Probability and Statistics

Module 1: Topic 4
Probability: Information content in data

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Outline

- Information of an event
- Information entropy ✓
- Mutual information ✓

Information theory

- It is a field of study concerned with quantifying information for communication
- A foundational concept from information is the quantification of the amount of information in things like events, random variables, and distributions
- Use the concepts from probability

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Information theory

- The sky colour is blue ✓
- The sky colour is red

Second statement provides more info.
(and indicates that something is abnormal)

- Person A always goes for jogging at 6 PM ✓
- Person B goes after the sunset ✓

This provides more information
– Season. (predict) !

Information of an event

How likely?
→ Define this in terms of a probability law.

- Quantifying information is the idea of measuring how much surprise there is in an event
- *"The basic intuition behind information theory is that learning that an unlikely event has occurred is more informative than learning that a likely event has occurred."*
- Low probability (rare) events: more surprising: **high** information
- High probability (common) events: less surprising: **low** information
- Shannon information or information h(x):

$$h(x) = -\log_2(p(x)) \quad bits$$

$[0, \infty)$ non-negative. $[0, 1]$

If $P(x) = 0$, $h(x) = \infty$

$P(x) = 1$, $h(x) = 0$

$\log_e$ / $\log_{10}$

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Information of an event

0, 1

Example 1: If p(x)=1 Certain event.

$$h(x) = -\,log_2(1) = 0 \quad bits$$

Example 2: coin toss (Fair)

Event (E): obtain a head

P(E) = 0.5 and $h(E) = -\,log_2(0.5) = 1.0 \quad bits$ → binary digits.

Example 3: Die roll (6-faced die, 6 outcomes)

Event (E): obtain a 6

P(E) = 1/6 and $h(E) = -\,log_2(1/6) = 2.585 \quad bits$

Die roll:

- possible outcomes

1 2 3 | $\boxed{4 \; \boxed{5} \; \boxed{6}}$

→ Naive way – 6 Qs.

1 Q.

- Divide and conquer.

(1) ≤ 3 ?

(2) ≤ 5 ?

} → 2 Q.

Information for an event

→ Outcome 6

$P(6) = 1/6$

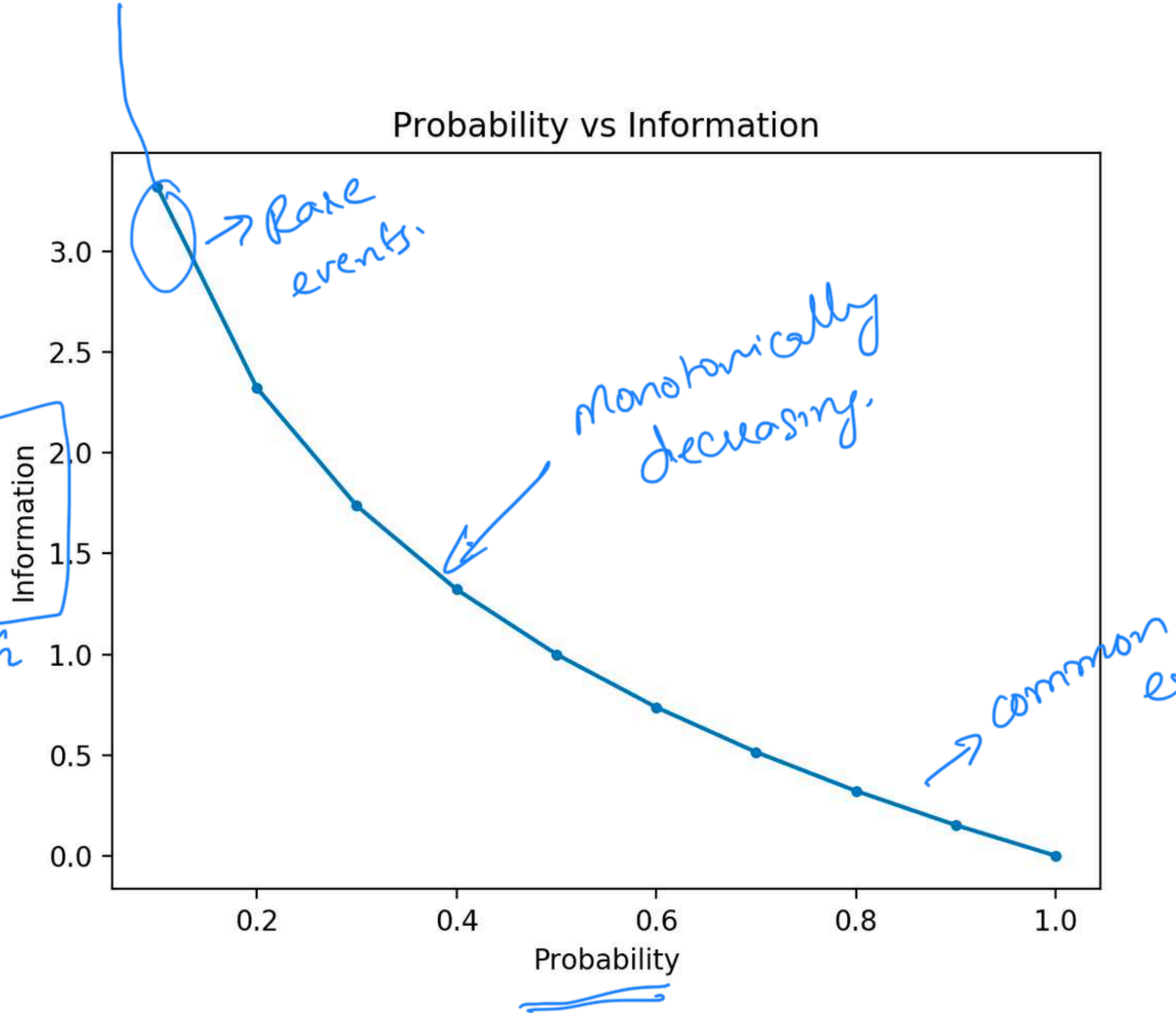
$h(x) = 2.585$

→ Outcome Even

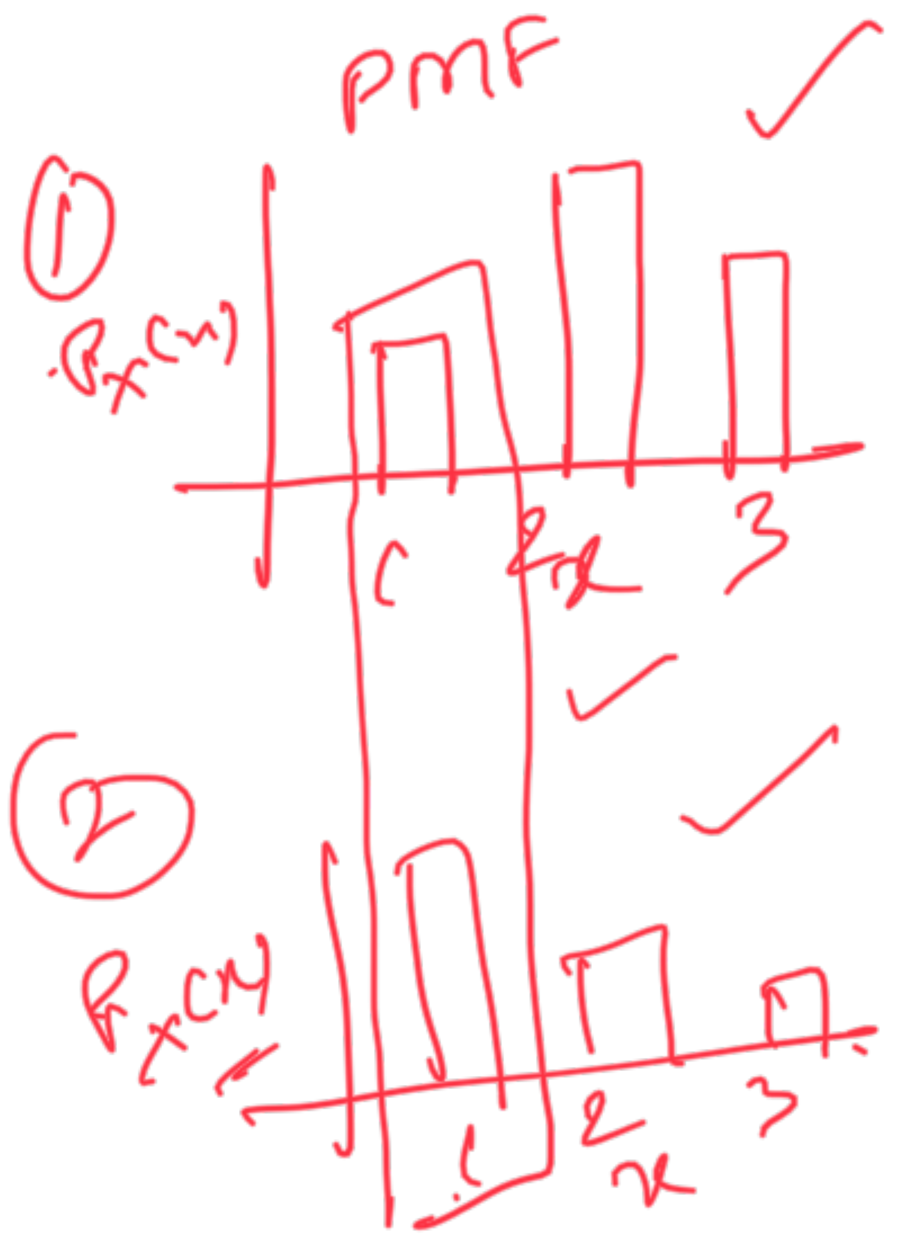
$P(\text{Even}) = \frac{1}{2}$

$h(\text{Even}) = -\log_2 \frac{1}{2}$

$= 1$



https://machinelearningmastery.com/what-is-information-entropy/



Information entropy

- Next we want to quantify how much information there is in a **random variable**
- Consider a random variable **X** with a probability distribution **P**
- Calculating information for a R.V. is called **information entropy or Shannon entropy** $\boxed{H(X)}$
- **Intuition:** it is the average number of bits required to represent or transmit an event drawn from the probability distribution for the R.V.
- If there are K discrete states (Cointoss → two discrete states)

$$H(X) = -\sum_{k=1}^{K} p(k) log_2(p(k)) \quad bits$$

Information entropy

Die roll

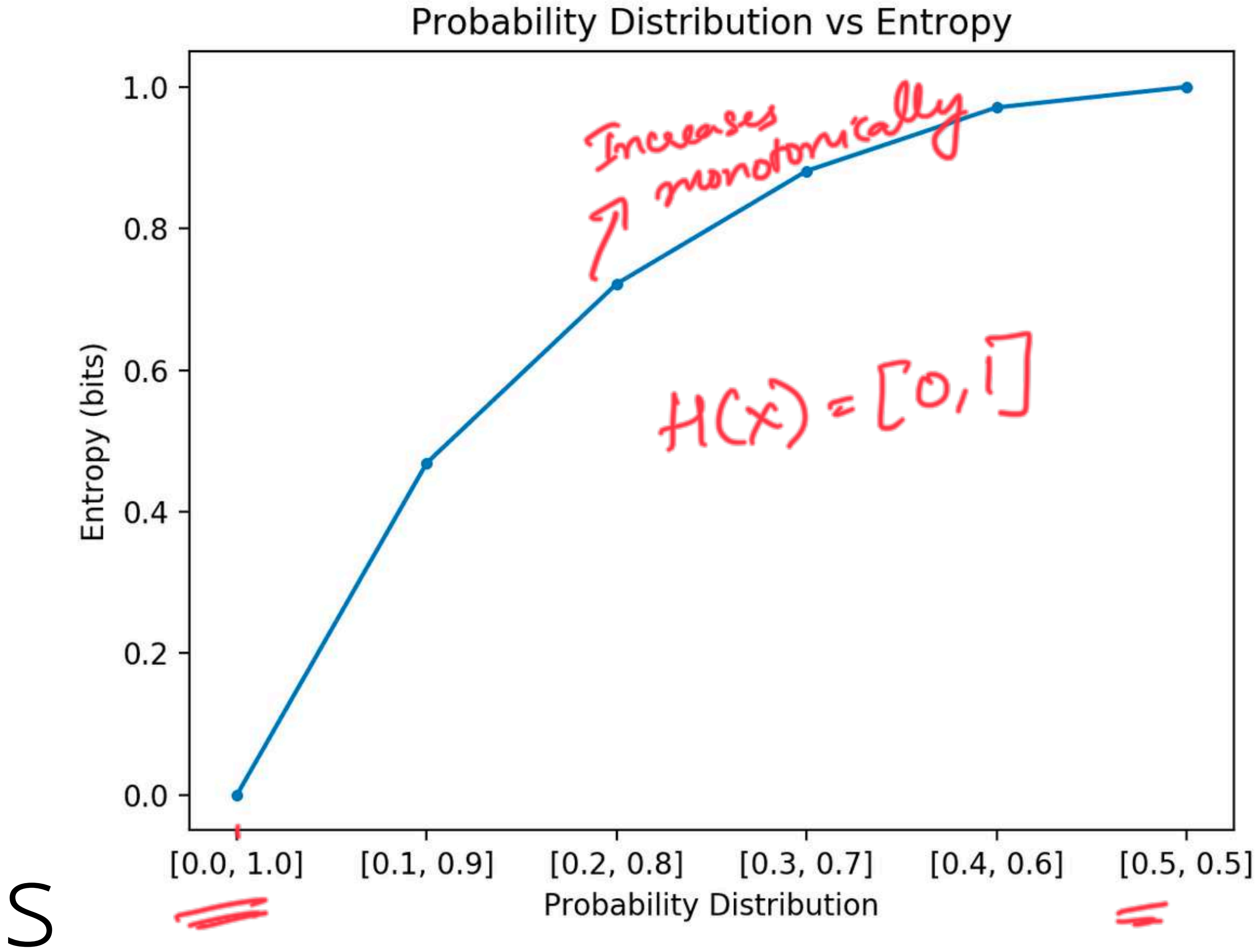
- Lowest entropy: R.V. that has a single event with a probability of 1.0
- Highest entropy: If all events are equally likely
- Example: Coin toss

$P(x=0) = 0.5$

$P(x=1) = 0.5$

  - Fair coin: X={0,1} —> no of heads

$$H(X) = -[0.5log_2(0.5) + 0.5log_2(0.5)] = 1.0 \quad bits$$



$P(x=1) = 0.75$ $P(x=0) = 0.25$

  - Biased coin (0.75,0.25): X={0,1} —> no of heads

H T

$$H(X) = -[\underbrace{0.25log_2(0.25)}_{X=0} + \underbrace{0.75log_2(0.75)}_{X=1}] = 0.8115 \quad bits$$

Skewed probability distribution: low entropy

Balanced probability distribution: high entropy

Information entropy

- Consider two random variables X and Y with a PMF $P_{X,Y}(x, y)$
- Joint entropy:

$$H(X, Y) = -\sum_{x,y} P_{X,Y}(x, y) log_2(P_{X,Y}(x, y)) \quad bits$$

Conditional entropy:

$$H(X|Y) = -\sum_{x,y} P_{X,Y}(x, y) log_2(P_{X,Y}(x|y)) \quad bits$$

The conditional entropy is a measure of how much uncertainty remains about the random variable X when we know the value of Y

Mutual information

- Entropy provides the basis for calculating the difference between two probability distributions
- Mutual information is calculated between two variables and measures the reduction in uncertainty for one variable given a known value of the other variable → Condition Y
- The mutual information between two random variables *X* and *Y*:

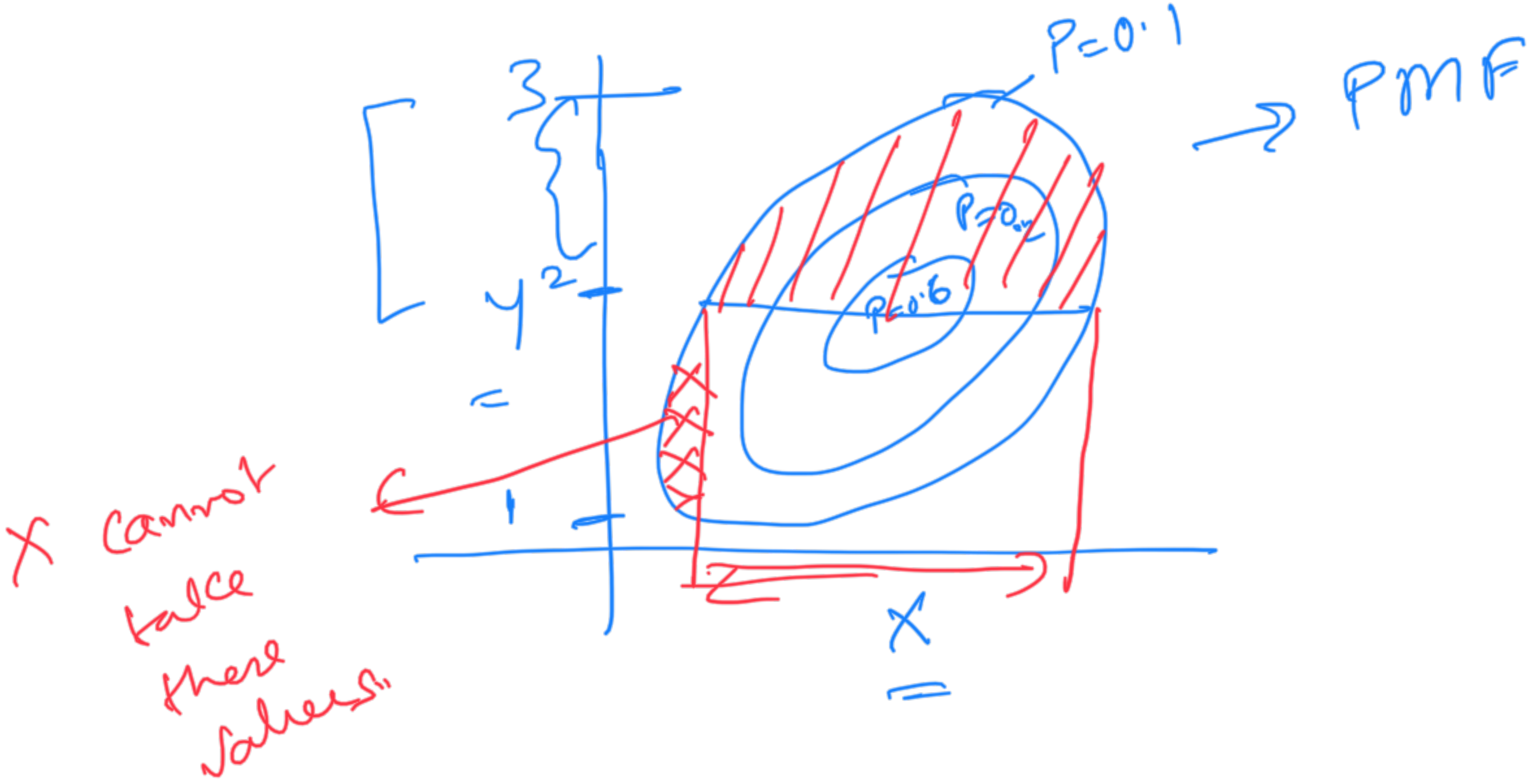
R.V.

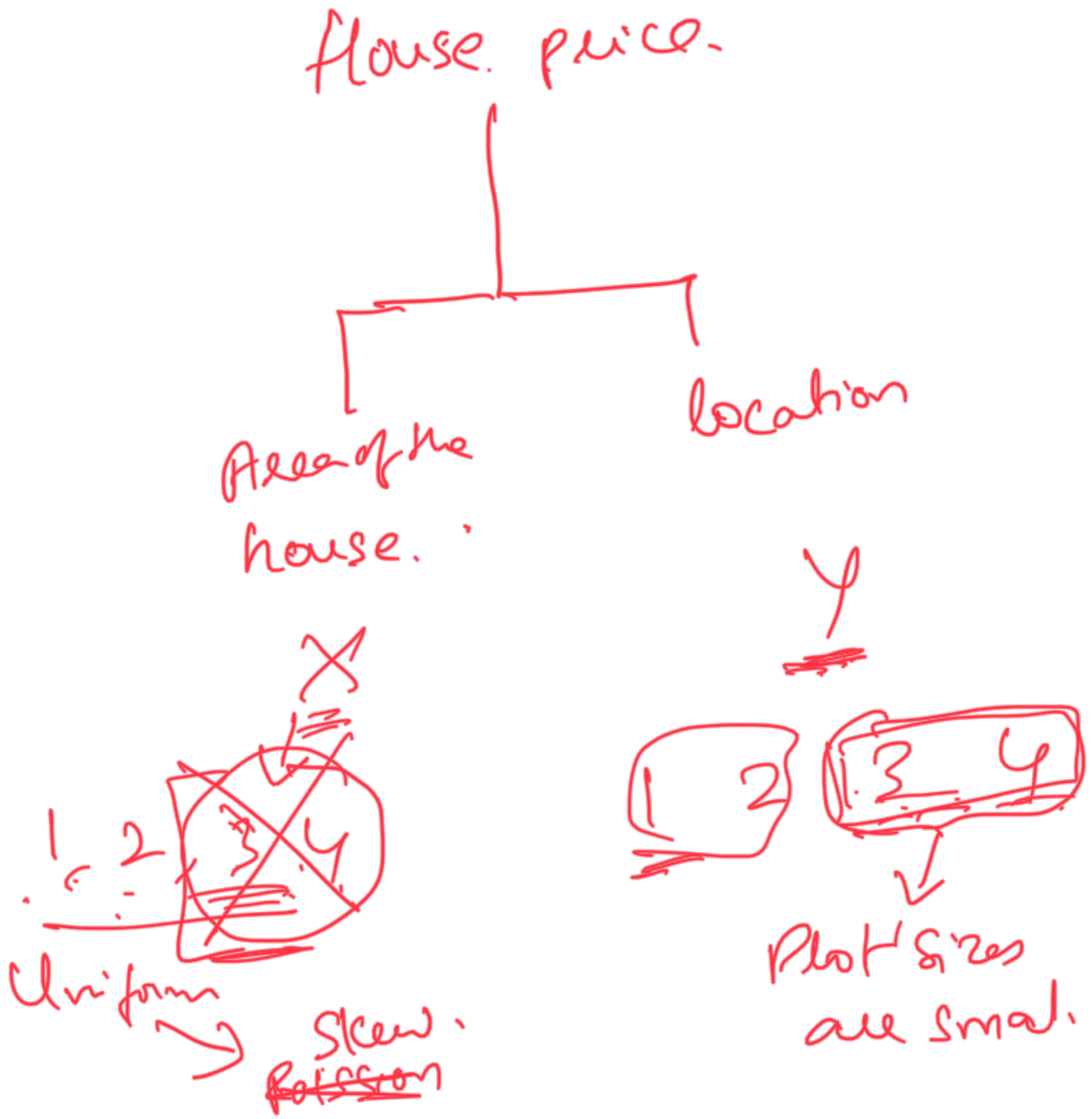
$$I(X;Y) = H(X) - H(X|Y)$$

→ H(X)

→ Conditional entropy

Independent $I(X:Y) = 0$


P=0.1
PMF
P=0.6
3
Y
2
1
X
X cannot take these values.


House price.
Area of the house.
location
X
Y
1 2 3 4
Uniform
Skew.
~~Poisson~~
1 2 3 4
Plot sizes are small.