CHAPTER 1. VECTOR SPACES

1.1. INTRODUCTION

Many familiar physical notions, such as forces, velocities, and accelerations, involve both a magnitude (the amount of the force, velocity, or acceleration) and a direction. Any such entity involving both magnitude and direction is called a vector. A vector is represented by an arrow whose length denotes the magnitude of the vector and whose direction represents the direction of the vector. In most physical situations involving vectors, only the magnitude and direction of the vector are significant; consequently, we regard vectors with the same magnitude and direction as being equal irrespective of their positions. In this section the geometry of vectors is discussed. This geometry is derived from physical experiments that test the manner in which two vectors interact.

Familiar situations suggest that when two like physical quantities act simultaneously at a point, the magnitude of their effect need not equal the sum of the magnitudes of the original quantities. For example, a swimmer swimming upstream at the rate of 2 miles per hour against a current of 1 mile per hour does not progress at the rate of 3 miles per hour. For in this instance the motions of the swimmer and current oppose each other, and the rate of progress of the swimmer is only 1 mile per hour upstream. If, however, the swimmer is moving downstream (with the current), then his or her rate of progress is 3 miles per hour downstream.

Experiments show that if two like quantities act together, their effect is predictable. In this case, the vectors used to represents the combined effects of the original quantities. This resultant vector is called the sum of the original vectors, and the rule for their combination is called the parallelogram law.

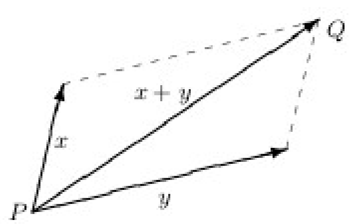

Figure 1.1.

Parallelogram Law for Vector Addition.

The sum of two vectors x and y that act at the same point P is the vector beginning at P that is represented by the diagonal of parallelogram having x and y as adjacent sides.

Since opposite sides of a parallelogram are parallel and of equal length, the endpoint Q of the arrow representing $x + y$ can also obtained by allowing x to act at P and then allowing y to act at the endpoint of x. Similarly, the endpoint of the vector $x + y$ can be obtained by first permitting y to act at P and then allowing x to act at the endpoint of y. Thus two vectors x and y that both act at the point P may be added tail-to-head; that is, either x or y may be applied at P and a vector having the same magnitude and direction as the other may be applied to the endpoint of the first. If this is done, the endpoint of the second vector is the endpoint of $x + y$.

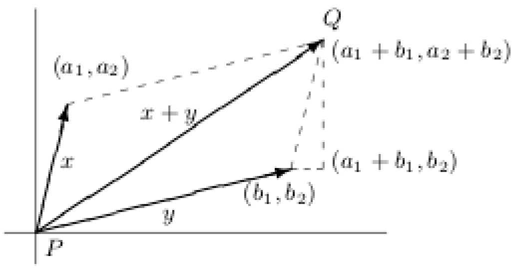

Figure 1.2.1.

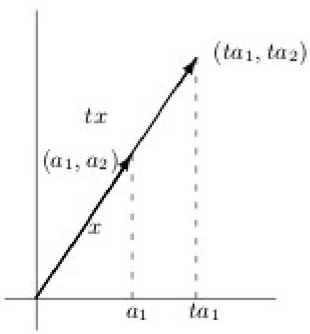

Figure 1.2.2.

The addition of vectors can be described algebraically with the use of analytic geometry. In the plane containing x and y, introduce a coordinate system with P at the origin. Let (a_1, a_2) denote the endpoint x and (b_1, b_2) denote the endpoint of y. Then as Figure 1.2.1 shows, the endpoint Q of $x + y$ is $(a_1 + b_1, a_2 + b_2)$. Henceforth, when a reference is made to the coordinates of the endpoint of a vector, the vector should be assumed to emanate from the origin. Moreover, since a vector beginning at the origin is completely determined by its endpoint, we sometimes refer to the point x rather than the endpoint of the vector x if x is a vector emanating from the origin.

Besides the operation of vector addition, there is another natural operation that can be performed on vectors – the length of a vector may be magnified or contracted. This operation, called scalar multiplication, consists of multiplying the vector by a real number. If the vector x is represented by an arrow, then for any real number t, the vector tx is represented by an arrow in the same direction if $t \geq 0$ and in the opposite direction if $t < 0$. The length of arrow tx is $|t|$ times the length of the arrow x. Two nonzero vectors x and y are called parallel if $y = tx$ for some nonzero real number t. Thus nonzero vectors having the same or opposite directions are parallel.

To describe scalar multiplication algebraically, again introduce a coordinate system into a plane containing the vector x so that x emanates from origin. If the endpoint of x has coordinates (a_1, a_2), then the coordinates of the endpoint of tx are easily seen to be (ta_1, ta_2).

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y, $x + y = y + x$.
2. For all vectors x, y, and z, $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted 0 such that $x + 0 = x$ for each vector x.
4. For each vector x, there is a vector y such that $x + y = 0$.
5. For each vector x, $1x = x$.
6. For each pair of real numbers a and b and each vector x, $(ab) x = a (bx)$.
7. For each real number a and each pair of vectors x and y, $a (x + y) = ax + by$.
8. For each pair of real numbers a and b and each vector x, $(a + b) x = ax + bx$.

Arguments similar to the preceding ones show that these eight properties, as well as the geometric interpretations of vector addition and scalar multiplication, are true also for vectors acting in space rather than in a plane. These results can be used to write equations of lines and planes in space.

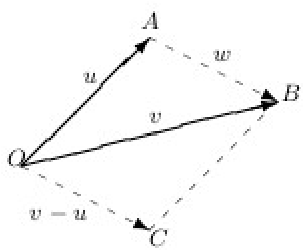

Figure 1.3.

Consider first the equation of a line in space that passes through two distinct points A and B. Let O denote the origin of a coordinate system in space, and let u and v denote the vectors that begin at O and end at A and B, respectively. If w denotes the vector beginning at A and ending at B, then tail-to-head addition shows that $u + w = v$, and hence $w = v – u$, where $-u$ denotes the vector $(-1)u$. Since a scalar multiple of w is parallel to w but possibly of a different length than w, any point on the line joining A and B may be obtained as the endpoint of a vector that begins at A and has the form tw for some real number t. Conversely, the endpoint of every vector of the form tw that begins at A lies on the line joining A and B. Thus an equation of the line through A an B is $x = u + tw = u + t (v – u)$, where t is a real number and x denotes an arbitrary point on the line. Notice also that the endpoint C of the vector $v – u$ in Figure 1.3 has coordinates equal to the difference of the coordinates of B and A.

Example

Let A and B be points having coordinates $(-2, 0, 1)$ and $(4, 5, 3)$, respectively. The endpoint C of the vector emanating from the origin and having the same direction as the vector beginning at A and terminating at B has coordinates $(4, 5, 3) - (-2, 0, 1) = (6, 5, 2)$. Hence the equation of the line through A and B is $y = (-2, 0, 1) + x (6, 5, 2)$.

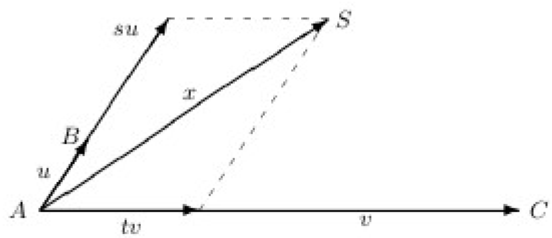

Figure 1.4.

Now let A, B, and C denote any three noncollinear points in space. These points determine a unique plane, and its equation can be found by use of our previous observations about vectors. Let u and v denote vectors beginning at A and ending at B and C, respectively. Observe that any point in the plane containing A, B, and C is the endpoint S of a vector x beginning at A and having the form $su + tv$ for some real numbers s and t. The endpoint of su is the point of intersection of the line through A and B with the line through S parallel to the line through A and C. A similar procedure locates the endpoint of tv. Moreover, for any real numbers s and t, the vector $su + tv$ lies in the plane containing A, B, and C. It follows that an equation of the plane containing A, B, and C is

$$y = A + su + tv$$

where s and t are arbitrary real numbers and x denotes an arbitrary point in the plane.

Example

Let A, B, and C be the points having coordinates $(1, 0, 2)$, $(-3, -2, 4)$, and $(1, 8, -5)$, respectively. The endpoint of the vector emanating from the origin and having the same length and direction as the vector beginning at A and terminate at B is $(-3, -2, 4) – (1, 0, 2) = (-4, -2, 2)$. Similarly, the endpoint of a vector emanating from the origin and having the same length and direction as the vector beginning at A and terminate at C is $(1, 8, -5) – (1, 0, 2) = (0, 8, -7)$. Hence the equation of the plane containing the three given points is $y = (1, 0, 2) + s(-4, -2, 2) + t(0, 8, -7)$.

Any mathematical structure possessing the eight properties on page 3 is called a vector space. In the next section we formally define a vector space and consider many examples of vector spaces other than the ones mentioned above.