1. FUNCTIONS AND LIMITS

1.1. FOUR WAYS TO REPRESENT A FUNCTION

A **function** *f* is a rule that assigns to each element *x* in a set *D* exactly one element, called *f(x),* in a set *E*.

We usually consider functions for which the sets *D* and *E* are sets of real numbers. The set *D* is called the **domain** of the function. The number *f(x)* is the **value of *f* at *x*** and is read "*f of x*". The **range** of *f* is the set of all possible values of *f(x)* as *x* varies throughout the domain. A symbol that represents an arbitrary number int the domain of a function *f* is called an **independent variable**. A symbol that represents a number in the range of *f* is called a **dependent variable**.

Example: Sketch the graph and find the domain and range of each function.

$f(x)=2x-1$

$f(x)=x^2$

Example: If $f(x) = 2x^{2}-5x + 1$ and $h \neq 0$, evaluate

$$\frac{f(a+h)-f(a)}{h}$$

Example: Find the domain of each function.

$f(x)=\sqrt{x+2}$

$$f(x)=\frac{1}{x^2-x}$$

1.1.1. PIECEWISE DEFINED FUNCTION

Example: A function *f* is defined by

$$f(x) = \begin{cases} 1-x, \text{ if } x \le -1 \\ x^2, \text{ if } x > -1 \end{cases}$$

Evaluate *f(-2), f(-1),* and *f(0)* and sketch the graph.

Example: Sketch the graph of the absolute value function

$$f(x) = |x|$$

1.1.2. SYMMETRY

If a function f satisfied *f(-x) = f(x)* for every number *x* in its domain, then *f* is called an **even function**.

If f satisfied *f(-x) = -f(x)* for every number *x* in its domain, then *f* is called an **odd function**.

Example: Determine whether each of the following functions is even, odd, or neither even nor odd.

$$f(x) = x^5 + x$$

$$f(x) = 1 - x^4$$

$$f(x) = 2x - x^2$$

1.1.3. INCREASING AND DECREASING FUNCTIONS

A function f is called **increasing** on an interval if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2$$

It is called **decreasing** on a interval if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2$$

1.2. MATHEMATICAL MODELS: A CATALOG OF ESSENTIAL FUNCTIONS

1.2.1. LINEAR MODELS

When we say that *y* is a **linear function** of *x*, we mean that the graph of the function is a line, so we can use the slope intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b$$

where *m* is the slope of the line and *b* is the *y* intercept.
A characteristic feature of linear functions is that they grow at a constant rate.

1.2.2. POLYNOMIALS

A function *f* is called a polynomial if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where *n* is a non-negative integer and the number $a_0, a_1, a_2, \ldots, a_n$ are constants called the coefficients of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is *n*.

A polynomial of degree 1 is of the form $f(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $f(x) = ax^2 + bx + c$ is called quadratic function. A polynomial of degree 3 is of the form $f(x) = ax^3+bx^2+cx+d$, $a \neq 0$ and is called a cubic function.

1.2.3. POWER FUNCTIONS

A function of the form $f(x) = x^a$, where *a* is a constant, is called a power function.

***a* = *n*, where *n* is a positive integer**

***a* = *1/n*, where *n* is a positive integer**

The function $f(x) = x^{1/n}$ is a root function.

a* = *-1

The graph of the reciprocal function $f(x) = x^{-1} = 1/x$.

1.2.4. RATIONAL FUNCTIONS

A rational function f is a ratio of two polynomials:

$$f(x)=\frac{g(x)}{h(x)}$$

1.2.5. ALGEBRAIC FUNCTIONS

A function *f* is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials.

1.2.6. TRIGONOMETRIC FUNCTIONS

$$-1 \le \sin x \le 1$$
$$-1 \le \cos x \le 1$$
$$|\sin x| \le 1$$
$$|\cos x| \le 1$$
$$\sin x = 0 \text{ when } x = n\pi \ \ (n \text{ an integer})$$
$$\sin(x+2\pi)=\sin x$$
$$\cos(x+2\pi)=\cos x$$
$$\tan x=\frac{\sin x}{\cos x}$$

It is undefined whenever $\cos x = 0$, that is, when $x = \dots, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, \dots$. Its range if $(-\infty, \infty)$. Notice that the tangent function has period π:

$$\tan(x+\pi)=\tan x$$

Example: What is the domain of the function

$$f(x)=\frac{1}{1-2\cos x}$$

1.2.7. EXPONENTIAL FUNCTIONS

The exponential functions are the function of the form $f(x) = b^x$, where the base b is a positive constant.

1.2.8. LOGARITHMIC FUNCTIONS

The logarithmic functions $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions.

Example: Classify the following functions as one of the types of functions that we have discussed.

$$f(x)=5^x$$

$$f(x)=x^5$$

$$f(x)=\frac{1+x}{1-\sqrt{x}}$$

$$f(x)=1-x+5x^4$$

1.3. NEW FUNCTIONS FROM OLD FUNCTIONS

1.3.1. TRANSFORMATIONS OF FUNCTIONS

TRANSLATIONS (VERTICAL AND HORIZONTAL SHIFTS)

Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

STRETCHING AND REFLECTING (VERTICAL AND HORIZONTAL STRETCHING AND REFLECTING)

Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y axis

Example: Given the graph of $y=\sqrt{x}$ **, use transformations to graph**

$$y=\sqrt{x}-2$$

$$y=\sqrt{x-2}$$

$$y=-\sqrt{x}$$

$$y=2\sqrt{x}$$

$$y=\sqrt{-x}$$

Example: Sketch the graph of the function

$$f(x)=x^2+6x+10$$

Example: Sketch the graphs of the following functions.

$$y = \sin 2x$$

$$y = 1 - \sin x$$

Example: Sketch the graph of the function

$$y = |x^2 - 1|$$

1.3.2. COMBINATIONS OF FUNCTIONS

Two functions *f* and *g* can be combined to form new functions *f* + *g*, *f* – *g*, *fg*, and *f/g* in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$

If the domain of *f* is *A* and the domain of *g* is *B*, then the domain of *f* + *g* is the intersection $A \cap B$ because both *f(x)* and *g(x)* have to be defined.

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)\,g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of *fg* is $A \cap B$. Because we can't divide by 0, the domain of f/g is therefore $\{x \in A \cap B \mid g(x) \neq 0\}$.

Given two functions *f* and *g*, the composite function *f* ∘ *g* (also called the composition of *f* and *g*) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of *f* ∘ *g* is the set of all *x* in the domain of *g* such that *g(x)* is in the domain of *f*. In other words, *(f* ∘ *g)(x)* is defined whenever both *g(x)* and *f(g(x))* are defined.

Example: If *f(x)* = *x2* and *g(x)* = *x* – *3*, find the composite functions *f* ∘ *g* and *g* ∘ *f*.

Example: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each of the following functions and their domains

$f \circ g$
$g \circ f$

$f \circ f$
$g \circ g$

Example: Find $f \circ g \circ h$ **if** $f(x) = x / (x+1)$, $g(x) = x^{10}$, **and** $h(x) = x + 3$.

Example: Given $F(x) = \cos^2(x+9)$, **find functions** f, g, **and** h **such that** $F = f \circ g \circ h$.

1.4. THE TANGENT AND VELOCITY PROBLEMS

1.5. THE LIMIT OF A FUNCTION

1.5.1. INTUITIVE DEFINITION OF A LIMIT

Suppose $f(x)$ is defined when x is near the number a. This means that f is defined on some open interval that contains a, except possibly at a itself. The we write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Example: Guess the value of

$$\lim_{x \to 1} \frac{x-1}{x^2-1}$$

Example: Estimate the value of

$$\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$$

Example: Guess the value of

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Example: Investigate

$$\lim_{x \to 0} \sin \frac{\pi}{x}$$

Example: Find

$$\lim_{x\to 0}\left(x^3+\frac{\cos 5x}{10,000}\right)$$

1.5.2. DEFINITION OF ONE-SIDE LIMITS

We write

$$\lim_{x\to a^-} f(x)=L$$

and say the left-hand limit of *f(x)* as *x* approaches *a* (or the limit of *f(x)* as *x* approaches *a* from the left) is equal to *L* if we can make the values of *f(x)* arbitrarily close to *L* by taking *x* to be sufficiently close to a with *x* less than *a*.

If we require that *x* be greater than *a*, we get "the right-hand limit of *f(x)* as *x* approaches *a* is equal to *L*" and we write

$$\lim_{x\to a^+} f(x)=L$$

1.5.3. INFINITE LIMITS

INTUITIVE DEFINITION OF AN INFINITE LIMIT

Let *f* be a function defined on both sides of *a*, except possibly at *a* itself. Then

$$\lim_{x\to a} f(x)=\infty$$

means that the value of *f(x)* can be made arbitrarily large (as large as we please) by taking *x* sufficiently close to *a*, but not equal to *a*.

Let *f* be a function defined on both sides of *a*, except possibly at a itself. Then

$$\lim_{x\to a} f(x)=-\infty$$

means that the values of *f(x)* can be made arbitrarily large negative by taking *x* sufficiently close to *a*, but not equal to *a*.

Example: Find

$$\lim_{x\to 3^+}\frac{2x}{x-3}$$

$$\lim_{x\to 3^-}\frac{2x}{x-3}$$

Example: Find the vertical asymptotes of

$f(x)=\tan x$

1.6. CALCULATING LIMITS USING THE LIMIT LAWS