# Contents

# Math 4241 Integration

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## 1 Introduction

This document covers all the topics of integration taught in the program.

## 2 Anti-derivatives

To find a function $F$ whose derivative is $f(x) = 3x^2$, we can use our knowledge of derivatives to conclude that

$$F(x) = x^3 + C$$

because

$$\frac{d}{dx}(x^3 + C) = 3x^2$$

where $C$ is the constant of integration.

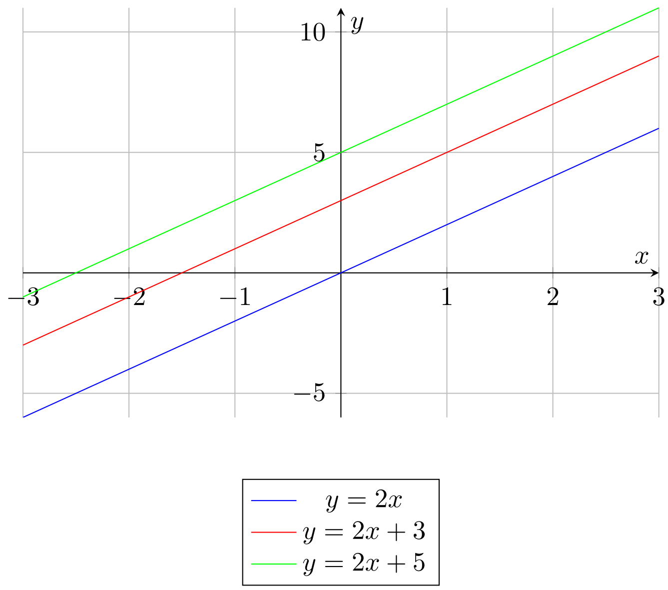
### 2.1 Definition of Anti-derivative

A function $F$ is an anti-derivative of $f$ on an interval $I$ when $F'(x) = f(x)$ for all $x$ in $I$.

#### 2.1.1 Theorem: Representation of Antiderivatives

If $F$ is an antiderivative of $f$ on an interval $I$, then $G$ is an antiderivative of $f$ on the interval $I$ if and only if $G(x) = F(x) + C$ for all $x$ in $I$, where $C$ is a constant.

Consider the function $f(x) = 2x$. The function $G(x) = x^2 + C$ represents a family of functions which is the general antiderivative of $f$, and $G(x) = x^2 + C$ is the general solution of the differential equation $G'(x) = 2x$.




This indicates that any two antiderivatives of $f$ are vertical translations of each other.

### 2.1.2 Example 1

Find the general solution and the particular solution that satisfies the initial condition $F(1) = 0$ where $f(x) = \frac{1}{x^2}$.

$$\begin{aligned} F(x) &= \int \frac{1}{x^2}\,dx \\ &= \int x^{-2}\,dx \\ &= -\frac{1}{x} + C, \quad x > 0 \end{aligned}$$

Using the initial condition $F(1) = 0$:

$$0 = -\frac{1}{1} + C \implies C = 1$$

Thus, the particular solution is:

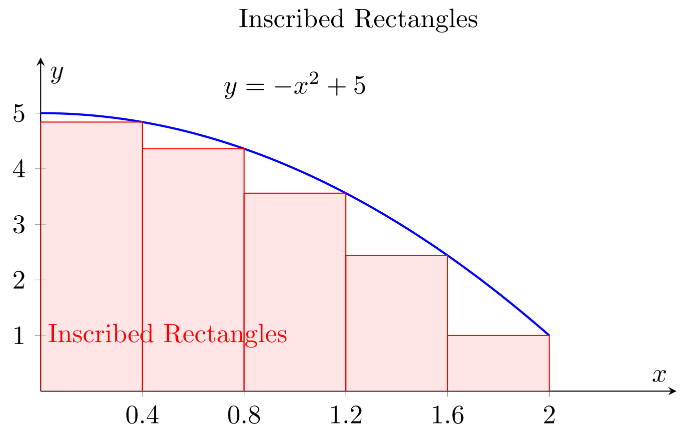
$$F(x) = -\frac{1}{x} + 1$$

# 3 Integration as Area

The fundamental idea of integration stems from the calculation of areas under curves. In this section, we explore this concept.

## 3.1 Approximating the Area of a Plane Region

Using five rectangles, we find two approximations of the area between the graph of $f(x) = -x^2 + 5$ and the $x$-axis on the interval $[0, 2]$.



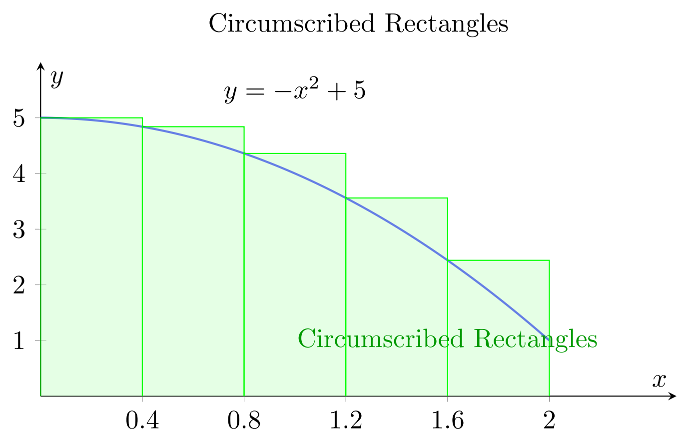




Figure 1: Riemann Sum approximations of $\int_0^2 (-x^2 + 5)\, dx$ with 5 rectangles

We can show the right endpoints of the five intervals as

$$\frac{2}{5} i$$

where $i = 1, 2, 3, 4, 5$

The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating $f$ at the right endpoint of each interval. Hence, the sum of the areas of these five rectangles:

$$\sum_{i=1}^{5} f\left(\frac{2i}{5}\right) \cdot \frac{2}{5} = \sum_{i=1}^{5} \left[-\left(\frac{2i}{5}\right)^2 + 5\right] \frac{2}{5} = \frac{162}{25} = 6.48$$

Similarly,

We can show the left endpoints as:

$$\frac{2}{5}(i-1)$$

where $i = 1, 2, 3, 4, 5$

The width of each rectangle is $\frac{2}{5}$ and the height of each rectangle can be obtained by evaluating $f$ at the left endpoint of each interval.

$$\sum_{i=1}^{5} f\left(\frac{2(i-1)}{5}\right) \cdot \frac{2}{5} = \sum_{i=1}^{5} \left[-\left(\frac{2(i-1)}{5}\right)^2 + 5\right] \frac{2}{5} = \frac{202}{25} = 8.08$$

## 4 Area by Limit Definition

Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$ as shown below. The region is bounded by $x$-axis and the left and right boundaries of the region are vertical lines $x = a$ and $x = b$.

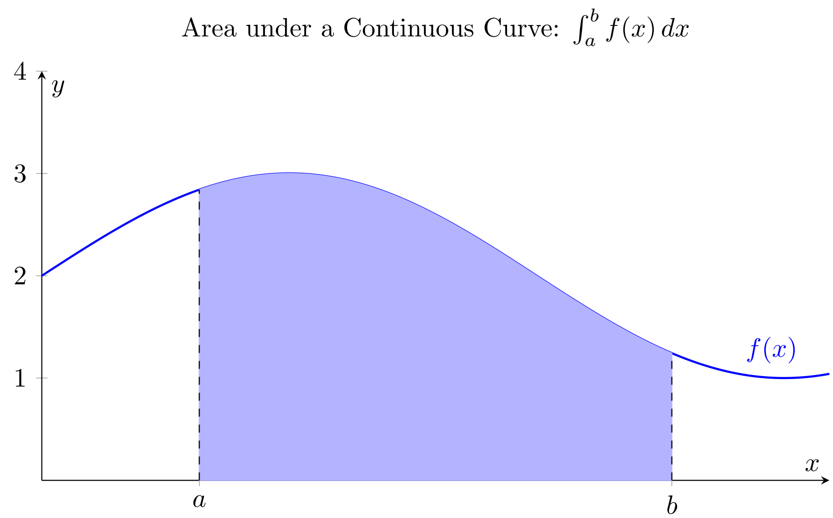



Figure 2: A continuous function with shaded integral region between $x = a$ and $x = b$

To approximate the area of the region, we can begin by dividing the interval $[a, b]$ into $n$ subintervals each of width:

$$\Delta x = \frac{b-a}{n}$$

Area under a Continuous Curve: $\int_a^b f(x)\,dx$

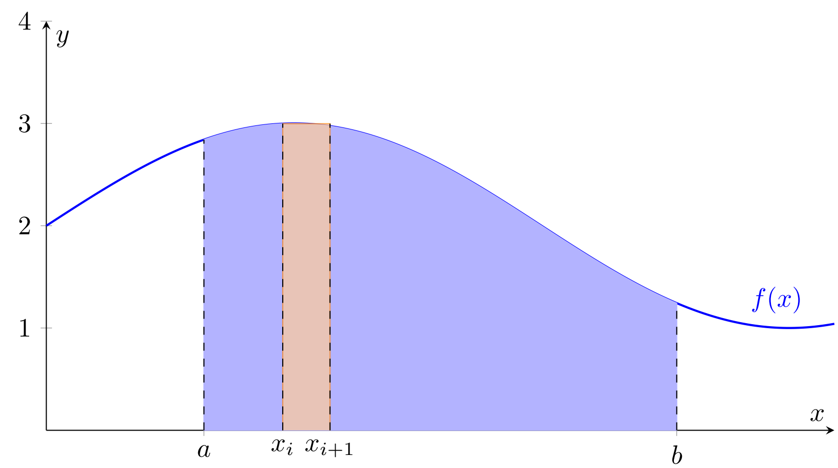



Figure 3: The interval [a,b] is divided into n subintervals of width $\Delta x = \frac{b-a}{n}$

The endpoints of the intervals are:

$$\underbrace{a + 0\Delta x}_{\text{a}=x_0} \quad < \quad \underbrace{a + 1\Delta x}_{x_1} \quad < \quad \underbrace{a + 2\Delta x}_{x_2} \quad < \quad \cdots \quad < \quad \underbrace{a + n\Delta x}_{x_0=\text{b}}$$

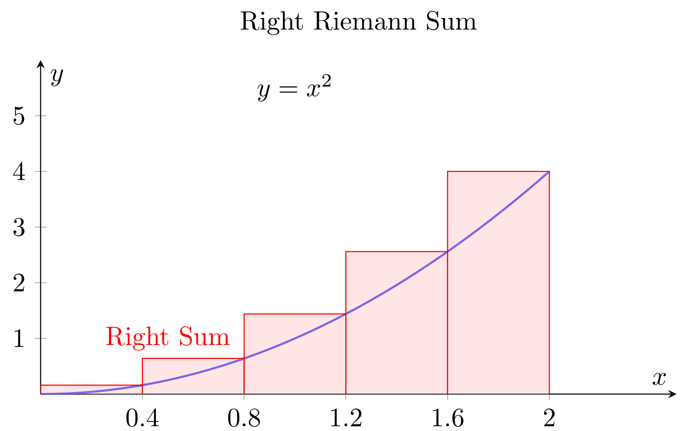
Because f is continuous on [a,b], the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in each subinterval.

Let us define an inscribed rectangle lying inside the $i-th$ subregion and a circumscribed rectangle extending outside that region. The height of the inscribed rectangle is $f(m_i)$ and the height of the circumscribed rectangle is $f(M_i)$

Lower sum $= s(n) = \sum_{i=1}^{n} f(m_i)\Delta x$ Upper sum $= S(n) = \sum_{i=1}^{n} f(M_i)\Delta x$

## 4.1 Example Problem

Find the Upper and Lower sum for the function $f(x) = x^2$ from 0 to 2 using limit definitiosn.



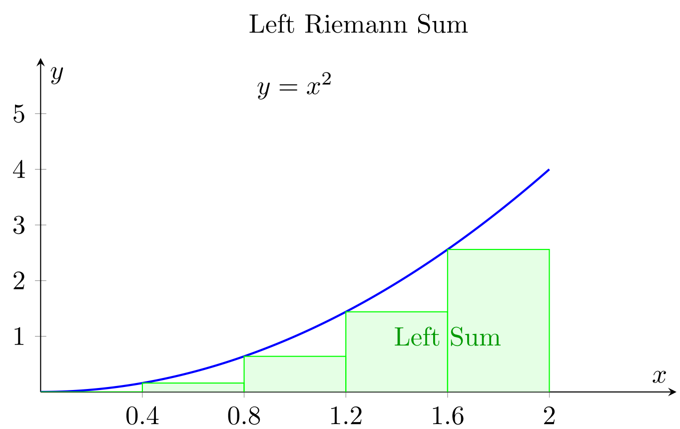




Figure 4: Left and Right Riemann Sum approximations of $\int_0^2 (x^2)\,dx$ with 5 rectangles

Using the left endpoints, we get:

$$
\begin{aligned}
s(n) = \sum_{i=1}^{n} f(m_i)\Delta x &= \sum_{i=1}^{n} f\left[\frac{2(i-1)}{n}\right]\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\left[\frac{2(i-1)}{n}\right]^2\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\frac{8(i-1)^2}{n^3} \\
&= \frac{8}{n^3}\sum_{i=1}^{n}(i-1)^2 \\
&= \frac{8}{n^3}\cdot\frac{(n-1)n(2n-1)}{6} \\
&= \frac{8(n-1)n(2n-1)}{6n^3}
\end{aligned}
$$

Using the right endpoints, we get:

$$
\begin{aligned}
s(n) = \sum_{i=1}^{n} f(m_i)\Delta x &= \sum_{i=1}^{n} f\left[\frac{2i}{n}\right]\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\left[\frac{2i}{n}\right]^2\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\frac{8i^2}{n^3} \\
&= \frac{8}{n^3}\sum_{i=1}^{n}i^2 \\
&= \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\
&= \frac{8n(n+1)(2n+1)}{6n^3}
\end{aligned}
$$

Since we have the left and right limits as the same, we can use the Squeeze Theorem to imply that the limit at that interval exists at that point.

## 4.2 Limits of Upper and Lower Sums

Let $f$ be continuous and nonnegative on the interval $[a, b$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\Delta x$$

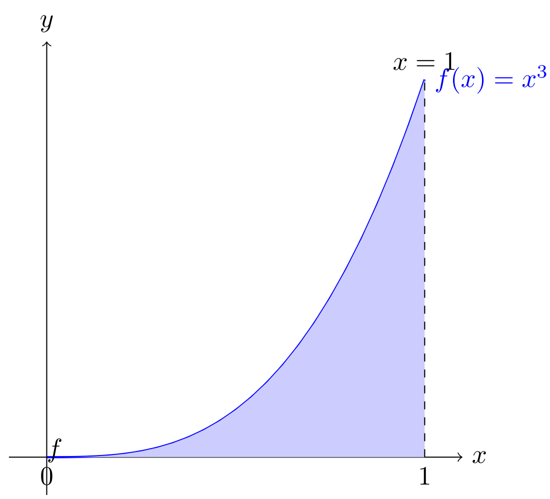
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\Delta x$$

$$\lim_{n\to\infty} S(n)$$

where $\Delta x = (b-a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of $f$ on the $i$th subinterval. That means that you are free to choose any arbitrary x-value in the $i$th subinterval, as shown in the definition of the are of a region in the plane.

### 4.2.1 Example Problem: Finding Area by the Limit Definition

Find the are of the region bounded by the graph of $f(x) = x^3$, and the x-axis, and the vertical lines $x = 0$ and $x = 1$, as shown in the figure below.




Begin by noting that $f$ is continuous and nonnegative on the interval $[0, 1]$. Next, partition the interval $[0, 1]$ into n subintervals, each of width $\Delta x = 1/n$. According to the definition of area, you can choose any x-value in the $i$th subinterval. For this example, the right endpoints $c_i = i/n$ are convenient to use.

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{n^4} = \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\cdot\left(\frac{n^2(n+1)^2}{4}\right)$$

$$= \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4} = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}$$

$$= \frac{1+0+0}{4} = \frac{1}{4}$$

$$\boxed{\text{Area} = \frac{1}{4}}$$

In general, a good value to choose is the midpoint of the interval $c_i = (x_{i-1}+x_i)/2$, and apply the Midpoints Rule.

$$Area = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

# 5 Riemann Sums

Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \le x \le 1$, as shown in the figure below. Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where $c_i$ is the right endpoint of the partition given by $c_i = \frac{i^2}{n^2}$ and $\Delta x_i$ is the width of the $i$th interval.

# Non-Uniform Riemann Sum with $f(x) = \sqrt{x}$ and $c_i = \frac{i^2}{n^2}$

We are given:

$$c_i = \frac{i^2}{n^2}, \quad f(x) = \sqrt{x}, \quad \text{so } f(c_i) = \sqrt{\frac{i^2}{n^2}} = \frac{i}{n}$$

To find the width of the $i$th subinterval, we calculate:

$$\Delta x_i = c_i - c_{i-1} = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} = \frac{i^2-(i-1)^2}{n^2}$$

$$= \frac{i^2-(i^2-2i+1)}{n^2} = \frac{2i-1}{n^2}$$

Now compute the Riemann sum:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{2i-1}{n^2} = \sum_{i=1}^{n} \frac{i(2i-1)}{n^3}$$

Expand and simplify:

$$\sum_{i=1}^{n} \frac{i(2i-1)}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n}(2i^2 - i) = \frac{1}{n^3}\left(2\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i\right)$$

Use formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

So:

$$= \frac{1}{n^3}\left[2 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right] = \frac{1}{n^3} \cdot \frac{n(n+1)}{6}\,(2(2n+1) - 3)$$

$$= \frac{1}{n^3} \cdot \frac{n(n+1)}{6} \cdot (4n-1) = \frac{n(n+1)(4n-1)}{6n^3}$$

Now take the limit as $n \to \infty$:

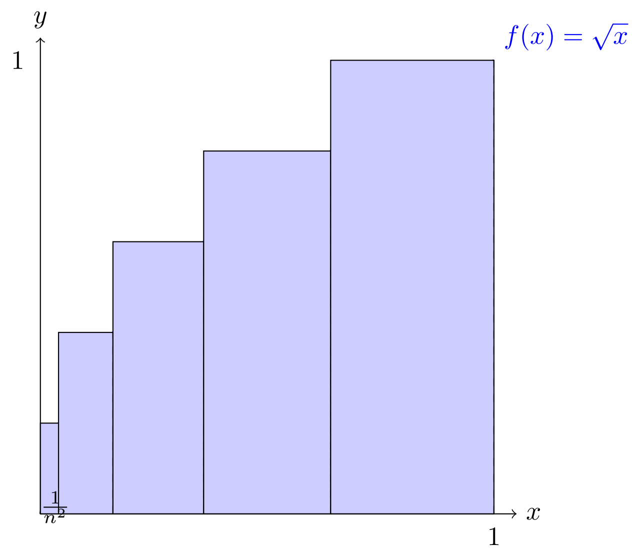
$$\lim_{n\to\infty} \frac{n(n+1)(4n-1)}{6n^3} = \lim_{n\to\infty} \frac{(n+1)(4n-1)}{6n^2}$$

Expand numerator:

$$(n+1)(4n-1) = 4n^2 + 3n - 1 \Rightarrow \lim_{n\to\infty} \frac{4n^2+3n-1}{6n^2} = \frac{4+0+0}{6} = \frac{2}{3}$$

$$\boxed{\text{Area} = \frac{2}{3}}$$

## Graphical Representation




### 5.1 Definition of the Riemann Sum

Let $f$ be defined on the closed interval $[a, b]$, and let $\Delta$ be on a partition of $[a, b]$ given by:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

where $\Delta x$ is the width of the $i$th subinterval.

$$[x_{i-1}, x_i]$$

If $c_i$ is any point in the $i$th subinterval, then the sum:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x$$

is called a Riemann sum of $f$ for the partition $\Delta$.

# 6 Definite Integrals

If $f$ is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions $\Delta$.

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_a^b f(x) dx$$

The limit is called the definite integral of $f$ from $a$ to $b$, that is, the lower limit to the upper limit, respectively.

### 6.0.1 Example Problem

Let $\Delta x = \frac{1-(-2)}{n} = \frac{3}{n}$, and choose the right endpoints of the subintervals: $x_i = -2 + i\Delta x = -2 + \frac{3i}{n}$. Then,

$$\int_{-2}^{1} 2x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} 2x_i\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} 2\left(-2+\frac{3i}{n}\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(-4+\frac{6i}{n}\right)\cdot\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{-12}{n}+\frac{18i}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{-12}{n}+\sum_{i=1}^{n}\frac{18i}{n^2}\right] = \lim_{n\to\infty}\left[-12+\frac{18}{n^2}\cdot\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[-12+\frac{18}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[-12+\frac{18(n+1)}{2n}\right]$$

$$= \lim_{n\to\infty}\left[-12+9\cdot\frac{n+1}{n}\right] = \lim_{n\to\infty}\left[-12+9\left(1+\frac{1}{n}\right)\right] = -12+9 = -3$$

Therefore, $\int_{-2}^{1} 2x\,dx = \boxed{-3}$.

# 7 Some Theorems to Take Home

## Fundamental Theorem of Calculus (Part I)

**Theorem.** Let $f$ be a continuous function on the interval $[a,b]$, and let

$$F(x) = \int_a^x f(t)\,dt$$

for $x \in [a,b]$. Then $F$ is continuous on $[a,b]$, differentiable on $(a,b)$, and

$$F'(x) = f(x).$$

## Mean Value Theorem for Integrals

**Theorem.** If $f$ is continuous on the closed interval $[a,b]$, then there exists a number $c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

## Average Value of a Function

**Definition.** If $f$ is integrable on $[a,b]$, then the *average value* of $f$ on the interval is given by

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

## Fundamental Theorem of Calculus (Part II)

**Theorem.** If $f$ is continuous on $[a, b]$, and $F$ is any antiderivative of $f$ on $[a, b]$ (i.e., $F' = f$), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

## Net Change Theorem

**Theorem.** If $F$ is a differentiable function whose derivative $F'$ is continuous on the interval $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

In other words, the integral of a rate of change is the total change over the interval.

## Chemical Flow into a Storage Tank Using the Net Change Theorem

**Problem.** A chemical flows into a storage tank at a rate of $R(t) = 180 + 3t$ liters per minute, where $t$ is the time in minutes and $0 \leq t \leq 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

**Solution.** By the **Net Change Theorem**, the total amount of chemical that flows into the tank from $t = 0$ to $t = 20$ is given by the integral of the rate function:

$$\text{Total amount} = \int_0^{20} R(t)\,dt = \int_0^{20} (180 + 3t)\,dt.$$

We compute the integral:

$$\int_0^{20} (180 + 3t)\,dt = \int_0^{20} 180\,dt + \int_0^{20} 3t\,dt = 180t\Big|_0^{20} + \frac{3t^2}{2}\Big|_0^{20}.$$

Evaluating at the bounds,

$$= 180(20) - 180(0) + \frac{3 \times (20)^2}{2} - \frac{3 \times 0^2}{2} = 3600 + \frac{3 \times 400}{2} = 3600 + 600 = 4200.$$

$$\boxed{\text{The total amount of chemical flowed in during the first 20 minutes is 4200 liters.}}$$

## Even and Odd Functions

**Definition:** A function $f(x)$ is called:

- **Even** if $f(-x) = f(x)$ for all $x$ in the domain.

- **Odd** if $f(-x) = -f(x)$ for all $x$ in the domain.

**Integration Properties:**

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

**Example (Even Function):** Let $f(x) = x^2$. Since $f(-x) = (-x)^2 = x^2 = f(x)$, it's even.

$$\int_{-2}^{2} x^2\,dx = 2\int_0^2 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^2 = 2 \cdot \frac{8}{3} = \frac{16}{3}$$

**Example (Odd Function):** Let $f(x) = x^3$. Since $f(-x) = (-x)^3 = -x^3 = -f(x)$, it's odd.

$$\int_{-2}^{2} x^3\,dx = 0$$

## Common Integration Formulas

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \csc x \cot x\,dx = -\csc x + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

## Integration by Parts

**Formula:**

$$\int u\,dv = uv - \int v\,du$$

**Example:** Evaluate $\int xe^x\,dx$
Let $u = x$, so $du = dx$ Let $dv = e^x dx$, so $v = e^x$
Then,

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C$$

## The DI Method

The DI (Derivative-Integral) method is a shortcut for repeated integration by parts, especially when one part is easily differentiable and the other easily integrable.

**Example:** $\int x^2 e^x \, dx$

| D (Derivative) | I (Integral) |
|---|---|
| $x^2$ | $e^x$ |
| $2x$ | $e^x$ |
| $2$ | $e^x$ |
| $0$ | |

Now apply alternating signs (starting with +) and multiply diagonally:

$$= (+)x^2 e^x - (1)2xe^x + (1)2e^x + C = e^x(x^2 - 2x + 2) + C$$

## Arc Length

**Arc Length Formula (Derivation):**

Let $y = f(x)$ be a smooth, continuous function on the interval $[a, b]$. The arc length $L$ of the curve from $x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

*Derivation:* From the distance formula in calculus, the length of a small segment is approximately:

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \, \Delta x$$

Taking the limit as $\Delta x \to 0$, this becomes the integral:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

**Example:** Let $f(x) = \frac{1}{2}x^2$, compute the arc length from $x = 0$ to $x = 1$.
First, compute the derivative:

$$f'(x) = x$$

Now apply the formula:

$$L = \int_0^1 \sqrt{1 + (x)^2} \, dx$$

This is a standard integral:

$$L = \int_0^1 \sqrt{1 + x^2} \, dx$$

Use a standard substitution $x = \sinh u$, or evaluate numerically if necessary. The exact integral is:

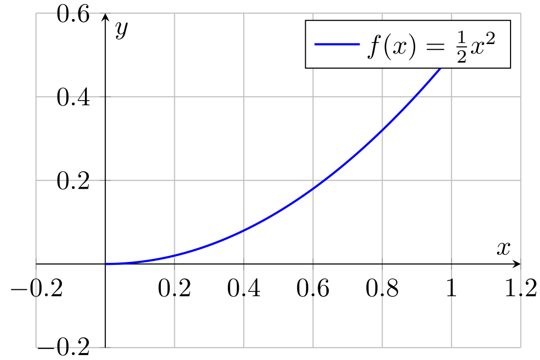
$$L = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right)\Bigg|_0^1$$

Evaluating:

$$= \frac{1}{2}(1)\sqrt{2} + \frac{1}{2}\ln(1+\sqrt{2}) - \left[0 + \frac{1}{2}\ln(0+\sqrt{1})\right]$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})$$

**Graph of** $f(x) = \frac{1}{2}x^2$ on $[0, 1]$:




# Reduction Formula

**Definition:** A *reduction formula* expresses an integral involving a positive integer power $n$ of a function in terms of a similar integral with a lower power. This allows us to evaluate complex integrals recursively.

**Example:** $\int x^n e^x \, dx$

We integrate by parts:

$$u = x^n, \quad dv = e^x dx \Rightarrow du = nx^{n-1}dx, \quad v = e^x$$

Then,

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$$

**Reduction Formula:**

$$\boxed{\int x^n e^x \, dx = x^n e^x - n\int x^{n-1}e^x \, dx}$$

**Reduction Formula for $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$**

Let:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

Using integration by parts or a standard identity:

$$I_n = \frac{n-1}{n} I_{n-2}, \quad \text{for } n \geq 2$$

With base cases:

$$I_0 = \frac{\pi}{2}, \quad I_1 = 1$$

**Reduction Formula:**

$$\boxed{I_n = \frac{n-1}{n} I_{n-2}}$$

**Reduction Formula for $\int_0^{\frac{\pi}{2}} \cos^n x \, dx$**

Let:

$$J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

Similarly,

$$J_n = \frac{n-1}{n} J_{n-2}$$

With:

$$J_0 = \frac{\pi}{2}, \quad J_1 = 1$$

**Reduction Formula:**

$$\boxed{J_n = \frac{n-1}{n} J_{n-2}}$$

**Reduction Formula for $\int_0^{\frac{\pi}{4}} \tan^n x \, dx$**

Let:

$$T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$$

We use the identity:

$$\tan^n x = \tan^{n-2} x \cdot \tan^2 x = \tan^{n-2} x (\sec^2 x - 1)$$

Then,

$$T_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1) \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

Let:

$$u = \tan^{n-2} x, \quad dv = \sec^2 x \, dx \Rightarrow du = (n-2)\tan^{n-3} x \sec^2 x \, dx, \quad v = \tan x$$

So:

$$T_n = \frac{1}{n-1} \tan^{n-1} x \Big|_0^{\frac{\pi}{4}} - \frac{1}{n-1} T_{n-2}$$

Since $\tan\left(\frac{\pi}{4}\right) = 1$, $\tan(0) = 0$:

$$T_n = \frac{1}{n-1} - \frac{1}{n-1} T_{n-2}$$

**Reduction Formula:**

$$\boxed{T_n = \frac{1}{n-1} - \frac{1}{n-1} T_{n-2}}, \quad T_0 = \frac{\pi}{4}, \quad T_1 = \ln\sqrt{2}$$

**Reduction Formula for $\int_0^{\frac{\pi}{4}} \sec^n x \, dx$**

Let:

$$S_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$$

For $n \geq 2$, use the standard reduction:

$$S_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} S_{n-2}$$

**Reduction Formula:**

$$\boxed{\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx}$$

Evaluating from 0 to $\frac{\pi}{4}$:

$$S_n = \left[\frac{\sec^{n-2} x \tan x}{n-1}\right]_0^{\frac{\pi}{4}} + \frac{n-2}{n-1} S_{n-2}$$

Since $\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$, $\tan\left(\frac{\pi}{4}\right) = 1$, we can plug in values to compute specific cases.

# Beta and Gamma Functions

## Gamma Function

The **Gamma function** is a generalization of the factorial function to real (and complex) numbers. It is defined for $x > 0$ by the improper integral:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, dt$$

**Key Properties:**

- $\Gamma(n) = (n-1)!$ for any positive integer $n$
- $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

## Beta Function

The **Beta function** is defined for $x > 0, y > 0$ as:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

**Symmetry:**

$$B(x,y) = B(y,x)$$

## Relationship Between Beta and Gamma Functions

There is a fundamental identity connecting the Beta and Gamma functions:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

**Proof Sketch:** Start with the product of two gamma functions:

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}dt \cdot \int_0^\infty s^{y-1}e^{-s}ds$$

Make the substitution $t = ru,\ s = r(1-u)$, change variables to polar coordinates or apply the transformation $r = t+s,\ u = \frac{t}{t+s}$, and evaluate the Jacobian. The full computation yields:

$$\Gamma(x)\Gamma(y) = \int_0^1 u^{x-1}(1-u)^{y-1}du \cdot \int_0^\infty r^{x+y-1}e^{-r}dr = B(x,y)\cdot\Gamma(x+y)$$

## Applications of Gamma and Beta Functions

**1. Integrals Involving Powers and Exponentials:**

Evaluate:

$$\int_0^\infty x^n e^{-ax^b}\, dx$$

Make the substitution $u = ax^b \Rightarrow x = \left(\frac{u}{a}\right)^{1/b},\ dx = \frac{1}{b}\left(\frac{u}{a}\right)^{(1-b)/b}\cdot\frac{1}{a}du$, and simplify. The result is expressed in terms of Gamma function:

$$\int_0^\infty x^n e^{-ax^b}\, dx = \frac{1}{b}a^{-(n+1)/b}\Gamma\left(\frac{n+1}{b}\right)$$

**2. Trigonometric Integrals:**

Evaluate:

$$\int_0^{\frac{\pi}{2}} \sin^{p-1}x\cos^{q-1}x\, dx = \frac{1}{2}B\left(\frac{p}{2},\frac{q}{2}\right)$$

This is useful for evaluating integrals involving powers of sine and cosine.

## Examples

**Example 1:** Evaluate $\Gamma(5)$

$$\Gamma(5) = (5-1)! = 4! = 24$$

**Example 2:** Evaluate $\int_0^1 \sqrt{t}(1-t)^2 dt$
This is in Beta form:

$$= \int_0^1 t^{1/2}(1-t)^2 dt = B\left(\frac{3}{2}, 3\right) = \frac{\Gamma(3/2)\Gamma(3)}{\Gamma(9/2)}$$

Use known values:

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \quad \Gamma(3) = 2!, \quad \Gamma(9/2) = \frac{105\sqrt{\pi}}{16}$$

$$= \frac{\frac{\sqrt{\pi}}{2} \cdot 2}{\frac{105\sqrt{\pi}}{16}} = \frac{16}{105}$$

**Answer:** $\boxed{\frac{16}{105}}$

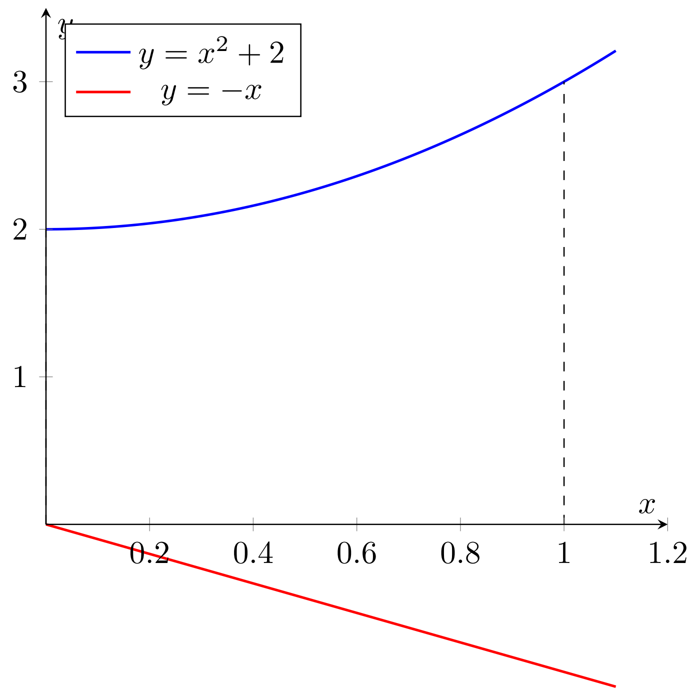
# Integration as Area Between Two Curves

## Definition

The area between two continuous curves $f(x)$ and $g(x)$ on the interval $[a, b]$, where $f(x) \geq g(x)$ for all $x \in [a, b]$, is given by:

$$\text{Area} = \int_a^b [f(x) - g(x)]\, dx$$

This represents the vertical distance between the curves, integrated across the interval.




**Example: Area Between $y = x^2 + 2$ and $y = -x$ from $x = 0$ to $x = 1$**

We are given two curves:

$$f(x) = x^2 + 2, \quad g(x) = -x, \quad \text{on } [0, 1]$$

The area between the curves is:

$$\text{Area} = \int_0^1 \left[(x^2 + 2) - (-x)\right] dx = \int_0^1 \left(x^2 + x + 2\right) dx$$

Compute the integral:

$$\int_0^1 (x^2 + x + 2)\, dx = \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1 = \left(\frac{1}{3} + \frac{1}{2} + 2\right) - 0 = \frac{1}{3} + \frac{1}{2} + 2$$

$$= \frac{2}{6} + \frac{3}{6} + \frac{12}{6} = \frac{17}{6}$$

**Final Answer:**

$$\boxed{\text{Area} = \frac{17}{6}}$$

# Volume Using Integration

Finding the volume of a solid of revolution involves slicing the object into thin disks or washers and summing their volumes via integration.

## Disk Method

The disk method is used when a solid is generated by revolving a region around an axis, and the cross-sections perpendicular to the axis of rotation are solid disks.

### Formulas

**1. Vertical Axis of Rotation (e.g., about the $x$-axis):**

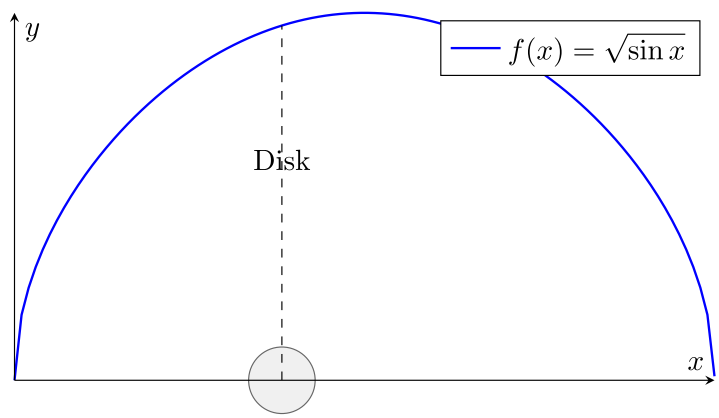
$$V = \pi \int_a^b [f(x)]^2 dx$$

**2. Horizontal Axis of Rotation (e.g., about the $y$-axis):**

$$V = \pi \int_c^d [g(y)]^2 dy$$

Where $x = g(y)$ is the inverse of the function.

### Diagram




## Example: Find Volume Using Disk Method

Let $f(x) = \sqrt{\sin x}$, and revolve the curve about the $x$-axis over the interval $[0, \pi]$.

$$V = \pi \int_0^\pi \left[\sqrt{\sin x}\right]^2 dx = \pi \int_0^\pi \sin x \, dx$$

$$= \pi[-\cos x]_0^\pi = \pi\left[-\cos(\pi) + \cos(0)\right] = \pi[1+1] = 2\pi$$

**Final Answer:** $\boxed{V = 2\pi}$

### Animation Option (External Tools)

Visit here: https://www.geogebra.org/3d/m66h3dcx to see the graph of sqrt(sinx) revolved around the x-axis. You can change the function and do it on your own using the 'Surface of Revolution' Tool.

# Washer Method for Volume

## What is the Washer Method?

The **washer method** is used to find the volume of a solid of revolution when the region being revolved has a *hole* in the middle — that is, the solid has an inner radius and an outer radius. It is essentially the disk method with a central part removed.

## Derivation of the Washer Formula

Let a region be bounded by two curves $f(x) \geq g(x) \geq 0$, and we revolve this region about the $x$-axis. Then, the volume of the solid is:

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx$$

Where: - $f(x)$ is the **outer radius** (distance from the axis to the outer curve), - $g(x)$ is the **inner radius** (distance from the axis to the inner curve).

This gives the volume of a washer-shaped cross section:

$$\text{Area of washer} = \pi\left([R]^2 - [r]^2\right)$$

## Example: Volume Formed by Rotating $y = \sqrt{\sin x}$ and $y = x^2$ Around the $x$-Axis

Let us consider the region bounded by:

$$y = \sqrt{\sin x}, \quad y = x^2, \quad x = 0, \quad x = \frac{\pi}{2}$$

To find the volume of the solid formed by revolving this region about the $x$-axis, we use the washer method.

**Step 1: Identify Outer and Inner Radii**

Since $\sqrt{\sin x} \geq x^2$ on $[0, \frac{\pi}{2}]$, we define:

$$R(x) = \sqrt{\sin x}, \quad r(x) = x^2$$

**Step 2: Apply the Washer Formula**

$$V = \pi \int_0^{\frac{\pi}{2}} \left( (\sqrt{\sin x})^2 - (x^2)^2 \right) dx = \pi \int_0^{\frac{\pi}{2}} \left( \sin x - x^4 \right) dx$$

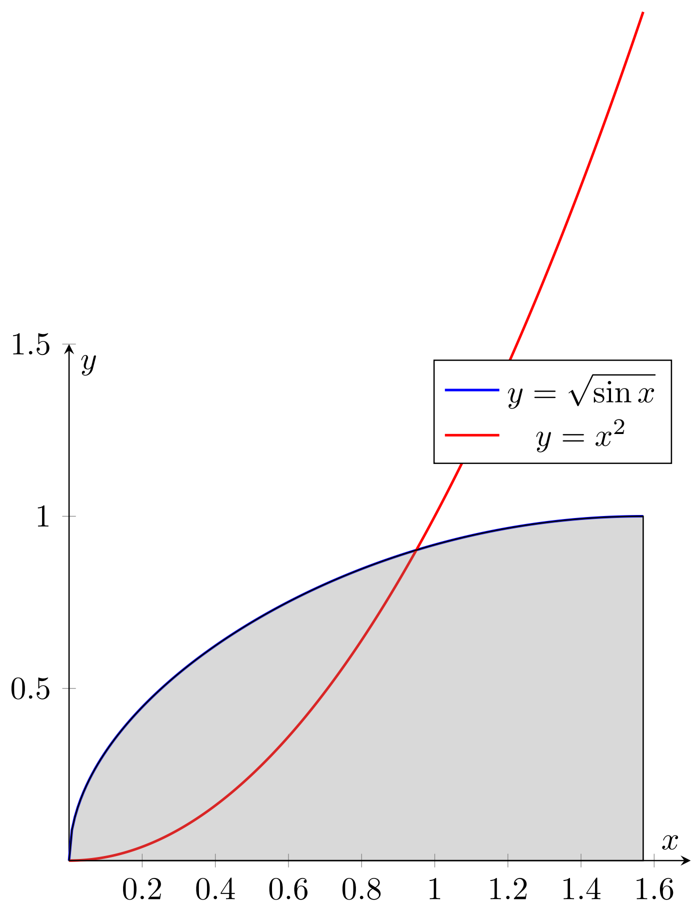
$$= \pi \left[ -\cos x - \frac{x^5}{5} \right]_0^{\frac{\pi}{2}} = \pi \left( -\cos\left(\frac{\pi}{2}\right) - \frac{(\frac{\pi}{2})^5}{5} + \cos(0) + 0 \right)$$

$$= \pi \left( 0 - \frac{\pi^5}{160} + 1 \right) = \pi \left( 1 - \frac{\pi^5}{160} \right)$$

**Final Answer:**

$$\boxed{V = \pi \left( 1 - \frac{\pi^5}{160} \right)}$$

## Diagram of Region and Rotation



**Animation Option (External Tools)**

Visit here: https://www.geogebra.org/3d/xbuv9fqz to see the intersection of $x^2$ and $\sqrt{sinx}$ revolved around the x-axis. You can change the function and do it on your own using the 'Surface of Revolution' Tool.

## Shell Method

The **Shell Method** is a technique for finding the volume of a solid of revolution, particularly useful when the axis of rotation is *parallel* to the slicing direction.

The idea is to sum up the volumes of thin cylindrical shells.

**General formula (Shell Method around the $y$-axis):**

$$V = 2\pi \int_a^b (\text{radius}) \cdot (\text{height}) \cdot dx$$

Here:

- **radius** = distance from the shell to the axis of rotation
- **height** = height of the shell (value of the function)
- **thickness** = $dx$ (infinitesimally thin width of each shell)

**Mnemonic:**

$$\boxed{V = 2\pi \cdot (\text{avg. radius}) \cdot (\text{height}) \cdot (\text{thickness})}$$

**Example:** Find the volume of the solid generated by revolving the region bounded by $y = x - x^3$ and the $x$-axis over the interval $[0, 1]$, about the $y$-axis.

**Step 1:** Set up the integral.

- Radius: $x$ (distance from $x$ to $y$-axis)
- Height: $x - x^3$

$$V = 2\pi \int_0^1 x(x - x^3)\, dx$$

$$= 2\pi \int_0^1 (x^2 - x^4)\, dx$$

**Step 2:** Integrate.

$$= 2\pi \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

$$= 2\pi \left( \frac{1}{3} - \frac{1}{5} \right) = 2\pi \cdot \frac{2}{15} = \frac{4\pi}{15}$$

**Final Answer:**

$$\boxed{V = \frac{4\pi}{15}}$$

# Arc Length

To compute the length of a smooth curve $y = f(x)$ over an interval $[a, b]$, we approximate the curve by a series of straight-line segments.

### Derivation of the Formula

Let the curve be divided into small segments with coordinates $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$. The length of each segment is approximately

$$\Delta s_i = \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}$$

Using the Mean Value Theorem, $f(x_{i+1}) - f(x_i) \approx f'(x_i)(x_{i+1} - x_i)$, so

$$\Delta s_i \approx \sqrt{1 + [f'(x_i)]^2} \cdot \Delta x$$

Summing over all intervals and taking the limit, we obtain the arc length formula:

### Arc Length Formula

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

—

## Surface Area of Revolution

When a curve $y = f(x)$ is revolved around an axis, the surface area of the resulting solid can be computed using the following formula:

**If the curve is rotated around the $x$-axis:**

$$\boxed{S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx}$$

**If the curve is rotated around the $y$-axis:**

$$\boxed{S = 2\pi \int_a^b x \sqrt{1 + [f'(x)]^2}\, dx}$$

—

**Example: Surface Area of $f(x) = x^3$ from 0 to 1 around the $x$-axis**

**Step 1:** Compute the derivative:

$$f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2$$

**Step 2:** Plug into the surface area formula:

$$S = 2\pi \int_0^1 x^3 \sqrt{1 + (3x^2)^2}\, dx = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx$$

This integral has no elementary antiderivative, but it can be approximated numerically or expressed in terms of standard functions if needed.

**Final Answer (exact form):**

$$\boxed{S = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx}$$

# Integration in Physics

Integration is used in physics to calculate various quantities, especially when a physical quantity changes continuously. One key application is in computing **work** done by a variable force.

## Work Done

The work $W$ done by a variable force $F(x)$ moving an object from position $a$ to $b$ is defined as:

$$\boxed{W = \int_a^b F(x)\, dx}$$

**Important Physical Laws:**

- **Hooke's Law:** The force required to compress or stretch a spring by a distance $x$ from its natural length is $F(x) = kx$, where $k$ is the spring constant.
- **Newton's Law of Gravitation:** The force of attraction between two masses is
$$F(r) = G\frac{m_1 m_2}{r^2}$$
where $G$ is the gravitational constant and $r$ is the distance between the masses.

- **Coulomb's Law:** The electric force between two point charges is

$$F(r) = k\frac{q_1 q_2}{r^2}$$

where $k$ is Coulomb's constant and $r$ is the distance between the charges.

### Problem 1: Spring Work Problem

**Problem:** A force of 30 N compresses a spring 0.3 m from its natural length of 1.5 m. Find the work done in compressing the spring an additional 0.3 m.

**Solution:**

From Hooke's Law, $F(x) = kx$. First, we find $k$:

$$30 = k(0.3) \quad \Rightarrow \quad k = \frac{30}{0.3} = 100$$

Now, we compute the work done to compress the spring from $x = 0.3$ to $x = 0.6$:

$$W = \int_{0.3}^{0.6} 100x\,dx = 100\left[\frac{x^2}{2}\right]_{0.3}^{0.6} = 100\left(\frac{0.6^2}{2} - \frac{0.3^2}{2}\right)$$

$$= 100\left(\frac{0.36 - 0.09}{2}\right) = 100 \cdot \frac{0.27}{2} = 100 \cdot 0.135 = 13.5 \text{ J}$$

**Answer:** $\boxed{13.5 \text{ J}}$

### Problem 2: Work to Pump Oil from a Spherical Tank

**Problem:** A spherical tank of radius 8 ft is half full of oil that weighs 50 lb/ft$^3$. Find the work done to pump all of the oil out through a hole in the top of the tank.

**Solution:**

We place the origin at the center of the sphere. The top of the tank is at $y = 8$, and oil fills from $y = -8$ to $y = 0$ (half full).

A horizontal slice at height $y$ has radius $r(y) = \sqrt{64 - y^2}$, and volume:

$$dV = \pi r(y)^2\,dy = \pi(64 - y^2)\,dy$$

Weight of the slice:

$$\text{Weight} = 50 \cdot dV = 50\pi(64 - y^2)\,dy$$

Distance to lift each slice to the top ($y = 8$) is $8 - y$, so work is:

$$W = \int_{-8}^{0} 50\pi(64 - y^2)(8 - y)\,dy$$

Factor out constants:

$$W = 50\pi \int_{-8}^{0} (64 - y^2)(8 - y)\, dy$$

Expand and integrate:

$$(64 - y^2)(8 - y) = 512 - 64y - 8y^2 + y^3$$

$$W = 50\pi \int_{-8}^{0} (512 - 64y - 8y^2 + y^3)\, dy$$

$$= 50\pi \left[ 512y - 32y^2 - \frac{8}{3}y^3 + \frac{1}{4}y^4 \right]_{-8}^{0}$$

Plug in values at $y = 0$ and $y = -8$:

$$= 50\pi \left( 0 - \left[ 512(-8) - 32(64) - \frac{8}{3}(-512) + \frac{1}{4}(4096) \right] \right)$$

$$= 50\pi \left( 4096 + 2048 + \frac{4096}{3} + 1024 \right) = 50\pi \left( 7168 + \frac{4096}{3} \right)$$

$$= 50\pi \left( \frac{21504 + 4096}{3} \right) = 50\pi \cdot \frac{25600}{3} = \frac{1280000\pi}{3} \text{ ft-lb}$$

**Answer:** $\boxed{\frac{1280000\pi}{3} \text{ ft-lb}}$

## Moments and Center of Mass

**Center of Mass (COM)** is the average position of the mass distribution of an object. It is the point where the object would balance if suspended.

**Moment** is a measure of the tendency of a mass to rotate about a given axis. For discrete systems, it is the sum of the products of the masses and their distances from the axis.

### Center of Mass in 1D (Discrete System)

If a system has point masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$, then the center of mass is:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

**Center of Mass in 2D (Discrete System)**

If the point masses are located at coordinates $(x_i, y_i)$, then:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

**Center of Mass of a Planar Lamina (Continuous Case)**

Suppose we have a lamina of uniform density $\rho$, bounded by curves $y = f(x)$ and $y = g(x)$, for $a \leq x \leq b$. Then:

**Area of the lamina:**

$$A = \int_a^b (f(x) - g(x))\, dx$$

**Moments:**

$$M_x = \rho \int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)\, dx \quad \text{(Moment about x-axis)}$$

$$M_y = \rho \int_a^b x(f(x) - g(x))\, dx \quad \text{(Moment about y-axis)}$$

**Center of Mass Coordinates:**

$$\bar{x} = \frac{M_y}{\rho A}, \quad \bar{y} = \frac{M_x}{\rho A}$$

—

**Example: Center of Mass of a Lamina Bounded by $y = 4 - x^2$**

**Problem:** Find the center of mass of the lamina of uniform density $\rho$, bounded by the graph $y = 4 - x^2$ and the x-axis.

**Step 1:** Determine the bounds.

The curve intersects the x-axis when $y = 0 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$

**Step 2:** Compute the area.

$$A = \int_{-2}^{2} (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(8 - \frac{8}{3}\right) - (-8 + \frac{8}{3}) = \frac{64}{3}$$

**Step 3:** Compute the moments.

**Moment about x-axis:**

$$M_x = \rho \int_{-2}^{2} \frac{1}{2}(4 - x^2)^2\, dx = \frac{\rho}{2} \int_{-2}^{2} (16 - 8x^2 + x^4)\, dx$$

$$= \frac{\rho}{2} \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} = \frac{\rho}{2} \left(64 - \frac{64}{3} + \frac{32}{5} + 64 - \frac{64}{3} + \frac{32}{5}\right) = \rho \left(64 - \frac{64}{3} + \frac{32}{5}\right)$$

$$= \rho\left(\frac{960 - 320 + 192}{15}\right) = \rho \cdot \frac{832}{15}$$

**Moment about y-axis:**

$$M_y = \rho \int_{-2}^{2} x(4 - x^2)\, dx = \rho \int_{-2}^{2} (4x - x^3)\, dx$$

Because this is an odd function over a symmetric interval:

$$M_y = 0$$

**Step 4:** Compute coordinates of center of mass:

$$\bar{x} = \frac{M_y}{\rho A} = \frac{0}{\rho \cdot \frac{64}{3}} = 0$$

$$\bar{y} = \frac{M_x}{\rho A} = \frac{\frac{832}{15}}{\frac{64}{3}} = \frac{832}{15} \cdot \frac{3}{64} = \frac{2496}{960} = \frac{26}{10} = 2.6$$

**Answer:** The center of mass is at

$$\boxed{\left(0, \frac{13}{5}\right)}$$