Chapter 2
Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Section 2.1 :Sets
Section 2.2 :Set Operations

Sets

- A set is an unordered collection of objects, called elements or members of the set.
- A set is said to contain its elements. We write,
  - $a \in A$ to denote that a is an element of the set A.
  - The notation $a \notin A$ denotes that a is not an element of the set A.
- For example,
  - The notation $\{a, b, c, d\}$ represents the set with the four elements $a, b, c,$ and d.

Sets(Contd.)

- Consider a set of all odd positive integers less than 10.

- **Roster Method:**
  - Using this method, the set can be written as $\{1,3,5,7,9\}$

- **Set Builder Method**
  - $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$,

Or,

  - $O = \{x \in Z^{+} \mid x \text{ is odd and } x < 10\}$.

Sets(Contd.)

- Few common notations

N	$\{0, 1, 2, 3, \ldots\}$, *the set of natural numbers*
Z	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, *the set of integers*
Z^+	$\{1, 2, 3, \ldots\}$, *the set of positive integers*
Q	$\{p/q \mid p \in Z, q \in Z, \text{and } q = 0\}$, *the set of rational numbers*
R	*the set of real numbers*
R^+	*the set of positive real numbers*
C	*the set of complex numbers*
U	*Universal Set*

Equal Sets

- Equal Sets
  - Two sets are equal if and only if they have the same elements.
  - Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.
  - We write $A = B$ if A and B are equal sets.
  - The order of elements and the repetition of the same element in two sets don't matter in order for them to be equal.
  - To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Equal Sets(Contd.)

- **Example 1:**
  - The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements.
  - Similarly, the sets $\{1, 3, 5\}$ and $\{3, 5, 5, 5, 5, 1\}$ are equal, because they have the same elements.

Empty & Singleton Sets

- Empty Set
  - A special set that has no elements. This set is called the *empty set*, or *null set*.
  - It is denoted by $\emptyset$ or $\{ \}$.

- Singleton Set
  - A set with a single element is known as a singleton set.

Empty & Singleton Sets(Contd.)

But WAIT!!!

Empty & Singleton Sets(Contd.)

- Let, $A = \{\ \}$ and $B = \{\emptyset\}$. Are both the sets empty?

- The Answer is $\boldsymbol{NO}$.
  - Because set B has an element which is the empty element itself. Thus, B is a singleton set and not an empty set.

Venn Diagram

- Venn Diagram:
  - Graphical representation of sets.
  - In Venn diagrams the universal set U, which contains all the objects under consideration, is represented by a rectangle.
  - Inside this rectangle, circles or other geometrical figures are used to represent sets.
  - Sometimes points are used to represent the particular elements of the set.

Subsets

- Subsets
  - The set A is a subset of B if and only if every element of A is also an element of B.
  - The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- Showing that A is a Subset of B
  - To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.
- Showing that A is Not a Subset of B
  - To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.
- Every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

Cardinality

- Size of a set
  - Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

- **Example:**
  - Let A be the set of odd positive integers less than 10. Then $|A| = 5$.

Power Sets

- Power Sets
  - Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.
  - The cardinality of a power set of any set A is 2^n, where, n is the cardinality of A.
  - Note that the empty set and the set itself are members of the set of subsets.

Power Sets(Contd.)

- **Example 1:**
  - What is the power set of the set $\{0, 1, 2\}$?

- **Solution:**
  - The power set $P(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence, $P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.

Power Sets(Contd.)

- **<u>Example 2:</u>**
  - What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

- **<u>Solution:</u>**
  - The empty set has exactly one subset, namely, itself. Consequently,
$$P(\emptyset) = \{\emptyset\}.$$
  - The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,
$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

Ordered Pairs

- The ordered $n - tuple$ $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element,…, and a_n as its n^{th} element.
- We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal.
  - In other words, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$,for $i = 1, 2, \ldots, n$.
- In particular, ordered 2-tuples are called ordered pairs.
  - The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. Note that (a, b) and (b, a) are not equal unless $a = b$.

Ordered Pairs(Contd.)

- **<u>Example 1:</u>**
  - What are the ordered pairs in the set R having the less than or equal to relation, which contains (a, b) if $a \le b$, on the set $\{0, 1, 2, 3\}$?

- **<u>Solution:</u>**

  The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \le b$. Consequently, the ordered pairs in R are

  $(0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2, 3), and\ (3, 3)$.

Cartesian Products

- Let A and B be two sets.
- The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Cartesian Products(Contd.)

- **<u>Example 1:</u>**
  - Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

- **<u>Solution:</u>**

  The Cartesian product $A \times B$ consists of all the ordered pairs of the form (a, b), where a is a student at the university and b is a course offered at the university. One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university.

Cartesian Products(Contd.)

- **Example 2:**
  - What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

- **Solution:**

  The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

Cartesian Products(Contd.)

- **Example 3:**
  - Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where A and B are $A = \{1, 2\}$ and $B = \{a, b, c\}$.

- **Solution:**

The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

The Cartesian product $B \times A$ is

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

Since Cartesian Product is the set of all ordered pairs, thus by definition of ordered pairs we can conclude that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$.

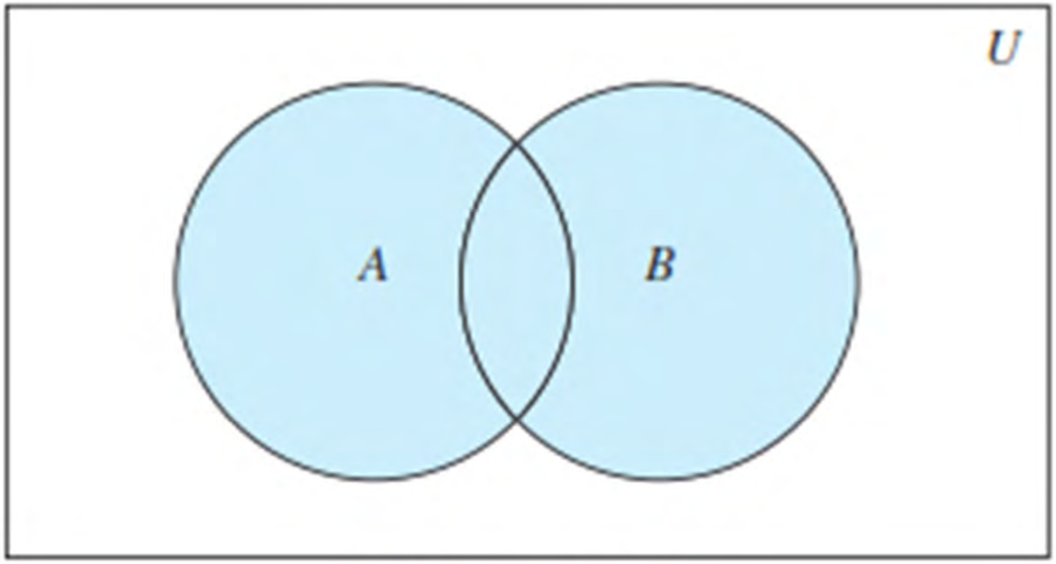
Section 2.2: Set Operations

Union of Sets

- Let A and B be sets.
- The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in $both$.
- An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B. This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

Union of Sets(Contd.)



$A \cup B$ is shaded.

FIGURE 1 **Venn Diagram of the Union of A and B.**

Union of Sets(Contd.)

- **Example 1:**
  - Find the union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

- **Solution:**

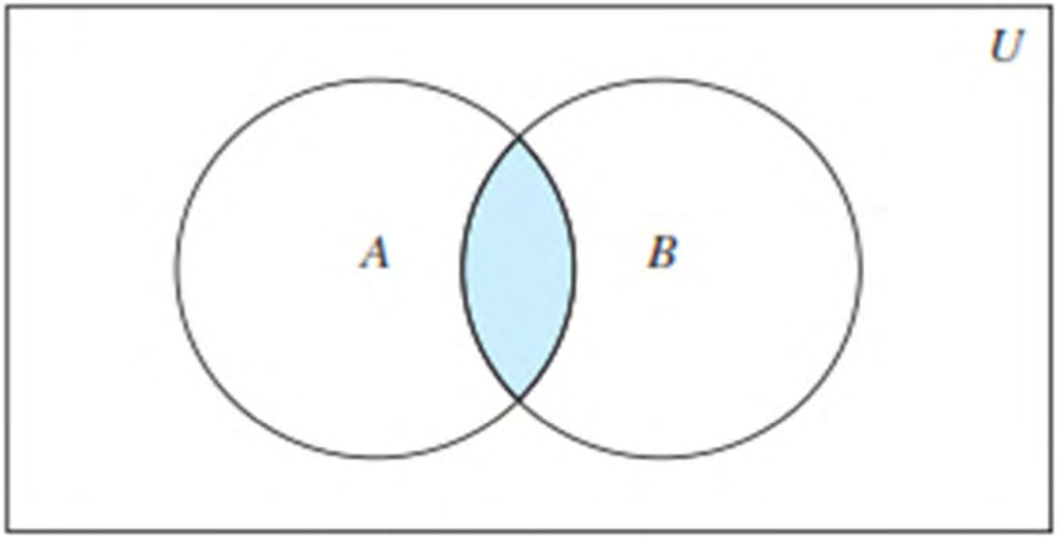
The union of the two sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ can be written as,

$$\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$$

Intersection of Sets

- Let A and B be sets.
- The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.
- An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B. This tells us that

Intersection of Sets(Contd.)



$A \cap B$ is shaded.

FIGURE 2 **Venn Diagram of the Intersection of A and B.**

Intersection of Sets(Contd.)

- **<u>Example 1:</u>**
  - Find the intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

- **<u>Solution:</u>**

The intersection of the two sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ can be written as,

$$\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$$

Disjoint Sets

- Two sets are called disjoint if their intersection is the empty set.

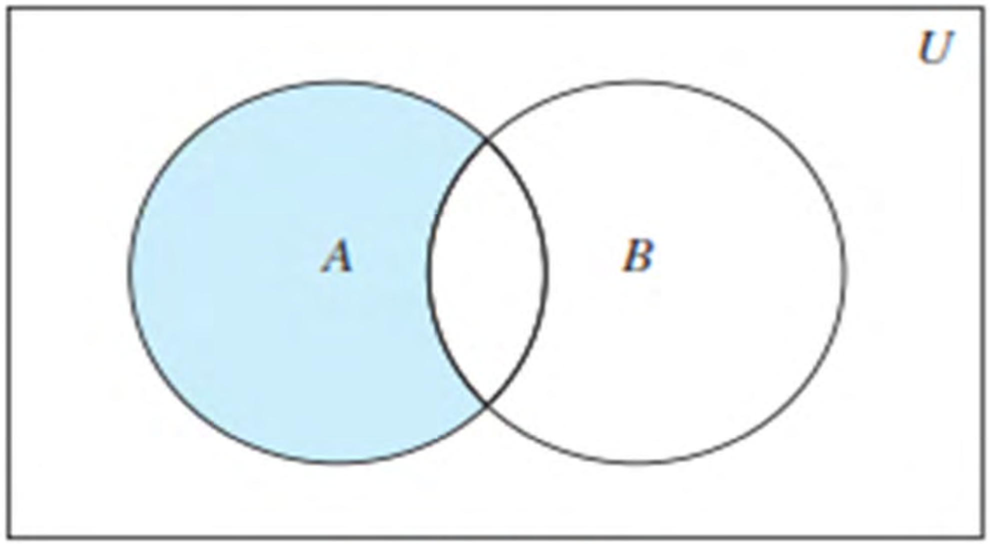
- **Example 1:**
  - Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$.
    Because $A \cap B = \emptyset$, A and B are disjoint.

Difference of Sets

- Let A and B be sets.
- The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.
- The difference of sets A and B is sometimes denoted by $A\backslash\mathrm{B}$.
- An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$. This tells us that

$$A - B = \{x \mid x \in A \land x \notin B\}.$$

Difference of Sets(Contd.)



$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of A and B.

Difference of Sets(Contd.)

- **Example 1:**
  - Find the difference of the sets $\{1,3,5\}$ and $\{1,2,3\}$.

- **Solution:**

  The difference of the two sets $\{1,3,5\}$ and $\{1,2,3\}$ can be written as,

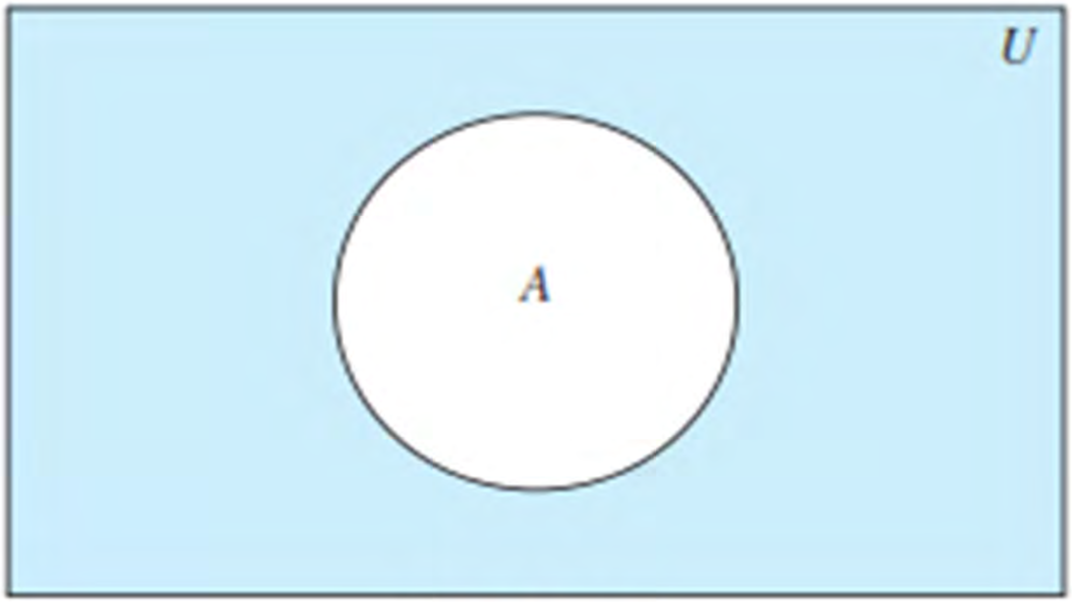
$$\{1,3,5\} - \{1,2,3\} = \{5\}$$

Complement of Sets

- Let U be the universal set. The complement of the set A, denoted by A, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.
- An element belongs to A if and only if $x \notin A$. This tells us that

$$A = \{x \in U \mid x \notin A\}$$

Complement of Sets(Contd.)



$\overline{A}$ is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set A.

Complement of Sets(Contd.)

- **Example 1:**
  - Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then

    $$\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}.$$

Set Identities

TABLE 1 Set Identities.	
Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Identities(Contd.)

- **<u>Example 1:</u>**
  - Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

- **<u>Solution:</u>**

Let,

$$x \in (\overline{A \cup B}) \equiv x \notin (A \cap B)$$
$$\equiv x \notin A \; or \; x \notin B$$
$$\equiv x \in \bar{A} \; or \; x \in \bar{B}$$
$$\equiv x \in (\bar{A} \cup \bar{B})$$

$$\therefore \overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$$

Again Let,

$$x \in (\bar{A} \cup \bar{B}) \equiv x \in \bar{A} \; or \; x \in \bar{B}$$
$$\equiv x \notin A \; or \; x \notin B$$
$$\equiv x \notin (A \cap B)$$
$$\equiv x \in (\overline{A \cup B})$$

$$\therefore \bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$$

$$\therefore \overline{A \cap B} = \bar{A} \cup \bar{B}$$

Set Identities(Contd.)

- **Example 2:**
  - Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

- **Solution:**

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid \neg(x \in (A \cap B))\} \\
&= \{x \mid \neg(x \in A \wedge x \in B) \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \bar{A} \vee x \in \bar{B}\} \\
&= \{x \mid x \in \bar{A} \cup \bar{B}\} \\
&= \bar{A} \cup \bar{B}
\end{aligned}$$

Computer Representation of Sets

- Assume that a universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used).
- First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$.
- Secondly, represent a subset A of U with the bit string of length n, where the i^{th} bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Computer Representation of Sets(Contd.)

- **Example 1:**
  - Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$.
    - What bit strings represent the subset of all odd integers in U?
    - What bit strings represent the subset of all even integers in U?
    - What bit strings represent the subset of integers not exceeding 5 in U?

Computer Representation of Sets(Contd.)

- **Solution:**

The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the $first, third, fifth, seventh\ and\ ninth$ positions, and a $zero$ elsewhere. It is

$$10\ 1010\ 1010.$$

Similarly, we represent the subset of all even integers in U, namely, $\{2, 4, 6, 8, 10\}$, by the string

$$01\ 0101\ 0101.$$

The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string

$$11\ 1110\ 0000.$$

Computer Representation of Sets(Contd.)

- **<u>Example 2:</u>**
  - Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$.
    - What bit strings represent the complement of the subset of all odd integers in U?

Computer Representation of Sets(Contd.)

- **Solution:**

The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the $first, third, fifth, seventh$ $and\ ninth$ positions, and a $zero$ elsewhere. It is

$$10\ 1010\ 1010.$$

The bit string for the complement of this set is obtained by replacing $0s$ with $1s$ and vice versa. This yields the string

$$01\ 0101\ 0101$$

which corresponds to the set $\{2, 4, 6, 8, 10\}$.

Computer Representation of Sets(Contd.)

- **Example 3:**
  - The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are $11\ 1110\ 0000$ and $10\ 1010\ 1010$, respectively. Use bit strings to find the union and intersection of these sets.

- **Solution:**

The bit string for the union of these sets is,

$$11\ 1110\ 0000 \ \vee\ 10\ 1010\ 1010 \ =\ 11\ 1110\ 1010,$$

which corresponds to the set $\{1, 2, 3, 4, 5, 7, 9\}$.

The bit string for the intersection of these sets is,

$$11\ 1110\ 0000 \ \wedge\ 10\ 1010\ 1010 \ =\ 10\ 1010\ 0000,$$

which corresponds to the set $\{1, 3, 5\}$.

THE END