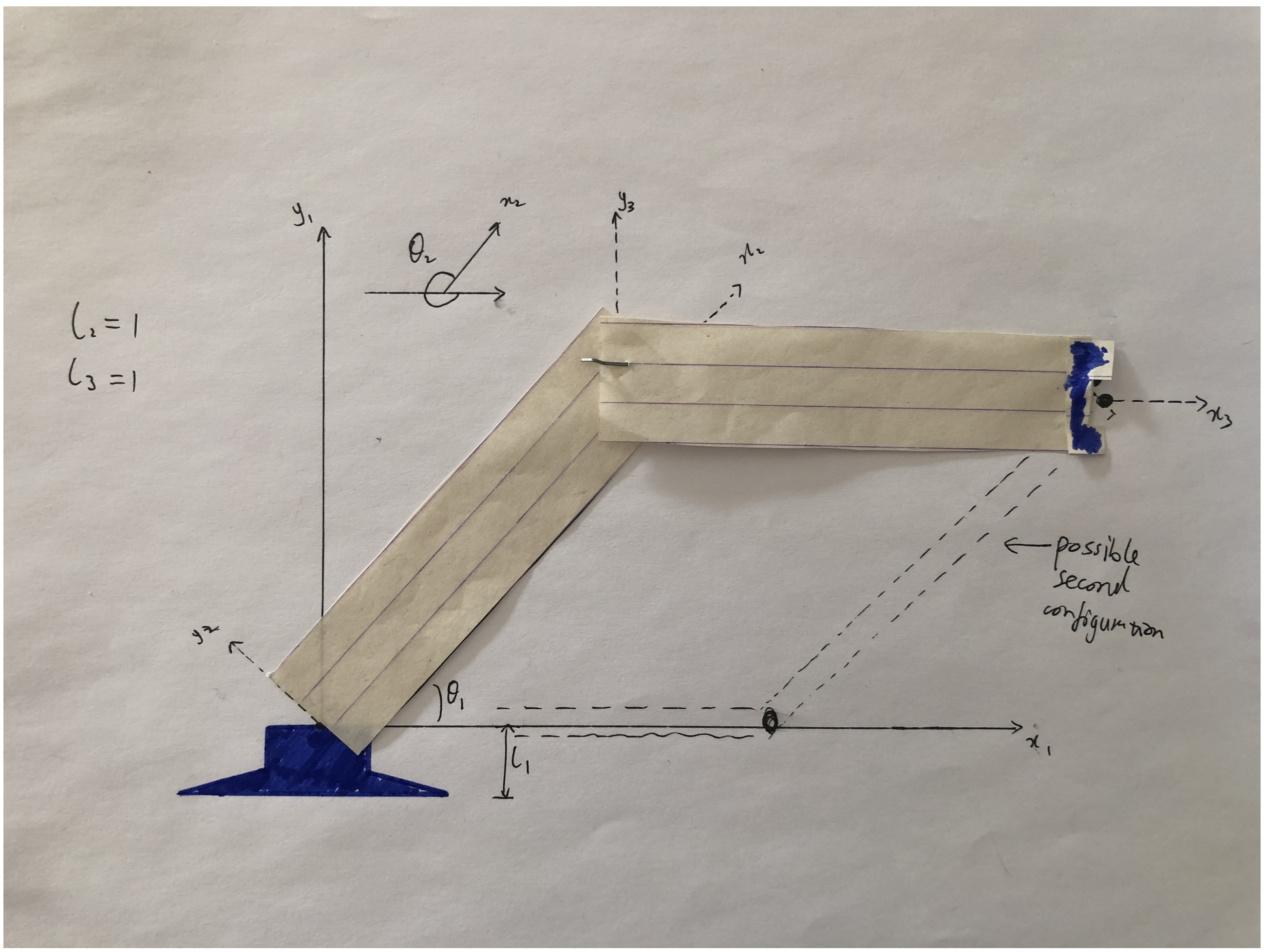
Robotic Arm Project

Yuan Chang

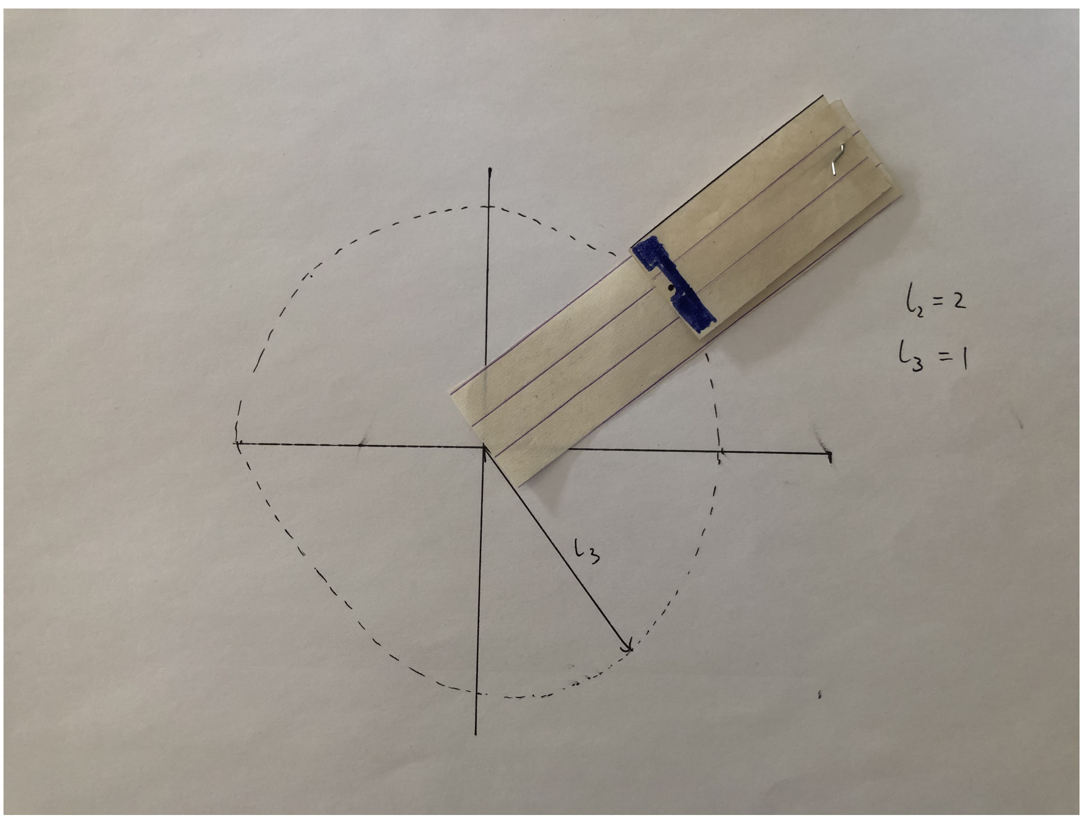
June 12, 2021



Contents

1 Introduction

There are many problems in engineering that integrate mathematics with computers and one particular example is the analysis of the possible configurations of a robotic arm with many joints through the lens of computational algebraic varieties. Computations without the help of a computer can be extremely tedious and arduous when solving this type of engineering problems.
In this project, I will explore some basic methods to describe the configuration of a robotic arm with connections to Groebner basis and affine varieties, discuss some special cases when calculating the forward/reverse kinematic problem and finally make some connections and geometric intuitions about the kinematic singularities that we might encounter when planning the position of the hand of the robot.
Furthermore, we will also explore two industry research papers on this topic and make some brief connections with this introductory project.
Below is an example of a special case that we might encounter when doing the reverse kinematic problem.



2 Robotic Problems

2.1 Groebner Basis - 6.3.1

- a) The Groebner basis of our 3 joints robot arm problem is as followed:
  There are typos in the 4th edition of the book, the below are the corrected formulae.

$$c_1 - \frac{2bl_2l_3}{2l_2(a^2+b^2)}s_2 - \frac{a(a^2+b^2+l_2^2-l_3^2)}{2l_2(a^2+b^2)},$$

$$s_1 + \frac{2al_2l_3}{2l_2(a^2+b^2)}s_2 - \frac{b(a^2+b^2+l_2^2-l_3^2)}{2l_2(a^2+b^2)},$$

$$c_2 - \frac{a^2+b^2-l_2^2-l_3^2}{2l_2l_3},$$

$$s_2^2 + \frac{(a^2+b^2)^2 - 2(a^2+b^2)(l_2^2+l_3^2) + (l_2^2-l_3^2)^2}{4l_2^2l_3^2}$$

  If we substitute $l_2 = l_3 = 1$, we have the following:

$$c_1 - \frac{2b}{2(a^2+b^2)}s_2 - \frac{a(a^2+b^2)}{2(a^2+b^2)},$$

$$s_1 + \frac{2a}{2(a^2+b^2)}s_2 - \frac{b(a^2+b^2)}{2(a^2+b^2)},$$

$$c_2 - \frac{a^2+b^2-2}{2},$$

$$s_2^2 + \frac{(a^2+b^2)^2 - 4(a^2+b^2)}{4}$$

  With some simplifying, we obtain equations (3) as followed:

$$c_1 - \frac{b}{(a^2+b^2)}s_2 - \frac{a}{2},$$

$$s_1 + \frac{a}{(a^2+b^2)}s_2 - \frac{b}{2},$$

$$c_2 - \frac{a^2+b^2-2}{2},$$

$$s_2^2 + \frac{(a^2+b^2)(a^2+b^2-4)}{4}$$

- b) The system of polynomial equations (1) is as followed:

$$a = l_3(c_1c_2 - s_1s_2) + l_2c_1,$$

$$b = l_3(c_1s_2 + c_2s_1) + l_2s_1,$$

$$0 = c_1^2 + s_1^2 - 1,$$
$$0 = c_2^2 + s_2^2 - 1$$

We can then compute the Groebner basis in Maple while setting $l_2 = l_3 = 1$. When doing this we need to manipulate the first and second equation into:

$$0 = l_3(c_1c_2 - s_1s_2) + l_2c_1 - a,$$

$$0 = l_3(c_1s_2 + c_2s_1) + l_2s_1 - b,$$

With Maple, we can compute the Groebner basis with the Groebner package.

```
l2 := 1
                                   l2 := 1
l3 := 1
                                   l3 := 1
F := [l3·(c1 c2 - s1 s2) + l2 c1 − a, l3·(c1 s2 + c2·s1) + l2 s1 − b, c1^2 + s1^2 − 1, c2^2 + s2^2 − 1]
              F := [c1 c2 − s1 s2 − a + c1, c1 s2 + s1 c2 − b + s1, c1^2 + s1^2 − 1, c2^2 + s2^2 − 1]
with(Groebner) :
Basis(F, grlex(c1, s1, c2, s2))
[-a^2 − b^2 + 2 c2 + 2, (2 a^2 + 2 b^2) s1 + 2 s2 a − a^2 b − b^3, (2 a^2 + 2 b^2) c1 − 2 s2 b − a^3 − a b^2, a^4 + 2 a^2 b^2 + b^4 − 4 a^2 − 4 b^2 + 4 s2^2]
```

We can do some algebraic manipulations:

$$-a^2 - b^2 + 2c_2 + 2 = c_2 - \frac{a^2 + b^2 - 2}{2},$$

$$(2a^2 + 2b^2)s_1 + 2s_2a - a^2b - b^3 = s_1 + \frac{a}{(a^2 + b^2)}s_2 + \frac{-a^2b - b^3}{2(a^2 + b^2)} = s_1 + \frac{a}{(a^2 + b^2)}s_2 - \frac{b}{2}$$

where $\frac{-a^2b-b^3}{a^2+b^2}$ can be simplified to $-b$.

$$(2a^2 + 2b^2)c_1 - 2s_2b - a^3 - ab^2 = c_1 - \frac{b}{(a^2 + b^2)}s_2 + \frac{-a^3 - ab^2}{2(a^2 + b^2)} = c_1 - \frac{b}{(a^2 + b^2)}s_2 - \frac{a}{2}$$

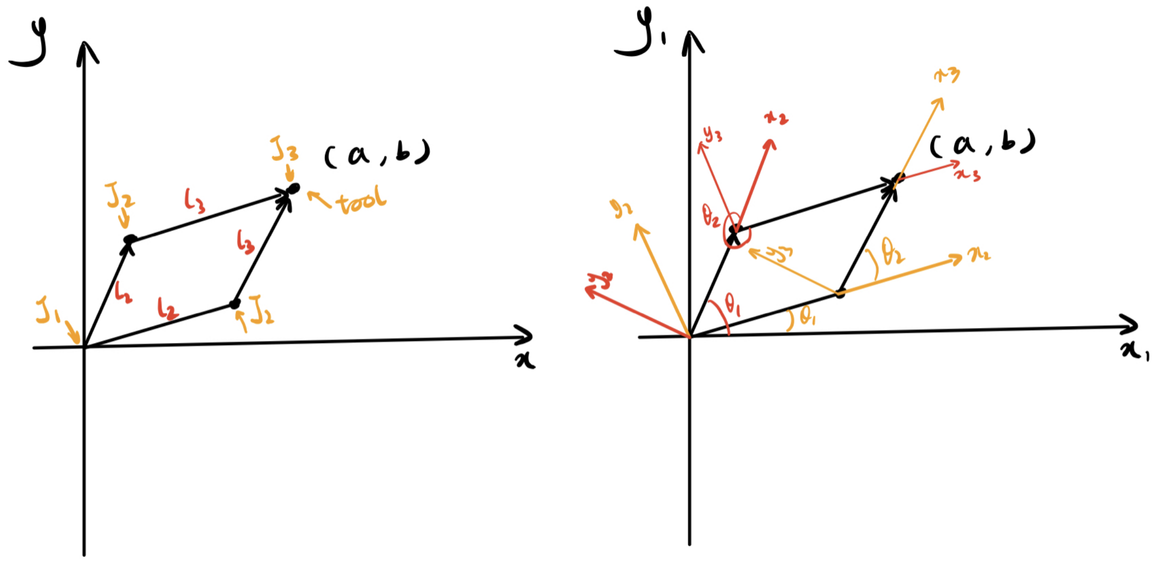
where $\frac{-ab^2-a^3}{a^2+b^2}$ can be simplified to $-a$.

$$a^4 + 2a^2b^2 + b^4 - 4a^2 - 4b^2 + 4s_2^2 = s_2^2 + \frac{(a^2 + b^2)(a^2 + b^2 - 4)}{4}$$

The results after simplifying matches the Groebner basis in part 1.

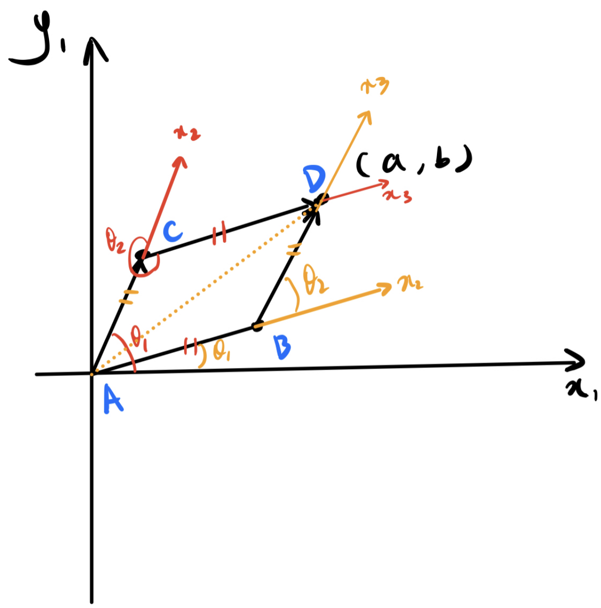
2.2 Reverse Kinematic Problem - 6.3.2

- a) If we have the constraints that $0 < a^2 + b^2 < 4$, this implies that the tool is within a circle with radius 4 centered at $(0, 0)$ in the global coordinate system and also not at the origin.



The figure on the left side indicates the two possible solutions when the tool is at a general location (a, b) where $0 < a^2 + b^2 < 4$. Although not to scale, the image is showing that $l_2 = l_3 = 1$. The difference in the configuration is again illustrated in the figure on the right side of the picture, indicating the different set of θ_i. Each colour represent one unique answer to the same problem.

- b) Geometrically, we know that the line AB is parallel to CD, and so are AC and BD.



With some geometry, we can easily find the inner angles of this rhombus $ABCD$ and we can see that θ_2 in both cases are just angles about this rhombus. They can be calculated with some trigonometry alone without the need to know θ_1, thus the equations for c_2 and s_2 do not involve c_1 and s_1. The θ_2 in the two cases actually sum up to 2π.

- c) Numerical examples:
  - $(a, b) = (0, 1)$
    If we set $a = 0, b = 1$ and then compute the Groebner basis again in Maple, we obtain the following set of solutions:

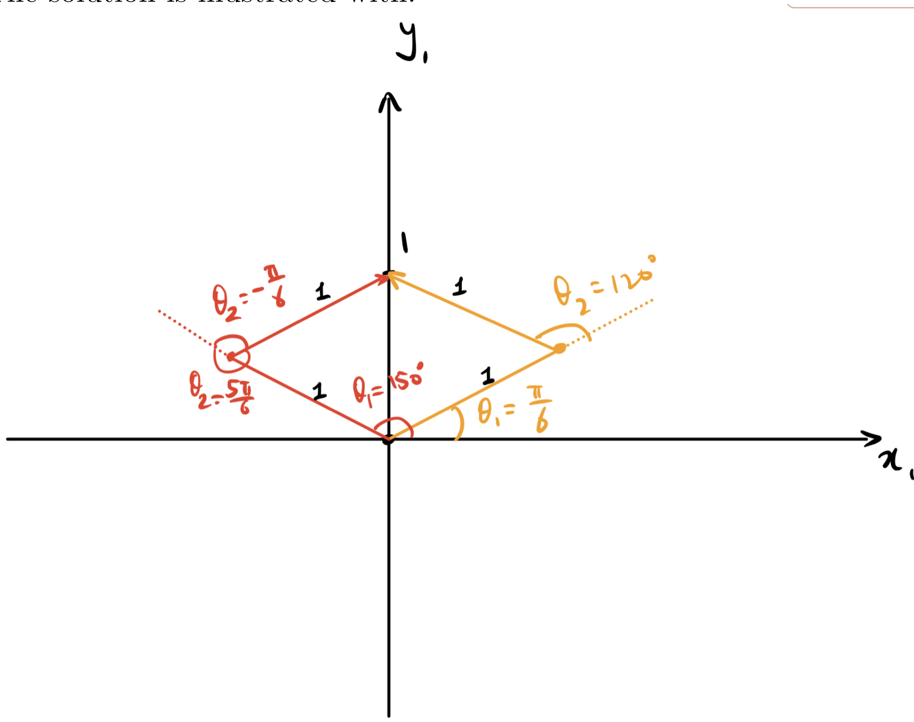
    $a := 0:$
    $b := 1:$
    $Basis(F, grlex(c1, s1, c2, s2))$

    $$[1 + 2\,c2, 2\,s1 - 1, c1 - s2, 4\,s2^2 - 3]$$

    Which is:

$$c_2 = \frac{-1}{2} = 120°$$
$$s_1 = \frac{1}{2} = 30°$$
$$s_2 = \pm\sqrt{\frac{3}{4}} = \pm 60°$$
$$c_1 = \pm\sqrt{\frac{3}{4}} = 30°/150°$$

    I loosely equate the c_i and s_i with an angle, but angles were calculated then taking the inverse trig function of their values. Such as $s_1 = 1$ implies that $\theta_1 = \arcsin(\frac{1}{2}) = 30°$.
    The solution is illustrated with:



– $(a, b) = (0, 2)$
If we set $a = 0, b = 2$ and then compute the Groebner basis again, we obtain the following set of solutions:

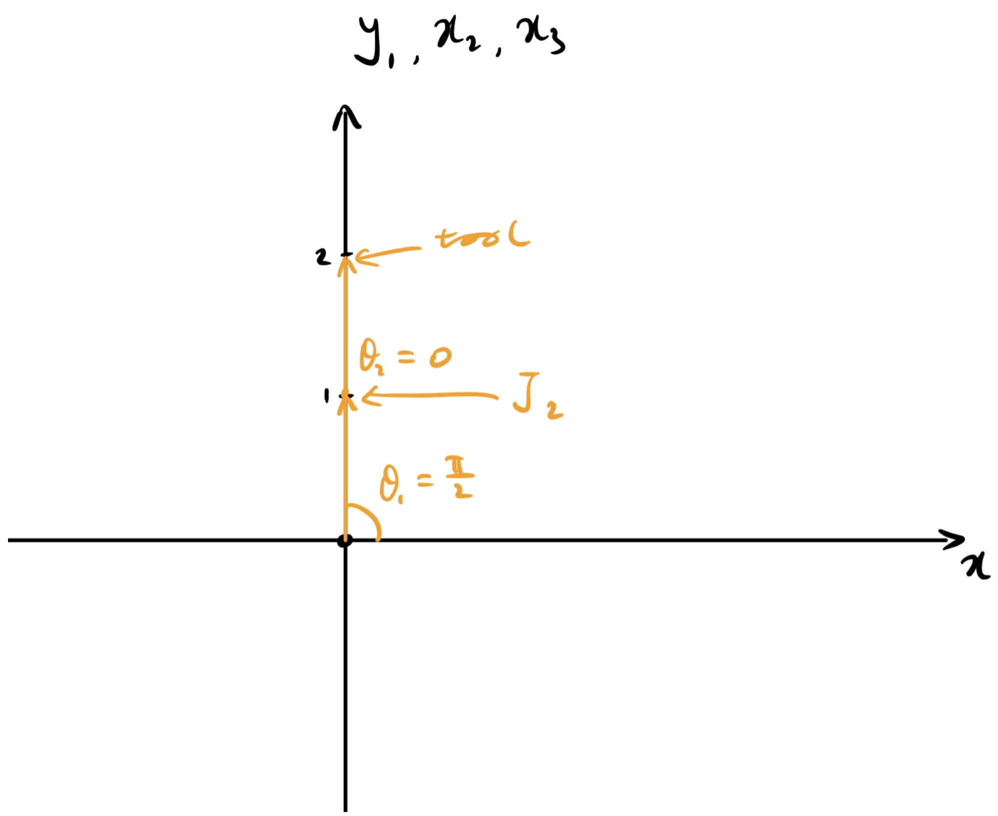
$a := 0 :$
$b := 2 :$
$Basis(F, grlex(c1, s1, c2, s2))$

$$[-1 + c2, s1 - 1, 2\,c1 - s2, s2^2]$$

Which is:

$$c_2 = 1 = 0^\circ$$
$$s_1 = 1 = 90^\circ$$
$$s_2 = 0 = 0^\circ$$
$$c_1 = 0 = 90^\circ$$

This implies that the only solution is when $\theta_1 = 90^\circ$ and $\theta_2 = 0$, where first arm is straight vertical up from the base and second arm is continuing the same direction of the first arm, thus having $\theta_2 = 0$. The solution is illustrated with the figure:



This example follows the idea when joint 1 to joint 3 is length 2, which is only reachable by setting $\theta_2 = 0$. This is from p307 of IVA.

2.3 Kinematic Singularity - 6.3.3

When we only have the assumption of $l_2 = l_3 = 1$, our Groebner basis was the following:

$$c_1 - \frac{b}{(a^2+b^2)}s_2 - \frac{a}{2},$$

$$s_1 + \frac{a}{(a^2+b^2)}s_2 - \frac{b}{2},$$

$$c_2 - \frac{a^2+b^2-2}{2},$$

$$s_2^2 + \frac{(a^2+b^2)(a^2+b^2-4)}{4}$$

When we extend our assumption with the hand position at $(0,0)$ for (a,b), our Groebner basis become:

$$c_1^2 + s_1^2 - 1$$

$$c_2 + 1$$

$$s_2$$

Kinematic singularity can occur with our robot's arms are aligned in space. In our case, to reach $(0,0)$, the only way is allowing $\theta_2 = \pi$, which we can see from second set of equations, where

$$c_2 = cos^{-1}(-1) = \pi$$

And θ_1 have the property of satisfying this equation $cos^2(\theta) + sin^2(\theta) = 1$. This equation will be satisfied with any θ, which gives an infinite amount of solutions for θ_1 and thus infinite amount of joint positions to reach $(0,0)$. This is one of the properties of kinematic singularity, which can cause infinite inverse kinematic solutions to exist.

The second set of equations differs from the first set in the case that the kinematic singularity is revealed differently. In the second case, the infinite amount of possible solutions indicate the presence of an internal singularity. In the first case, our first two equations would be undefined if we assume the hand position is $(0,0)$, since we would have 0 as denominators. This indicates that our robot cannot reach point $(0,0)$. However, our robot can, just with infinite different ways. This is the ultimately the property of kinematic singularity.
Furthermore, one observation is that our Groebner basis solution in the first case indicates that s_2 has degree of 2, implies 2 possible solutions. Whereas in the second case, we have s_2 with degree 1. This inconsistency in the same solution is also caused by the kinematic singularity.

2.4 Specialization Problem - 6.3.4

- a) If we substitute $l_2 = 1, l_3 = 2$ into (2), we have the following:

$$c_1 - \frac{4b}{2(a^2+b^2)}s_2 - \frac{a(a^2+b^2-3)}{2(a^2+b^2)},$$

$$s_1 + \frac{4a}{2(a^2+b^2)}s_2 - \frac{b(a^2+b^2-3)}{2(a^2+b^2)},$$

$$c_2 - \frac{a^2+b^2-5}{4},$$

$$s_2^2 + \frac{(a^2+b^2)^2 - 10(a^2+b^2) + 9}{8}$$

In this case we still consider the case where $a^2 + b^2 \neq 0$. If we want to solve the last equation in the above solution set, we get

$$s_2 = \pm\frac{\sqrt{-2(a^2+b^2-1)(a^2+b^2-9)}}{4}$$

This implies that the solutions for this configuration are real if and only if $1 \leq a^2+b^2 \leq 9$. When we have the hand position at (a, b) where a^2+b^2 is either 1 or 9, we have only a unique solution. For hand positions where $l_2 + l_3 = 3$, we will set $\theta_2 = 0$, where the arms will fully extend to the furthest it can reach. For hand positions where $l_2+l_3 = 1$, the first arm must move away from the position we want and second arm, having set $\theta_2 = \pi$, will bend back and reach the point we want. For example, if we want to each the point $(1, 0)$, θ_1 must be π, namely away from the point, and with second arm having $\theta_2 = \pi$ again, we bend back and reaching the point $(1, 0)$. Also, in these two cases, there is a double root for s_2. In addition to this, one key difference between this and setting $l_2 = l_3 = 1$ is that we do not have the kinematic singularity at $(0, 0)$ where infinite number of solutions exists since $(0, 0)$ is no longer reachable, also there will be a concentric circle with radius 1 in the configuration $l_2 = 1, l_3 = 2$ that is unreachable. So, to sum up, in this configuration, we can see that we have two possible lengths that the hand can be from the origin where a unique solution exist, these are whenever the hand is directly 1 or 9 away from the origin. Also, whenever the hand is in the concentric circle of radius 1 centered at the origin, the robot cannot reach the point. If the hand is at (a, b), where $1 < a^2 + b^2 < 9$, we will have two possible solutions.

- b) If we substitute $l_2 = 2, l_3 = 1$ into (2), we have the following:

$$c_1 - \frac{4b}{4(a^2+b^2)}s_2 - \frac{a(a^2+b^2-3)}{4(a^2+b^2)},$$

$$s_1 + \frac{4a}{4(a^2+b^2)}s_2 - \frac{b(a^2+b^2-3)}{4(a^2+b^2)},$$

$$c_2 - \frac{a^2+b^2-5}{4},$$

$$s_2^2 + \frac{(a^2+b^2)^2 - 10(a^2+b^2) + 9}{8}$$

When solving for s_2, we yielded a similar equation:

$$s_2 = \pm \frac{\sqrt{-(a^2+b^2-1)(a^2+b^2-9)}}{4}$$

We still have the similar constraints where the solutions of the robot arms are only real if and only if $1 \leq a^2 + b^2 \leq 9$. There will still be a unique solution for hand position where $\sqrt{a^2+b^2} = 3$, we will have $\theta_2 = 0$. When we want to reach the points along the concentric circle centered at origin and having radius of 1, we will use the similar bending idea from part a, however, now the first arm should move towards the point instead of away form the point in the previous configuration. For example, to reach $(1,0)$, arm 1 will have $\theta_1 = 0$ that moves along the trajectory of the intended position. Then, second arm will have $\theta_2 = \pi$, bending it back to reach the point $(1,0)$. In comparison to the previous configuration, our robot cannot reach points where $\sqrt{a^2+b^2} < 1$ anymore, and thus also not having $(0,0)$ as a kinematic singularity. Also, if the hand is at (a,b), where $1 < a^2+b^2 < 9$, we will have two possible solutions.

2.5 Forward Kinematics Problem

Finally, if $l_1 = 1$, $l_2 = 2$, $l_3 = 3$, $\theta_1 = \pi/3$, and $\theta_2 = 3\pi/2$, give (a_i, b_i) where each (a_i, b_i) represents the coordinate of the hand position in $x_i y_i$-coordinates.
We can assume that at $(x_4, y_4) = (0,0)$, since the hand can only rotate at this point. For each of the following (x_i, y_i) we will apply the affine transformation matrix formula:

$$A_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & l_i \\ \sin(\theta_i) & \cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For (x_3, y_3), since the angle between x_3, x_4 axis is $0°$,our transformation matrix is

$$\begin{bmatrix} \cos(0) & -\sin(0) & 3 \\ \sin(0) & \cos(0) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we multiply our transformation matrix by our position vector $(x_4, y_4, 1)$, we get the new position

$$\begin{bmatrix} \cos(0) & -\sin(0) & 3 \\ \sin(0) & \cos(0) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

This shows that in the x_3, y_3 system, the hand is at position $(3,0)$. For x_2, y_2 system, we apply the same formula with transformation matrix being:

$$\begin{bmatrix} \cos(\frac{3\pi}{2}) & -\sin(\frac{3\pi}{2}) & 2 \\ \sin(\frac{3\pi}{2}) & \cos(\frac{3\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the new coordinate is:

$$\begin{bmatrix} \cos(\frac{3\pi}{2}) & -\sin(\frac{3\pi}{2}) & 2 \\ \sin(\frac{3\pi}{2}) & \cos(\frac{3\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

For x_2, y_2 system, the hand is at $(2, -3)$.
To reach the x_1, y_1 system, we iterate one more time with A_1 being:

$$\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) & 1 \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the new coordinate is:

$$\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) & 1 \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 + \frac{3\sqrt{3}}{2} \\ \sqrt{3} - \frac{3}{2} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 4.6 \\ 0.23 \\ 1 \end{bmatrix}$$

For x_1, y_1 system, the hand is at $(4.6, 0.23)$.

To sum up the methodology, start with the coordinate system of x_4, y_4 and with a particular affine transformation matrix, we can transform our coordinate system back to x_{i-1}, y_{i-1} with rotating by θ_i degrees and translating with l_i length. Eventually, to the global coordinate. In our case, we are considering the global coordinate system to be reached after 3 transformations back to the base.
The above calculations were carried out in Maple, showing the example for calculating the last part:

$$A1 := \begin{bmatrix} \cos(\text{theta}) & -\sin(\text{theta}) & 1 \\ \sin(\text{theta}) & \cos(\text{theta}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A1 := \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A1 . \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 + \frac{3\sqrt{3}}{2} \\ \sqrt{3} - \frac{3}{2} \\ 1 \end{bmatrix}$$

3 Example Research Paper on Robotic Arms

3.1 *"Robotic Arm Movement Optimization Using Soft Computing"*

by Surender Kumar, Kavita Rani, V.K.Banga

This paper was published in 2017, it discussed some of the optimization problems in robotic arm movement, such as optimal control for movement and trajectory planning using soft computing techniques. The paper starts with a basic introduction of robotic arm, very similar to chapter 6 of IVA, with less constraints than our example. They considered robotic arms that can have rotary joints that can have rotational movements up to 360 degrees. Although we studied the idea of a linear translational joint, we did not take it into consideration of our robot design. Their design include both rotational and linear translational joints in 3D.

They approached the inverse kinematics problem in a similar manner and a higher degree of accuracy. The paper pointed out that when using computer to solve trigonometric functions, such as arccos, the results can be very inaccurate for small angles, they proposed to further convert the trig functions down to the 2-argument arctangent functions with 2 variables x, y. This also introduced new constraints ranges for the function such as $(x > 0), (x < 0, y \geq 0), (x = 0, y > 0) \ldots$.

In computing the reverse kinematic problem, the paper uses the solution of denavit-hartenberg representation, which looks very similar to our transformation matrices but with more dimensions in the matrix. The computing work of their solution was not shown in full detail, but the end results looks like a simplified results using the technique of Groebner basis to solve a system of equations with multiple unknowns.

With the terminologies explained in chapter 6.1-6.3, I think the content of this paper is on the similar level, if not higher, than our book. Also, they use slightly different math ideas to solve kinematic problems.

The high level conceptual ideas about optimization using soft computing might be a far reach for me at the moment, as it would include topics from machine learning and neural networks. The applications of this paper should be aiming for robotic arm movements problems in highly unpredictable or autonomous environment.

Below are page 7 and 9 of the paper illustrating the way the paper models the robotic arm and the way they define their affine transformation matrices A_i respectively.

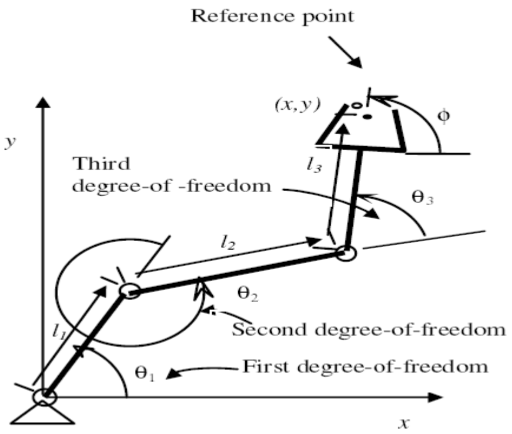


Figure 6. Three link robotic arm [20]

4.1. Modeling of Robotics Arm

The inverse kinematics problem is much more interesting and its solution is more useful. “Given the desired position of the robot's hand, what must be the angles at all of the robots joints?” Humans solve this problem all the time without even thinking about it. How most robots have to solve the problem.

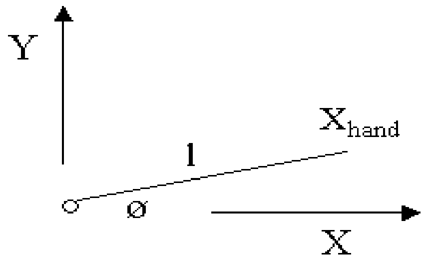


Figure 7. Single link manipulator [20]

The Figure 7 above is a schematic of a simple robot lying in the X-Y plane. The robot has one link of length l and one joint with angle Ø. The position of the robot's hand is Xhand. The inverse kinematics problem (at the position level) for this robots as follows: Given Xhand what is the joint angle Ø? We'll start the solution to this problem by writing down the forward position equation, and then solve for Ø.

Xhand=lcosØ (forward position solution)

CosØ=Xhand / l

Ø=cos-1(Xhand/l)

To finish the solution let's say that this robot's link has a length of 1 foot and we want the robot's hand to be at X= .7071 feet. That gives:

Ø=cos-1(.7071)=+/- 45 degrees

There are two solutions to the inverse kinematics problem: one at plus 45 degrees and one at minus 45 degrees! The existence of multiple solutions adds to the challenge of the inverse kinematics problem. Typically we will need to know which of the solutions is correct. All programming languages that I know of supply a trigonometric function called A Tan 2 that will find the proper quadrant when given both the X and Y arguments: Ø=A Tan 2(Y/X). There is one more interesting inverse kinematics problem. Two link manipulator as shown in Figure 8.

$$T_e = R_Z(\Theta_i)\, D_Z(d_i)\, D_x(a_i)\, R_x(\alpha_i) \quad (28)$$

$$\begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & \alpha_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & \alpha_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (29)$$

where R_X and R_Z present rotation, D_X and D_Z denote translation, and $C\theta_i$ and $S\theta_i$ are the short hands of $\cos\theta_i$ and $\sin\theta_i$, respectively. The forward kinematics of the end-effectors with respect to the base frame is determined by multiplying all of the T_5 matrices. The four quantities θ_i, a_i, d_i, α_i are parameters associated with link i and joint i. The four parameters a_i, α_i, d_i, and θ_i are generally given the names link length, link twist, link offset, and joint angle respectively.

An alternative representation of T_e can be written as (30):

$$T_e = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (30)$$

where r_{kj}'s represent the rotational elements of transformation matrix (k and j=1, 2 and 3). p_x, p_y, and p_z denote the elements of the position vector. For a five robotic arm, the position and orientation of the end-effector with respect to the base is given by (31-32)

$${}^0T_5 = {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5 \quad (31)$$

$${}^0T_1 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2T_3 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3 * C_3 \\ S_3 & C_3 & 0 & a_3 * C_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3T_4 = \begin{bmatrix} C_4 & -S_4 & 0 & a_4 * C_4 \\ S_4 & C_4 & 0 & a_4 * C_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^4T_5 = \begin{bmatrix} C_5 & 0 & 0 & 0 \\ S_5 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_5 = {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5$$

3.2 *"A kinematic analysis of the gmf a-510 robot: An introduction and application of Groebner basis theory"*

by Kimberly D. Kendricks

This paper was published in 2013, it discussed the geometric and algebraic modeling of the gmf a-510 robot, and using a similar program to Maple, MAGMA, it explores some of the difficulties when modeling the robot's kinematics without the help of a computer program.

The robot discussed in this paper is more complex than the robot demonstrated in this project. Not only it is in 3D with 3 separate revolute joints having a total rotation of only 300°, namely, these joints can only rotate in the range of $-150°, 150°$. In addition to this, the base of a-510 has a total rotation of 540°, but it is ignored for the simplicity of calculations.

The notations of this paper, maybe because it is also an introductory paper, is very similar to the notations used in chapter 6 of IVA. It starts off with a in depth description of the kinematic model of a-510 with concepts such as Denavit-Hartenberg matrices as their affine transformation matrix. Although the math is done mostly by hand at this stage, it is not difficult to follow. The only difficulty is to understand how the revolute joints' 300° total rotation will affect the possible solutions. In addition, they also discussed the number of possible solutions in different cases, because it a spatial robot instead of planar, the number of solutions might be none, one, two, three, ... , etc. The fact that their robot has very specific restrictions, there needed to be case studies in order to analyze the geometric nature of the edge cases, namely the cases where absurd or potential erroneous cases.

Here was the natural flow where the paper introduced Groebner basis theory to help with the computations during the case studies, and they were very easy to follow. The acknowledgement of the simplification that Groebner basis theory made to their algebraic model was align with chapter 6 of IVA.

Overall, this paper read like a natural extension to the problems of chapter 6.1-6.3 of IVA, where more constraints were introduced and we have to consider rotational joints in 3D.

Below are page 18 and 19 of the paper illustrating the working of their computation when utilizing Groebner basis theory to analyze the kinematic model of a-510.

By examining the discriminant, there are two real solutions for s_i, denoted s_{11}, and s_{12}, and for each, a corresponding value for c_1, s_2, c_2, found by using back substitution. Furthermore, we conclude that there are two unique joint settings when

$$0 \leq 4(a^2+b^2)^2 + \frac{3504640000 - 2a^2 * 27700 + 2b^2 * 59200}{1640(a^2+b^2)}$$
$$\leq \frac{a^2b + b^3 + 592000}{24272000^2}$$

Now we can consider two remaining cases:

Case 1: $a = 0$ and $b = 0$

The Groebner basis for this case is 1. Thus, there are no joint settings to place the robot arm at the point (a, b). It is obvious to see by looking at the geometry of the robot arm.

Case 2: $a = 0$ and $b \neq 0$

The Groebner basis for this case is

$$c_3 + \frac{L_3^2 + L_1^2 - b^2}{2L_3 * L_1}$$

$$s_3 - \frac{b}{L_3}c_1$$

$$c_1^2 + \frac{(L_3^4 + L_1^4 + b^4) - 2(L_3^2 + L_1^2 + b^2(L_3^2 + L_1^2))}{4L_1^2 b^2}$$

$$s_1 + \frac{L_3^2 - L_1^2 - b^2}{2L_1 b}$$

So, we have a Groebner basis when $L_1, L_3, b \neq 0$. We can immediately solve for s_1 and c_3. Solving for these aligns each joint link pair. Note that the third element in the basis is a quadratic polynomial in terms of c_1. By solving for c_1, we nd two solutions c_{11} and c_{12}, and for each, there is a corresponding value for s_3. This means that we can rotate about θ_1, given those values of s_3 that will keep each joint link pair collinear. Thus, there is only one unique joint setting to place the robot arm at the point (a, b) when $a = 0$ and $b \neq 0$.

4.3. *Discussion of groebner basis theory*

By analyzing the above, we nd the following results:

(1) There are at most two real solutions (i.e., two joint angle con gurations) when (a,b) satis es

$$0 \leq \frac{4(a^2+b^2)^2 + 3504640000 - 554000a^2 + 118400b^2}{1640(a^2+b^2)} \leq \left(\frac{a^2b+b^3+59200}{24272000}\right)^2.$$

(2) From Case 1, no solutions when $a = b = 0$.

(3) From Case 2, one real solution (i.e., one joint angle con guration) when $(0,b)$ satis es

$$0 \leq b\sqrt{554000 - b^2} \leq 3504640000.$$

(4) Those points (a,b) that do not satisfy any of the above are outside of the robot's reachable workspace. These points represent no solutions.

By using Groebner Basis Theory and MAGMA, we have found real solutions (see above) to the inverse kinematics robotics problem. In fact, we have found all of the possible formations to place the robot hand at the point (a,b). These solutions are more precise because we determine the set of points that have two solutions, the set of points that have one solution, and the set of points that have no solution. Thus, the inverse kinematics robotics problem is solved. We explain our results further in the next section.

5. Comparative analysis

When using the Denavit-Hartenberg Matrix, we were only able to develop an equation that could nd two solutions for θ_1 and θ_3 within the given domain: $\frac{-\pi}{2} \leq \theta_1 \leq \frac{\pi}{2}$ and $\frac{-\pi}{2} \leq \theta_3 \leq \frac{\pi}{2}$. Since θ_1 and θ_3 rotated beyond $\frac{-\pi}{2}$ to $\frac{\pi}{2}$, there was the possibility that more solutions existed. By manipulating the equations found using the Denavit-Hartenberg Matrix, we expanded our domain for θ_3 from 0 to π, allowing to rotate

4 Conclusion

It is important to have a solid understanding of the connection between math and computers, and allowing computational algebraic tools to enable us to solve engineering problems much easier. If we were to approach the problems of finding the forward/reverse kinematic problem of our robotic arms by hand, it would involve huge amount of arduous algebraic calculation and prone to errors. With the help of Groebner basis theory, we can quickly solve a specialized form of the problem and gain some knowledge of where this can lead us through the lens of industry research papers on this topic.

Although the problems explored within this project are elementary compare to the industry problems, they serve as the foundation for understanding this engineering problem with more constraints and complexity, also allows the more specialized robotic analysis to feel as a natural extension of this project.