Rado's Theorem

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July 20, 2021

Introduction

- Last time Jack talked about Schur's theorem and Schur's Number.
- Schur's theorem was taken further by his Ph.D student Richard Rado.
- In 1930, Rado determined that linear equations in the form of $\sum_{i=1}^{k} c_i x_i = 0$ are guaranteed to have monochromatic solutions under any finite coloring of $\mathbb{Z}^+$.

Definitions

Definition (Coloring)

An $r-$coloring of a set S is a function $\chi : S \to C$, where $|C| = r$.

We can think of an $r-$coloring of a set S as a partition of S into r subsets $S_1, S_2, \ldots, S_r$, by associating the subset S_i with the set $\{x \in S : \chi(x) = i\}$.

Definition (Monochromatic)

A coloring χ is monochromatic on a set S if χ is constant on S.

Definitions cont'd

Definition (Regularity)

For $r \geq 1$, a linear equation D is called $r-$regular if there exists $n = n(D; r)$ such that for every $r-$coloring of $[1, n]$ there is a monochromatic solution to D. The equation D is *regular* if it is $r-$regular for all $r \geq 1$.

Definition (Rado Number)

The Rado Number of an equation D is a theoretical quantity associated to D. For any equation D, the Rado Number $R_r(D)$ is the smallest N such that any r coloring $\chi : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, r\}$ must induce a monochromatic solution to D.

Rado's Theorem

One of the initial discoveries of Rado was the following theorem:

Theorem

Let $k3$ and $c_i \in \mathbb{Z}^+ - \{0\}$ for $i = 1, 2, \ldots, k$. Let D represent the equation $\sum_{i=1}^{k} c_i x_i = 0$. If there exist some $i, j \in \{1, 2, \ldots, k\}$ such that $c_i < 0$ and $c_j > 0$, then D is 2-regular.

Proof

Proof

We start by rewriting D as $\sum_{i=1}^{m} \alpha_i y_i = \sum_{i=1}^{n} \beta_i z_i$, where $m \geq 2, n \geq 1$, all of the coefficients are positive.
We consider a subset of the solution to D, where $y = y_1 = y_2 = \cdots = y_{m-1}, w = y_m$, and $z = z_1 = z_2 = \cdots = z_n$.
The equation D now can be rewritten as $ay + bw = cz$, where $a = \sum_{i=1}^{m-1} \alpha_i, b = a_m, c = \sum_{i=1}^{n} \beta_i$.
We will now show that with any 2-coloring of $\mathbb{Z}^+$, there must exists a monochromatic solution to D.
Let's consider all possible solution to D, for each of the solution (y, w, z), we want to determine the $\max(y, w, z)$. Let $(\bar{y}, \bar{w}, \bar{z})$ be solution where this maximum is minimal and we define $A = \max(\bar{y}, \bar{w}, \bar{z})$.
Note that $[1, \ldots, A]$ contains a solution to D.

Proof Cont'd

Proof

Assume for the sake of contradiction, that there exists a 2-coloring of $\mathbb{Z}^+$ with no monochromatic solution to D. For the two colors, let them be red and blue.
Let $l = \text{lcm}(\frac{a}{gcd(a,b)}, \frac{c}{gcd(b,c)})$ so that $\frac{bl}{a}, \frac{bl}{c}$ are positive integers.
Without the loss of generality, we can assume that l is colored red.
Let s be the smallest element in $\{i \cdot l : i = 2, \ldots, A\}$ that is blue. s must exists since $\{i \cdot n : i = 1, 2, \ldots, A\}$ is *not* monochromatic for any $n \in \mathbb{Z}^+$, otherwise, $(n\bar{y}, n\bar{w}, n\bar{z})$ will be a monochromatic solution, contradict our assumption.
Now, for some $p \in \mathbb{Z}^+$, we have $t = \frac{b}{a}(s - l)p$ is blue, otherwise contradicting the above equation since $\frac{b}{a}(s - l)p$ is a positive integer.
Now, $q = \frac{b}{c}((s - l)p + s)$ must be red, otherwise (t, s, q) would be a solution to D and it is monochromatic(blue).

Proof Cont'd

Proof

To see this, let's plug (t, s, q) into D.

$$ay + bw = cz$$

$$a(\frac{b}{a}(s-l)p) + bs = c(\frac{b}{c}((s-l)p+s))$$
$$bsp - blp + bs = bsp - blp + bs$$

With similar reasoning, since we know that both l, q are red, $\frac{b}{a}(s-l)(p+1)$ must be blue, for otherwise $(\frac{b}{a}(s-l)(p+1), l, q)$ is another monochromatic(red) solution.

$$a(\frac{b}{a}(s-l)(p+1)) + bl = c(\frac{b}{c}((s-l)p+s))$$

$$bsp - blp + bs - bl + bl = bsp - blp + bs$$

Proof Cont'd

Proof.

From before, we know that $t = \frac{b}{a}(s-l)p$ and $\frac{b}{a}(s-l)(p+1)$. Since p was chosen randomly in $\mathbb{Z}^+$, we can see that $\{i \cdot \frac{b}{a}(s-l) : i = p, p+1, \dots\}$ is monochromatic. In particular, we arrive at the following:

$$\{i \cdot \frac{b}{a}(s-l) : i = 1, 2, \dots, A\}$$

is monochromatic, which contradicting our assumption. $\square$

Rado's Theorems

- The above theorem cannot be extended to 3 colors, since it is known that $x + 2y - 4z = 0$ is not 3−regular.
- This hints that there must exists a stronger condition on the equation's coefficients to guarantee regularity.

Theorem (Rado's Single Equation Theorem)

Let $k \geq 2$, Let $c_i \in \mathbb{Z} \setminus \{0\}$, for all $i \in \{1, 2, \ldots, k\}$, be constants. Then

$$\sum_{i=1}^{k} c_i x_i = 0$$

is regular if and only if there exists a nonempty subset $C_s \subset \{c_i : 1 \leq i \leq k\}$ such that $\sum_{d \in C_s} d = 0$.

Rado's Theorem

- In Rado's original paper, he conjectured that for all $r \in \mathbb{Z}^+$, there must exists equations that are $r-$regular but not $(r+1)-$regular.
- This conjecture has been resolved with a very recent paper with the following theorem:

Theorem

For every $r \in \mathbb{Z}^+$, the equation

$$\sum_{i=1}^{r} \frac{2^i}{2^i - 1} x_i = \left(\sum_{i=1}^{r} \frac{2^i}{2^i - 1} - 1\right) x_r + 1$$

is $r-$regular but not $(r+1)-$regular.

Another View to Rado's Theorem

- We can also view Rado's theorem in a different way.

Definition

Let A be an $m \times n$ matrix with rational entries. A is partition regular(PR) if whenever $\mathbb{N}$ is finitely colored, there exists a monochromatic solution $x \in \mathbb{N}^n$ where $Ax = 0$.

Example

$A = [1, 1, -1]$ is partition regular. Let $x = [a, b, c]$

$$\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a + b - c = 0.$$

By Schur's theorem, there will exists solution x that is monochromatic, thus A is PR.

Another View to Rado's Theorem

Lemma

A is PR if and only if λA is PR for all $\lambda \in \mathbb{Q} \setminus \{0\}$.

Proof.

It is easy to see that assume A is PR, then there will exists a monochromatic solution $x \in \mathbb{N}^n$ such that $Ax = 0$. The associative law implies that $(\lambda A)x = \lambda(Ax) = 0$. Thus, λA is PR. The other direction can be proven simply replacing multiplication with division. □

- There are some consequences of this lemma, hinting that if a particular equation having x as its Rado Number, then any constant multiple to this equation would have the same Rado Number. As long as the constant is in $\mathbb{Q} \setminus \{0\}$.
- Jack will talk more about computed Rado's number and some symmetries within them.

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Rado numbers $R_3(a(x - y) = bz)$

ba	1	2	3	4	5	6	7	8
1	14	14	27	64	125	216	343	512
2	43	*14*	31	*14*	125	*27*	343	*64*
3	94	61	*14*	73	125	*14*	343	512
4	173	*43*	109	*14*	141	*31*	343	*14*
5	286	181	186	180	*14*	241	343	512
6	439	*94*	*43*	*61*	300	*14*	379	*73*
7	638	428	442	456	470	462	*14*	561
8	889	*173*	633	*43*	665	*109*	644	*14*
9	1198	856	*94*	892	910	*61*	896	896
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Italicized numbers denote equations that are multiples of an equation whose Rado number is already computed.

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[Myers '15] $R_3(x-y=bz)=(b+2)^3-(b+2)^2-(b+2)-1$

Generalized Schur numbers

Definition

The *generalized Schur number* $S(k,m)$ is the Rado number

$$R_k(x_1 + x_2 + \cdots + x_{m-1} = x_m).$$

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- Myers's conjecture implies the conjecture for generalized Schur numbers