1 Nonlinear dynamics

The purpose of this project is to familiarize you with the basics of nonlinear dynamics: fixed points, stability, and linearization technique. **The tasks should be finished till 24.10.2024**

1. Find the fixed points ($f(x^*) = 0$) for the equation:

$$\dot{x} = f(x) = x(x-1)(x-2),$$

   and analyze their stability by solving the equation numerically. For this purpose, choose initial conditions from the vicinity of x^* and observe the behavior of $x(t)$. Use the simplest *Euler method* (its only advantage is simplicity),

$$x_{n+1} = x_n + \Delta t\, f(x_n),$$

   where Δt is the time step. To verify the correctness of the solutions, repeat the simulations with different time steps.

2. Implement an approximate method for drawing the **phase portraits**, and apply it to the following equations: $\ddot{x} + x = 0$, $\ddot{x} + \sin(x) = 0$, $\ddot{x} = -x + x^3$, $\ddot{x} = x - x^3$.

   In the first step, by substituting $\dot{x} = y$, reduce the second-order equation to a system of two first-order equations. In subsequent steps, for given initial conditions (x_0, y_0), i.e., the starting point in the phase space, we numerically calculate the trajectory $(x(t), y(t))$. Use the so-called *midpoint method* (one of the second-order Runge-Kutta methods):

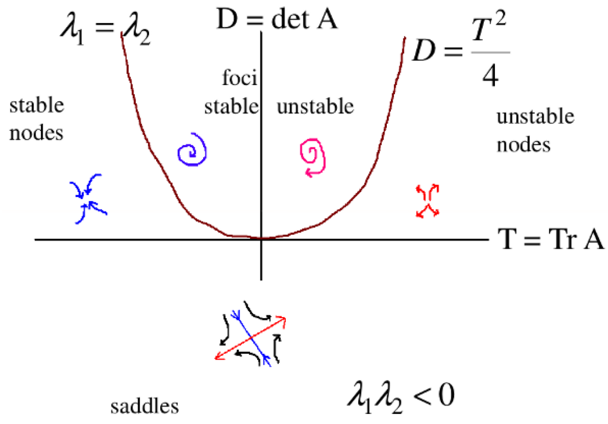
$$\begin{aligned} k_x &= \Delta t\, f_x(x_n, y_n), \\ k_y &= \Delta t\, f_y(x_n, y_n), \\ x_{n+1} &= x_n + \Delta t\, f_x\left(x_n + \frac{k_x}{2}, y_n + \frac{k_y}{2}\right), \\ y_{n+1} &= y_n + \Delta t\, f_y\left(x_n + \frac{k_x}{2}, y_n + \frac{k_y}{2}\right). \end{aligned}$$

   Repeat the entire procedure for different, evenly distributed starting points in the phase space. For the obtained phase portraits, determine the **isoclines** and describe how trajectories behave in their vicinity. What property do points at the intersection of isoclines have?

3. Use the implemented method for drawing phase portraits to analyze the behavior of trajectories in the vicinity of fixed points for the **linear** equation $\dot{x} = Ax$, where $x \in \mathbb{R}^2$ and A is given by the following matrices:

$$\text{a) } \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}, \text{ b) } \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}, \text{ c) } \begin{pmatrix} -3 & -2 \\ -1 & -3 \end{pmatrix}, \text{ d) } \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

   Please verify the obtained behavior of the trajectories according to the diagram by calculating the determinant (det A) and the trace (tr A):



4. Sketch the phase portrait for the **nonlinear** system:

$$\begin{cases} \dot{x} = x(3 - x - 2y), \\ \dot{y} = y(2 - x - y). \end{cases} \tag{1}$$

Find the fixed points and determine the behavior of the trajectories in their vicinity. To illustrate the orientation of the trajectories, you can change the color of the curve based on the value of the progressing time.

Lotka-Volterra model: Let $x, y \geqslant 0$. Modify the system (equations) to obtain a stable nonzero population for both species.