

Beta and Gamma Function

- Beta and gamma functions are popular functions in mathematics. Gamma is a single variable function while beta is a dual variable function.
- The Gamma function and Beta functions belong to the category of the special transcendental functions and are defined in terms of improper definite integrals.
- These functions are very useful in many areas like asymptotic series, Riemann-zeta function, number theory, etc. and also have many applications in engineering and physics.
- The Gamma function was first introduced by Swiss mathematician Leonhard Euler(1707-1783).

Gamma Function

Let n be any positive number. Then the definite integral $\int_0^{\infty} x^{n-1} e^{-x} dx$, for $n > 0$ is called gamma function of n which is denoted by Γn and it is defined as

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx, \text{ for } n > 0$$

Example 1-Gamma Function

Prove that $\Gamma(1)=1$

Proof: We know by definition of Gamma function

$$\begin{aligned}\Gamma(n) &= \int_0^{\infty} x^{n-1} e^{-x} dx \\ \therefore \Gamma(1) &= \int_0^{\infty} x^{1-1} e^{-x} dx \\ &= \int_0^{\infty} x^0 e^{-x} dx \quad \left[\because e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0 \right] \\ &= \left[-e^{-x} \right]_0^{\infty} = -\{0 - 1\} = 1\end{aligned}$$

Example 2-Gamma Function

Prove that $\Gamma(n+1) = n!$

Proof: We know by definition of Gamma function

$$\begin{aligned}\Gamma(n) &= \int_0^{\infty} x^{n-1} e^{-x} dx \\ \therefore \Gamma(n+1) &= \int_0^{\infty} x^n e^{-x} dx \\ &= \left[-x^n e^{-x} \right]_0^{\infty} - \int_0^{\infty} -e^{-x} nx^{n-1} dx \\ &= 0 + \int_0^{\infty} e^{-x} nx^{n-1} dx = n \int_0^{\infty} x^{n-1} e^{-x} dx \\ &= n\Gamma(n)\end{aligned}$$

Example 2-Gamma Function

$$\therefore \Gamma(n+1) = n\Gamma(n)$$

Replacing n by $n-1, n-2, n-3, \dots, 3, 2, 1$ we get

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$

$$\Gamma(n-2) = (n-3)\Gamma(n-3)$$

.....

$$\Gamma 4 = 3\Gamma(3)$$

$$\Gamma 3 = 2\Gamma(2)$$

$$\Gamma 2 = 1\Gamma(1) = 1$$

$$\therefore \Gamma(1) = 1$$

$$\therefore \Gamma(n+1) = n(n-1)(n-2)(n-3)\dots 3.2.1 = n!$$

Example 3-Gamma Function

Prove that $\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{2}$

Proof: We know by definition of Gamma function

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^{\infty} x^{\frac{1}{2}-1} e^{-x} dx = \int_0^{\infty} x^{-\frac{1}{2}} e^{-x} dx \\ &= \int_0^{\infty} e^{-t^2} t^{-1} \cdot 2t dt; \text{ by putting } x = t^2 \\ &= 2 \int_0^{\infty} e^{-t^2} dt \qquad \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \\ &= 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}\end{aligned}$$

Beta Function

The integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$, for $m > 0, n > 0$

is called beta function of m, n. We write as

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \text{ for } m > 0, n > 0$$

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Example 4-Beta Function

Prove that $\beta(m, n) = \beta(n, m)$

Proof: We know by definition of Beta function

$$\begin{aligned}\beta(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\&= \int_0^1 (1-x)^{m-1} (1-(1-x))^{n-1} dx && \because \int_0^a f(x) dx = \int_0^a f(a-x) dx \\&= \int_0^1 (1-x)^{m-1} x^{n-1} dx \\&= \int_0^1 x^{n-1} (1-x)^{m-1} dx \\&= \beta(n, m)\end{aligned}$$

Example 5-Beta Function

Show that $\beta(m, n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$

Proof: We know by definition of Beta function

$$\begin{aligned}\beta(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\ &= \int_{\infty}^0 \left(\frac{1}{1+y} \right)^{m-1} \left(\frac{y}{1+y} \right)^{n-1} \left\{ -\frac{1}{(1+y)^2} dy \right\} \\ &= - \int_{\infty}^0 \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2} dy \\ &= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy\end{aligned}$$

$$\text{Let, } x = \frac{1}{1+y}$$

$$\therefore dx = -\frac{1}{(1+y)^2} dy$$

$$\text{and, } 1-x = 1 - \frac{1}{1+y} = \frac{y}{1+y}$$

x	1	0
y	0	∞

Example 6-Beta Function

Prove that $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof: We know by definition of Gamma function

$$\begin{aligned}\Gamma(n) &= \int_0^{\infty} x^{n-1} e^{-x} dx \\ &= \int_0^{\infty} (yz)^{n-1} e^{-yz} z dy \\ &= \int_0^{\infty} y^{n-1} z^n e^{-yz} dy\end{aligned}$$

Let, $x = zy$
 $\therefore dx = z dy$

Multiplying both sides by $z^{m-1}e^{-z}$ and then integrating w.r.to z from 0 to ∞

$$\Gamma(n) \int_0^{\infty} z^{m-1} e^{-z} dz = \int_0^{\infty} \left[z^{m-1} e^{-z} \int_0^{\infty} y^{n-1} z^n e^{-yz} dy \right] dz$$

Example 7-Beta Function

$$\Gamma(n) \int_0^{\infty} z^{m-1} e^{-z} dz = \int_0^{\infty} \left[z^{m-1} e^{-z} \int_0^{\infty} y^{n-1} z^n e^{-yz} dy \right] dz$$

$$\Gamma(n)\Gamma(m) = \int_0^{\infty} \left[\int_0^{\infty} z^{m+n-1} e^{-z(1+y)} dz \right] y^{n-1} dy$$

$$\Gamma(n)\Gamma(m) = \int_0^{\infty} \frac{1}{(1+y)^{m+n}} \Gamma(m+n) y^{n-1} dy$$

$$\frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)} = \int_0^{\infty} \frac{1}{(1+y)^{m+n}} y^{n-1} dy$$

$$\frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)} = \beta(m, n)$$

Let, $z(1+y) = u$

$$dz = \frac{1}{1+y} du$$

$$\begin{aligned} & \int_0^{\infty} z^{m+n-1} e^{-z(1+y)} dz \\ &= \int_0^{\infty} \left(\frac{u}{1+y} \right)^{m+n-1} e^{-u} \frac{1}{1+y} du \\ &= \frac{1}{(1+y)^{m+n}} \int_0^{\infty} u^{(m+n)-1} e^{-u} du \\ &= \frac{1}{(1+y)^{m+n}} \Gamma(m+n) \end{aligned}$$



Example 8-Beta Function

Evaluate $\int_0^{\infty} \frac{x^8}{(1+x)^{24}} dx$

Solution: $\int_0^{\infty} \frac{x^8}{(1+x)^{24}} dx$ $\left[\because \beta(m, n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy \right]$

$$= \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{15+9}} dx$$

$$= \beta(15, 9)$$

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\Gamma(9)\Gamma(15)}{\Gamma(9+15)}$$

$$= \frac{\Gamma(9)\Gamma(15)}{\Gamma(24)} = \frac{8!14!}{23!}$$

Example 9-Beta Function

Evaluate

$$\int_0^{\infty} \frac{x^8 (1 - x^6)}{(1 + x)^{24}} dx$$

Solution:

$$\begin{aligned} \int_0^{\infty} \frac{x^8 (1 - x^6)}{(1 + x)^{24}} dx &= \int_0^{\infty} \frac{x^8 - x^{14}}{(1 + x)^{24}} dx = \int_0^{\infty} \frac{x^8}{(1 + x)^{24}} dx - \int_0^{\infty} \frac{x^{14}}{(1 + x)^{24}} dx \\ &= \int_0^{\infty} \frac{x^{9-1}}{(1 + x)^{15+9}} dx - \int_0^{\infty} \frac{x^{15-1}}{(1 + x)^{9+15}} dx \\ &= \beta(15, 9) - \beta(9, 15) \quad \left[\because \beta(m, n) = \int_0^{\infty} \frac{y^{n-1}}{(1 + y)^{m+n}} dy \right] \\ &= \beta(9, 15) - \beta(9, 15) \quad \left[\because \beta(m, n) = \beta(n, m) \right] \\ &= 0 \end{aligned}$$

Example 11-Gamma Function

Prove that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof: we know that

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\frac{\pi}{2}} (\sin \theta)^p (\cos \theta)^q d\theta$$

$$\frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{0+0+2}{2}\right)} = \int_0^{\frac{\pi}{2}} (\sin \theta)^0 (\cos \theta)^0 d\theta$$

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2}{2}\right)} = \int_0^{\frac{\pi}{2}} d\theta \Rightarrow \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 2 \cdot [\theta]_0^{\frac{\pi}{2}} = 2 \cdot \frac{\pi}{2} = \pi$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 12-Gamma Function

Evaluate: $\Gamma\left(\frac{7}{2}\right)$

we know that

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma\left(\frac{7}{2}\right) = \left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$$

$$\Gamma\left(\frac{7}{2}\right) = \left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)$$

$$\Gamma\left(\frac{7}{2}\right) = \left(\frac{5}{2}\right)\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)$$

$$\Gamma\left(\frac{7}{2}\right) = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)$$

$$\Gamma\left(\frac{7}{2}\right) = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$$

$$\Gamma\left(\frac{7}{2}\right) = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

Example 13-Gamma Function

Evaluate: $\Gamma(9)$

we know that

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(9) = (9-1)\Gamma(9-1)$$

$$\Gamma(9) = (8)\Gamma(8)$$

$$\Gamma(9) = (8)(8-1)\Gamma(8-1)$$

$$\Gamma(9) = (8)(7)\Gamma(7)$$

$$\Gamma(9) = 8.7.6.5.4.3.2.1.\Gamma(1)$$

$$\Gamma(9) = 8.7.6.5.4.3.2.1 = 8!$$

Example 14-Gamma Function

Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^7 x \, dx$

Solⁿ: we know that $\int_0^{\frac{\pi}{2}} (\sin^p \theta)(\cos^q \theta) \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^6 x \cos^7 x \, dx = \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{2\Gamma\left(\frac{6+7+2}{2}\right)}$$

$$\begin{aligned} &= \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{8}{2}\right)}{2\Gamma\left(\frac{15}{2}\right)} = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma(4)}{2 \cdot \frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \Gamma\left(\frac{7}{2}\right)} \\ &= \frac{48}{9009} \end{aligned}$$

Try Yourself

$$\begin{array}{ll} (1) \int_0^a x^4 (a^2 - x^2)^{1/2} dx & [\text{Hints: } x = a \sin \theta] \\ (2) \int_0^\infty \frac{x^4}{(1+x^2)^4} dx & [\text{Hints: } x = \tan \theta] \\ (3) \int_0^\infty \frac{x^2}{(1+x^4)} dx & [\text{Hints: } x^2 = \tan \theta] \\ (4) \int_0^1 \frac{x^7}{(1-x^4)^{1/2}} dx & [\text{Hints: } x^2 = \sin \theta] \end{array} \quad \begin{array}{l} (5) \int_0^1 \frac{x}{(1-x^5)^{1/2}} dx \quad [\text{Hints: } x^5 = t] \\ (6) \int_0^2 x \sqrt[3]{(8-x^3)} dx \quad [\text{Hints: } x^3 = 8t] \\ (7) \int_0^4 x^{3/2} \sqrt{(16-x^2)} dx \quad [\text{Hints: } x^2 = 16t] \\ (8) \int_{-\infty}^\infty e^{-x^2} dx \quad [\text{Hints: } x^2 = t] \end{array}$$

Thanks a lot ...