\frac{n-1}{n} \right) \left(\frac{n-3}{n-2} \right) I_{n-4} = \left(\frac{n-1}{n} \right) \left(\frac{n-3}{n-2} \right) \left(\frac{n-5}{n-4} \right) I_{n-6} = \dots\dots\dots$$

Example 2-Using Reduction Formula

Solution:

$$=\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\left(\frac{n-7}{n-6}\right)\ldots\frac{5}{6}\frac{3}{4}\frac{1}{2}I_0$$

$$=\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\left(\frac{n-7}{n-6}\right)\ldots\frac{5}{6}\frac{3}{4}\frac{1}{2}\int_0^{\pi/2} 1dx$$

$$=\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\left(\frac{n-7}{n-6}\right)\ldots\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2};\text{if } n \text{ is even}$$

Practice: $\int_0^{\pi/2} \mathrm{Cos}^n x\,dx$

Example 2-Using Reduction Formula

Using Reduction Formula Evaluate the Definite Integral:

$$\int_0^{\pi/2} \sin^5 x dx$$

Solution:

$$\text{We have } \int \sin^n x dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$\therefore \int_0^{\pi/2} \sin^n x dx = \left[\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx; n \ge 2$$

$$\therefore \int_0^{\pi/2} \sin^5 x dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x dx = \frac{4}{5}.\frac{2}{3}\int_0^{\pi/2} \sin x dx = \frac{8}{15}(-\cos x)_0^{\pi/2} = \frac{8}{15}$$

Example 3-Using Reduction Formula

Evaluate the Definite Integral:

$$\int_0^\infty \frac{dx}{\left(1+x^2\right)^4}$$

Solution: Let, $x = \tan\theta; dx = \sec^2\theta d\theta$

$when, x \to 0; \theta \to 0, \text{and}\, when, x \to \infty; \theta \to \pi/2$

$$\therefore \text{Given integral becomes, } \int_0^{\pi/2} \frac{\sec^2\theta}{\left(1+\tan^2\theta\right)^4} d\theta = \int_0^{\pi/2} \frac{\sec^2\theta}{\left(\sec^2\theta\right)^4} d\theta = \int_0^{\pi/2} \frac{1}{\sec^6\theta} d\theta$$

$$= \int_0^{\pi/2} \cos^6\theta d\theta = \frac{(6-1)(6-3)(6-5)}{6(6-2)(6-4)} \frac{\pi}{2} = \frac{15\pi}{32}$$

Thanks a lot …