Answers to Odd-Numbered Exercises

Chapter 1

Section 1.1, page 10

1. The solution is $(x_1, x_2) = (-8, 3)$, or simply $(-8, 3)$.

3. $(4/7, 9/7)$

5. Replace row 2 by its sum with 3 times row 3, and then replace row 1 by its sum with -5 times row 3.

7. The solution set is empty.

9. $(4, 8, 5, 2)$ **11.** Inconsistent

13. $(5, 3, -1)$ **15.** Consistent

17. The three lines have one point in common.

19. $h \neq 2$ **21.** All h

23. Mark a statement True only if the statement is *always* true. Giving you the answers here would defeat the purpose of the true–false questions, which is to help you learn to read the text carefully. The *Study Guide* will tell you where to look for the answers, but you should not consult it until you have made an honest attempt to find the answers yourself.

25. $k + 2g + h = 0$

27. The row reduction of $\begin{bmatrix} 1 & 3 & f \\ c & d & g \end{bmatrix}$ to $\begin{bmatrix} 1 & 3 & f \\ 0 & d-3c & g-cf \end{bmatrix}$ shows that $d - 3c$ must be nonzero, since f and g are arbitrary. Otherwise, for some choices of f and g the second row could correspond to an equation of the form $0 = b$, where b is nonzero. Thus $d \neq 3c$.

29. Swap row 1 and row 2; swap row 1 and row 2.

31. Replace row 3 by row 3 + (-4) row 1; replace row 3 by row 3 + (4) row 1.

33. $$\begin{aligned} 4T_1 - T_2 - T_4 &= 30 \\ -T_1 + 4T_2 - T_3 &= 60 \\ -T_2 + 4T_3 - T_4 &= 70 \\ -T_1 - T_3 + 4T_4 &= 40 \end{aligned}$$

Section 1.2, page 21

1. Reduced echelon form: a and b. Echelon form: d. Not echelon: c.

3. $\begin{bmatrix} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Pivot cols 1 and 2: $\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 4 & \boxed{5} & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$.

5. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$, $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$

7. $\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$ **9.** $\begin{cases} x_1 = 4 + 5x_3 \\ x_2 = 5 + 6x_3 \\ x_3 \text{ is free} \end{cases}$

11. $\begin{cases} x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

13. $\begin{cases} x_1 = 5 + 3x_5 \\ x_2 = 1 + 4x_5 \\ x_3 \text{ is free} \\ x_4 = 4 - 9x_5 \\ x_5 \text{ is free} \end{cases}$

Note: The *Study Guide* discusses the common mistake $x_3 = 0$.

15. **a.** Consistent, with a unique solution
b. Inconsistent

17. $h = 7/2$

19. **a.** Inconsistent when $h = 2$ and $k \neq 8$
b. A unique solution when $h \neq 2$
c. Many solutions when $h = 2$ and $k = 8$

21. Read the text carefully, and write your answers before you consult the *Study Guide*. Remember, a statement is true only if it is true in all cases.

23. Yes. The system is consistent because with three pivots, there must be a pivot in the third (bottom) row of the coefficient matrix. The reduced echelon form cannot contain a row of the form $[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$.

25. If the coefficient matrix has a pivot position in every row, then there is a pivot position in the bottom row, and there is no room for a pivot in the augmented column. So, the system is consistent, by Theorem 2.

27. If a linear system is consistent, then the solution is unique if and only if *every column in the coefficient matrix is a pivot column; otherwise, there are infinitely many solutions*.

29. An underdetermined system always has more variables than equations. There cannot be more basic variables than there are equations, so there must be at least one free variable. Such a variable may be assigned infinitely many different values. If the system is consistent, each different value of a free variable will produce a different solution.

31. Yes, a system of linear equations with more equations than unknowns can be consistent. The following system has a solution ($x_1 = x_2 = 1$):

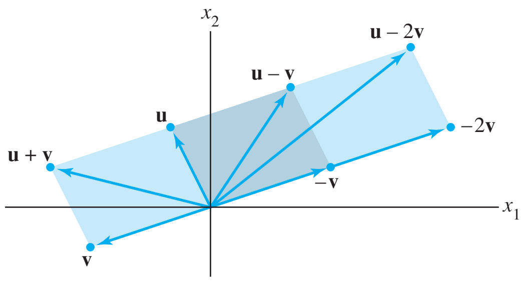
$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 0 \\ 3x_1 + 2x_2 &= 5 \end{aligned}$$

33. [**M**] $p(t) = 7 + 6t - t^2$

Section 1.3, page 32

1. $\begin{bmatrix} -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

3.



5. $x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$,

$$\begin{bmatrix} 6x_1 \\ -x_1 \\ 5x_1 \end{bmatrix} + \begin{bmatrix} -3x_2 \\ 4x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}, \begin{bmatrix} 6x_1 - 3x_2 \\ -x_1 + 4x_2 \\ 5x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$$

$$\begin{aligned} 6x_1 - 3x_2 &= 1 \\ -x_1 + 4x_2 &= -7 \\ 5x_1 &= -5 \end{aligned}$$

Usually the intermediate steps are not displayed.

7. $\mathbf{a} = \mathbf{u} - 2\mathbf{v}$, $\mathbf{b} = 2\mathbf{u} - 2\mathbf{v}$, $\mathbf{c} = 2\mathbf{u} - 3.5\mathbf{v}$, $\mathbf{d} = 3\mathbf{u} - 4\mathbf{v}$

9. $x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

11. Yes, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

13. No, $\mathbf{b}$ is *not* a linear combination of the columns of A.

15. Noninteger weights are acceptable, of course, but some simple choices are $0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 = \mathbf{0}$, and

$$1 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}, 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

$$1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, 1 \cdot \mathbf{v}_1 - 1 \cdot \mathbf{v}_2 = \begin{bmatrix} 12 \\ -2 \\ -6 \end{bmatrix}$$

17. $h = -17$

19. Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of points on the line through $\mathbf{v}_1$ and $\mathbf{0}$.

21. *Hint:* Show that $\begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix}$ is consistent for all h and k. Explain what this calculation shows about Span $\{\mathbf{u}, \mathbf{v}\}$.

23. Before you consult your *Study Guide*, read the entire section carefully. Pay special attention to definitions and theorem statements, and note any remarks that precede or follow them.

25. **a.** No, three **b.** Yes, infinitely many

c. $\mathbf{a}_1 = 1 \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + 0 \cdot \mathbf{a}_3$

27. **a.** $5\mathbf{v}_1$ is the output of 5 day's operation of mine #1.

b. The total output is $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$, so x_1 and x_2 should satisfy $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$.

c. [**M**] 1.5 days for mine #1 and 4 days for mine #2

29. $(1.3, .9, 0)$

31. **a.** $\begin{bmatrix} 10/3 \\ 2 \end{bmatrix}$

b. Add 3.5 g at $(0, 1)$, add .5 g at $(8, 1)$, and add 2 g at $(2, 4)$.

33. Review Practice Problem 1 and then *write* a solution. The *Study Guide* has a solution.

Section 1.4, page 40

1. The product is not defined because the number of columns (2) in the 3×2 matrix does not match the number of entries (3) in the vector.

3. $A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2 \cdot \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} - 3 \cdot \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 12 \\ -8 \\ 14 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$, and

$A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \cdot 2 + 5 \cdot (-3) \\ (-4) \cdot 2 + (-3) \cdot (-3) \\ 7 \cdot 2 + 6 \cdot (-3) \end{bmatrix}$

$= \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$. Show your work here and for Exercises 4–6, but thereafter perform the calculations mentally.

5. $5\cdot\begin{bmatrix}5\\-2\end{bmatrix}-1\cdot\begin{bmatrix}1\\-7\end{bmatrix}+3\cdot\begin{bmatrix}-8\\3\end{bmatrix}-2\cdot\begin{bmatrix}4\\-5\end{bmatrix}=\begin{bmatrix}-8\\16\end{bmatrix}$

7. $\begin{bmatrix}4&-5&7\\-1&3&-8\\7&-5&0\\-4&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}6\\-8\\0\\-7\end{bmatrix}$

9. $x_1\begin{bmatrix}3\\0\end{bmatrix}+x_2\begin{bmatrix}1\\1\end{bmatrix}+x_3\begin{bmatrix}-5\\4\end{bmatrix}=\begin{bmatrix}9\\0\end{bmatrix}$ and

$$\begin{bmatrix}3&1&-5\\0&1&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}9\\0\end{bmatrix}$$

11. $\begin{bmatrix}1&2&4&-2\\0&1&5&2\\-2&-4&-3&9\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\-3\\1\end{bmatrix}$

13. Yes. (Justify your answer.)

u are here

15. The equation $A\mathbf{x}=\mathbf{b}$ is not consistent when $3b_1+b_2$ is nonzero. (Show your work.) The set of $\mathbf{b}$ for which the equation *is* consistent is a line through the origin—the set of all points (b_1,b_2) satisfying $b_2=-3b_1$.

17. Only three rows contain a pivot position. The equation $A\mathbf{x}=\mathbf{b}$ does *not* have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$, by Theorem 4.

19. The work in Exercise 17 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbb{R}^4$ can be written as a linear combination of the columns of A. Also, the columns of A do *not* span $\mathbb{R}^4$.

21. The matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ does not have a pivot in each row, so the columns of the matrix do not span $\mathbb{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ does not span $\mathbb{R}^4$.

23. Read the text carefully and try to mark each exercise statement True or False before you consult the *Study Guide*. Several parts of Exercises 23 and 24 are *implications* of the form

"If ⟨statement 1⟩, then ⟨statement 2⟩"

or equivalently,

"⟨statement 2⟩, if ⟨statement 1⟩"

Mark such an implication as True if ⟨statement 2⟩ is true in all cases when ⟨statement 1⟩ is true.

25. $c_1=-3, c_2=-1, c_3=2$

27. $Q\mathbf{x}=\mathbf{v}$, where $Q=[\mathbf{q}_1\ \ \mathbf{q}_2\ \ \mathbf{q}_3]$ and $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

Note: If your answer is the equation $A\mathbf{x}=\mathbf{b}$, you must specify what A and $\mathbf{b}$ are.

29. *Hint:* Start with any 3×3 matrix B in echelon form that has three pivot positions.

31. *Write* your solution before you check the *Study Guide*.

33. *Hint:* How many pivot columns does A have? Why?

35. Given $A\mathbf{x}_1=\mathbf{y}_1$ and $A\mathbf{x}_2=\mathbf{y}_2$, you are asked to show that the equation $A\mathbf{x}=\mathbf{w}$ has a solution, where $\mathbf{w}=\mathbf{y}_1+\mathbf{y}_2$. Observe that $\mathbf{w}=A\mathbf{x}_1+A\mathbf{x}_2$ and use Theorem 5(a) with $\mathbf{x}_1$ and $\mathbf{x}_2$ in place of $\mathbf{u}$ and $\mathbf{v}$, respectively. That is, $\mathbf{w}=A\mathbf{x}_1+A\mathbf{x}_2=A(\mathbf{x}_1+\mathbf{x}_2)$. So the vector $\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2$ is a solution of $\mathbf{w}=A\mathbf{x}$.

37. **[M]** The columns do not span $\mathbb{R}^4$.

39. **[M]** The columns span $\mathbb{R}^4$.

41. **[M]** Delete column 4 of the matrix in Exercise 39. It is also possible to delete column 3 instead of column 4.

Section 1.5, page 48

1. The system has a nontrivial solution because there is a free variable, x_3.

3. The system has a nontrivial solution because there is a free variable, x_3.

5. $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}5\\-2\\1\end{bmatrix}$

7. $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}-9\\4\\1\\0\end{bmatrix}+x_4\begin{bmatrix}8\\-5\\0\\1\end{bmatrix}$

9. $\mathbf{x}=x_2\begin{bmatrix}3\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-2\\0\\1\end{bmatrix}$

11. *Hint:* The system derived from the *reduced* echelon form is

$$\begin{aligned}x_1-4x_2\qquad\quad +5x_6&=0\\ x_3\qquad -x_6&=0\\ x_5-4x_6&=0\\ 0&=0\end{aligned}$$

The basic variables are x_1, x_3, and x_5. The remaining variables are free. The *Study Guide* discusses two mistakes that are often made on this type of problem.

13. $\mathbf{x}=\begin{bmatrix}5\\-2\\0\end{bmatrix}+x_3\begin{bmatrix}4\\-7\\1\end{bmatrix}=\mathbf{p}+x_3\mathbf{q}$. Geometrically, the solution set is the line through $\begin{bmatrix}5\\-2\\0\end{bmatrix}$ parallel to $\begin{bmatrix}4\\-7\\1\end{bmatrix}$.

15. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$. The solution set is the line through $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 5.

17. Let $\mathbf{u} = \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$. The solution of the homogeneous equation is $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$, the plane through the origin spanned by $\mathbf{u}$ and $\mathbf{v}$. The solution set of the nonhomogeneous system is $\mathbf{x} = \mathbf{p} + x_2\mathbf{u} + x_3\mathbf{v}$, the plane through $\mathbf{p}$ parallel to the solution set of the homogeneous equation.

19. $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, where t represents a parameter, or $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \end{bmatrix}$, or $\begin{cases} x_1 = -2 - 5t \\ x_2 = 3t \end{cases}$

21. $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} 2 \\ -5 \end{bmatrix} + t\begin{bmatrix} -5 \\ 6 \end{bmatrix}$

23. It is important to read the text carefully and write your answers. After that, check the *Study Guide*, if necessary.

25. $A\mathbf{v}_h = A(\mathbf{w} - \mathbf{p}) = A\mathbf{w} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$

27. When A is the 3×3 zero matrix, *every* $\mathbf{x}$ in $\mathbb{R}^3$ satisfies $A\mathbf{x} = \mathbf{0}$. So the solution set is all vectors in $\mathbb{R}^3$.

29. **a.** When A is a 3×3 matrix with three pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence has no nontrivial solution.

b. With three pivot positions, A has a pivot position in each of its three rows. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$. The word "possible" in the exercise means that the only vectors considered in this case are those in $\mathbb{R}^3$, because A has three rows.

31. **a.** When A is a 3×2 matrix with two pivot positions, each column is a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence no nontrivial solution.

b. With two pivot positions and three rows, A cannot have a pivot in every row. So the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbb{R}^3$), by Theorem 4 in Section 1.4.

33. One answer: $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

35. Your example should have the property that the sum of the entries in each row is zero. Why?

37. One answer is $A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$. The *Study Guide* shows how to analyze the problem in order to construct A. If $\mathbf{b}$ is any vector *not* a multiple of the first column of A, then the solution set of $A\mathbf{x} = \mathbf{b}$ is empty and thus cannot be formed by translating the solution set of $A\mathbf{x} = \mathbf{b}$. This does not contradict Theorem 6, because that theorem applies when the equation $A\mathbf{x} = \mathbf{b}$ has a nonempty solution set.

39. If c is a scalar, then $A(c\mathbf{u}) = cA\mathbf{u}$, by Theorem 5(b) in Section 1.4. If $\mathbf{u}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $A\mathbf{u} = \mathbf{0}$, $cA\mathbf{u} = c \cdot \mathbf{0} = \mathbf{0}$, and so $A(c\mathbf{u}) = \mathbf{0}$.

Section 1.6, page 55

1. The general solution is $p_{\text{Goods}} = .875 p_{\text{Services}}$, with p_{Services} free. One equilibrium solution is $p_{\text{Services}} = 1000$ and $p_{\text{Goods}} = 875$. Using fractions, the general solution could be written $p_{\text{Goods}} = (7/8) p_{\text{Services}}$, and a natural choice of prices might be $p_{\text{Services}} = 80$ and $p_{\text{Goods}} = 70$. Only the *ratio* of the prices is important. The economic equilibrium is unaffected by a proportional change in prices.

3. **a.**

	Distribution of Output From:				
	C&M	F&P	Mach.		
Output	↓	↓	↓	Input	Purchased By:
	.2	.8	.4	→	C&M
	.3	.1	.4	→	F&P
	.5	.1	.2	→	Mach.

b. $\begin{bmatrix} .8 & -.8 & -.4 & 0 \\ -.3 & .9 & -.4 & 0 \\ -.5 & -.1 & .8 & 0 \end{bmatrix}$

c. **[M]** $p_{\text{Chemicals}} = 141.7$, $p_{\text{Fuels}} = 91.7$, $p_{\text{Machinery}} = 100$. To two significant figures, $p_{\text{Chemicals}} = 140$, $p_{\text{Fuels}} = 92$, $p_{\text{Machinery}} = 100$.

5. $B_2S_3 + 6H_2O \rightarrow 2H_3BO_3 + 3H_2S$

7. $3NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + 3H_2O + 3CO_2$

9. **[M]** $15PbN_6 + 44CrMn_2O_8 \rightarrow$
$5Pb_3O_4 + 22Cr_2O_3 + 88MnO_2 + 90NO$

11. $\begin{cases} x_1 = 20 - x_3 \\ x_2 = 60 + x_3 \\ x_3 \text{ is free} \\ x_4 = 60 \end{cases}$ The largest value of x_3 is 20.

13. **a.** $\begin{cases} x_1 = x_3 - 40 \\ x_2 = x_3 + 10 \\ x_3 \text{ is free} \\ x_4 = x_6 + 50 \\ x_5 = x_6 + 60 \\ x_6 \text{ is free} \end{cases}$ **b.** $\begin{cases} x_2 = 50 \\ x_3 = 40 \\ x_4 = 50 \\ x_5 = 60 \end{cases}$

Section 1.7, page 61

Justify your answers to Exercises 1–22.

1. Lin. indep. **3.** Lin. depen.

5. Lin. indep. **7.** Lin. depen.

9. **a.** No h **b.** All h

11. $h = 6$ **13.** All h

15. Lin. depen. **17.** Lin. depen. **19.** Lin. indep.

21. If you consult your *Study Guide* before you make a good effort to answer the true-false questions, you will destroy most of their value.

23. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$ **25.** $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

27. All five columns of the 7×5 matrix A must be pivot columns. Otherwise, the equation $A\mathbf{x} = \mathbf{0}$ would have a free variable, in which case the columns of A would be linearly dependent.

29. A: Any 3×2 matrix with two nonzero columns such that neither column is a multiple of the other. In this case, the columns are linearly independent, and so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
B: Any 3×2 matrix with one column a multiple of the other.

31. $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

33. True, by Theorem 7. (The *Study Guide* adds another justification.)

35. False. The vector $\mathbf{v}_1$ could be the zero vector.

37. True. A linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ may be extended to a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ by placing a zero weight on $\mathbf{v}_4$.

39. You should be able to work this important problem without help. *Write* your solution before you consult the *Study Guide*.

41. [**M**] $B = \begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$. Other choices are possible.

43. [**M**] Each column of A that is not a column of B is in the set spanned by the columns of B.

Section 1.8, page 69

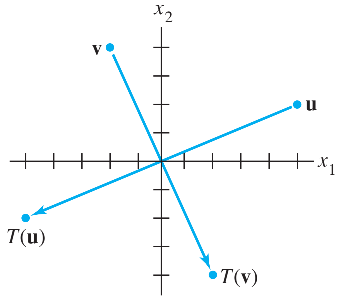
1. $\begin{bmatrix} 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 2a \\ 2b \end{bmatrix}$ **3.** $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, unique solution

5. $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, not unique **7.** $a = 5, b = 6$

9. $\mathbf{x} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

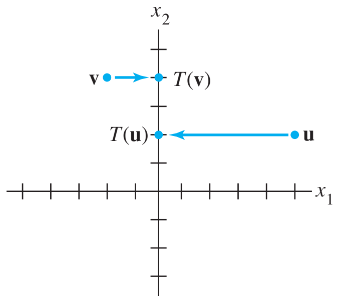
11. Yes, because the system represented by $[\,A \quad \mathbf{b}\,]$ is consistent.

13.



A reflection through the origin

15.



A projection onto the x_2-axis.

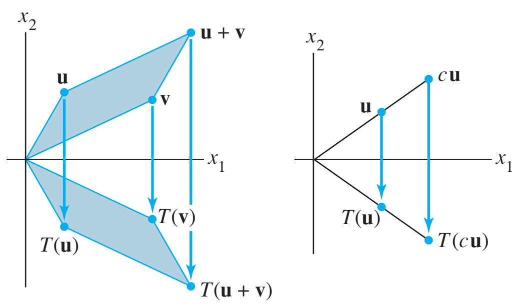
17. $\begin{bmatrix} 6 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \end{bmatrix}$ **19.** $\begin{bmatrix} 13 \\ 7 \end{bmatrix}, \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$

21. Read the text carefully and write your answers before you check the *Study Guide*. Notice that Exercise 21(e) is a sentence of the form

"$\langle$statement 1$\rangle$ if and only if $\langle$statement 2$\rangle$"

Mark such a sentence as True if $\langle$statement 1$\rangle$ is true whenever $\langle$statement 2$\rangle$ is true *and* also $\langle$statement 2$\rangle$ is true whenever $\langle$statement 1$\rangle$ is true.

23.



25. *Hint:* Show that the image of a line (that is, the set of images of all points on a line) can be represented by the parametric equation of a line.

27. a. The line through $\mathbf{p}$ and $\mathbf{q}$ is parallel to $\mathbf{q} - \mathbf{p}$. (See Exercises 21 and 22 in Section 1.5.) Since $\mathbf{p}$ is on the line, the equation of the line is $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$. Rewrite this as $\mathbf{x} = \mathbf{p} - t\mathbf{p} + t\mathbf{q}$ and $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$.

b. Consider $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$ for t such that $0 \le t \le 1$. Then, by linearity of T, for $0 \le t \le 1$

$$T(\mathbf{x}) = T((1-t)\mathbf{p} + t\mathbf{q}) = (1-t)T(\mathbf{p}) + tT(\mathbf{q}) \quad (*)$$

If $T(\mathbf{p})$ and $T(\mathbf{q})$ are distinct, then (*) is the equation for the line segment between $T(\mathbf{p})$ and $T(\mathbf{q})$, as shown in part (a). Otherwise, the set of images is just the single point $T(\mathbf{p})$, because

$$(1-t)T(\mathbf{p}) + tT(\mathbf{q}) = (1-t)T(\mathbf{p}) + tT(\mathbf{p}) = T(\mathbf{p})$$

29. a. When $b = 0$, $f(x) = mx$. In this case, for all x, y in $\mathbb{R}$ and all scalars c and d,

$$\begin{aligned} f(cx + dy) &= m(cx + dy) = mcx + mdy \\ &= c(mx) + d(my) = c \cdot f(x) + d \cdot f(y) \end{aligned}$$

This shows that f is linear.

b. When $f(x) = mx + b$, with b nonzero, $f(0) = m(0) + b = b \neq 0$.

c. In calculus, f is called a "linear function" because the graph of f is a line.

31. *Hint:* Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, you can write a certain equation and work with it.

33. One possibility is to show that T does not map the zero vector into the zero vector, something that every linear transformation *does* do: $T(0, 0) = (0, 4, 0)$.

35. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ and let c and d be scalars. Then

$$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$$

The transformation T is linear because

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, cu_2 + dv_2, -(cu_3 + dv_3)) \\ &= (cu_1 + dv_1, cu_2 + dv_2, -cu_3 - dv_3) \\ &= (cu_1, cu_2, -cu_3) + (dv_1, dv_2, -dv_3) \\ &= c(u_1, u_2, -u_3) + d(v_1, v_2, -v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

37. [**M**] All multiples of $(7, 9, 0, 2)$

39. [**M**] Yes. One choice for $\mathbf{x}$ is $(4, 7, 1, 0)$.

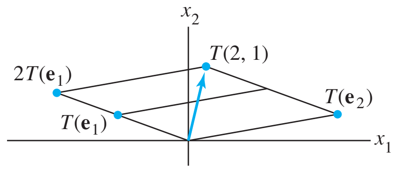
Section 1.9, page 79

1. $\begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$ **3.** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

7. $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ **9.** $\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$

11. The described transformation T maps $\mathbf{e}_1$ into $-\mathbf{e}_1$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_2$. A rotation through π radians also maps $\mathbf{e}_1$ into $-\mathbf{e}_1$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_2$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

13.



15. $\begin{bmatrix} 3 & 0 & -2 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ **17.** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$ **21.** $\mathbf{x} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

23. Answer the questions before checking the *Study Guide*.

Justify your answers to Exercises 25–28.

25. Not one-to-one and does not map $\mathbb{R}^4$ onto $\mathbb{R}^4$

27. Not one-to-one but maps $\mathbb{R}^3$ onto $\mathbb{R}^2$

29. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

31. n. (Explain why, and then check the *Study Guide*).

33. *Hint:* If $\mathbf{e}_j$ is the jth column of I_n, then $B\mathbf{e}_j$ is the jth column of B.

35. *Hint:* Is it possible that $m > n$? What about $m < n$?

37. [**M**] No. (Explain why.)

39. [**M**] No. (Explain why.)

Section 1.10, page 87

1. a. $x_1\begin{bmatrix} 110 \\ 4 \\ 20 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 130 \\ 3 \\ 18 \\ 5 \end{bmatrix} = \begin{bmatrix} 295 \\ 9 \\ 48 \\ 8 \end{bmatrix}$, where x_1 is the number of servings of Cheerios and x_2 is the number of servings of 100% Natural Cereal.

b. $\begin{bmatrix} 110 & 130 \\ 4 & 3 \\ 20 & 18 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 295 \\ 9 \\ 48 \\ 8 \end{bmatrix}$. Mix 1.5 servings of Cheerios together with 1 serving of 100% Natural Cereal.

3. a. She should mix .99 serving of Mac and Cheese, 1.54 servings of broccoli, and .79 serving of chicken to get her desired nutritional content.

b. She should mix 1.09 servings of shells and white cheddar, .88 serving of broccoli, and 1.03 servings of chicken to get her desired nutritional content. Notice that this mix contains significantly less broccoli, so she should like it better.

5. $R\mathbf{i} = \mathbf{v}$, $\begin{bmatrix} 11 & -5 & 0 & 0 \\ -5 & 10 & -1 & 0 \\ 0 & -1 & 9 & -2 \\ 0 & 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 50 \\ -40 \\ 30 \\ -30 \end{bmatrix}$

[**M**]: $\mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 3.68 \\ -1.90 \\ 2.57 \\ -2.49 \end{bmatrix}$

7. $R\mathbf{i} = \mathbf{v}$, $\begin{bmatrix} 12 & -7 & 0 & -4 \\ -7 & 15 & -6 & 0 \\ 0 & -6 & 14 & -5 \\ -4 & 0 & -5 & 13 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ -10 \end{bmatrix}$

[**M**]: $\mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 11.43 \\ 10.55 \\ 8.04 \\ 5.84 \end{bmatrix}$

9. $\mathbf{x}_{k+1} = M\mathbf{x}_k$ for $k = 0, 1, 2, \dots$, where

$M = \begin{bmatrix} .93 & .05 \\ .07 & .95 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 800{,}000 \\ 500{,}000 \end{bmatrix}$.

The population in 2017 (for $k = 2$) is $\mathbf{x}_2 = \begin{bmatrix} 741{,}720 \\ 558{,}280 \end{bmatrix}$.

11. a. $M = \begin{bmatrix} .98033 & .00179 \\ .01967 & .99821 \end{bmatrix}$

b. [**M**] $\mathbf{x}_{10} = \begin{bmatrix} 35.729 \\ 278.18 \end{bmatrix}$

13. [**M**]

a. The population of the city decreases. After 7 years, the populations are about equal, but the city population continues to decline. After 20 years, there are only 417,000 persons in the city (417,456 rounded off). However, the changes in population seem to grow smaller each year.

b. The city population is increasing slowly, and the suburban population is decreasing. After 20 years, the city population has grown from 350,000 to about 370,000.

Chapter 1 Supplementary Exercises, page 89

1.

a. F	**b.** F	**c.** T	**d.** F	**e.** T	
f. T	**g.** F	**h.** F	**i.** T	**j.** F	
k. T	**l.** F	**m.** T	**n.** T	**o.** T	
p. T	**q.** F	**r.** T	**s.** F	**t.** T	
u. F	**v.** F	**w.** T	**x.** T	**y.** T	**z.** F

3. a. Any consistent linear system whose echelon form is

$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

or $\begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b. Any consistent linear system whose reduced echelon form is I_3.

c. Any inconsistent linear system of three equations in three variables.

5. a. The solution set: (i) is empty if $h = 12$ and $k \neq 2$; (ii) contains a unique solution if $h \neq 12$; (iii) contains infinitely many solutions if $h = 12$ and $k = 2$.

b. The solution set is empty if $k + 3h = 0$; otherwise, the solution set contains a unique solution.

7. a. Set $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. "Determine if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$."
Solution: No.

b. Set $A = \begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix}$. "Determine if the columns of A span $\mathbb{R}^3$."

c. Define $T(\mathbf{x}) = A\mathbf{x}$. "Determine if T maps $\mathbb{R}^3$ onto $\mathbb{R}^3$."

9. $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{4}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{7}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 7/3 \\ 14/3 \end{bmatrix}$

10. *Hint:* Construct a "grid" on the x_1x_2-plane determined by $\mathbf{a}_1$ and $\mathbf{a}_2$.

11. A solution set is a line when the system has one free variable. If the coefficient matrix is 2×3, then two of the columns should be pivot columns. For instance, take $\begin{bmatrix} 1 & 2 & * \\ 0 & 3 & * \end{bmatrix}$. Put anything in column 3. The resulting matrix will be in echelon form. Make one row replacement operation on the second row to create a matrix *not* in echelon form, such as $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$.

12. *Hint:* How many free variables are in the equation $A\mathbf{x} = \mathbf{0}$?

13. $E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

15. a. If the three vectors are linearly independent, then a, c, and f must all be nonzero.

b. The numbers $a, \dots, f$ can have any values.

16. *Hint:* List the columns from right to left as $\mathbf{v}_1, \dots, \mathbf{v}_4$.

17. *Hint:* Use Theorem 7.

19. Let M be the line through the origin that is parallel to the line through $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ are both on M. So one of these two vectors is a multiple of the other, say $\mathbf{v}_2 - \mathbf{v}_1 = k(\mathbf{v}_3 - \mathbf{v}_1)$. This equation produces a linear dependence relation: $(k - 1)\mathbf{v}_1 + \mathbf{v}_2 - k\mathbf{v}_3 = \mathbf{0}$.

A second solution: A parametric equation of the line is $\mathbf{x} = \mathbf{v}_1 + t(\mathbf{v}_2 - \mathbf{v}_1)$. Since $\mathbf{v}_3$ is on the line, there is some t_0 such that $\mathbf{v}_3 = \mathbf{v}_1 + t_0(\mathbf{v}_2 - \mathbf{v}_1) = (1 - t_0)\mathbf{v}_1 + t_0\mathbf{v}_2$. So $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

21. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **23.** $a = 4/5$ and $b = -3/5$

25. a. The vector lists the number of three-, two-, and one-bedroom apartments provided when x_1 floors of plan A are constructed.

b. $x_1 \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix}$

c. [**M**] Use 2 floors of plan A and 15 floors of plan B. Or, use 6 floors of plan A, 2 floors of plan B, and 8 floors of plan C. These are the only feasible solutions. There are other mathematical solutions, but they require a negative number of floors of one or two of the plans, which makes no physical sense.

Chapter 2

Section 2.1, page 102

1. $\begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}$, $\begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}$, not defined, $\begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}$, $\begin{bmatrix} 12 & -3 \\ 15 & -6 \end{bmatrix}$

5. a. $A\mathbf{b}_1 = \begin{bmatrix} -7 \\ 7 \\ 12 \end{bmatrix}$, $A\mathbf{b}_2 = \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix}$,

$AB = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$

b. $AB = \begin{bmatrix} -1 \cdot 3 + 2(-2) & -1(-2) + 2 \cdot 1 \\ 5 \cdot 3 + 4(-2) & 5(-2) + 4 \cdot 1 \\ 2 \cdot 3 - 3(-2) & 2(-2) - 3 \cdot 1 \end{bmatrix}$

$= \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$

7. 3×7 **9.** $k = 5$

11. $AD = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$, $DA = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$

Right-multiplication (that is, multiplication on the right) by D multiplies each *column* of A by the corresponding diagonal entry of D. Left-multiplication by D multiplies each *row* of A by the corresponding diagonal entry of D. The *Study Guide* tells how to make $AB = BA$, but you should try this yourself before looking there.

13. *Hint:* One of the two matrices is Q.

15. Answer the questions before looking in the *Study Guide*.

17. $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$

19. The third column of AB is the sum of the first two columns of AB. Here's why. Write $B = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,]$. By definition, the third column of AB is $A\mathbf{b}_3$. If $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$, then $A\mathbf{b}_3 = A(\mathbf{b}_1 + \mathbf{b}_2) = A\mathbf{b}_1 + A\mathbf{b}_2$, by a property of matrix-vector multiplication.

21. The columns of A are linearly dependent. Why?

23. *Hint:* Suppose $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, and show that $\mathbf{x}$ must be $\mathbf{0}$.

25. *Hint:* Use the results of Exercises 23 and 24, and apply the associative law of multiplication to the product CAD.

27. $\mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u} = -2a + 3b - 4c$,

$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$,

$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$

29. *Hint:* For Theorem 2(b), show that the (i, j)-entry of $A(B + C)$ equals the (i, j)-entry of $AB + AC$.

31. *Hint:* Use the definition of the product $I_m A$ and the fact that $I_m\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^m$.

33. *Hint:* First write the (i, j)-entry of $(AB)^T$, which is the (j, i)-entry of AB. Then, to compute the (i, j)-entry in B^TA^T, use the facts that the entries in row i of B^T are $b_{1i}, \dots, b_{ni}$, because they come from column i of B, and the entries in column j of A^T are $a_{j1}, \dots, a_{jn}$, because they come from row j of A.

35. [**M**] The answer here depends on the choice of matrix program. For MATLAB, use the `help` command to read about `zeros`, `ones`, `eye`, and `diag`.

37. [**M**] Display your results and report your conclusions.

39. [**M**] The matrix S "shifts" the entries in a vector (a, b, c, d, e) to yield $(b, c, d, e, 0)$. S^5 is the 5×5 zero matrix. So is S^6.

Section 2.2, page 111

1. $\begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$ **3.** $-\frac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix}$

5. $x_1 = 7$ and $x_2 = -9$

7. a and b: $\begin{bmatrix} -9 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 11 \\ -5 \end{bmatrix}$, $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 13 \\ -5 \end{bmatrix}$

9. Write out your answers before checking the *Study Guide*.

11. The proof can be modeled after the proof of Theorem 5.

13. $AB = AC \Rightarrow A^{-1}AB = A^{-1}AC \Rightarrow IB = IC \Rightarrow B = C$. No, in general, B and C can be different when A is not invertible. See Exercise 10 in Section 2.1.

15. $D = C^{-1}B^{-1}A^{-1}$. Show that D works.

17. $A = BCB^{-1}$

19. After you find $X = CB - A$, show that X is a solution.

21. *Hint:* Consider the equation $A\mathbf{x} = \mathbf{0}$.

23. *Hint:* If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then there are no free variables in the equation $A\mathbf{x} = \mathbf{0}$, and each column of A is a pivot column.

25. *Hint:* Consider the case $a = b = 0$. Then consider the vector $\begin{bmatrix} -b \\ a \end{bmatrix}$, and use the fact that $ad - bc = 0$.

27. *Hint:* For part (a), interchange A and B in the box following Example 6 in Section 2.1, and then replace B by the identity matrix. For parts (b) and (c), begin by writing

$$A = \begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ \text{row}_3(A) \end{bmatrix}$$

29. $\begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$ **31.** $\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

33. $A^{-1} = B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$. *Hint:* For $j = 1, \dots, n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$ denote the jth columns of A, B, and I, respectively. Use the facts that $\mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$ and $\mathbf{b}_j = \mathbf{e}_j - \mathbf{e}_{j+1}$ for $j = 1, \dots, n-1$, and $\mathbf{a}_n = \mathbf{b}_n = \mathbf{e}_n$.

35. $\begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$. Find this by row reducing $[\ A \quad \mathbf{e}_3\ ]$.

37. $C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

39. .27, .30, and .23 inch, respectively

41. [**M**] 12, 1.5, 21.5, and 12 newtons, respectively

Section 2.3, page 117

The abbreviation IMT (here and in the *Study Guide*) denotes the Invertible Matrix Theorem (Theorem 8).

1. Invertible, by the IMT. Neither column of the matrix is a multiple of the other column, so they are linearly independent. Also, the matrix is invertible by Theorem 4 in Section 2.2 because the determinant is nonzero.

3. Invertible, by the IMT. The matrix row reduces to $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and has 3 pivot positions.

5. Not invertible, by the IMT. The matrix row reduces to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ and is not row equivalent to I_3.

7. Invertible, by the IMT. The matrix row reduces to $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and has four pivot positions.

9. [**M**] The 4×4 matrix has four pivot positions, so it is invertible by the IMT.

11. The *Study Guide* will help, but first try to answer the questions based on your careful reading of the text.

13. A square upper triangular matrix is invertible if and only if all the entries on the diagonal are nonzero. Why?

Note: The answers below for Exercises 15–29 mention the IMT. In many cases, part or all of an acceptable answer could also be based on results that were used to establish the IMT.

15. If A has two identical columns then its columns are linearly dependent. Part (e) of the IMT shows that A cannot be invertible.

17. If A is invertible, so is A^{-1}, by Theorem 6 in Section 2.2. By (e) of the IMT applied to A^{-1}, the columns of A^{-1} are linearly independent.

19. By (e) of the IMT, D is invertible. Thus the equation $D\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^7$, by (g) of the IMT. Can you say more?

21. The matrix G cannot be invertible, by Theorem 5 in Section 2.2 or by the paragraph following the IMT. So (g) of the IMT is false and so is (h). The columns of G do not span $\mathbb{R}^n$.

23. Statement (b) of the IMT is false for K, so statements (e) and (h) are also false. That is, the columns of K are linearly *dependent* and the columns do *not* span $\mathbb{R}^n$.

25. *Hint:* Use the IMT first.

27. Let W be the inverse of AB. Then $ABW = I$ and $A(BW) = I$. Unfortunately, this equation by itself does not prove that A is invertible. Why not? Finish the proof before you check the *Study Guide*.

29. Since the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one, statement (f) of the IMT is false. Then (i) is also false and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ does not map $\mathbb{R}^n$ onto $\mathbb{R}^n$. Also, A is not invertible, which implies that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not invertible, by Theorem 9.

31. *Hint:* If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$, then A has a pivot in each row (Theorem 4 in Section 1.4). Could there be free variables in an equation $A\mathbf{x} = \mathbf{b}$?

33. *Hint:* First show that the standard matrix of T is invertible. Then use a theorem or theorems to show that $T^{-1}(\mathbf{x}) = B\mathbf{x}$, where $B = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$.

35. *Hint:* To show that T is one-to-one, suppose that $T(\mathbf{u}) = T(\mathbf{v})$ for some vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Deduce that $\mathbf{u} = \mathbf{v}$. To show that T is onto, suppose $\mathbf{y}$ represents an arbitrary vector in $\mathbb{R}^n$ and use the inverse S to produce an $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{y}$. A second proof can be given using Theorem 9 together with a theorem from Section 1.9.

37. *Hint:* Consider the standard matrices of T and U.

39. Given any $\mathbf{v}$ in $\mathbb{R}^n$, we may write $\mathbf{v} = T(\mathbf{x})$ for some $\mathbf{x}$, because T is an onto mapping. Then, the assumed properties of S and U show that $S(\mathbf{v}) = S(T(\mathbf{x})) = \mathbf{x}$ and $U(\mathbf{v}) = U(T(\mathbf{x})) = \mathbf{x}$. So $S(\mathbf{v})$ and $U(\mathbf{v})$ are equal for each $\mathbf{v}$. That is, S and U are the same function from $\mathbb{R}^n$ into $\mathbb{R}^n$.

41. [**M**] **a.** The exact solution of (3) is $x_1 = 3.94$ and $x_2 = .49$. The exact solution of (4) is $x_1 = 2.90$ and $x_2 = 2.00$.

b. When the solution of (4) is used as an approximation for the solution in (3), the error in using the value of 2.90 for x_1 is about 26%, and the error in using 2.0 for x_2 is about 308%.

c. The condition number of the coefficient matrix is 3363. The percentage change in the solution from (3) to (4) is about 7700 times the percentage change in the right side of the equation. This is the same order of magnitude as the condition number. The condition number gives a rough measure of how sensitive the solution of $A\mathbf{x} = \mathbf{b}$ can be to changes in $\mathbf{b}$. Further information about the condition number is given at the end of Chapter 6 and in Chapter 7.

43. [**M**] cond$(A) \approx 69{,}000$, which is between 10^4 and 10^5. So about 4 or 5 digits of accuracy may be lost. Several experiments with MATLAB should verify that $\mathbf{x}$ and $\mathbf{x}_1$ agree to 11 or 12 digits.

45. [**M**] Some versions of MATLAB issue a warning when asked to invert a Hilbert matrix of order about 12 or larger using floating-point arithmetic. The product AA^{-1} should have several off-diagonal entries that are far from being zero. If not, try a larger matrix.

Section 2.4, page 123

1. $\begin{bmatrix} A & B \\ EA + C & EB + D \end{bmatrix}$ **3.** $\begin{bmatrix} Y & Z \\ W & X \end{bmatrix}$

5. $Y = B^{-1}$ (explain why), $X = -B^{-1}A$, $Z = C$

7. $X = A^{-1}$ (why?), $Y = -BA^{-1}$, $Z = 0$ (why?)

9. $X = -A_{21}A_{11}^{-1}$, $Y = -A_{31}A_{11}^{-1}$, $B_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}$

11. You can check your answers in the *Study Guide*.

13. *Hint:* Suppose A is invertible, and let $A^{-1} = \begin{bmatrix} D & E \\ F & G \end{bmatrix}$. Show that $BD = I$ and $CG = I$. This implies that B and C are invertible. (Explain why!) Conversely, suppose B and C are invertible. To prove that A is invertible, guess what A^{-1} must be and check that it works.

15. $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} =$
$$\begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$
with $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

17. $$G_{k+1} = \begin{bmatrix} X_k & \mathbf{x}_{k+1} \end{bmatrix}\begin{bmatrix} X_k^T \\ \mathbf{x}_{k+1}^T \end{bmatrix} = X_kX_k^T + \mathbf{x}_{k+1}\mathbf{x}_{k+1}^T = G_k + \mathbf{x}_{k+1}\mathbf{x}_{k+1}^T$$
Only the outer product matrix $\mathbf{x}_{k+1}\mathbf{x}_{k+1}^T$ needs to be computed (and then added to G_k).

19. $W(s) = I_m - C(A - sI_n)^{-1}B$. This is the Schur complement of $A - sI_n$ in the system matrix.

21. **a.** $$A^2 = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 3-3 & 0+(-1)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b. $$M^2 = \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix}\begin{bmatrix} A & 0 \\ I & -A \end{bmatrix} = \begin{bmatrix} A^2+0 & 0+0 \\ A-A & 0+(-A)^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

23. If A_1 and B_1 are $(k+1) \times (k+1)$ and lower triangular, then we can write $A_1 = \begin{bmatrix} a & \mathbf{0}^T \\ \mathbf{v} & A \end{bmatrix}$ and $B_1 = \begin{bmatrix} b & \mathbf{0}^T \\ \mathbf{w} & B \end{bmatrix}$, where A and B are $k \times k$ and lower triangular, $\mathbf{v}$ and $\mathbf{w}$ are in $\mathbb{R}^k$, and a and b are suitable scalars. Assume that the product of $k \times k$ lower triangular matrices is lower triangular, and compute the product A_1B_1. What do you conclude?

25. Use Example 5 to find the inverse of a matrix of the form $B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$, where B_{11} is $p \times p$, B_{22} is $q \times q$ and B is invertible. Partition the matrix A, and apply your result twice to find that
$$A^{-1} = \begin{bmatrix} -5 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -5/2 & 7/2 \end{bmatrix}$$

27. **a, b.** [**M**] The commands to be used in these exercises will depend on the matrix program.

c. The algebra needed comes from the block matrix equation
$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$
where $\mathbf{x}_1$ and $\mathbf{b}_1$ are in $\mathbb{R}^{20}$ and $\mathbf{x}_2$ and $\mathbf{b}_2$ are in $\mathbb{R}^{30}$. Then $A_{11}\mathbf{x}_1 = \mathbf{b}_1$, which can be solved to produce $\mathbf{x}_1$. The equation $A_{21}\mathbf{x}_1 + A_{22}\mathbf{x}_2 = \mathbf{b}_2$ yields

$A_{22}\mathbf{x}_2 = \mathbf{b}_2 - A_{21}\mathbf{x}_1$, which can be solved for $\mathbf{x}_2$ by row reducing the matrix $[A_{22} \quad \mathbf{c}]$, where $\mathbf{c} = \mathbf{b}_2 - A_{21}\mathbf{x}_1$.

Section 2.5, page 131

1. $L\mathbf{y} = \mathbf{b} \Rightarrow \mathbf{y} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$, $U\mathbf{x} = \mathbf{y} \Rightarrow \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$

3. $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$ **5.** $\mathbf{y} = \begin{bmatrix} 1 \\ 5 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 2 \\ -3 \end{bmatrix}$

7. $LU = \begin{bmatrix} 1 & 0 \\ -3/2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 0 & 7/2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 5 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

17. $U^{-1} = \begin{bmatrix} 1/4 & 3/8 & 1/4 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$,

$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$,

$A^{-1} = \begin{bmatrix} 1/8 & 3/8 & 1/4 \\ -3/2 & -1/2 & 1/2 \\ -1 & 0 & 1/2 \end{bmatrix}$

19. *Hint:* Think about row reducing $[A \quad I]$.

21. *Hint:* Represent the row operations by a sequence of elementary matrices.

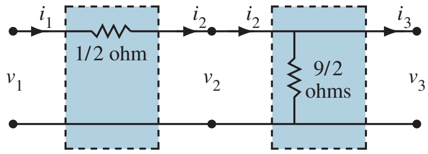
23. a. Denote the rows of D as transposes of column vectors. Then partitioned matrix multiplication yields

$$A = CD = [\mathbf{c}_1 \quad \cdots \quad \mathbf{c}_4]\begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_4^T \end{bmatrix} = \mathbf{c}_1\mathbf{v}_1^T + \cdots + \mathbf{c}_4\mathbf{v}_4^T$$

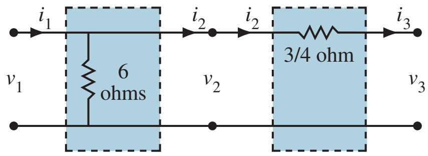
b. A has 40,000 entries. Since C has 1600 entries and D has 400 entries, together they occupy only 5% of the memory needed to store A.

25. Explain why U, D, and V^T are invertible. Then use a theorem on the inverse of a product of invertible matrices.

27. a.

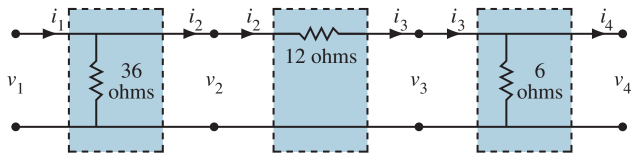


b.



29. a. $\begin{bmatrix} 1 + R_2/R_1 & -R_2 \\ -1/R_1 - R_2/(R_1R_3) - 1/R_3 & 1 + R_2/R_3 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 \\ -1/6 & 1 \end{bmatrix}\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/36 & 1 \end{bmatrix}$



31. [M]

a. $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.25 & -.0667 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.2667 & -.2857 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.2679 & -.0833 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.2917 & -.2921 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.2697 & -.0861 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.2948 & -.2931 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.75 & -.25 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.7333 & -1.0667 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.4286 & -.2857 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.7083 & -1.0833 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.3919 & -.2921 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.7052 & -1.0861 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.3868 \end{bmatrix}$

b. $\mathbf{x} = (3.9569, 6.5885, 4.2392, 7.3971, 5.6029, 8.7608, 9.4115, 12.0431)$

c. $A^{-1} = \begin{bmatrix} .2953 & .0866 & .0945 & .0509 & .0318 & .0227 & .0100 & .0082 \\ .0866 & .2953 & .0509 & .0945 & .0227 & .0318 & .0082 & .0100 \\ .0945 & .0509 & .3271 & .1093 & .1045 & .0591 & .0318 & .0227 \\ .0509 & .0945 & .1093 & .3271 & .0591 & .1045 & .0227 & .0318 \\ .0318 & .0227 & .1045 & .0591 & .3271 & .1093 & .0945 & .0509 \\ .0227 & .0318 & .0591 & .1045 & .1093 & .3271 & .0509 & .0945 \\ .0100 & .0082 & .0318 & .0227 & .0945 & .0509 & .2953 & .0866 \\ .0082 & .0100 & .0227 & .0318 & .0509 & .0945 & .0866 & .2953 \end{bmatrix}$

Obtain A^{-1} directly and then compute $A^{-1} - U^{-1}L^{-1}$ to compare the two methods for inverting a matrix.

Section 2.6, page 138

1. $C = \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & 0 \\ .30 & .10 & .10 \end{bmatrix}$, $\begin{Bmatrix} \text{intermediate} \\ \text{demand} \end{Bmatrix} = \begin{bmatrix} 60 \\ 20 \\ 10 \end{bmatrix}$

3. $\mathbf{x} = \begin{bmatrix} 40 \\ 15 \\ 15 \end{bmatrix}$ **5.** $\mathbf{x} = \begin{bmatrix} 110 \\ 120 \end{bmatrix}$

7. a. $\begin{bmatrix} 1.6 \\ 1.2 \end{bmatrix}$ **b.** $\begin{bmatrix} 111.6 \\ 121.2 \end{bmatrix}$

9. $\mathbf{x} = \begin{bmatrix} 82.8 \\ 131.0 \\ 110.3 \end{bmatrix}$

11. *Hint:* Use properties of transposes to obtain $\mathbf{p}^T = \mathbf{p}^T C + \mathbf{v}^T$, so that $\mathbf{p}^T\mathbf{x} = (\mathbf{p}^T C + \mathbf{v}^T)\mathbf{x} = \mathbf{p}^T C\mathbf{x} + \mathbf{v}^T\mathbf{x}$. Now compute $\mathbf{p}^T\mathbf{x}$ from the production equation.

13. [M] $\mathbf{x} = (99576, 97703, 51231, 131570, 49488, 329554, 13835)$. The entries in $\mathbf{x}$ suggest more precision in the answer than is warranted by the entries in $\mathbf{d}$, which appear to be accurate only to perhaps the nearest thousand. So a more realistic answer for $\mathbf{x}$ might be $\mathbf{x} = 1000 \times (100, 98, 51, 132, 49, 330, 14)$.

15. [M] $\mathbf{x}^{(12)}$ is the first vector whose entries are accurate to the nearest thousand. The calculation of $\mathbf{x}^{(12)}$ takes about 1260 flops, while row reduction of $\begin{bmatrix} (I - C) & \mathbf{d} \end{bmatrix}$ takes only about 550 flops. If C is larger than 20×20, then fewer flops are needed to compute $\mathbf{x}^{(12)}$ by iteration than to compute the equilibrium vector $\mathbf{x}$ by row reduction. As the size of C grows, the advantage of the iterative method increases. Also, because C becomes more sparse for larger models of the economy, fewer iterations are needed for reasonable accuracy.

Section 2.7, page 146

1. $\begin{bmatrix} 1 & .25 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & \sqrt{2} \\ \sqrt{2}/2 & \sqrt{2}/2 & 2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 3 + 4\sqrt{3} \\ \sqrt{3}/2 & 1/2 & 4 - 3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$

See the Practice Problem.

9. $A(BD)$ requires 1600 multiplications. $(AB)D$ requires 808 multiplications. The first method uses about twice as many multiplications. If D had 20,000 columns, the counts would be 160,000 and 80,008, respectively.

11. Use the fact that

$$\sec\varphi - \tan\varphi \sin\varphi = \frac{1}{\cos\varphi} - \frac{\sin^2\varphi}{\cos\varphi} = \cos\varphi$$

13. $\begin{bmatrix} A & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$. First apply the linear transformation A, and then translate by $\mathbf{p}$.

15. $(12, -6, 3)$ **17.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

19. The triangle with vertices at $(7, 2, 0)$, $(7.5, 5, 0)$, $(5, 5, 0)$

21. [M] $\begin{bmatrix} 2.2586 & -1.0395 & -.3473 \\ -1.3495 & 2.3441 & .0696 \\ .0910 & -.3046 & 1.2777 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} R \\ G \\ B \end{bmatrix}$

Section 2.8, page 153

1. The set is closed under sums but not under multiplication by a negative scalar. (Sketch an example.)

3. The set is not closed under sums or scalar multiples. The subset consisting of the points on the line $x_2 = x_1$ is a subspace, so any "counterexample" must use at least one point not on this line.

5. No. The system corresponding to $[\ \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{w}\ ]$ is inconsistent.

7. a. The three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$

b. Infinitely many vectors

c. Yes, because $A\mathbf{x} = \mathbf{p}$ has a solution.

9. No, because $A\mathbf{p} \neq \mathbf{0}$.

11. $p = 4$ and $q = 3$. Nul A is a subspace of $\mathbb{R}^4$ because solutions of $A\mathbf{x} = \mathbf{0}$ must have four entries, to match the columns of A. Col A is a subspace of $\mathbb{R}^3$ because each column vector has three entries.

13. For Nul A, choose $(1, -2, 1, 0)$ or $(-1, 4, 0, 1)$, for example. For Col A, select any column of A.

15. Yes. Let A be the matrix whose columns are the vectors given. Then A is invertible because its determinant is nonzero, and so its columns form a basis for $\mathbb{R}^2$, by the IMT (or by Example 5). (Other reasons for the invertibility of A could be given.)

17. Yes. Let A be the matrix whose columns are the vectors given. Row reduction shows three pivots, so A is invertible. By the IMT, the columns of A form a basis for $\mathbb{R}^3$.

19. No. Let A be the 3×2 matrix whose columns are the vectors given. The columns of A cannot possibly span $\mathbb{R}^3$ because A cannot have a pivot in every row. So the columns are not a basis for $\mathbb{R}^3$. (They are a basis for a plane in $\mathbb{R}^3$.)

21. Read the section carefully, and write your answers before checking the *Study Guide*. This section has terms and key concepts that you must learn now before going on.

23. Basis for Col A: $\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$

Basis for Nul A: $\begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$

25. Basis for Col A: $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$

Basis for Nul A: $\begin{bmatrix} 2 \\ -2.5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ .5 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

27. Construct a nonzero 3×3 matrix A, and construct $\mathbf{b}$ to be almost any convenient linear combination of the columns of A.

29. *Hint:* You need a nonzero matrix whose columns are linearly dependent.

31. If Col $F \neq \mathbb{R}^5$, then the columns of F do not span $\mathbb{R}^5$. Since F is square, the IMT shows that F is not invertible and the equation $F\mathbf{x} = \mathbf{0}$ has a nontrivial solution. That is, Nul F contains a nonzero vector. Another way to describe this is to write Nul $F \neq \{\mathbf{0}\}$.

33. If Col $Q = \mathbb{R}^4$, then the columns of Q span $\mathbb{R}^4$. Since Q is square, the IMT shows that Q is invertible and the equation $Q\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^4$. Also, each solution is unique, by Theorem 5 in Section 2.2.

35. If the columns of B are linearly independent, then the equation $B\mathbf{x} = \mathbf{0}$ has only the trivial (zero) solution. That is, Nul $B = \{\mathbf{0}\}$.

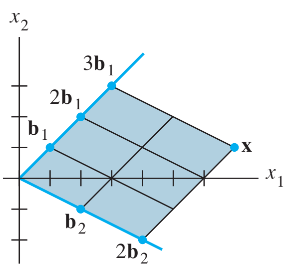
37. **[M]** Display the reduced echelon form of A, and select the pivot columns of A as a basis for Col A. For Nul A, write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

Basis for Col A : $\begin{bmatrix} 3 \\ -7 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ 7 \\ -7 \end{bmatrix}$

Basis for Nul A : $\begin{bmatrix} -2.5 \\ -1.5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4.5 \\ 2.5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3.5 \\ -1.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Section 2.9, page 159

1. $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2 = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$



3. $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ **5.** $\begin{bmatrix} 1/4 \\ -5/4 \end{bmatrix}$

7. $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1.5 \\ .5 \end{bmatrix}$

9. Basis for Col A: $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$; dim Col $A = 3$

Basis for Nul A: $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$; dim Nul $A = 1$

11. Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix}$; dim Col $A = 3$ Basis for Nul A: $\begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$; dim Nul $A = 2$

13. Columns 1, 3, and 4 of the original matrix form a basis for H, so dim $H = 3$.

15. Col $A = \mathbb{R}^3$, because A has a pivot in each row, and so the columns of A span $\mathbb{R}^3$. Nul A *cannot* equal $\mathbb{R}^2$, because Nul A is a subspace of $\mathbb{R}^5$. It is true, however, that Nul A is two-dimensional. Reason: The equation $A\mathbf{x} = \mathbf{0}$ has two free variables, because A has five columns and only three of them are pivot columns.

17. See the *Study Guide* after you write your justifications.

19. The fact that the solution space of $A\mathbf{x} = \mathbf{0}$ has a basis of three vectors means that dim Nul $A = 3$. Since a 5×7 matrix A has seven columns, the Rank Theorem shows that rank $A = 7 -$ dim Nul $A = 4$. See the *Study Guide* for a justification that does not explicitly mention the Rank Theorem.

21. A 7×6 matrix has six columns. By the Rank Theorem, dim Nul $A = 6 -$ rank A. Since the rank is four, dim Nul $A = 2$. That is, the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is two.

23. A 3×4 matrix A with a two-dimensional column space has two pivot columns. The remaining two columns will correspond to free variables in the equation $A\mathbf{x} = \mathbf{0}$. So the desired construction is possible. There are six possible locations for the two pivot columns, one of which is $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A simple construction is to take two vectors in $\mathbb{R}^3$ that are obviously not linearly dependent and place them in a matrix along with a copy of each vector, in any order. The resulting matrix will obviously have a two-dimensional column space. There is no need to worry about whether Nul A has the correct dimension, since this is guaranteed by the Rank Theorem: dim Nul $A = 4 -$ rank A.

25. The p columns of A span Col A by definition. If $\dim \operatorname{Col} A = p$, then the spanning set of p columns is automatically a basis for Col A, by the Basis Theorem. In particular, the columns are linearly independent.

27. a. *Hint:* The columns of B span W, and each vector $\mathbf{a}_j$ is in W. The vector $\mathbf{c}_j$ is in $\mathbb{R}^p$ because B has p columns.

b. *Hint:* What is the size of C?

c. *Hint:* How are B and C related to A?

29. [**M**] Your calculations should show that the matrix $[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}\,]$ corresponds to a consistent system. The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $(-5/3, 8/3)$.

Chapter 2 Supplementary Exercises, page 162

1. **a.** T **b.** F **c.** T **d.** F
e. F **f.** F **g.** T **h.** T
i. T **j.** F **k.** T **l.** F
m. F **n.** T **o.** F **p.** T

3. I

5. $A^2 = 2A - I$. Multiply by A: $A^3 = 2A^2 - A$. Substitute $A^2 = 2A - I$: $A^3 = 2(2A - I) - A = 3A - 2I$.
Multiply by A again: $A^4 = A(3A - 2I) = 3A^2 - 2A$.
Substitute the identity $A^2 = 2A - I$ again:
$A^4 = 3(2A - I) - 2A = 4A - 3I$.

7. $\begin{bmatrix} 10 & -1 \\ 9 & 10 \\ -5 & -3 \end{bmatrix}$ **9.** $\begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}$

11. a.
$$\begin{aligned} p(x_i) &= c_0 + c_1 x_i + \cdots + c_{n-1} x_i^{n-1} \\ &= \operatorname{row}_i(V) \cdot \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix} = \operatorname{row}_i(V\mathbf{c}) = y_i \end{aligned}$$

b. Suppose $x_1, \ldots, x_n$ are distinct, and suppose $V\mathbf{c} = \mathbf{0}$ for some vector $\mathbf{c}$. Then the entries in $\mathbf{c}$ are the coefficients of a polynomial whose value is zero at the distinct points $x_1, \ldots, x_n$. However, a nonzero polynomial of degree $n - 1$ cannot have n zeros, so the polynomial must be identically zero. That is, the entries in $\mathbf{c}$ must all be zero. This shows that the columns of V are linearly independent.

c. *Hint:* When $x_1, \ldots, x_n$ are distinct, there is a vector $\mathbf{c}$ such that $V\mathbf{c} = \mathbf{y}$. Why?

13. a. $P^2 = (\mathbf{u}\mathbf{u}^T)(\mathbf{u}\mathbf{u}^T) = \mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T = \mathbf{u}(1)\mathbf{u}^T = P$

b. $P^T = (\mathbf{u}\mathbf{u}^T)^T = \mathbf{u}^{TT}\mathbf{u}^T = \mathbf{u}\mathbf{u}^T = P$

c.
$$\begin{aligned} Q^2 &= (I - 2P)(I - 2P) \\ &= I - I(2P) - 2PI + 2P(2P) \\ &= I - 4P + 4P^2 = I, \text{ because of part (a).} \end{aligned}$$

15. Left-multiplication by an elementary matrix produces an elementary row operation:

$$B \sim E_1 B \sim E_2 E_1 B \sim E_3 E_2 E_1 B = C$$

So B is row equivalent to C. Since row operations are reversible, C is row equivalent to B. (Alternatively, show C being changed into B by row operations using the inverses of the E_i.)

17. Since B is 4×6 (with more columns than rows), its six columns are linearly dependent and there is a nonzero $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. Thus $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$, which shows that the matrix AB is not invertible, by the Invertible Matrix Theorem.

19. [**M**] To four decimal places, as k increases,

$$A^k \to \begin{bmatrix} .2857 & .2857 & .2857 \\ .4286 & .4286 & .4286 \\ .2857 & .2857 & .2857 \end{bmatrix} \quad \text{and}$$

$$B^k \to \begin{bmatrix} .2022 & .2022 & .2022 \\ .3708 & .3708 & .3708 \\ .4270 & .4270 & .4270 \end{bmatrix}$$

or, in rational format,

$$A^k \to \begin{bmatrix} 2/7 & 2/7 & 2/7 \\ 3/7 & 3/7 & 3/7 \\ 2/7 & 2/7 & 2/7 \end{bmatrix} \quad \text{and}$$

$$B^k \to \begin{bmatrix} 18/89 & 18/89 & 18/89 \\ 33/89 & 33/89 & 33/89 \\ 38/89 & 38/89 & 38/89 \end{bmatrix}$$

Chapter 3

Section 3.1, page 169

1. 1 **3.** 0 **5.** -24 **7.** 4

9. 15. Start with row 3.

11. -18. Start with column 1 or row 4.

13. 6. Start with row 2 or column 2.

15. 24 **17.** -10

19. $ad - bc$, $cb - da$. Interchanging two rows changes the sign of the determinant.

21. $ad - bc$, $akd - bkc = k(ad - bc)$. Scaling a row by a constant k multiplies the determinant by k.

23. $7a - 14b + 7c$. $-7a + 14b - 7c$. Interchanging two rows changes the sign of the determinant.

25. 1 **27.** 1 **29.** k

31. 1. The matrix is upper or lower triangular, with only 1's on the diagonal. The determinant is 1, the product of the diagonal entries.

33.
$$\begin{aligned} \det EA &= \det \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix} \\ &= (a + kc)d - (b + kd)c \\ &= ad + kcd - bc - kdc = (+1)(ad - bc) \\ &= (\det E)(\det A) \end{aligned}$$

35.
$$\begin{aligned} \det EA &= \det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cb - ad = (-1)(ad - bc) \\ &= (\det E)(\det A) \end{aligned}$$

37. $5A = \begin{bmatrix} 15 & 5 \\ 20 & 10 \end{bmatrix}$; no

39. Hints are in the *Study Guide*.

41. The area of the parallelogram and the determinant of $\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$ both equal 6. If $\mathbf{v} = \begin{bmatrix} x \\ 2 \end{bmatrix}$ for any x, the area is still 6. In each case the base of the parallelogram is unchanged, and the altitude remains 2 because the second coordinate of $\mathbf{v}$ is always 2.

43. [**M**] In general, $\det A^{-1} = 1/\det A$ as long as $\det A$ is nonzero.

45. [**M**] You can check your conjectures when you get to Section 3.2.

Section 3.2, page 177

1. Interchanging two rows reverses the sign of the determinant.

3. Multiplying a row by 3 multiplies the determinant by 3.

5. -3 **7.** 0 **9.** -28 **11.** -48

13. 6 **15.** 21 **17.** 7 **19.** 14

21. Not invertible **23.** Invertible

25. Linearly independent **27.** See the *Study Guide*.

29. 16

31. *Hint:* Show that $(\det A)(\det A^{-1}) = 1$.

33. *Hint:* Use Theorem 6.

35. *Hint:* Use Theorem 6 and another theorem.

37. $\det AB = \det \begin{bmatrix} 6 & 0 \\ 17 & 4 \end{bmatrix} = 24$; $(\det A)(\det B) = 3 \cdot 8 = 24$

39. **a.** -12 **b.** -375 **c.** 4 **d.** $-\frac{1}{3}$ **e.** -27

41.
$$\begin{aligned} \det A &= (a+e)d - (b+f)c = ad + ed - bc - fc \\ &= (ad - bc) + (ed - fc) = \det B + \det C \end{aligned}$$

43. *Hint:* Compute $\det A$ by a cofactor expansion down column 3.

45. [**M**] See the *Study Guide* after you have made a conjecture about A^TA and AA^T.

Section 3.3, page 186

1. $\begin{bmatrix} 5/6 \\ -1/6 \end{bmatrix}$ **3.** $\begin{bmatrix} 4/5 \\ -3/10 \end{bmatrix}$ **5.** $\begin{bmatrix} 1/4 \\ 11/4 \\ 3/8 \end{bmatrix}$

7. $s \neq \pm\sqrt{3}$; $x_1 = \dfrac{5s+4}{6(s^2-3)}$, $x_2 = \dfrac{-4s-15}{4(s^2-3)}$

9. $s \neq 0, 1$; $x_1 = \dfrac{7}{3(s-1)}$, $x_2 = \dfrac{4s+3}{6s(s-1)}$

11. $\operatorname{adj} A = \begin{bmatrix} 0 & 1 & 0 \\ -5 & -1 & -5 \\ 5 & 2 & 10 \end{bmatrix}$, $A^{-1} = \dfrac{1}{5}\begin{bmatrix} 0 & 1 & 0 \\ -5 & -1 & -5 \\ 5 & 2 & 10 \end{bmatrix}$

13. $\operatorname{adj} A = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$, $A^{-1} = \dfrac{1}{6}\begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$

15. $\operatorname{adj} A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -5 & 0 \\ -1 & -15 & 5 \end{bmatrix}$, $A^{-1} = \dfrac{-1}{5}\begin{bmatrix} -1 & 0 & 0 \\ -1 & -5 & 0 \\ -1 & -15 & 5 \end{bmatrix}$

17. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $C_{11} = d$, $C_{12} = -c$, $C_{21} = -b$, $C_{22} = a$. The adjugate matrix is the transpose of cofactors:

$$\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Following Theorem 8, we divide by $\det A$; this produces the formula from Section 2.2.

19. 8 **21.** 3 **23.** 23

25. A 3×3 matrix A is not invertible if and only if its columns are linearly dependent (by the Invertible Matrix Theorem). This happens if and only if one of the columns is in the plane spanned by the other two columns, which is equivalent to the condition that the parallelepiped determined by these columns has zero volume, which in turn is equivalent to the condition that $\det A = 0$.

27. 12 **29.** $\frac{1}{2}\left|\det\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\right|$

31. **a.** See Example 5. **b.** $4\pi abc/3$

33. [**M**] In MATLAB, the entries in $B - \text{inv}(A)$ are approximately 10^{-15} or smaller. See the *Study Guide* for suggestions that may save you keystrokes as you work.

35. [**M**] MATLAB Student Version 4.0 uses 57,771 flops for $\text{inv}(A)$, and 14,269,045 flops for the inverse formula. The `inv(A)` command requires only about 0.4% of the operations for the inverse formula. The *Study Guide* shows how to use the `flops` command.

Chapter 3 Supplementary Exercises, page 188

1. **a.** T **b.** T **c.** F **d.** F
e. F **f.** F **g.** T **h.** T
i. F **j.** F **k.** T **l.** F
m. F **n.** T **o.** F **p.** T

The solution for Exercise 3 is based on the fact that if a matrix contains two rows (or two columns) that are multiples of each other, then the determinant of the matrix is zero, by Theorem 4, because the matrix cannot be invertible.

3. Make two row replacement operations, and then factor out a common multiple in row 2 and a common multiple in row 3.

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{vmatrix}$$
$$= (b-a)(c-a)\begin{vmatrix} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$
$$= 0$$

5. -12

7. When the determinant is expanded by cofactors of the first row, the equation has the form $ax + by + c = 0$, where at least one of a and b is not zero. This is the equation of a line. It is clear that (x_1, y_1) and (x_2, y_2) are on the line, because when the coordinates of one of the points are substituted for x and y, two rows of the matrix are equal and so the determinant is zero.

9. $T \sim \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$. Thus, by Theorem 3,

$$\det T = (b-a)(c-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{bmatrix}$$
$$= (b-a)(c-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{bmatrix}$$
$$= (b-a)(c-a)(c-b)$$

11. Area $= 12$. If one vertex is subtracted from all four vertices, and if the new vertices are $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, then the translated figure (and hence the original figure) will be a parallelogram if and only if one of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is the sum of the other two vectors.

13. By the Inverse Formula, $(\text{adj } A) \cdot \dfrac{1}{\det A} A = A^{-1}A = I$. By the Invertible Matrix Theorem, adj A is invertible and $(\text{adj } A)^{-1} = \dfrac{1}{\det A} A$.

15. **a.** $X = CA^{-1}$, $Y = D - CA^{-1}B$. Now use Exercise 14(c).

b. From part (a), and the multiplicative property of determinants,

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det[A(D - CA^{-1}B)]$$
$$= \det[AD - ACA^{-1}B]$$
$$= \det[AD - CAA^{-1}B]$$
$$= \det[AD - CB]$$

where the equality $AC = CA$ was used in the third step.

17. First consider the case $n = 2$, and prove that the result holds by directly computing the determinants of B and C. Now assume that the formula holds for all $(k-1) \times (k-1)$ matrices, and let A, B, and C be $k \times k$ matrices. Use a cofactor expansion along the first column and the inductive hypothesis to find det B. Use row replacement operations on C to create zeros below the first pivot and produce a triangular matrix. Find the determinant of this matrix and add to det B to get the result.

19. [**M**] Compute:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 1, \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix} = 1,$$
$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{vmatrix} = 1$$

Conjecture:

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & & 2 \\ 1 & 2 & 3 & & 3 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{vmatrix} = 1$$

To confirm the conjecture, use row replacement operations to create zeros below the first pivot, then the second pivot, and so on. The resulting matrix is

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & & 1 \\ 0 & 0 & 1 & & 1 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

which is an upper triangular matrix with determinant 1.

Chapter 4

Section 4.1, page 197

1. **a.** $\mathbf{u} + \mathbf{v}$ is in V because its entries will both be nonnegative.

b. *Example:* If $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $c = -1$, then $\mathbf{u}$ is in V, but $c\mathbf{u}$ is not in V.

3. *Example:* If $\mathbf{u} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$ and $c = 4$, then $\mathbf{u}$ is in H, but $c\mathbf{u}$ is not in H.

5. Yes, by Theorem 1, because the set is Span $\{t^2\}$.

7. No, the set is not closed under multiplication by scalars that are not integers.

9. $H = \text{Span}\{\mathbf{v}\}$, where $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. By Theorem 1, H is a subspace of $\mathbb{R}^3$.

11. $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. By Theorem 1, W is a subspace of $\mathbb{R}^3$.

13. a. There are only three vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and $\mathbf{w}$ is not one of them.

b. There are infinitely many vectors in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

c. $\mathbf{w}$ is in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

15. Not a vector space because the zero vector is not in W

17. $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

19. *Hint:* Use Theorem 1.

Warning: Although the *Study Guide* has complete solutions for *every* odd-numbered exercise whose answer here is only a "Hint," you *must* really try to work the solution yourself. Otherwise, you will not benefit from the exercise.

21. Yes. The conditions for a subspace are obviously satisfied: The zero matrix is in H, the sum of two upper triangular matrices is upper triangular, and any scalar multiple of an upper triangular matrix is again upper triangular.

23. See the *Study Guide* after you have written your answers.

25. 4 **27. a.** 8 **b.** 3 **c.** 5 **d.** 4

29.
$$\begin{aligned} \mathbf{u} + (-1)\mathbf{u} &= 1\mathbf{u} + (-1)\mathbf{u} && \text{Axiom 10} \\ &= [1 + (-1)]\mathbf{u} && \text{Axiom 8} \\ &= 0\mathbf{u} = \mathbf{0} && \text{Exercise 27} \end{aligned}$$
From Exercise 26, it follows that $(-1)\mathbf{u} = -\mathbf{u}$.

31. Any subspace H that contains $\mathbf{u}$ and $\mathbf{v}$ must also contain all scalar multiples of $\mathbf{u}$ and $\mathbf{v}$ and hence must contain all sums of scalar multiples of $\mathbf{u}$ and $\mathbf{v}$. Thus H must contain Span $\{\mathbf{u}, \mathbf{v}\}$.

33. *Hint:* For part of the solution, consider $\mathbf{w}_1$ and $\mathbf{w}_2$ in $H + K$, and write $\mathbf{w}_1$ and $\mathbf{w}_2$ in the form $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are in H, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are in K.

35. [**M**] The reduced echelon form of $[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{w}\,]$ shows that $\mathbf{w} = \mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$.

37. [**M**] The functions are $\cos 4t$ and $\cos 6t$. See Exercise 34 in Section 4.5.

Section 4.2, page 207

1. $\begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $\mathbf{w}$ is in Nul A.

3. $\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}$

7. W is not a subspace of $\mathbb{R}^3$ because the zero vector $(0, 0, 0)$ is not in W.

9. W is a subspace of $\mathbb{R}^4$ because W is the set of solutions of the system
$$\begin{aligned} a - 2b - 4c \phantom{{}-3d} &= 0 \\ 2a \phantom{{}-2b} - c - 3d &= 0 \end{aligned}$$

11. W is not a subspace because $\mathbf{0}$ is not in W. *Justification:* If a typical element $(b - 2d, 5 + d, b + 3d, d)$ were zero, then $5 + d = 0$ and $d = 0$, which is impossible.

13. $W = \text{Col}\, A$ for $A = \begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, so W is a vector space by Theorem 3.

15. $\begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$

17. a. 2 **b.** 4 **19. a.** 5 **b.** 2

21. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in Nul A, $\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$ in Col A. Other answers possible.

23. $\mathbf{w}$ is in both Nul A and Col A.

25. See the *Study Guide*. By now you should know how to use it properly.

27. Let $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -3 & -3 \\ -2 & 4 & 2 \\ -1 & 5 & 7 \end{bmatrix}$. Then $\mathbf{x}$ is in Nul A. Since Nul A is a subspace of $\mathbb{R}^3$, $10\mathbf{x}$ is in Nul A.

29. a. $A\mathbf{0} = \mathbf{0}$, so the zero vector is in Col A.

b. By a property of matrix multiplication, $A\mathbf{x} + A\mathbf{w} = A(\mathbf{x} + \mathbf{w})$, which shows that $A\mathbf{x} + A\mathbf{w}$ is a linear combination of the columns of A and hence is in Col A.

c. $c(A\mathbf{x}) = A(c\mathbf{x})$, which shows that $c(A\mathbf{x})$ is in Col A for all scalars c.

31. a. For arbitrary polynomials $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_2$ and any scalar c,
$$\begin{aligned} T(\mathbf{p} + \mathbf{q}) &= \begin{bmatrix} (\mathbf{p} + \mathbf{q})(0) \\ (\mathbf{p} + \mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) + \mathbf{q}(0) \\ \mathbf{p}(1) + \mathbf{q}(1) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q}) \\ T(c\mathbf{p}) &= \begin{bmatrix} c\mathbf{p}(0) \\ c\mathbf{p}(1) \end{bmatrix} = c\begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = cT(\mathbf{p}) \end{aligned}$$
So T is a linear transformation from $\mathbb{P}_2$ into $\mathbb{P}_2$.

b. Any quadratic polynomial that vanishes at 0 and 1 must be a multiple of $\mathbf{p}(t) = t(t - 1)$. The range of T is $\mathbb{R}^2$.

33. **a.** For A, B in $M_{2\times 2}$ and any scalar c,

$$\begin{aligned} T(A+B) &= (A+B)+(A+B)^T \\ &= A+B+A^T+B^T \quad \text{Transpose property} \\ &= (A+A^T)+(B+B^T) = T(A)+T(B) \\ T(cA) &= (cA)+(cA)^T = cA+cA^T \\ &= c(A+A^T) = cT(A) \end{aligned}$$

So T is a linear transformation from $M_{2\times 2}$ into $M_{2\times 2}$.

b. If B is any element in $M_{2\times 2}$ with the property that $B^T = B$, and if $A = \frac{1}{2}B$, then

$$T(A) = \tfrac{1}{2}B + \left(\tfrac{1}{2}B\right)^T = \tfrac{1}{2}B + \tfrac{1}{2}B = B$$

c. Part (b) showed that the range of T contains all B such that $B^T = B$. So it suffices to show that any B in the range of T has this property. If $B = T(A)$, then by properties of transposes,

$$B^T = (A+A^T)^T = A^T + A^{TT} = A^T + A = B$$

d. The kernel of T is $\left\{\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \text{ real}\right\}$.

35. *Hint:* Check the three conditions for a subspace. Typical elements of $T(U)$ have the form $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are in U.

37. **[M]** $\mathbf{w}$ is in Col A but not in Nul A. (Explain why.)

39. **[M]** The reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Section 4.3, page 215

1. Yes, the 3×3 matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has 3 pivot positions. By the Invertible Matrix Theorem, A is invertible and its columns form a basis for $\mathbb{R}^3$. (See Example 3.)

3. No, the vectors are linearly dependent and do not span $\mathbb{R}^3$.

5. No, the set is linearly dependent because the zero vector is in the set. However,

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ -3 & 9 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

The matrix has pivots in each row and hence its columns span $\mathbb{R}^3$.

7. No, the vectors are linearly independent because they are not multiples. (More precisely, neither vector is a multiple of the other.) However, the vectors do not span $\mathbb{R}^3$. The matrix $\begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 5 \end{bmatrix}$ can have at most two pivots since it has only two columns. So there will not be a pivot in each row.

9. $\begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$ **11.** $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

13. Basis for Nul A: $\begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$

Basis for Col A: $\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}$

15. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ **17.** **[M]** $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

19. The three simplest answers are $\{\mathbf{v}_1, \mathbf{v}_2\}$ or $\{\mathbf{v}_1, \mathbf{v}_3\}$ or $\{\mathbf{v}_2, \mathbf{v}_3\}$. Other answers are possible.

21. See the *Study Guide* for hints.

23. *Hint:* Use the Invertible Matrix Theorem.

25. No. (Why is the set not a basis for H?)

27. $\{\cos \omega t, \sin \omega t\}$

29. Let A be the $n \times k$ matrix $[\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_k\,]$. Since A has fewer columns than rows, there cannot be a pivot position in each row of A. By Theorem 4 in Section 1.4, the columns of A do not span $\mathbb{R}^n$ and hence are not a basis for $\mathbb{R}^n$.

31. *Hint:* If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent, then there exist $c_1, \ldots, c_p$, not all zero, such that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. Use this equation.

33. Neither polynomial is a multiple of the other polynomial, so $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a linearly independent set in $\mathbb{P}_3$.

35. Let $\{\mathbf{v}_1, \mathbf{v}_3\}$ be any linearly independent set in the vector space V, and let $\mathbf{v}_2$ and $\mathbf{v}_4$ be linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_3$. Then $\{\mathbf{v}_1, \mathbf{v}_3\}$ is a basis for Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

37. **[M]** You could be clever and find special values of t that produce several zeros in (5), and thereby create a system of equations that can be solved easily by hand. Or, you could use values of t such as $t = 0, .1, .2, \ldots$ to create a system of equations that you can solve with a matrix program.

Section 4.4, page 224

1. $\begin{bmatrix} 3 \\ -7 \end{bmatrix}$ **3.** $\begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$ **5.** $\begin{bmatrix} 8 \\ -5 \end{bmatrix}$ **7.** $\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}$ **11.** $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ **13.** $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$

15. The *Study Guide* has hints.

17. $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\mathbf{v}_1 - 2\mathbf{v}_2 = 10\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$ (infinitely many answers)

19. *Hint:* By hypothesis, the zero vector has a unique representation as a linear combination of elements of S.

21. $\begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$

23. *Hint:* Suppose that $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$ for some $\mathbf{u}$ and $\mathbf{w}$ in V, and denote the entries in $[\mathbf{u}]_{\mathcal{B}}$ by $c_1, \ldots, c_n$. Use the definition of $[\mathbf{u}]_{\mathcal{B}}$.

25. One possible approach: First, show that if $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are linearly *dependent*, then $[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}$ are linearly dependent. Second, show that if $[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}$ are linearly dependent, then $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are linearly *dependent*. Use the two equations displayed in the exercise. A slightly different proof is given in the *Study Guide*.

27. Linearly independent. (Justify answers to Exercises 27–34.)

29. Linearly dependent

31. a. The coordinate vectors $\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ do not span $\mathbb{R}^3$. Because of the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the corresponding polynomials do not span $\mathbb{P}_2$.

b. The coordinate vectors $\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ span $\mathbb{R}^3$. Because of the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the corresponding polynomials span $\mathbb{P}_2$.

33. **[M]** The coordinate vectors $\begin{bmatrix} 3 \\ 7 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 16 \\ -6 \\ 2 \end{bmatrix}$ are a linearly dependent subset of $\mathbb{R}^4$. Because of the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the corresponding polynomials form a linearly dependent subset of $\mathbb{P}_3$, and thus cannot be a basis for $\mathbb{P}_3$.

35. **[M]** $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -5/3 \\ 8/3 \end{bmatrix}$ **37.** **[M]** $\begin{bmatrix} 1.3 \\ 0 \\ 0.8 \end{bmatrix}$

Section 4.5, page 231

1. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$; dim is 2

3. $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$; dim is 3

5. $\begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix}$; dim is 2

7. No basis; dim is 0 **9.** 2 **11.** 2 **13.** 2, 3

15. 2, 2 **17.** 0, 3 **19.** See the *Study Guide*.

21. *Hint:* You need only show that the first four Hermite polynomials are linearly independent. Why?

23. $[\mathbf{p}]_{\mathcal{B}} = \left(3, 3, -2, \frac{3}{2}\right)$

25. *Hint:* Suppose S does span V, and use the Spanning Set Theorem. This leads to a contradiction, which shows that the spanning hypothesis is false.

27. *Hint:* Use the fact that each $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

29. Justify each answer.

a. True **b.** True **c.** True

31. *Hint:* Since H is a nonzero subspace of a finite-dimensional space, H is finite-dimensional and has a basis, say, $\mathbf{v}_1, \ldots, \mathbf{v}_p$. First show that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ spans $T(H)$.

33. **[M]** **a.** One basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_2, \mathbf{e}_3\}$. In fact, any two of the vectors $\mathbf{e}_2, \ldots, \mathbf{e}_5$ will extend $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to a basis of $\mathbb{R}^5$.

Section 4.6, page 238

1. rank $A = 2$; dim Nul $A = 2$;

Basis for Col A: $\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$

Basis for Row A: $(1, 0, -1, 5), (0, -2, 5, -6)$

Basis for Nul A: $\begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

3. rank $A = 3$; dim Nul $A = 2$;

Basis for Col A: $\begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix}$

Row A: $(2, -3, 6, 2, 5), (0, 0, 3, -1, 1), (0, 0, 0, 1, 3)$

Basis for Nul A: $\begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix}$

5. 5, 3, 3

7. Yes; no. Since Col A is a four-dimensional subspace of $\mathbb{R}^4$, it coincides with $\mathbb{R}^4$. The null space cannot be $\mathbb{R}^3$, because the vectors in Nul A have 7 entries. Nul A is a three-dimensional subspace of $\mathbb{R}^7$, by the Rank Theorem.

9. 2 **11.** 3

13. 5, 5. In both cases, the number of pivots cannot exceed the number of columns or the number of rows.

15. 2 **17.** See the *Study Guide*.

19. Yes. Try to write an explanation before you consult the *Study Guide*.

21. No. Explain why.

23. Yes. Only six homogeneous linear equations are necessary.

25. No. Explain why.

27. Row A and Nul A are in $\mathbb{R}^n$; Col A and Nul A^T are in $\mathbb{R}^m$. There are only four distinct subspaces because Row A^T = Col A and Col A^T = Row A.

29. Recall that $\dim \text{Col } A = m$ precisely when $\text{Col } A = \mathbb{R}^m$, or equivalently, when the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$. By Exercise 28(b), $\dim \text{Col } A = m$ precisely when $\dim \text{Nul } A^T = 0$, or equivalently, when the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

31. $\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$. The columns are all multiples of $\mathbf{u}$, so Col $\mathbf{u}\mathbf{v}^T$ is one-dimensional, unless $a = b = c = 0$.

33. *Hint:* Let $A = [\,\mathbf{u} \quad \mathbf{u}_2 \quad \mathbf{u}_3\,]$. If $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{u}$ is a basis for Col A. Why?

35. **[M]** **a.** Many answers are possible. Here are the "canonical" choices, for $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_7\,]$:

$$C = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_4 \quad \mathbf{a}_6\,], \quad N = \begin{bmatrix} -13/2 & -5 & 3 \\ -11/2 & -1/2 & -2 \\ 1 & 0 & 0 \\ 0 & 11/2 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

b. $M = [\,2 \quad 41 \quad 0 \quad -28 \quad 11\,]^T$. The matrix $[\,R^T \quad N\,]$ is 7×7 because the columns of R^T and N are in $\mathbb{R}^7$, and $\dim \text{Row } A + \dim \text{Nul } A = 7$. The matrix $[\,C \quad M\,]$ is 5×5 because the columns of C and M are in $\mathbb{R}^5$ and $\dim \text{Col } A + \dim \text{Nul } A^T = 5$, by Exercise 28(b). The invertibility of these matrices follows from the fact that their columns are linearly independent, which can be proved from Theorem 3 in Section 6.1.

37. **[M]** The C and R given for Exercise 35 work here, and $A = CR$.

Section 4.7, page 244

1. a. $\begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ **3.** (ii)

5. a. $\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$

7. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$

9. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

11. See the *Study Guide*.

13. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$, $[-1 + 2t]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

15. a. $\mathcal{B}$ is a basis for V.

b. The coordinate mapping is a linear transformation.

c. The product of a matrix and a vector

d. The coordinate vector of $\mathbf{v}$ relative to $\mathcal{B}$

17. a. **[M]**

$$P^{-1} = \frac{1}{32}\begin{bmatrix} 32 & 0 & 16 & 0 & 12 & 0 & 10 \\ & 32 & 0 & 24 & 0 & 20 & 0 \\ & & 16 & 0 & 16 & 0 & 15 \\ & & & 8 & 0 & 10 & 0 \\ & & & & 4 & 0 & 6 \\ & & & & & 2 & 0 \\ & & & & & & 1 \end{bmatrix}$$

b. P is the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$. So P^{-1} is the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, by equation (5), and the columns of this matrix are the $\mathcal{C}$-coordinate vectors of the basis vectors in $\mathcal{B}$, by Theorem 15.

19. **[M]** *Hint:* Let $\mathcal{C}$ be the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Then the columns of P are $[\mathbf{u}_1]_{\mathcal{C}}$, $[\mathbf{u}_2]_{\mathcal{C}}$, and $[\mathbf{u}_3]_{\mathcal{C}}$. Use the definition of $\mathcal{C}$-coordinate vectors and matrix algebra to compute $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. The solution method is discussed in the *Study Guide*. Here are the numerical answers:

a. $\mathbf{u}_1 = \begin{bmatrix} -6 \\ -5 \\ 21 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -6 \\ -9 \\ 32 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$

b. $\mathbf{w}_1 = \begin{bmatrix} 28 \\ -9 \\ -3 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 38 \\ -13 \\ 2 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} 21 \\ -7 \\ 3 \end{bmatrix}$

Section 4.8, page 253

1. If $y_k = 2^k$, then $y_{k+1} = 2^{k+1}$ and $y_{k+2} = 2^{k+2}$. Substituting these formulas into the left side of the equation gives

$$\begin{aligned} y_{k+2} + 2y_{k+1} - 8y_k &= 2^{k+2} + 2\cdot 2^{k+1} - 8\cdot 2^k \\ &= 2^k(2^2 + 2\cdot 2 - 8) \\ &= 2^k(0) = 0 \quad \text{for all } k \end{aligned}$$

Since the difference equation holds for all k, 2^k is a solution. A similar calculation works for $y_k = (-4)^k$.

3. The signals 2^k and $(-4)^k$ are linearly independent because neither is a multiple of the other. For instance, there is no scalar c such that $2^k = c(-4)^k$ *for all* k. By Theorem 17, the solution set H of the difference equation in Exercise 1 is two-dimensional. By the Basis Theorem in Section 4.5, the two linearly independent signals 2^k and $(-4)^k$ form a basis for H.

5. If $y_k = (-3)^k$, then

$$\begin{aligned} y_{k+2} + 6y_{k+1} + 9y_k &= (-3)^{k+2} + 6(-3)^{k+1} + 9(-3)^k \\ &= (-3)^k[(-3)^2 + 6(-3) + 9] \\ &= (-3)^k(0) = 0 \quad \text{for all } k \end{aligned}$$

Similarly, if $y_k = k(-3)^k$, then

$$\begin{aligned} & y_{k+2} + 6y_{k+1} + 9y_k \\ &\quad = (k+2)(-3)^{k+2} + 6(k+1)(-3)^{k+1} + 9k(-3)^k \\ &\quad = (-3)^k[(k+2)(-3)^2 + 6(k+1)(-3) + 9k] \\ &\quad = (-3)^k[9k + 18 - 18k - 18 + 9k] \\ &\quad = (-3)^k(0) \quad \text{for all } k \end{aligned}$$

Thus both $(-3)^k$ and $k(-3)^k$ are in the solution space H of the difference equation. Also, there is no scalar c such that $k(-3)^k = c(-3)^k$ *for all* k, because c must be chosen independently of k. Likewise, there is no scalar c such that $(-3)^k = ck(-3)^k$ for all k. So the two signals are linearly independent. Since dim $H = 2$, the signals form a basis for H, by the Basis Theorem.

7. Yes **9.** Yes

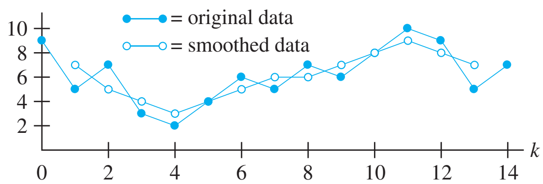
11. No, two signals cannot span the three-dimensional solution space.

13. $\left(\frac{1}{3}\right)^k, \left(\frac{2}{3}\right)^k$ **15.** $5^k, (-5)^k$

17. $Y_k = c_1(.8)^k + c_2(.5)^k + 10 \to 10 \quad \text{as } k \to \infty$

19. $y_k = c_1(-2 + \sqrt{3})^k + c_2(-2 - \sqrt{3})^k$

21. 7, 5, 4, 3, 4, 5, 6, 6, 7, 8, 9, 8, 7; see figure below.



23. a. $y_{k+1} - 1.01y_k = -450$, $y_0 = 10{,}000$

b. [**M**] MATLAB code:

```
pay = 450, y = 10000, m = 0
table = [0 ; y]
while y > 450
      y = 1.01*y - pay
      m = m + 1
      table = [table [m ; y] ]
                %append new column
end
m, y
```

c. [**M**] At month 26, the last payment is \$114.88. The total paid by the borrower is \$11,364.88.

25. $k^2 + c_1 \cdot (-4)^k + c_2$ **27.** $2 - 2k + c_1 \cdot 4^k + c_2 \cdot 2^{-k}$

29. $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & -6 & -8 & 6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{bmatrix}$$

31. The equation holds for all k, so it holds with k replaced by $k - 1$, which transforms the equation into

$$y_{k+2} + 5y_{k+1} + 6y_k = 0 \quad \text{for all } k$$

The equation is of order 2.

33. For all k, the Casorati matrix $C(k)$ is not invertible. In this case, the Casorati matrix gives no information about the linear independence/dependence of the set of signals. In fact, neither signal is a multiple of the other, so they are linearly independent.

35. *Hint:* Verify the two properties that define a linear transformation. For $\{y_k\}$ and $\{z_k\}$ in $\mathbb{S}$, study $T(\{y_k\} + \{z_k\})$. Note that if r is any scalar, then the kth term of $r\{y_k\}$ is ry_k; so $T(r\{y_k\})$ is the sequence $\{w_k\}$ given by

$$w_k = ry_{k+2} + a(ry_{k+1}) + b(ry_k)$$

37. $(TD)(y_0, y_1, y_2, \ldots) = T(D(y_0, y_1, y_2, \ldots)) = T(0, y_0, y_1, y_2, \ldots) = (y_0, y_1, y_2, \ldots) = I(y_0, y_1, y_2, \ldots)$, while $(DT)(y_0, y_1, y_2, \ldots) = D(T(y_0, y_1, y_2, \ldots)) = D(y_1, y_2, y_3, \ldots) = (0, y_1, y_2, y_3, \ldots)$.

Section 4.9, page 262

1. a. From:

$$\begin{array}{cc} \text{N} & \text{M} \\ \end{array} \quad \text{To:}$$

$$\begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix} \begin{array}{l} \text{News} \\ \text{Music} \end{array}$$

b. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **c.** 33%

3. a. From:

$$\begin{array}{cc} \text{H} & \text{I} \\ \end{array} \quad \text{To:}$$

$$\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{array}{l} \text{Healthy} \\ \text{Ill} \end{array}$$

b. 15%, 12.5%

c. .925; use $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

5. $\begin{bmatrix} .4 \\ .6 \end{bmatrix}$ **7.** $\begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$

9. Yes, because P^2 has all positive entries.

11. a. $\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$ **b.** 2/3

13. a. $\begin{bmatrix} .9 \\ .1 \end{bmatrix}$ **b.** .10, no

15. [**M**] About 17.3% of the United States population

17. a. The entries in a column of P sum to 1. A column in the matrix $P - I$ has the same entries as in P except that one of the entries is decreased by 1. Hence each column sum is 0.

b. By (a), the bottom row of $P - I$ is the negative of the sum of the other rows.

c. By (b) and the Spanning Set Theorem, the bottom row of $P - I$ can be removed and the remaining $(n - 1)$ rows will still span the row space. Alternatively, use (a) and the fact that row operations do not change the row space. Let A be the matrix obtained from $P - I$ by adding to the bottom row all the other rows. By (a), the row space is spanned by the first $(n - 1)$ rows of A.

d. By the Rank Theorem and (c), the dimension of the column space of $P - I$ is less than n, and hence the null space is nontrivial. Instead of the Rank Theorem, you may use the Invertible Matrix Theorem, since $P - I$ is a square matrix.

19. a. The product $S\mathbf{x}$ equals the sum of the entries in $\mathbf{x}$. For a probability vector, this sum must be 1.

b. $P = [\,\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n\,]$, where the $\mathbf{p}_i$ are probability vectors. By matrix multiplication and part (a),

$$SP = [\,S\mathbf{p}_1 \quad S\mathbf{p}_2 \quad \cdots \quad S\mathbf{p}_n\,] = [\,1 \quad 1 \quad \cdots \quad 1\,] = S$$

c. By part (b), $S(P\mathbf{x}) = (SP)\mathbf{x} = S\mathbf{x} = 1$. Also, the entries in $P\mathbf{x}$ are nonnegative (because P and $\mathbf{x}$ have nonnegative entries). Hence, by (a), $P\mathbf{x}$ is a probability vector.

21. [M]

a. To four decimal places,

$$P^4 = P^5 = \begin{bmatrix} .2816 & .2816 & .2816 & .2816 \\ .3355 & .3355 & .3355 & .3355 \\ .1819 & .1819 & .1819 & .1819 \\ .2009 & .2009 & .2009 & .2009 \end{bmatrix},$$

$$\mathbf{q} = \begin{bmatrix} .2816 \\ .3355 \\ .1819 \\ .2009 \end{bmatrix}$$

Note that, due to round-off, the column sums are not 1.

b. To four decimal places,

$$Q^{80} = \begin{bmatrix} .7354 & .7348 & .7351 \\ .0881 & .0887 & .0884 \\ .1764 & .1766 & .1765 \end{bmatrix},$$

$$Q^{116} = Q^{117} = \begin{bmatrix} .7353 & .7353 & .7353 \\ .0882 & .0882 & .0882 \\ .1765 & .1765 & .1765 \end{bmatrix},$$

$$\mathbf{q} = \begin{bmatrix} .7353 \\ .0882 \\ .1765 \end{bmatrix}$$

c. Let P be an $n \times n$ regular stochastic matrix, $\mathbf{q}$ the steady-state vector of P, and $\mathbf{e}_1$ the first column of the identity matrix. Then $P^k\mathbf{e}_1$ is the first column of P^k. By Theorem 18, $P^k\mathbf{e}_1 \to \mathbf{q}$ as $k \to \infty$. Replacing $\mathbf{e}_1$ by the other columns of the identity matrix, we conclude that each column of P^k converges to $\mathbf{q}$ as $k \to \infty$. Thus $P^k \to [\,\mathbf{q} \quad \mathbf{q} \quad \cdots \quad \mathbf{q}\,]$.

Chapter 4 Supplementary Exercises, page 264

1. a. T **b.** T **c.** F **d.** F **e.** T **f.** T
g. F **h.** F **i.** T **j.** F **k.** F **l.** F
m. T **n.** F **o.** T **p.** T **q.** F **r.** T
s. T **t.** F

3. The set of all (b_1, b_2, b_3) satisfying $b_1 + 2b_2 + b_3 = 0$.

5. The vector $\mathbf{p}_1$ is not zero and $\mathbf{p}_2$ is not a multiple of $\mathbf{p}_1$, so keep both of these vectors. Since $\mathbf{p}_3 = 2\mathbf{p}_1 + 2\mathbf{p}_2$, discard $\mathbf{p}_3$. Since $\mathbf{p}_4$ has a t^2 term, it cannot be a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$, so keep $\mathbf{p}_4$. Finally, $\mathbf{p}_5 = \mathbf{p}_1 + \mathbf{p}_4$, so discard $\mathbf{p}_5$. The resulting basis is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$.

7. You would have to know that the solution set of the homogeneous system is spanned by two solutions. In this case, the null space of the 18×20 coefficient matrix A is at most two-dimensional. By the Rank Theorem, $\dim \operatorname{Col} A \geq 20 - 2 = 18$, which means that $\operatorname{Col} A = \mathbb{R}^{18}$, because A has 18 rows, and every equation $A\mathbf{x} = \mathbf{b}$ is consistent.

9. Let A be the standard $m \times n$ matrix of the transformation T.

a. If T is one-to-one, then the columns of A are linearly independent (Theorem 12 in Section 1.9), so $\dim \operatorname{Nul} A = 0$. By the Rank Theorem, $\dim \operatorname{Col} A = \operatorname{rank} A = n$. Since the range of T is $\operatorname{Col} A$, the dimension of the range of T is n.

b. If T is onto, then the columns of A span $\mathbb{R}^m$ (Theorem 12 in Section 1.9), so $\dim \operatorname{Col} A = m$. By the Rank Theorem, $\dim \operatorname{Nul} A = n - \dim \operatorname{Col} A = n - m$. Since the kernel of T is $\operatorname{Nul} A$, the dimension of the kernel of T is $n - m$.

11. If S is a finite spanning set for V, then a subset of S—say S'—is a basis for V. Since S' must span V, S' cannot be a proper subset of S because of the minimality of S. Thus $S' = S$, which proves that S is a basis for V.

12. a. *Hint:* Any $\mathbf{y}$ in $\operatorname{Col} AB$ has the form $\mathbf{y} = AB\mathbf{x}$ for some $\mathbf{x}$.

13. By Exercise 12, $\operatorname{rank} PA \leq \operatorname{rank} A$, and $\operatorname{rank} A = \operatorname{rank} P^{-1}PA \leq \operatorname{rank} PA$. Thus $\operatorname{rank} PA = \operatorname{rank} A$.

15. The equation $AB = 0$ shows that each column of B is in $\operatorname{Nul} A$. Since $\operatorname{Nul} A$ is a subspace, all linear combinations of the columns of B are in $\operatorname{Nul} A$, so $\operatorname{Col} B$ is a subspace of $\operatorname{Nul} A$. By Theorem 11 in Section 4.5, $\dim \operatorname{Col} B \leq \dim \operatorname{Nul} A$. Applying the Rank Theorem, we find that

$$n = \operatorname{rank} A + \dim \operatorname{Nul} A \geq \operatorname{rank} A + \operatorname{rank} B$$

17. a. Let A_1 consist of the r pivot columns in A. The columns of A_1 are linearly independent. So A_1 is an $m \times r$ with rank r.

b. By the Rank Theorem applied to A_1, the dimension of Row A is r, so A_1 has r linearly independent rows. Use them to form A_2. Then A_2 is $r \times r$ with linearly independent rows. By the Invertible Matrix Theorem, A_2 is invertible.

19. $$[\,B \quad AB \quad A^2B\,] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -.9 & .81 \\ 1 & .5 & .25 \end{bmatrix} \sim \begin{bmatrix} 1 & -.9 & .81 \\ 0 & 1 & 0 \\ 0 & 0 & -.56 \end{bmatrix}$$

This matrix has rank 3, so the pair (A, B) is controllable.

21. [**M**] rank $[\,B \quad AB \quad A^2B \quad A^3B\,] = 3$. The pair (A, B) is not controllable.

Chapter 5

Section 5.1, page 273

1. Yes **3.** No **5.** Yes, $\lambda = 0$ **7.** Yes, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

9. $\lambda = 1$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\lambda = 5$: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **11.** $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

13. $\lambda = 1$: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $\lambda = 2$: $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$; $\lambda = 3$: $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

15. $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ **17.** $0, 2, -1$

19. 0. Justify your answer.

21. See the *Study Guide*, after you have written your answers.

23. *Hint:* Use Theorem 2.

25. *Hint:* Use the equation $A\mathbf{x} = \lambda\mathbf{x}$ to find an equation involving A^{-1}.

27. *Hint:* For any λ, $(A - \lambda I)^T = A^T - \lambda I$. By a theorem (which one?), $A^T - \lambda I$ is invertible if and only if $A - \lambda I$ is invertible.

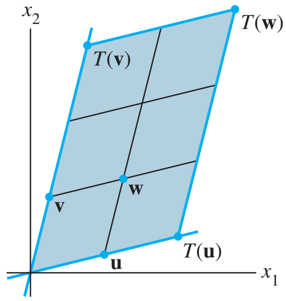
29. Let $\mathbf{v}$ be the vector in $\mathbb{R}^n$ whose entries are all 1's. Then $A\mathbf{v} = s\mathbf{v}$.

31. *Hint:* If A is the standard matrix of T, look for a nonzero vector $\mathbf{v}$ (a point in the plane) such that $A\mathbf{v} = \mathbf{v}$.

33. a. $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}$

b.
$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}) \\ &= c_1\lambda^k A\mathbf{u} + c_2\mu^k A\mathbf{v} && \text{Linearity} \\ &= c_1\lambda^k\lambda\mathbf{u} + c_2\mu^k\mu\mathbf{v} && \mathbf{u} \text{ and } \mathbf{v} \text{ are eigenvectors.} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

35.



37. [**M**] $\lambda = 3$: $\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$; $\lambda = 13$: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. You can speed up your calculations with the program `nulbasis` discussed in the *Study Guide*.

39. [**M**] $\lambda = -2$: $\begin{bmatrix} -2 \\ 7 \\ -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -5 \\ 0 \\ 5 \end{bmatrix}$;

$\lambda = 5$: $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Section 5.2, page 281

1. $\lambda^2 - 4\lambda - 45$; $9, -5$ **3.** $\lambda^2 - 2\lambda - 1$; $1 \pm \sqrt{2}$

5. $\lambda^2 - 6\lambda + 9$; 3 **7.** $\lambda^2 - 9\lambda + 32$; no real eigenvalues

9. $-\lambda^3 + 4\lambda^2 - 9\lambda - 6$ **11.** $-\lambda^3 + 9\lambda^2 - 26\lambda + 24$

13. $-\lambda^3 + 18\lambda^2 - 95\lambda + 150$ **15.** $4, 3, 3, 1$

17. $3, 3, 1, 1, 0$

19. *Hint:* The equation given holds for all λ.

21. The *Study Guide* has hints.

23. *Hint:* Find an invertible matrix P so that $RQ = P^{-1}AP$.

25. a. $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = .3$

b. $\mathbf{x}_0 = \mathbf{v}_1 - \frac{1}{14}\mathbf{v}_2$

c. $\mathbf{x}_1 = \mathbf{v}_1 - \frac{1}{14}(.3)\mathbf{v}_2$, $\mathbf{x}_2 = \mathbf{v}_1 - \frac{1}{14}(.3)^2\mathbf{v}_2$, and $\mathbf{x}_k = \mathbf{v}_1 - \frac{1}{14}(.3)^k\mathbf{v}_2$. As $k \to \infty$, $(.3)^k \to 0$ and $\mathbf{x}_k \to \mathbf{v}_1$.

27. a. $A\mathbf{v}_1 = \mathbf{v}_1$, $A\mathbf{v}_2 = .5\mathbf{v}_2$, $A\mathbf{v}_3 = .2\mathbf{v}_3$. (This also shows that the eigenvalues of A are 1, .5, and .2.)

b. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent because the eigenvectors correspond to distinct eigenvalues (Theorem 2). Since there are 3 vectors in the set, the set is a basis for $\mathbb{R}^3$. So there exist (unique) constants such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Then

$$\mathbf{w}^T\mathbf{x}_0 = c_1\mathbf{w}^T\mathbf{v}_1 + c_2\mathbf{w}^T\mathbf{v}_2 + c_3\mathbf{w}^T\mathbf{v}_3 \qquad (*)$$

Since $\mathbf{x}_0$ and $\mathbf{v}_1$ are probability vectors and since the entries in $\mathbf{v}_2$ and in $\mathbf{v}_3$ each sum to 0, $(*)$ shows that $1 = c_1$.

c. By (b),

$$\mathbf{x}_0 = \mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Using (a),

$$\begin{aligned}\mathbf{x}_k &= A^k\mathbf{x}_0 = A^k\mathbf{v}_1 + c_2A^k\mathbf{v}_2 + c_3A^k\mathbf{v}_3\\ &= \mathbf{v}_1 + c_2(.5)^k\mathbf{v}_2 + c_3(.2)^k\mathbf{v}_3\\ &\to \mathbf{v}_1 \text{ as } k \to \infty\end{aligned}$$

29. [**M**] Report your results and conclusions. You can avoid tedious calculations if you use the program `gauss` discussed in the *Study Guide*.

Section 5.3, page 288

1. $\begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$ **3.** $\begin{bmatrix} a^k & 0 \\ 3(a^k - b^k) & b^k \end{bmatrix}$

5. $\lambda = 5$: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\lambda = 1$: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

When an answer involves a diagonalization, $A = PDP^{-1}$, the factors P and D are not unique, so your answer may differ from that given here.

7. $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **9.** Not diagonalizable

11. $P = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. $P = \begin{bmatrix} -1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

15. $P = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

17. Not diagonalizable

19. $P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

21. See the *Study Guide*. **23.** Yes. (Explain why.)

25. No, A must be diagonalizable. (Explain why.)

27. *Hint:* Write $A = PDP^{-1}$. Since A is invertible, 0 is not an eigenvalue of A, so D has nonzero entries on its diagonal.

29. One answer is $P_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, whose columns are eigenvectors corresponding to the eigenvalues in D_1.

31. *Hint:* Construct a suitable 2×2 triangular matrix.

33. [**M**] $P = \begin{bmatrix} 2 & 2 & 1 & 6 \\ 1 & -1 & 1 & -3 \\ -1 & -7 & 1 & 0 \\ 2 & 2 & 0 & 4 \end{bmatrix}$,

$D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

35. [**M**] $P = \begin{bmatrix} 6 & 3 & 2 & 4 & 3 \\ -1 & -1 & -1 & -3 & -1 \\ -3 & -3 & -4 & -2 & -4 \\ 3 & 0 & -1 & 5 & 0 \\ 0 & 3 & 4 & 0 & 5 \end{bmatrix}$,

$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Section 5.4, page 295

1. $\begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}$

3. a. $T(\mathbf{e}_1) = -\mathbf{b}_2 + \mathbf{b}_3$, $T(\mathbf{e}_2) = -\mathbf{b}_1 - \mathbf{b}_3$, $T(\mathbf{e}_3) = \mathbf{b}_1 - \mathbf{b}_2$

b. $[T(\mathbf{e}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $[T(\mathbf{e}_2)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$,

$[T(\mathbf{e}_3)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

5. a. $10 - 3t + 4t^2 + t^3$

b. For any $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_2$ and any scalar c,

$$\begin{aligned}T[\mathbf{p}(t) + \mathbf{q}(t)] &= (t+5)[\mathbf{p}(t) + \mathbf{q}(t)]\\ &= (t+5)\mathbf{p}(t) + (t+5)\mathbf{q}(t)\\ &= T[\mathbf{p}(t)] + T[\mathbf{q}(t)]\\ T[c\cdot\mathbf{p}(t)] &= (t+5)[c\cdot\mathbf{p}(t)] = c\cdot(t+5)\mathbf{p}(t)\\ &= c\cdot T[\mathbf{p}(t)]\end{aligned}$$

c. $\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

9. a. $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

b. *Hint:* Compute $T(\mathbf{p}+\mathbf{q})$ and $T(c\cdot\mathbf{p})$ for arbitrary $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_2$ and an arbitrary scalar c.

c. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ **13.** $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

15. $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

17. a. $A\mathbf{b}_1 = 2\mathbf{b}_1$, so $\mathbf{b}_1$ is an eigenvector of A. However, A has only one eigenvalue, $\lambda = 2$, and the eigenspace is only one-dimensional, so A is not diagonalizable.

b. $\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

19. By definition, if A is similar to B, there exists an invertible matrix P such that $P^{-1}AP = B$. (See Section 5.2.) Then B is invertible because it is the product of invertible matrices. To show that A^{-1} is similar to B^{-1}, use the equation $P^{-1}AP = B$. See the *Study Guide*.

21. *Hint:* Review Practice Problem 2.

23. *Hint:* Compute $B(P^{-1}\mathbf{x})$.

25. *Hint:* Write $A = PBP^{-1} = (PB)P^{-1}$, and use the trace property.

27. For each j, $I(\mathbf{b}_j) = \mathbf{b}_j$. Since the standard coordinate vector of any vector in $\mathbb{R}^n$ is just the vector itself, $[I(\mathbf{b}_j)]_{\mathcal{E}} = \mathbf{b}_j$. Thus the matrix for I relative to $\mathcal{B}$ and the standard basis $\mathcal{E}$ is simply $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$. This matrix is precisely the *change-of-coordinates* matrix $P_{\mathcal{B}}$ defined in Section 4.4.

29. The $\mathcal{B}$-matrix for the identity transformation is I_n, because the $\mathcal{B}$-coordinate vector of the jth basis vector $\mathbf{b}_j$ is the jth column of I_n.

31. **[M]** $\begin{bmatrix} -7 & -2 & -6 \\ 0 & -4 & -6 \\ 0 & 0 & -1 \end{bmatrix}$

Section 5.5, page 302

1. $\lambda = 2+i, \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$; $\lambda = 2-i, \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

3. $\lambda = 2+3i, \begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$; $\lambda = 2-3i, \begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$

5. $\lambda = 2+2i, \begin{bmatrix} 1 \\ 2+2i \end{bmatrix}$; $\lambda = 2-2i, \begin{bmatrix} 1 \\ 2-2i \end{bmatrix}$

7. $\lambda = \sqrt{3} \pm i, \varphi = \pi/6$ radian, $r = 2$

9. $\lambda = -\sqrt{3}/2 \pm (1/2)i, \varphi = -5\pi/6$ radians, $r = 1$

11. $\lambda = .1 \pm .1i, \varphi = -\pi/4$ radian, $r = \sqrt{2}/10$

In Exercises 13–20, other answers are possible. Any P that makes $P^{-1}AP$ equal to the given C or to C^T is a satisfactory answer. First find P; then compute $P^{-1}AP$.

13. $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

15. $P = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$

17. $P = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}, C = \begin{bmatrix} -.6 & -.8 \\ .8 & -.6 \end{bmatrix}$

19. $P = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}, C = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$

21. $\mathbf{y} = \begin{bmatrix} 2 \\ -1+2i \end{bmatrix} = \dfrac{-1+2i}{5}\begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$

23. (a) Properties of conjugates and the fact that $\overline{\mathbf{x}}^T = \overline{\mathbf{x}^T}$; (b) $\overline{A\mathbf{x}} = A\overline{\mathbf{x}}$ and A is real; (c) because $\mathbf{x}^T A\overline{\mathbf{x}}$ is a scalar and hence may be viewed as a 1×1 matrix; (d) properties of transposes; (e) $A^T = A$, definition of q

25. *Hint:* First write $\mathbf{x} = \operatorname{Re}\mathbf{x} + i(\operatorname{Im}\mathbf{x})$.

27. **[M]** $P = \begin{bmatrix} 1 & -1 & -2 & 0 \\ -4 & 0 & 0 & 2 \\ 0 & 0 & -3 & -1 \\ 2 & 0 & 4 & 0 \end{bmatrix}$,

$C = \begin{bmatrix} .2 & -.5 & 0 & 0 \\ .5 & .2 & 0 & 0 \\ 0 & 0 & .3 & -.1 \\ 0 & 0 & .1 & .3 \end{bmatrix}$

Other choices are possible, but C must equal $P^{-1}AP$.

Section 5.6, page 311

1. a. *Hint:* Find c_1, c_2 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Use this representation and the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A to compute $\mathbf{x}_1 = \begin{bmatrix} 49/3 \\ 41/3 \end{bmatrix}$.

b. In general, $\mathbf{x}_k = 5(3)^k\mathbf{v}_1 - 4(\frac{1}{3})^k\mathbf{v}_2$ for $k \geq 0$.

3. When $p = .2$, the eigenvalues of A are .9 and .7, and

$$\mathbf{x}_k = c_1(.9)^k\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(.7)^k\begin{bmatrix} 2 \\ 1 \end{bmatrix} \to \mathbf{0} \quad \text{as } k \to \infty$$

The higher predation rate cuts down the owls' food supply, and eventually both predator and prey populations perish.

5. If $p = .325$, the eigenvalues are 1.05 and .55. Since $1.05 > 1$, both populations will grow at 5% per year. An eigenvector for 1.05 is (6, 13), so eventually there will be approximately 6 spotted owls to every 13 (thousand) flying squirrels.

7. a. The origin is a saddle point because A has one eigenvalue larger than 1 and one smaller than 1 (in absolute value).

b. The direction of greatest attraction is given by the eigenvector corresponding to the eigenvalue $1/3$, namely, $\mathbf{v}_2$. All vectors that are multiples of $\mathbf{v}_2$ are attracted to the origin. The direction of greatest repulsion is given by the eigenvector $\mathbf{v}_1$. All multiples of $\mathbf{v}_1$ are repelled.

c. See the *Study Guide*.

9. Saddle point; eigenvalues: 2, .5; direction of greatest repulsion: the line through $(0,0)$ and $(-1,1)$; direction of greatest attraction: the line through $(0,0)$ and $(1,4)$

11. Attractor; eigenvalues: .9, .8; greatest attraction: line through $(0,0)$ and $(5,4)$

13. Repellor; eigenvalues: 1.2, 1.1; greatest repulsion: line through $(0,0)$ and $(3,4)$

15. $\mathbf{x}_k = \mathbf{v}_1 + .1(.5)^k \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + .3(.2)^k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \to \mathbf{v}_1$ as $k \to \infty$

17. a. $A = \begin{bmatrix} 0 & 1.6 \\ .3 & .8 \end{bmatrix}$

b. The population is growing because the largest eigenvalue of A is 1.2, which is larger than 1 in magnitude. The eventual growth rate is 1.2, which is 20% per year. The eigenvector $(4,3)$ for $\lambda_1 = 1.2$ shows that there will be 4 juveniles for every 3 adults.

c. **[M]** The juvenile–adult ratio seems to stabilize after about 5 or 6 years. The *Study Guide* describes how to construct a matrix program to generate a data matrix whose columns list the numbers of juveniles and adults each year. Graphing the data is also discussed.

Section 5.7, page 319

1. $\mathbf{x}(t) = \frac{5}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} - \frac{3}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$

3. $-\frac{5}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{t} + \frac{9}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$. The origin is a saddle point. The direction of greatest attraction is the line through $(-1,1)$ and the origin. The direction of greatest repulsion is the line through $(-3,1)$ and the origin.

5. $-\frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t} + \frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$. The origin is a repellor. The direction of greatest repulsion is the line through $(1,1)$ and the origin.

7. Set $P = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$. Then $A = PDP^{-1}$. Substituting $\mathbf{x} = P\mathbf{y}$ into $\mathbf{x}' = A\mathbf{x}$, we have

$$\frac{d}{dt}(P\mathbf{y}) = A(P\mathbf{y})$$
$$P\mathbf{y}' = PDP^{-1}(P\mathbf{y}) = PD\mathbf{y}$$

Left-multiplying by P^{-1} gives

$$\mathbf{y}' = D\mathbf{y}, \quad \text{or} \quad \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

9. (complex solution):

$$c_1\begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(-2+i)t} + c_2\begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(-2-i)t}$$

(real solution):

$$c_1\begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} e^{-2t} + c_2\begin{bmatrix} \sin t - \cos t \\ \sin t \end{bmatrix} e^{-2t}$$

The trajectories spiral in toward the origin.

11. (complex): $c_1\begin{bmatrix} -3+3i \\ 2 \end{bmatrix} e^{3it} + c_2\begin{bmatrix} -3-3i \\ 2 \end{bmatrix} e^{-3it}$

(real):

$$c_1\begin{bmatrix} -3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} + c_2\begin{bmatrix} -3\sin 3t + 3\cos 3t \\ 2\sin 3t \end{bmatrix}$$

The trajectories are ellipses about the origin.

13. (complex): $c_1\begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{(1+3i)t} + c_2\begin{bmatrix} 1-i \\ 2 \end{bmatrix} e^{(1-3i)t}$

(real): $c_1\begin{bmatrix} \cos 3t - \sin 3t \\ 2\cos 3t \end{bmatrix} e^{t} + c_2\begin{bmatrix} \sin 3t + \cos 3t \\ 2\sin 3t \end{bmatrix} e^{t}$

The trajectories spiral out, away from the origin.

15. **[M]** $\mathbf{x}(t) = c_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2\begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix} e^{-t} + c_3\begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} e^{t}$

The origin is a saddle point. A solution with $c_3 = 0$ is attracted to the origin. A solution with $c_1 = c_2 = 0$ is repelled.

17. **[M]** (complex):

$$c_1\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2\begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix} e^{(5+2i)t} + c_3\begin{bmatrix} 23+34i \\ -9-14i \\ 3 \end{bmatrix} e^{(5-2i)t}$$

(real): $c_1\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2\begin{bmatrix} 23\cos 2t + 34\sin 2t \\ -9\cos 2t - 14\sin 2t \\ 3\cos 2t \end{bmatrix} e^{5t} + c_3\begin{bmatrix} 23\sin 2t - 34\cos 2t \\ -9\sin 2t + 14\cos 2t \\ 3\sin 2t \end{bmatrix} e^{5t}$

The origin is a repellor. The trajectories spiral outward, away from the origin.

19. **[M]** $A = \begin{bmatrix} -2 & 3/4 \\ 1 & -1 \end{bmatrix}$,

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.5t} - \frac{1}{2}\begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-2.5t}$$

21. **[M]** $A = \begin{bmatrix} -1 & -8 \\ 5 & -5 \end{bmatrix}$,

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -20\sin 6t \\ 15\cos 6t - 5\sin 6t \end{bmatrix} e^{-3t}$$

Section 5.8, page 326

1. Eigenvector: $\mathbf{x}_4 = \begin{bmatrix} 1 \\ .3326 \end{bmatrix}$, or $A\mathbf{x}_4 = \begin{bmatrix} 4.9978 \\ 1.6652 \end{bmatrix}$; $\lambda \approx 4.9978$

3. Eigenvector: $\mathbf{x}_4 = \begin{bmatrix} .5188 \\ 1 \end{bmatrix}$, or $A\mathbf{x}_4 = \begin{bmatrix} .4594 \\ .9075 \end{bmatrix}$; $\lambda \approx .9075$

5. $\mathbf{x} = \begin{bmatrix} -.7999 \\ 1 \end{bmatrix}$, $A\mathbf{x} = \begin{bmatrix} 4.0015 \\ -5.0020 \end{bmatrix}$; estimated $\lambda = -5.0020$

7. **[M]** $\mathbf{x}_k$: $\begin{bmatrix} .75 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ .9565 \end{bmatrix}, \begin{bmatrix} .9932 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ .9990 \end{bmatrix}, \begin{bmatrix} .9998 \\ 1 \end{bmatrix}$

μ_k: 11.5, 12.78, 12.96, 12.9948, 12.9990

9. **[M]** $\mu_5 = 8.4233$, $\mu_6 = 8.4246$; actual value: 8.42443 (accurate to 5 places)

11. μ_k: 5.8000, 5.9655, 5.9942, 5.9990 $(k = 1, 2, 3, 4)$; $R(\mathbf{x}_k)$: 5.9655, 5.9990, 5.99997, 5.9999993

13. Yes, but the sequences may converge very slowly.

15. *Hint:* Write $A\mathbf{x} - \alpha\mathbf{x} = (A - \alpha I)\mathbf{x}$, and use the fact that $(A - \alpha I)$ is invertible when α is *not* an eigenvalue of A.

17. **[M]** $\nu_0 = 3.3384$, $\nu_1 = 3.32119$ (accurate to 4 places with rounding), $\nu_2 = 3.3212209$. Actual value: 3.3212201 (accurate to 7 places)

19. **a.** $\mu_6 = 30.2887 = \mu_7$ to four decimal places. To six places, the largest eigenvalue is 30.288685, with eigenvector (.957629, .688937, 1, .943782).

b. The inverse power method (with $\alpha = 0$) produces $\mu_1^{-1} = .010141$, $\mu_2^{-1} = .010150$. To seven places, the smallest eigenvalue is .0101500, with eigenvector $(-.603972, 1, -.251135, .148953)$. The reason for the rapid convergence is that the next-to-smallest eigenvalue is near .85.

21. **a.** If the eigenvalues of A are all less than 1 in magnitude, and if $\mathbf{x} \neq \mathbf{0}$, then $A^k\mathbf{x}$ is approximately an eigenvector for large k.

b. If the strictly dominant eigenvalue is 1, and if $\mathbf{x}$ has a component in the direction of the corresponding eigenvector, then $\{A^k\mathbf{x}\}$ will converge to a multiple of that eigenvector.

c. If the eigenvalues of A are all greater than 1 in magnitude, and if $\mathbf{x}$ is not an eigenvector, then the distance from $A^k\mathbf{x}$ to the nearest eigenvector will *increase* as $k \to \infty$.

Chapter 5 Supplementary Exercises, page 328

1.

a. T	**b.** F	**c.** T	**d.** F	**e.** T
f. T	**g.** F	**h.** T	**i.** F	**j.** T
k. F	**l.** F	**m.** F	**n.** T	**o.** F
p. T	**q.** F	**r.** T	**s.** F	**t.** T
u. T	**v.** T	**w.** F	**x.** T	

3. **a.** Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$. Then $(5I - A)\mathbf{x} = 5\mathbf{x} - A\mathbf{x} = 5\mathbf{x} - \lambda\mathbf{x} = (5 - \lambda)\mathbf{x}$. The eigenvalue is $5 - \lambda$.

b. $(5I - 3A + A^2)\mathbf{x} = 5\mathbf{x} - 3A\mathbf{x} + A(A\mathbf{x}) = 5\mathbf{x} - 3\lambda\mathbf{x} + \lambda^2\mathbf{x} = (5 - 3\lambda + \lambda^2)\mathbf{x}$. The eigenvalue is $5 - 3\lambda + \lambda^2$.

5. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$. Then

$$\begin{aligned} p(A)\mathbf{x} &= (c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n)\mathbf{x} \\ &= c_0\mathbf{x} + c_1 A\mathbf{x} + c_2 A^2\mathbf{x} + \cdots + c_n A^n\mathbf{x} \\ &= c_0\mathbf{x} + c_1\lambda\mathbf{x} + c_2\lambda^2\mathbf{x} + \cdots + c_n\lambda^n\mathbf{x} = p(\lambda)\mathbf{x} \end{aligned}$$

So $p(\lambda)$ is an eigenvalue of the matrix $p(A)$.

7. If $A = PDP^{-1}$, then $p(A) = Pp(D)P^{-1}$, as shown in Exercise 6. If the (j, j) entry in D is λ, then the (j, j) entry in D^k is λ^k, and so the (j, j) entry in $p(D)$ is $p(\lambda)$. If p is the characteristic polynomial of A, then $p(\lambda) = 0$ for each diagonal entry of D, because these entries in D are the eigenvalues of A. Thus $p(D)$ is the zero matrix. Thus $p(A) = P \cdot 0 \cdot P^{-1} = 0$.

9. If $I - A$ were not invertible, then the equation $(I - A)\mathbf{x} = \mathbf{0}$ would have a nontrivial solution $\mathbf{x}$. Then $\mathbf{x} - A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = 1 \cdot \mathbf{x}$, which shows that A would have 1 as an eigenvalue. This cannot happen if all the eigenvalues are less than 1 in magnitude. So $I - A$ must be invertible.

11. **a.** Take $\mathbf{x}$ in H. Then $\mathbf{x} = c\mathbf{u}$ for some scalar c. So $A\mathbf{x} = A(c\mathbf{u}) = c(A\mathbf{u}) = c(\lambda\mathbf{u}) = (c\lambda)\mathbf{u}$, which shows that $A\mathbf{x}$ is in H.

b. Let $\mathbf{x}$ be a nonzero vector in K. Since K is one-dimensional, K must be the set of all scalar multiples of $\mathbf{x}$. If K is invariant under A, then $A\mathbf{x}$ is in K and hence $A\mathbf{x}$ is a multiple of $\mathbf{x}$. Thus $\mathbf{x}$ is an eigenvector of A.

13. 1, 3, 7

15. Replace a by $a - \lambda$ in the determinant formula from Exercise 16 in Chapter 3 Supplementary Exercises:

$$\det(A - \lambda I) = (a - b - \lambda)^{n-1}[a - \lambda + (n - 1)b]$$

This determinant is zero only if $a - b - \lambda = 0$ or $a - \lambda + (n - 1)b = 0$. Thus λ is an eigenvalue of A if and only if $\lambda = a - b$ or $\lambda = a + (n - 1)$. From the formula for $\det(A - \lambda I)$ above, the algebraic multiplicity is $n - 1$ for $a - b$ and 1 for $a + (n - 1)b$.

17. $\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A$. Use the quadratic formula to solve the characteristic equation:

$$\lambda = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}}{2}$$

The eigenvalues are both real if and only if the discriminant is nonnegative, that is, $(\operatorname{tr} A)^2 - 4\det A \geq 0$. This inequality simplifies to $(\operatorname{tr} A)^2 \geq 4\det A$ and $\left(\frac{\operatorname{tr} A}{2}\right)^2 \geq \det A$.

19. $C_p = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$; $\det(C_p - \lambda I) = 6 - 5\lambda + \lambda^2 = p(\lambda)$

21. If p is a polynomial of order 2, then a calculation such as in Exercise 19 shows that the characteristic polynomial of C_p is $p(\lambda) = (-1)^2 p(\lambda)$, so the result is true for $n = 2$. Suppose the result is true for $n = k$ for some $k \geq 2$, and consider a polynomial p of degree $k + 1$. Then expanding $\det(C_p - \lambda I)$ by cofactors down the first column, the determinant of $C_p - \lambda I$ equals

$$(-\lambda)\det\begin{bmatrix} -\lambda & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 1 \\ -a_1 & -a_2 & \cdots & -a_k - \lambda \end{bmatrix} + (-1)^{k+1}a_0$$

The $k \times k$ matrix shown is $C_q - \lambda I$, where $q(t) = a_1 + a_2 t + \cdots + a_k t^{k-1} + t^k$. By the induction assumption, the determinant of $C_q - \lambda I$ is $(-1)^k q(\lambda)$. Thus

$$\begin{aligned}\det(C_p - \lambda I) &= (-1)^{k+1}a_0 + (-\lambda)(-1)^k q(\lambda) \\ &= (-1)^{k+1}[a_0 + \lambda(a_1 + \cdots + a_k\lambda^{k-1} + \lambda^k)] \\ &= (-1)^{k+1}p(\lambda)\end{aligned}$$

So the formula holds for $n = k + 1$ when it holds for $n = k$. By the principle of induction, the formula for $\det(C_p - \lambda I)$ is true for all $n \geq 2$.

23. From Exercise 22, the columns of the Vandermonde matrix V are eigenvectors of C_p, corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (the roots of the polynomial p). Since these eigenvalues are distinct, the eigenvectors form a linearly independent set, by Theorem 2 in Section 5.1. Thus V has linearly independent columns and hence is invertible, by the Invertible Matrix Theorem. Finally, since the columns of V are eigenvectors of C_p, the Diagonalization Theorem (Theorem 5 in Section 5.3) shows that $V^{-1}C_pV$ is diagonal.

25. [**M**] If your matrix program computes eigenvalues and eigenvectors by iterative methods rather than symbolic calculations, you may have some difficulties. You should find that $AP - PD$ has extremely small entries and PDP^{-1} is close to A. (This was true just a few years ago, but the situation could change as matrix programs continue to improve.) If you constructed P from the program's eigenvectors, check the condition number of P. This may indicate that you do not really have three linearly independent eigenvectors.

Chapter 6

Section 6.1, page 338

1. $5, 8, \frac{8}{5}$ **3.** $\begin{bmatrix} 3/35 \\ -1/35 \\ -1/7 \end{bmatrix}$ **5.** $\begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$

7. $\sqrt{35}$ **9.** $\begin{bmatrix} -.6 \\ .8 \end{bmatrix}$ **11.** $\begin{bmatrix} 7/\sqrt{69} \\ 2/\sqrt{69} \\ 4/\sqrt{69} \end{bmatrix}$

13. $5\sqrt{5}$ **15.** Not orthogonal **17.** Orthogonal

19. Refer to the *Study Guide* after you have written your answers.

21. *Hint:* Use Theorems 3 and 2 from Section 2.1.

23. $\mathbf{u}\cdot\mathbf{v} = 0$, $\|\mathbf{u}\|^2 = 30$, $\|\mathbf{v}\|^2 = 101$, $\|\mathbf{u}+\mathbf{v}\|^2 = (-5)^2 + (-9)^2 + 5^2 = 131 = 30 + 101$

25. The set of all multiples of $\begin{bmatrix} -b \\ a \end{bmatrix}$ (when $\mathbf{v} \neq \mathbf{0}$)

27. *Hint:* Use the definition of orthogonality.

29. *Hint:* Consider a typical vector $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ in W.

31. *Hint:* If $\mathbf{x}$ is in $W^\perp$, then $\mathbf{x}$ is orthogonal to every vector in W.

33. [**M**] State your conjecture and verify it algebraically.

Section 6.2, page 346

1. Not orthogonal **3.** Not orthogonal **5.** Orthogonal

7. Show $\mathbf{u}_1\cdot\mathbf{u}_2 = 0$, mention Theorem 4, and observe that two linearly independent vectors in $\mathbb{R}^2$ form a basis. Then obtain

$$\mathbf{x} = \tfrac{39}{13}\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \tfrac{26}{52}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

9. Show $\mathbf{u}_1\cdot\mathbf{u}_2 = 0$, $\mathbf{u}_1\cdot\mathbf{u}_3 = 0$, and $\mathbf{u}_2\cdot\mathbf{u}_3 = 0$. Mention Theorem 4, and observe that three linearly independent vectors in $\mathbb{R}^3$ form a basis. Then obtain

$$\mathbf{x} = \tfrac{5}{2}\mathbf{u}_1 - \tfrac{27}{18}\mathbf{u}_2 + \tfrac{18}{9}\mathbf{u}_3 = \tfrac{5}{2}\mathbf{u}_1 - \tfrac{3}{2}\mathbf{u}_2 + 2\mathbf{u}_3$$

11. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **13.** $\mathbf{y} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$

15. $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} .6 \\ -.8 \end{bmatrix}$, distance is 1

17. $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

19. Orthonormal **21.** Orthonormal

23. See the *Study Guide*.

25. *Hint:* $\|U\mathbf{x}\|^2 = (U\mathbf{x})^T(U\mathbf{x})$. Also, parts (a) and (c) follow from (b).

27. *Hint:* You need two theorems, one of which applies only to *square* matrices.

29. *Hint:* If you have a candidate for an inverse, you can check to see whether the candidate works.

31. Suppose $\hat{\mathbf{y}} = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$. Replace $\mathbf{u}$ by $c\mathbf{u}$ with $c \neq 0$; then

$$\frac{\mathbf{y}\cdot(c\mathbf{u})}{(c\mathbf{u})\cdot(c\mathbf{u})}(c\mathbf{u}) = \frac{c(\mathbf{y}\cdot\mathbf{u})}{c^2\mathbf{u}\cdot\mathbf{u}}(c)\mathbf{u} = \hat{\mathbf{y}}$$

33. Let $L = \text{Span}\{\mathbf{u}\}$, where $\mathbf{u}$ is nonzero, and let $T(\mathbf{x}) = \text{proj}_L\,\mathbf{x}$. By definition,

$$T(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = (\mathbf{x}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{u})^{-1}\mathbf{u}$$

For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and any scalars c and d, properties of the inner product (Theorem 1) show that

$$\begin{aligned} T(c\mathbf{x}+d\mathbf{y}) &= [(c\mathbf{x}+d\mathbf{y})\cdot\mathbf{u}](\mathbf{u}\cdot\mathbf{u})^{-1}\mathbf{u} \\ &= [c(\mathbf{x}\cdot\mathbf{u}) + d(\mathbf{y}\cdot\mathbf{u})](\mathbf{u}\cdot\mathbf{u})^{-1}\mathbf{u} \\ &= c(\mathbf{x}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{u})^{-1}\mathbf{u} + d(\mathbf{y}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{u})^{-1}\mathbf{u} \\ &= cT(\mathbf{x}) + dT(\mathbf{y}) \end{aligned}$$

Thus T is linear.

Section 6.3, page 354

1. $\mathbf{x} = -\frac{8}{9}\mathbf{u}_1 - \frac{2}{9}\mathbf{u}_2 + \frac{2}{3}\mathbf{u}_3 + 2\mathbf{u}_4$; $\quad \mathbf{x} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ **5.** $\begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} = \mathbf{y}$

7. $\mathbf{y} = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} + \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$ **9.** $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$

11. $\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ **13.** $\begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix}$ **15.** $\sqrt{40}$

17. a. $U^TU = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$UU^T = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$

b. $\text{proj}_W\,\mathbf{y} = 6\mathbf{u}_1 + 3\mathbf{u}_2 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$, and $(UU^T)\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$

19. Any multiple of $\begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix}$, such as $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

21. Write your answers before checking the *Study Guide*.

23. *Hint:* Use Theorem 3 and the Orthogonal Decomposition Theorem. For the uniqueness, suppose $A\mathbf{p} = \mathbf{b}$ and $A\mathbf{p}_1 = \mathbf{b}$, and consider the equations $\mathbf{p} = \mathbf{p}_1 + (\mathbf{p} - \mathbf{p}_1)$ and $\mathbf{p} = \mathbf{p} + \mathbf{0}$.

25. [**M**] U has orthonormal columns, by Theorem 6 in Section 6.2, because $U^TU = I_4$. The closest point to $\mathbf{y}$ in Col U is the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto Col U. From Theorem 10,

$\hat{\mathbf{y}} = UU^T\mathbf{y} = (1.2, .4, 1.2, 1.2, .4, 1.2, .4, .4)$

Section 6.4, page 360

1. $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$ **3.** $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$

5. $\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$ **7.** $\begin{bmatrix} 2/\sqrt{30} \\ -5/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

9. $\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$

13. $R = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$

15. $Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$,

$R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

17. See the *Study Guide*.

19. Suppose $\mathbf{x}$ satisfies $R\mathbf{x} = \mathbf{0}$; then $QR\mathbf{x} = Q\,\mathbf{0} = \mathbf{0}$, and $A\mathbf{x} = \mathbf{0}$. Since the columns of A are linearly independent, $\mathbf{x}$ must be zero. This fact, in turn, shows that the columns of R are linearly independent. Since R is square, it is invertible, by the Invertible Matrix Theorem.

21. Denote the columns of Q by $\mathbf{q}_1, \ldots, \mathbf{q}_n$. Note that $n \le m$, because A is $m \times n$ and has linearly independent columns. Use the fact that the columns of Q can be extended to an orthonormal basis for $\mathbb{R}^m$, say, $\{\mathbf{q}_1, \ldots, \mathbf{q}_m\}$. (The *Study Guide* describes one method.) Let $Q_0 = \begin{bmatrix} \mathbf{q}_{n+1} & \cdots & \mathbf{q}_m \end{bmatrix}$ and $Q_1 = \begin{bmatrix} Q & Q_0 \end{bmatrix}$. Then, using partitioned matrix multiplication, $Q_1\begin{bmatrix} R \\ 0 \end{bmatrix} = QR = A$.

23. *Hint:* Partition R as a 2×2 block matrix.

25. [**M**] The diagonal entries of R are 20, 6, 10.3923, and 7.0711, to four decimal places.

Section 6.5, page 368

1. a. $\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

3. a. $\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$

5. $\hat{\mathbf{x}} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ **7.** $2\sqrt{5}$

9. **a.** $\hat{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 2/7 \\ 1/7 \end{bmatrix}$

11. **a.** $\hat{\mathbf{b}} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix}$ **b.** $\hat{\mathbf{x}} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$

13. $A\mathbf{u} = \begin{bmatrix} 11 \\ -11 \\ 11 \end{bmatrix}$, $A\mathbf{v} = \begin{bmatrix} 7 \\ -12 \\ 7 \end{bmatrix}$,
$\mathbf{b} - A\mathbf{u} = \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix}$, $\mathbf{b} - A\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$. No, $\mathbf{u}$ could not possibly be a least-squares solution of $A\mathbf{x} = \mathbf{b}$. Why?

15. $\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ **17.** See the *Study Guide*.

19. **a.** If $A\mathbf{x} = \mathbf{0}$, then $A^TA\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$. This shows that Nul A is contained in Nul A^TA.

b. If $A^TA\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0$. So $(A\mathbf{x})^T(A\mathbf{x}) = 0$ (which means that $\|A\mathbf{x}\|^2 = 0$), and hence $A\mathbf{x} = \mathbf{0}$. This shows that Nul A^TA is contained in Nul A.

21. *Hint:* For part (a), use an important theorem from Chapter 2.

23. By Theorem 14, $\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^TA)^{-1}A^T\mathbf{b}$. The matrix $A(A^TA)^{-1}A^T$ occurs frequently in statistics, where it is sometimes called the *hat-matrix*.

25. The normal equations are $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$, whose solution is the set of (x, y) such that $x + y = 3$. The solutions correspond to points on the line midway between the lines $x + y = 2$ and $x + y = 4$.

Section 6.6, page 376

1. $y = .9 + .4x$ **3.** $y = 1.1 + 1.3x$

5. If two data points have different x-coordinates, then the two columns of the design matrix X cannot be multiples of each other and hence are linearly independent. By Theorem 14 in Section 6.5, the normal equations have a unique solution.

7. **a.** $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{y} = \begin{bmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{bmatrix}$, $X = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{bmatrix}$,

$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$, $\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$

b. [**M**] $y = 1.76x - .20x^2$

9. $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{y} = \begin{bmatrix} 7.9 \\ 5.4 \\ -.9 \end{bmatrix}$, $X = \begin{bmatrix} \cos 1 & \sin 1 \\ \cos 2 & \sin 2 \\ \cos 3 & \sin 3 \end{bmatrix}$,

$\boldsymbol{\beta} = \begin{bmatrix} A \\ B \end{bmatrix}$, $\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$

11. [**M**] $\beta = 1.45$ and $e = .811$; the orbit is an ellipse. The equation $r = \beta/(1 - e \cdot \cos\vartheta)$ produces $r = 1.33$ when $\vartheta = 4.6$.

13. [**M**] **a.** $y = -.8558 + 4.7025t + 5.5554t^2 - .0274t^3$

b. The velocity function is $v(t) = 4.7025 + 11.1108t - .0822t^2$, and $v(4.5) = 53.0$ ft/sec.

15. *Hint:* Write X and $\mathbf{y}$ as in equation (1), and compute X^TX and $X^T\mathbf{y}$.

17. **a.** The mean of the x-data is $\bar{x} = 5.5$. The data in mean-deviation form are $(-3.5, 1)$, $(-.5, 2)$, $(1.5, 3)$, and $(2.5, 3)$. The columns of X are orthogonal because the entries in the second column sum to 0.

b. $\begin{bmatrix} 4 & 0 \\ 0 & 21 \end{bmatrix}\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 9 \\ 7.5 \end{bmatrix}$,
$y = \frac{9}{4} + \frac{5}{14}x^* = \frac{9}{4} + \frac{5}{14}(x - 5.5)$

19. *Hint:* The equation has a nice geometric interpretation.

Section 6.7, page 384

1. **a.** $3, \sqrt{105}, 225$ **b.** All multiples of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

3. 28 **5.** $5\sqrt{2}, 3\sqrt{3}$ **7.** $\frac{56}{25} + \frac{14}{25}t$

9. **a.** Constant polynomial, $p(t) = 5$

b. $t^2 - 5$ is orthogonal to p_0 and p_1; values: $(4, -4, -4, 4)$; answer: $q(t) = \frac{1}{4}(t^2 - 5)$

11. $\frac{17}{5}t$

13. Verify each of the four axioms. For instance:

1.	$\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{u})\cdot(A\mathbf{v})$	Definition
	$= (A\mathbf{v})\cdot(A\mathbf{u})$	Property of the dot product
	$= \langle \mathbf{v}, \mathbf{u} \rangle$	Definition

15.

$\langle \mathbf{u}, c\mathbf{v} \rangle = \langle c\mathbf{v}, \mathbf{u} \rangle$	Axiom 1
$= c\langle \mathbf{v}, \mathbf{u} \rangle$	Axiom 3
$= c\langle \mathbf{u}, \mathbf{v} \rangle$	Axiom 1

17. *Hint:* Compute 4 times the right-hand side.

19. $\langle \mathbf{u}, \mathbf{v} \rangle = \sqrt{a}\sqrt{b} + \sqrt{b}\sqrt{a} = 2\sqrt{ab}$,
$\|\mathbf{u}\|^2 = (\sqrt{a})^2 + (\sqrt{b})^2 = a + b$. Since a and b are nonnegative, $\|\mathbf{u}\| = \sqrt{a + b}$. Similarly, $\|\mathbf{v}\| = \sqrt{b + a}$. By Cauchy–Schwarz, $2\sqrt{ab} \le \sqrt{a + b}\sqrt{b + a} = a + b$. Hence, $\sqrt{ab} \le \dfrac{a + b}{2}$.

21. 0 **23.** $2/\sqrt{5}$ **25.** $1, t, 3t^2 - 1$

27. [**M**] The new orthogonal polynomials are multiples of $-17t + 5t^3$ and $72 - 155t^2 + 35t^4$. Scale these polynomials so their values at $-2, -1, 0, 1$, and 2 are small integers.

Section 6.8, page 391

1. $y = 2 + \frac{3}{2}t$

3. $p(t) = 4p_0 - .1p_1 - .5p_2 + .2p_3$
$= 4 - .1t - .5(t^2 - 2) + .2\left(\frac{5}{6}t^3 - \frac{17}{6}t\right)$
(This polynomial happens to fit the data exactly.)

5. Use the identity

$$\sin mt \sin nt = \tfrac{1}{2}[\cos(mt - nt) - \cos(mt + nt)]$$

7. Use the identity $\cos^2 kt = \dfrac{1 + \cos 2kt}{2}$.

9. $\pi + 2\sin t + \sin 2t + \frac{2}{3}\sin 3t$ [*Hint:* Save time by using the results from Example 4.]

11. $\frac{1}{2} - \frac{1}{2}\cos 2t$ (Why?)

13. *Hint:* Take functions f and g in $C[0, 2\pi]$, and fix an integer $m \geq 0$. Write the Fourier coefficient of $f + g$ that involves $\cos mt$, and write the Fourier coefficient that involves $\sin mt\,(m > 0)$.

15. [**M**] The cubic curve is the graph of $g(t) = -.2685 + 3.6095t + 5.8576t^2 - .0477t^3$. The velocity at $t = 4.5$ seconds is $g'(4.5) = 53.4$ ft/sec. This is about .7% faster than the estimate obtained in Exercise 13 in Section 6.6.

Chapter 6 Supplementary Exercises, page 392

1.

a.	F	**b.**	T	**c.**	T	**d.**	F	**e.**	F	**f.**	T
g.	T	**h.**	T	**i.**	F	**j.**	T	**k.**	T	**l.**	F
m.	T	**n.**	F	**o.**	F	**p.**	T	**q.**	T	**r.**	F
s.	F										

2. *Hint:* If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal set and $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then the vectors $c_1\mathbf{v}_1$ and $c_2\mathbf{v}_2$ are orthogonal, and

$$\begin{aligned}\|\mathbf{x}\|^2 &= \|c_1\mathbf{v}_1 + c_2\mathbf{v}_2\|^2 = \|c_1\mathbf{v}_1\|^2 + \|c_2\mathbf{v}_2\|^2 \\ &= (|c_1|\|\mathbf{v}_1\|)^2 + (|c_2|\|\mathbf{v}_2\|)^2 = |c_1|^2 + |c_2|^2\end{aligned}$$

(Explain why.) So the stated equality holds for $p = 2$. Suppose that the equality holds for $p = k$, with $k \geq 2$, let $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$ be an orthonormal set, and consider $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} = \mathbf{u}_k + c_{k+1}\mathbf{v}_{k+1}$, where $\mathbf{u}_k = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$.

3. Given $\mathbf{x}$ and an orthonormal set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$, let $\hat{\mathbf{x}}$ be the orthogonal projection of $\mathbf{x}$ onto the subspace spanned by $\mathbf{v}_1, \dots, \mathbf{v}_p$. By Theorem 10 in Section 6.3,

$$\hat{\mathbf{x}} = (\mathbf{x}\cdot\mathbf{v}_1)\mathbf{v}_1 + \cdots + (\mathbf{x}\cdot\mathbf{v}_p)\mathbf{v}_p$$

By Exercise 2, $\|\hat{\mathbf{x}}\|^2 = |\mathbf{x}\cdot\mathbf{v}_1|^2 + \cdots + |\mathbf{x}\cdot\mathbf{v}_p|^2$. Bessel's inequality follows from the fact that $\|\hat{\mathbf{x}}\|^2 \leq \|\mathbf{x}\|^2$, noted before the statement of the Cauchy–Schwarz inequality, in Section 6.7.

5. Suppose $(U\mathbf{x})\cdot(U\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, and let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis for $\mathbb{R}^n$. For $j = 1, \dots, n$, $U\mathbf{e}_j$ is the jth column of U. Since $\|U\mathbf{e}_j\|^2 = (U\mathbf{e}_j)\cdot(U\mathbf{e}_j) = \mathbf{e}_j\cdot\mathbf{e}_j = 1$, the columns of U are unit vectors; since $(U\mathbf{e}_j)\cdot(U\mathbf{e}_k) = \mathbf{e}_j\cdot\mathbf{e}_k = 0$ for $j \neq k$, the columns are pairwise orthogonal.

7. *Hint:* Compute Q^TQ, using the fact that $(\mathbf{u}\mathbf{u}^T)^T = \mathbf{u}^{TT}\mathbf{u}^T = \mathbf{u}\mathbf{u}^T$.

9. Let $W = \operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$. Given $\mathbf{z}$ in $\mathbb{R}^n$, let $\hat{\mathbf{z}} = \operatorname{proj}_W \mathbf{z}$. Then $\hat{\mathbf{z}}$ is in Col A, where $A = \begin{bmatrix}\mathbf{u} & \mathbf{v}\end{bmatrix}$, say, $\hat{\mathbf{z}} = A\hat{\mathbf{x}}$ for some $\hat{\mathbf{x}}$ in $\mathbb{R}^2$. So $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{z}$. The normal equations can be solved to produce $\hat{\mathbf{x}}$, and then $\hat{\mathbf{z}}$ is found by computing $A\hat{\mathbf{x}}$.

11. *Hint:* Let $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$, and $A = \begin{bmatrix} \mathbf{v}^T \\ \mathbf{v}^T \\ \mathbf{v}^T \end{bmatrix} = \begin{bmatrix} 1 & -2 & 5 \\ 1 & -2 & 5 \\ 1 & -2 & 5 \end{bmatrix}$. The given set of equations is $A\mathbf{x} = \mathbf{b}$, and the set of all least-squares solutions coincides with the set of solutions of $A^TA\mathbf{x} = A^T\mathbf{b}$ (Theorem 13 in Section 6.5). Study this equation, and use the fact that $(\mathbf{v}\mathbf{v}^T)\mathbf{x} = \mathbf{v}(\mathbf{v}^T\mathbf{x}) = (\mathbf{v}^T\mathbf{x})\mathbf{v}$, because $\mathbf{v}^T\mathbf{x}$ is a scalar.

13. **a.** The row–column calculation of $A\mathbf{u}$ shows that each row of A is orthogonal to every $\mathbf{u}$ in Nul A. So each row of A is in $(\operatorname{Nul} A)^\perp$. Since $(\operatorname{Nul} A)^\perp$ is a subspace, it must contain all linear combinations of the rows of A; hence $(\operatorname{Nul} A)^\perp$ contains Row A.

b. If rank $A = r$, then $\dim \operatorname{Nul} A = n - r$, by the Rank Theorem. By Exercise 24(c) in Section 6.3,

$$\dim \operatorname{Nul} A + \dim(\operatorname{Nul} A)^\perp = n$$

So $\dim(\operatorname{Nul} A)^\perp$ must be r. But Row A is an r-dimensional subspace of $(\operatorname{Nul} A)^\perp$, by the Rank Theorem and part (a). Therefore, Row A must coincide with $(\operatorname{Nul} A)^\perp$.

c. Replace A by A^T in part (b) and conclude that Row A^T coincides with $(\operatorname{Nul} A^T)^\perp$. Since Row A^T = Col A, this proves (c).

15. If $A = URU^T$ with U orthogonal, then A is similar to R (because U is invertible and $U^T = U^{-1}$) and so A has the same eigenvalues as R (by Theorem 4 in Section 5.2), namely, the n real numbers on the diagonal of R.

17. [**M**] $\dfrac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} = .4618$,

$$\operatorname{cond}(A) \times \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} = 3363 \times (1.548 \times 10^{-4}) = .5206.$$

Observe that $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$ almost equals cond(A) times $\|\Delta\mathbf{b}\|/\|\mathbf{b}\|$.

19. [**M**] $\dfrac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} = 7.178 \times 10^{-8}$, $\dfrac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} = 2.832 \times 10^{-4}$.
Observe that the relative change in $\mathbf{x}$ is *much* smaller than the relative change in $\mathbf{b}$. In fact, since

$$\text{cond}(A) \times \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} = 23{,}683 \times (2.832 \times 10^{-4}) = 6.707$$

the theoretical bound on the relative change in $\mathbf{x}$ is 6.707 (to four significant figures). This exercise shows that even when a condition number is large, the relative error in a solution need not be as large as you might expect.

Chapter 7

Section 7.1, page 401

1. Symmetric **3.** Not symmetric **5.** Symmetric

7. Orthogonal, $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

9. Orthogonal, $\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$

11. Not orthogonal

13. $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

15. $P = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix}$

17. $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$,

$D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

19. $P = \begin{bmatrix} -1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix}$,

$D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

21. $P = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/2 & 1/2 \\ 1/\sqrt{2} & 0 & -1/2 & 1/2 \\ -1/\sqrt{2} & 0 & -1/2 & 1/2 \end{bmatrix}$,

$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

23. $P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$,

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

25. See the *Study Guide*.

27. $(A\mathbf{x})\cdot\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T\mathbf{y} = \mathbf{x}^T A\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y})$, because $A^T = A$.

29. *Hint:* Use an orthogonal diagonalization of A, or appeal to Theorem 2.

31. The Diagonalization Theorem in Section 5.3 says that the columns of P are (linearly independent) eigenvectors corresponding to the eigenvalues of A listed on the diagonal of D. So P has exactly k columns of eigenvectors corresponding to λ. These k columns form a basis for the eigenspace.

33. $A = 8\mathbf{u}_1\mathbf{u}_1^T + 6\mathbf{u}_2\mathbf{u}_2^T + 3\mathbf{u}_3\mathbf{u}_3^T$

$$= 8\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ 6\begin{bmatrix} 1/6 & 1/6 & -2/6 \\ 1/6 & 1/6 & -2/6 \\ -2/6 & -2/6 & 4/6 \end{bmatrix}$$

$$+ 3\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

35. *Hint:* $(\mathbf{u}\mathbf{u}^T)\mathbf{x} = \mathbf{u}(\mathbf{u}^T\mathbf{x}) = (\mathbf{u}^T\mathbf{x})\mathbf{u}$, because $\mathbf{u}^T\mathbf{x}$ is a scalar.

37. [**M**] $P = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$,

$D = \begin{bmatrix} 19 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -11 \end{bmatrix}$

39. [**M**] $P = \begin{bmatrix} 1/\sqrt{2} & 3/\sqrt{50} & -2/5 & -2/5 \\ 0 & 4/\sqrt{50} & -1/5 & 4/5 \\ 0 & 4/\sqrt{50} & 4/5 & -1/5 \\ 1/\sqrt{2} & -3/\sqrt{50} & 2/5 & 2/5 \end{bmatrix}$

$D = \begin{bmatrix} .75 & 0 & 0 & 0 \\ 0 & .75 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.25 \end{bmatrix}$

Section 7.2, page 408

1. a. $5x_1^2 + \frac{2}{3}x_1x_2 + x_2^2$ **b.** 185 **c.** 16

3. a. $\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$

5. a. $\begin{bmatrix} 3 & -3 & 4 \\ -3 & 2 & -2 \\ 4 & -2 & -5 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 3 & 2 \\ 3 & 0 & -5 \\ 2 & -5 & 0 \end{bmatrix}$

7. $\mathbf{x} = P\mathbf{y}$, where $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{y}^T D\mathbf{y} = 6y_1^2 - 4y_2^2$

In Exercises 9–14, other answers (change of variables and new quadratic form) are possible.

9. Positive definite; eigenvalues are 6 and 2

Change of variable: $\mathbf{x} = P\mathbf{y}$, with $P = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

New quadratic form: $6y_1^2 + 2y_2^2$

11. Indefinite; eigenvalues are 3 and -2

Change of variable: $\mathbf{x} = P\mathbf{y}$, with $P = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$

New quadratic form: $3y_1^2 - 2y_2^2$

13. Positive semidefinite; eigenvalues are 10 and 0

Change of variable: $\mathbf{x} = P\mathbf{y}$, with $P = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

New quadratic form: $10y_1^2$

15. **[M]** Negative definite; eigenvalues are $-13, -9, -7, -1$

Change of variable: $\mathbf{x} = P\mathbf{y}$;

$$P = \begin{bmatrix} 0 & -1/2 & 0 & 3/\sqrt{12} \\ 0 & 1/2 & -2/\sqrt{6} & 1/\sqrt{12} \\ -1/\sqrt{2} & 1/2 & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{2} & 1/2 & 1/\sqrt{6} & 1/\sqrt{12} \end{bmatrix}$$

New quadratic form: $-13y_1^2 - 9y_2^2 - 7y_3^2 - y_4^2$

17. **[M]** Positive definite; eigenvalues are 1 and 21.

Change of variable: $\mathbf{x} = P\mathbf{y}$;

$$P = \frac{1}{\sqrt{50}}\begin{bmatrix} 4 & 3 & 4 & -3 \\ -5 & 0 & 5 & 0 \\ 3 & -4 & 3 & 4 \\ 0 & 5 & 0 & 5 \end{bmatrix}$$

New quadratic form: $y_1^2 + y_2^2 + 21y_3^2 + 21y_4^2$

19. 8 **21.** See the *Study Guide*.

23. Write the characteristic polynomial in two ways:

$$\begin{aligned} \det(A - \lambda I) &= \det\begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix} \\ &= \lambda^2 - (a+d)\lambda + ad - b^2 \end{aligned}$$

and

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Equate coefficients to obtain $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1\lambda_2 = ad - b^2 = \det A$.

25. Exercise 28 in Section 7.1 showed that B^TB is symmetric. Also, $\mathbf{x}^TB^TB\mathbf{x} = (B\mathbf{x})^TB\mathbf{x} = \|B\mathbf{x}\|^2 \geq 0$, so the quadratic form is positive semidefinite, and we say that the matrix B^TB is positive semidefinite. *Hint:* To show that B^TB is positive definite when B is square and invertible, suppose that $\mathbf{x}^TB^TB\mathbf{x} = 0$ and deduce that $\mathbf{x} = \mathbf{0}$.

27. *Hint:* Show that $A + B$ is symmetric and the quadratic form $\mathbf{x}^T(A+B)\mathbf{x}$ is positive definite.

Section 7.3, page 415

1. $\mathbf{x} = P\mathbf{y}$, where $P = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$

3. **a.** 9 **b.** $\pm\begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$ **c.** 6

5. **a.** 6 **b.** $\pm\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ **c.** -4

7. $\pm\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ **9.** $5 + \sqrt{5}$ **11.** 3

13. *Hint:* If $m = M$, take $\alpha = 0$ in the formula for $\mathbf{x}$. That is, let $\mathbf{x} = \mathbf{u}_n$, and verify that $\mathbf{x}^TA\mathbf{x} = m$. If $m < M$ and if t is a number between m and M, then $0 \leq t - m \leq M - m$ and $0 \leq (t-m)/(M-m) \leq 1$. So let $\alpha = (t-m)/(M-m)$. Solve the expression for α to see that $t = (1-\alpha)m + \alpha M$. As α goes from 0 to 1, t goes from m to M. Construct $\mathbf{x}$ as in the statement of the exercise, and verify its properties.

15. **[M]** **a.** 9 **b.** $\begin{bmatrix} -2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ **c.** 3

17. **[M]** **a.** 34 **b.** $\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$ **c.** 26

Section 7.4, page 425

1. 3, 1 **3.** 4, 1

The answers in Exercises 5–13 are not the only possibilities.

5. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$
$\times\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

9. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$
$\times\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

11. $\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\times\begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

13. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

$\times \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$

15. a. rank $A = 2$

b. Basis for Col A: $\begin{bmatrix} .40 \\ .37 \\ -.84 \end{bmatrix}, \begin{bmatrix} -.78 \\ -.33 \\ -.52 \end{bmatrix}$

Basis for Nul A: $\begin{bmatrix} .58 \\ -.58 \\ .58 \end{bmatrix}$

(Remember that V^T appears in the SVD.)

17. If U is an orthogonal matrix then det $U = \pm 1$. If $A = U\Sigma V^T$ and A is square, then so are U, Σ, and V. Hence det A = det U det Σ det V^T $= \pm 1$ det $\Sigma = \pm\sigma_1 \cdots \sigma_n$

19. *Hint:* Since U and V are orthogonal,

$$\begin{aligned} A^TA &= (U\Sigma V^T)^T U\Sigma V^T = V\Sigma^T U^T U\Sigma V^T \\ &= V(\Sigma^T\Sigma)V^{-1} \end{aligned}$$

Thus V diagonalizes A^TA. What does this tell you about V?

21. The right singular vector $\mathbf{v}_1$ is an eigenvector for the largest eigenvalue λ_1 of A^TA. By Theorem 7 in Section 7.3, the largest eigenvalue, λ_2, is the maximum of $\mathbf{x}^T(A^TA)\mathbf{x}$ over all unit vectors orthogonal to $\mathbf{v}_1$. Since $\mathbf{x}^T(A^TA)\mathbf{x} = ||A\mathbf{x}||^2$, the square root of λ_2, which is the second largest eigenvalue, is the maximum of $||A\mathbf{x}||$ over all unit vector orthogonal to $\mathbf{v}_1$.

23. *Hint:* Use a column–row expansion of $(U\Sigma)V^T$.

25. *Hint:* Consider the SVD for the standard matrix of T—say, $A = U\Sigma V^T = U\Sigma V^{-1}$. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be bases constructed from the columns of V and U, respectively. Compute the matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$, as in Section 5.4. To do this, you must show that $V^{-1}\mathbf{v}_j = \mathbf{e}_j$, the jth column of I_n.

27. **[M]** $\begin{bmatrix} -.57 & -.65 & -.42 & .27 \\ .63 & -.24 & -.68 & -.29 \\ .07 & -.63 & .53 & -.56 \\ -.51 & .34 & -.29 & -.73 \end{bmatrix}$

$\times \begin{bmatrix} 16.46 & 0 & 0 & 0 & 0 \\ 0 & 12.16 & 0 & 0 & 0 \\ 0 & 0 & 4.87 & 0 & 0 \\ 0 & 0 & 0 & 4.31 & 0 \end{bmatrix}$

$\times \begin{bmatrix} -.10 & .61 & -.21 & -.52 & .55 \\ -.39 & .29 & .84 & -.14 & -.19 \\ -.74 & -.27 & -.07 & .38 & .49 \\ .41 & -.50 & .45 & -.23 & .58 \\ -.36 & -.48 & -.19 & -.72 & -.29 \end{bmatrix}$

29. **[M]** 25.9343, 16.7554, 11.2917, 1.0785, .00037793; $\sigma_1/\sigma_5 = 68{,}622$

Section 7.5, page 432

1. $M = \begin{bmatrix} 12 \\ 10 \end{bmatrix}$; $B = \begin{bmatrix} 7 & 10 & -6 & -9 & -10 & 8 \\ 2 & -4 & -1 & 5 & 3 & -5 \end{bmatrix}$;

$S = \begin{bmatrix} 86 & -27 \\ -27 & 16 \end{bmatrix}$

3. $\begin{bmatrix} .95 \\ -.32 \end{bmatrix}$ for $\lambda = 95.2$, $\begin{bmatrix} .32 \\ .95 \end{bmatrix}$ for $\lambda = 6.8$

5. **[M]** (.130, .874, .468), 75.9% of the variance

7. $y_1 = .95x_1 - .32x_2$; y_1 explains 93.3% of the variance.

9. $c_1 = 1/3, c_2 = 2/3, c_3 = 2/3$; the variance of y is 9.

11. a. If $\mathbf{w}$ is the vector in $\mathbb{R}^N$ with a 1 in each position, then

$$[\,\mathbf{X}_1 \;\cdots\; \mathbf{X}_N\,]\mathbf{w} = \mathbf{X}_1 + \cdots + \mathbf{X}_N = \mathbf{0}$$

because the $\mathbf{X}_k$ are in mean-deviation form. Then

$$\begin{aligned} &[\,\mathbf{Y}_1 \;\cdots\; \mathbf{Y}_N\,]\mathbf{w} \\ &\quad = [\,P^T\mathbf{X}_1 \;\cdots\; P^T\mathbf{X}_N\,]\mathbf{w} \qquad \text{By definition} \\ &\quad = P^T[\,\mathbf{X}_1 \;\cdots\; \mathbf{X}_N\,]\mathbf{w} = P^T\mathbf{0} = \mathbf{0} \end{aligned}$$

That is, $\mathbf{Y}_1 + \cdots + \mathbf{Y}_N = \mathbf{0}$, so the $\mathbf{Y}_k$ are in mean-deviation form.

b. *Hint:* Because the $\mathbf{X}_j$ are in mean-deviation form, the covariance matrix of the $\mathbf{X}_j$ is

$$1/(N-1)[\,\mathbf{X}_1 \;\cdots\; \mathbf{X}_N\,][\,\mathbf{X}_1 \;\cdots\; \mathbf{X}_N\,]^T$$

Compute the covariance matrix of the $\mathbf{Y}_j$, using part (a).

13. If $B = [\,\hat{\mathbf{X}}_1 \;\cdots\; \hat{\mathbf{X}}_N\,]$, then

$$\begin{aligned} S &= \frac{1}{N-1}BB^T = \frac{1}{N-1}[\,\hat{\mathbf{X}}_1 \;\cdots\; \hat{\mathbf{X}}_n\,]\begin{bmatrix} \hat{\mathbf{X}}_1^T \\ \vdots \\ \hat{\mathbf{X}}_N^T \end{bmatrix} \\ &= \frac{1}{N-1}\sum_1^N \hat{\mathbf{X}}_k\hat{\mathbf{X}}_k^T = \frac{1}{N-1}\sum_1^N (\mathbf{X}_k - \mathbf{M})(\mathbf{X}_k - \mathbf{M})^T \end{aligned}$$

Chapter 7 Supplementary Exercises, page 434

1. a. T **b.** F **c.** T **d.** F **e.** F **f.** F
g. F **h.** T **i.** F **j.** F **k.** F **l.** F
m. T **n.** F **o.** T **p.** T **q.** F

3. If rank $A = r$, then dim Nul $A = n - r$, by the Rank Theorem. So 0 is an eigenvalue of multiplicity $n - r$. Hence, of the n terms in the spectral decomposition of A, exactly $n - r$ are zero. The remaining r terms (corresponding to the nonzero eigenvalues) are all rank 1 matrices, as mentioned in the discussion of the spectral decomposition.

5. If $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero λ, then $\mathbf{v} = \lambda^{-1}A\mathbf{v} = A(\lambda^{-1}\mathbf{v})$, which shows that $\mathbf{v}$ is a linear combination of the columns of A.

7. *Hint:* If $A = R^T R$, where R is invertible, then A is positive definite, by Exercise 25 in Section 7.2. Conversely, suppose that A is positive definite. Then by Exercise 26 in Section 7.2, $A = B^T B$ for some positive definite matrix B. Explain why B admits a QR factorization, and use it to create the Cholesky factorization of A.

9. If A is $m \times n$ and $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x}^T A^T A \mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$. Thus $A^T A$ is positive semidefinite. By Exercise 22 in Section 6.5, rank $A^T A =$ rank A.

11. *Hint:* Write an SVD of A in the form $A = U\Sigma V^T = PQ$, where $P = U\Sigma U^T$ and $Q = UV^T$. Show that P is symmetric and has the same eigenvalues as Σ. Explain why Q is an orthogonal matrix.

13. **a.** If $\mathbf{b} = A\mathbf{x}$, then $\mathbf{x}^+ = A^+\mathbf{b} = A^+ A\mathbf{x}$. By Exercise 12(a), $\mathbf{x}^+$ is the orthogonal projection of $\mathbf{x}$ onto Row A.

b. From (a) and then Exercise 12(c), $A\mathbf{x}^+ = A(A^+ A\mathbf{x}) = (AA^+ A)\mathbf{x} = A\mathbf{x} = \mathbf{b}$.

c. Since $\mathbf{x}^+$ is the orthogonal projection onto Row A, the Pythagorean Theorem shows that $\|\mathbf{u}\|^2 = \|\mathbf{x}^+\|^2 + \|\mathbf{u} - \mathbf{x}^+\|^2$. Part (c) follows immediately.

15. [**M**] $A^+ = \dfrac{1}{40} \cdot \begin{bmatrix} -2 & -14 & 13 & 13 \\ -2 & -14 & 13 & 13 \\ -2 & 6 & -7 & -7 \\ 2 & -6 & 7 & 7 \\ 4 & -12 & -6 & -6 \end{bmatrix}$, $\hat{\mathbf{x}} = \begin{bmatrix} .7 \\ .7 \\ -.8 \\ .8 \\ .6 \end{bmatrix}$

The reduced echelon form of $\begin{bmatrix} A \\ \mathbf{x}^T \end{bmatrix}$ is the same as the reduced echelon form of A, except for an extra row of zeros. So adding scalar multiples of the rows of A to $\mathbf{x}^T$ can produce the zero vector, which shows that $\mathbf{x}^T$ is in Row A.

Basis for Nul A: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

Chapter 8

Section 8.1, page 444

1. Some possible answers: $\mathbf{y} = 2\mathbf{v}_1 - 1.5\mathbf{v}_2 + .5\mathbf{v}_3$, $\mathbf{y} = 2\mathbf{v}_1 - 2\mathbf{v}_3 + \mathbf{v}_4$, $\mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 7\mathbf{v}_3 + 3\mathbf{v}_4$

3. $\mathbf{y} = -3\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3$. The weights sum to 1, so this is an affine sum.

5. **a.** $\mathbf{p}_1 = 3\mathbf{b}_1 - \mathbf{b}_2 - \mathbf{b}_3 \in$ aff S since the coefficients sum to 1.

b. $\mathbf{p}_2 = 2\mathbf{b}_1 + 0\mathbf{b}_2 + \mathbf{b}_3 \notin$ aff S since the coefficients do not sum to 1.

c. $\mathbf{p}_3 = -\mathbf{b}_1 + 2\mathbf{b}_2 + 0\mathbf{b}_3 \in$ aff S since the coefficients sum to 1.

7. **a.** $\mathbf{p}_1 \in$ Span S, but $\mathbf{p}_1 \notin$ aff S

b. $\mathbf{p}_2 \in$ Span S, and $\mathbf{p}_2 \in$ aff S

c. $\mathbf{p}_3 \notin$ Span S, so $\mathbf{p}_3 \notin$ aff S

9. $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Other answers are possible.

11. See the *Study Guide*.

13. Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is a plane if and only if $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is linearly independent. Suppose c_2 and c_3 satisfy $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{0}$. Show that this implies $c_2 = c_3 = 0$.

15. Let $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$. To show that S is affine, it suffices to show that S is a flat, by Theorem 3. Let $W = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$. Then W is a subspace of $\mathbb{R}^n$, by Theorem 2 in Section 4.2 (or Theorem 12 in Section 2.8). Since $S = W + \mathbf{p}$, where $\mathbf{p}$ satisfies $A\mathbf{p} = \mathbf{b}$, by Theorem 6 in Section 1.5, S is a translate of W, and hence S is a flat.

17. A suitable set consists of any three vectors that are not collinear and have 5 as their third entry. If 5 is their third entry, they lie in the plane $z = 5$. If the vectors are not collinear, their affine hull cannot be a line, so it must be the plane.

19. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. Given any $t \in \mathbb{R}$, we must show that $\mathbf{z} = (1-t)\mathbf{p} + t\mathbf{q}$ is in $f(S)$. Now use definitions of $\mathbf{p}$ and $\mathbf{q}$, and the fact that f is linear. The complete proof is presented in the *Study Guide*.

21. Since B is affine, Theorem 2 implies that B contains all affine combinations of points of B. Hence B contains all affine combinations of points of A. That is, aff $A \subset B$.

23. Since $A \subset (A \cup B)$, it follows from Exercise 22 that aff $A \subset$ aff $(A \cup B)$. Similarly, aff $B \subset$ aff $(A \cup B)$, so [aff $A \cup$ aff B] $\subset$ aff $(A \cup B)$.

25. To show that $D \subset E \cap F$, show that $D \subset E$ and $D \subset F$. The complete proof is presented in the *Study Guide*.

Section 8.2, page 454

1. Affinely dependent and $2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$

3. The set is affinely independent. If the points are called $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ and $\mathbf{v}_4 = 16\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3$, but the weights in the linear combination do not sum to 1.

5. $-4\mathbf{v}_1 + 5\mathbf{v}_2 - 4\mathbf{v}_3 + 3\mathbf{v}_4 = \mathbf{0}$

7. The barycentric coordinates are $(-2, 4, -1)$.

9. See the *Study Guide*.

11. When a set of five points is translated by subtracting, say, the first point, the new set of four points must be linearly dependent, by Theorem 8 in Section 1.7, because the four points are in $\mathbb{R}^3$. By Theorem 5, the original set of five points is affinely dependent.

13. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent, then there exist c_1 and c_2, not both zero, such that $c_1 + c_2 = 0$ and $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Show that this implies $\mathbf{v}_1 = \mathbf{v}_2$. For the converse, suppose $\mathbf{v}_1 = \mathbf{v}_2$ and select specific c_1 and c_2 that show their affine dependence. The details are in the *Study Guide*.

15. a. The vectors $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are not multiples and hence are linearly independent. By Theorem 5, S is affinely independent.

b. $\mathbf{p}_1 \leftrightarrow \left(-\frac{6}{8}, \frac{9}{8}, \frac{5}{8}\right)$, $\mathbf{p}_2 \leftrightarrow \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{p}_3 \leftrightarrow \left(\frac{14}{8}, -\frac{5}{8}, -\frac{1}{8}\right)$, $\mathbf{p}_4 \leftrightarrow \left(\frac{6}{8}, -\frac{5}{8}, \frac{7}{8}\right)$, $\mathbf{p}_5 \leftrightarrow \left(\frac{1}{4}, \frac{1}{8}, \frac{5}{8}\right)$

c. $\mathbf{p}_6$ is $(-, -, +)$, $\mathbf{p}_7$ is $(0, +, -)$, and $\mathbf{p}_8$ is $(+, +, -)$.

17. Suppose $S = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is an affinely independent set. Then equation (7) has a solution, because $\mathbf{p}$ is in aff S. Hence equation (8) has a solution. By Theorem 5, the homogeneous forms of the points in S are linearly independent. Thus (8) has a unique solution. Then (7) also has a unique solution, because (8) encodes both equations that appear in (7).

The following argument mimics the proof of Theorem 7 in Section 4.4. If $S = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is an affinely independent set, then scalars $c_1, \ldots, c_k$ exist that satisfy (7), by definition of aff S. Suppose $\mathbf{x}$ also has the representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_k\mathbf{b}_k \quad \text{and} \quad d_1 + \cdots + d_k = 1 \tag{7a}$$

for scalars $d_1, \ldots, d_k$. Then subtraction produces the equation

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_k - d_k)\mathbf{b}_k \tag{7b}$$

The weights in (7b) sum to 0 because the c's and the d's separately sum to 1. This is impossible, unless each weight in (8) is 0, because S is an affinely independent set. This proves that $c_i = d_i$ for $i = 1, \ldots, k$.

19. If $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an affinely dependent set, then there exist scalars c_1, c_2, and c_3, not all zero, such that $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ and $c_1 + c_2 + c_3 = 0$. Now use the linearity of f.

21. Let $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Then $\det[\,\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}\,] = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} =$ $\det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{bmatrix}$, by the transpose property of the determinant (Theorem 5 in Section 3.2). By Exercise 30 in Section 3.3, this determinant equals 2 times the area of the triangle with vertices at $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

23. If $[\,\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}\,]\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \tilde{\mathbf{p}}$, then Cramer's rule gives $r = \det[\,\tilde{\mathbf{p}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}\,] / \det[\,\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}\,]$. By Exercise 21, the numerator of this quotient is twice the area of $\triangle\mathbf{pbc}$, and the denominator is twice the area of $\triangle\mathbf{abc}$. This proves the formula for r. The other formulas are proved using Cramer's rule for s and t.

25. The intersection point is $\mathbf{x}(4) =$
$$-.1\begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} + .6\begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix} + .5\begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 6.0 \\ -3.4 \end{bmatrix}.$$
It is not inside the triangle.

Section 8.3, page 461

1. See the *Study Guide*.

3. None are in conv S.

5. $\mathbf{p}_1 = -\frac{1}{6}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$, so $\mathbf{p}_1 \notin$ conv S.
$\mathbf{p}_2 = \frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{1}{6}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$, so $\mathbf{p}_2 \in$ conv S.

7. a. The barycentric coordinates of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ are, respectively, $\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{4}, \frac{3}{4}\right)$, and $\left(\frac{1}{2}, \frac{3}{4}, -\frac{1}{4}\right)$.

b. $\mathbf{p}_3$ and $\mathbf{p}_4$ are outside conv T. $\mathbf{p}_1$ is inside conv T. $\mathbf{p}_2$ is on the edge $\overline{\mathbf{v}_2\mathbf{v}_3}$ of conv T.

9. $\mathbf{p}_1$ and $\mathbf{p}_3$ are outside the tetrahedron conv S. $\mathbf{p}_2$ is on the face containing the vertices $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. $\mathbf{p}_4$ is inside conv S. $\mathbf{p}_5$ is on the edge between $\mathbf{v}_1$ and $\mathbf{v}_3$.

11. See the *Study Guide*.

13. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. The goal is to show that the line segment $\mathbf{y} = (1 - t)\mathbf{p} + t\mathbf{q}$, for $0 \le t \le 1$, is in $f(S)$. Use the linearity of f and the convexity of S to show that $\mathbf{y} = f(\mathbf{w})$ for some $\mathbf{w}$ in S. This will show that $\mathbf{y}$ is in $f(S)$ and that $f(S)$ is convex.

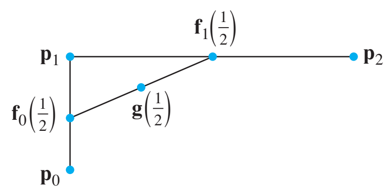
15. $\mathbf{p} = \frac{1}{6}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_4$ and $\mathbf{p} = \frac{1}{2}\mathbf{v}_1 + \frac{1}{6}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3$.

17. Suppose $A \subset B$, where B is convex. Then, since B is convex, Theorem 7 implies that B contains all convex combinations of points of B. Hence B contains all convex combinations of points of A. That is, conv $A \subset B$.

19. a. Use Exercise 18 to show that conv A and conv B are both subsets of conv $(A \cup B)$. This will imply that their union is also a subset of conv $(A \cup B)$.

b. One possibility is to let A be two adjacent corners of a square and let B be the other two corners. Then what is (conv A) $\cup$ (conv B), and what is conv $(A \cup B)$?

21.



23.
$$\begin{aligned} \mathbf{g}(t) &= (1 - t)\mathbf{f}_0(t) + t\mathbf{f}_1(t) \\ &= (1 - t)[(1 - t)\mathbf{p}_0 + t\mathbf{p}_1] + t[(1 - t)\mathbf{p}_1 + t\mathbf{p}_2] \\ &= (1 - t)^2\mathbf{p}_0 + 2t(1 - t)\mathbf{p}_1 + t^2\mathbf{p}_2. \end{aligned}$$

The sum of the weights in the linear combination for $\mathbf{g}$ is $(1-t)^2 + 2t(1-t) + t^2$, which equals $(1-2t+t^2) + (2t-2t^2) + t^2 = 1$. The weights are each between 0 and 1 when $0 \le t \le 1$, so $\mathbf{g}(t)$ is in $\text{conv}\,\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.

Section 8.4, page 469

1. $f(x_1, x_2) = 3x_1 + 4x_2$ and $d = 13$

3. **a.** Open **b.** Closed **c.** Neither
d. Closed **e.** Closed

5. **a.** Not compact, convex
b. Compact, convex
c. Not compact, convex
d. Not compact, not convex
e. Not compact, convex

7. **a.** $\mathbf{n} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ or a multiple
b. $f(\mathbf{x}) = 2x_2 + 3x_3,\ d = 11$

9. **a.** $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ or a multiple
b. $f(\mathbf{x}) = 3x_1 - x_2 + 2x_3 + x_4,\ d = 5$

11. $\mathbf{v}_2$ is on the same side as $\mathbf{0}$, $\mathbf{v}_1$ is on the other side, and $\mathbf{v}_3$ is in H.

13. One possibility is $\mathbf{p} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix},$
$\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$

15. $f(x_1, x_2, x_3, x_4) = x_1 - 3x_2 + 4x_3 - 2x_4$, and $d = 5$

17. $f(x_1, x_2, x_3) = x_1 - 2x_2 + x_3$, and $d = 0$

19. $f(x_1, x_2, x_3) = -5x_1 + 3x_2 + x_3$, and $d = 0$

21. See the *Study Guide*.

23. $f(x_1, x_2) = 3x_1 - 2x_2$ with d satisfying $9 < d < 10$ is one possibility.

25. $f(x, y) = 4x + y$. A natural choice for d is 12.75, which equals $f(3, .75)$. The point $(3, .75)$ is three-fourths of the distance between the center of $B(\mathbf{0}, 3)$ and the center of $B(\mathbf{p}, 1)$.

27. Exercise 2(a) in Section 8.3 gives one possibility. Or let $S = \{(x, y) : x^2y^2 = 1 \text{ and } y > 0\}$. Then $\text{conv}\,S$ is the upper (open) half-plane.

29. Let $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, \delta)$ and suppose $\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}$, where $0 \le t \le 1$. Then show that

$$\begin{aligned} \|\mathbf{z} - \mathbf{p}\| &= \|[(1-t)\mathbf{x} + t\mathbf{y}] - \mathbf{p}\| \\ &= \|(1-t)(\mathbf{x} - \mathbf{p}) + t(\mathbf{y} - \mathbf{p})\| < \delta. \end{aligned}$$

Section 8.5, page 481

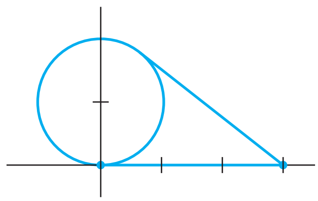
1. **a.** $m = 1$ at the point $\mathbf{p}_1$ **b.** $m = 5$ at the point $\mathbf{p}_2$
c. $m = 5$ at the point $\mathbf{p}_3$

3. **a.** $m = -3$ at the point $\mathbf{p}_3$
b. $m = 1$ on the set $\text{conv}\,\{\mathbf{p}_1, \mathbf{p}_3\}$
c. $m = -3$ on the set $\text{conv}\,\{\mathbf{p}_1, \mathbf{p}_2\}$

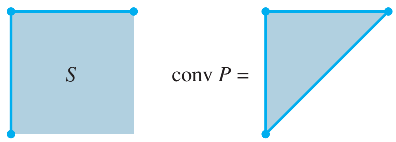
5. $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix} \right\}$

7. $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix} \right\}$

9. The origin is an extreme point, but it is not a vertex. Explain why.



11. One possibility is to let S be a square that includes part of the boundary but not all of it. For example, include just two adjacent edges. The convex hull of the profile P is a triangular region.



13. **a.** $f_0(C^5) = 32$, $f_1(C^5) = 80$, $f_2(C^5) = 80$, $f_3(C^5) = 40$, $f_4(C^5) = 10$, and $32 - 80 + 80 - 40 + 10 = 2$.

b.

	f_0	f_1	f_2	f_3	f_4
C^1	2				
C^2	4	4			
C^3	8	12	6		
C^4	16	32	24	8	
C^5	32	80	80	40	10

For a general formula, see the *Study Guide*.

15. **a.** $f_0(P^n) = f_0(Q) + 1$
b. $f_k(P^n) = f_k(Q) + f_{k-1}(Q)$
c. $f_{n-1}(P^n) = f_{n-2}(Q) + 1$

17. See the *Study Guide*.

19. Let S be convex and let $\mathbf{x} \in cS + dS$, where $c > 0$ and $d > 0$. Then there exist $\mathbf{s}_1$ and $\mathbf{s}_2$ in S such that $\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2$. But then

$$\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2 = (c+d)\left(\frac{c}{c+d}\mathbf{s}_1 + \frac{d}{c+d}\mathbf{s}_2\right).$$

Now show that the expression on the right side is a member of $(c+d)S$.

For the converse, pick a typical point in $(c+d)S$ and show it is in $cS + dS$.

21. *Hint:* Suppose A and B are convex. Let $\mathbf{x}, \mathbf{y} \in A + B$. Then there exist $\mathbf{a}, \mathbf{c} \in A$ and $\mathbf{b}, \mathbf{d} \in B$ such that $\mathbf{x} = \mathbf{a} + \mathbf{b}$ and $\mathbf{y} = \mathbf{c} + \mathbf{d}$. For any t such that $0 \le t \le 1$, show that

$$\mathbf{w} = (1-t)\mathbf{x} + t\mathbf{y} = (1-t)(\mathbf{a}+\mathbf{b}) + t(\mathbf{c}+\mathbf{d})$$

represents a point in $A + B$.

Section 8.6, page 492

1. The control points for $\mathbf{x}(t) + \mathbf{b}$ should be $\mathbf{p}_0 + \mathbf{b}$, $\mathbf{p}_1 + \mathbf{b}$, and $\mathbf{p}_3 + \mathbf{b}$. Write the Bézier curve through these points, and show algebraically that this curve is $\mathbf{x}(t) + \mathbf{b}$. See the *Study Guide*.

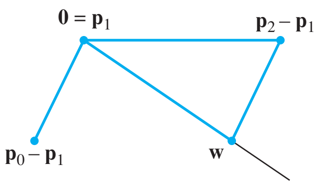
3. a. $\mathbf{x}'(t) = (-3 + 6t - 3t^2)\mathbf{p}_0 + (3 - 12t + 9t^2)\mathbf{p}_1 + (6t - 9t^2)\mathbf{p}_2 + 3t^2\mathbf{p}_3$, so $\mathbf{x}'(0) = -3\mathbf{p}_0 + 3\mathbf{p}_1 = 3(\mathbf{p}_1 - \mathbf{p}_0)$, and $\mathbf{x}'(1) = -3\mathbf{p}_2 + 3\mathbf{p}_3 = 3(\mathbf{p}_3 - \mathbf{p}_2)$. This shows that the tangent vector $\mathbf{x}'(0)$ points in the direction from $\mathbf{p}_0$ to $\mathbf{p}_1$ and is three times the length of $\mathbf{p}_1 - \mathbf{p}_0$. Likewise, $\mathbf{x}'(1)$ points in the direction from $\mathbf{p}_2$ to $\mathbf{p}_3$ and is three times the length of $\mathbf{p}_3 - \mathbf{p}_2$. In particular, $\mathbf{x}'(1) = \mathbf{0}$ if and only if $\mathbf{p}_3 = \mathbf{p}_2$.

b. $\mathbf{x}''(t) = (6 - 6t)\mathbf{p}_0 + (-12 + 18t)\mathbf{p}_1 + (6 - 18t)\mathbf{p}_2 + 6t\mathbf{p}_3$, so that

$\mathbf{x}''(0) = 6\mathbf{p}_0 - 12\mathbf{p}_1 + 6\mathbf{p}_2 = 6(\mathbf{p}_0 - \mathbf{p}_1) + 6(\mathbf{p}_2 - \mathbf{p}_1)$ and

$\mathbf{x}''(1) = 6\mathbf{p}_1 - 12\mathbf{p}_2 + 6\mathbf{p}_3 = 6(\mathbf{p}_1 - \mathbf{p}_2) + 6(\mathbf{p}_3 - \mathbf{p}_2)$

For a picture of $\mathbf{x}''(0)$, construct a coordinate system with the origin at $\mathbf{p}_1$, temporarily, label $\mathbf{p}_0$ as $\mathbf{p}_0 - \mathbf{p}_1$, and label $\mathbf{p}_2$ as $\mathbf{p}_2 - \mathbf{p}_1$. Finally, construct a line from this new origin through the sum of $\mathbf{p}_0 - \mathbf{p}_1$ and $\mathbf{p}_2 - \mathbf{p}_1$, extended out a bit. That line points in the direction of $\mathbf{x}''(0)$.



$\mathbf{w} = (\mathbf{p}_0 - \mathbf{p}_1) + (\mathbf{p}_2 - \mathbf{p}_1) = \frac{1}{6}\mathbf{x}''(0)$

5. a. From Exercise 3(a) or equation (9) in the text,

$$\mathbf{x}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2)$$

Use the formula for $\mathbf{x}'(0)$, with the control points from $\mathbf{y}(t)$, and obtain

$$\mathbf{y}'(0) = -3\mathbf{p}_3 + 3\mathbf{p}_4 = 3(\mathbf{p}_4 - \mathbf{p}_3)$$

For C^1 continuity, $3(\mathbf{p}_3 - \mathbf{p}_2) = 3(\mathbf{p}_4 - \mathbf{p}_3)$, so $\mathbf{p}_3 = (\mathbf{p}_4 + \mathbf{p}_2)/2$, and $\mathbf{p}_3$ is the midpoint of the line segment from $\mathbf{p}_2$ to $\mathbf{p}_4$.

b. If $\mathbf{x}'(1) = \mathbf{y}'(0) = \mathbf{0}$, then $\mathbf{p}_2 = \mathbf{p}_3$ and $\mathbf{p}_3 = \mathbf{p}_4$. Thus, the "line segment" from $\mathbf{p}_2$ to $\mathbf{p}_4$ is just the point $\mathbf{p}_3$. [*Note:* In this case, the combined curve is still C^1 continuous, by definition. However, some choices of the other "control" points, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_5$, and $\mathbf{p}_6$, can produce a curve with a visible corner at $\mathbf{p}_3$, in which case the curve is not G^1 continuous at $\mathbf{p}_3$.]

7. *Hint:* Use $\mathbf{x}''(t)$ from Exercise 3 and adapt this for the second curve to see that

$$\mathbf{y}''(t) = 6(1-t)\mathbf{p}_3 + 6(-2+3t)\mathbf{p}_4 + 6(1-3t)\mathbf{p}_5 + 6t\mathbf{p}_6$$

Then set $\mathbf{x}''(1) = \mathbf{y}''(0)$. Since the curve is C^1 continuous at $\mathbf{p}_3$, Exercise 5(a) says that the point $\mathbf{p}_3$ is the midpoint of the segment from $\mathbf{p}_2$ to $\mathbf{p}_4$. This implies that $\mathbf{p}_4 - \mathbf{p}_3 = \mathbf{p}_3 - \mathbf{p}_2$. Use this substitution to show that $\mathbf{p}_4$ and $\mathbf{p}_5$ are uniquely determined by $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. Only $\mathbf{p}_6$ can be chosen arbitrarily.

9. Write a vector of the polynomial weights for $\mathbf{x}(t)$, expand the polynomial weights, and factor the vector as $M_B\mathbf{u}(t)$:

$$\begin{bmatrix} 1 - 4t + 6t^2 - 4t^3 + t^4 \\ 4t - 12t^2 + 12t^3 - 4t^4 \\ 6t^2 - 12t^3 + 6t^4 \\ 4t^3 - 4t^4 \\ t^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{bmatrix},$$

$$M_B = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

11. See the *Study Guide*.

13. a. *Hint:* Use the fact that $\mathbf{q}_0 = \mathbf{p}_0$.

b. Multiply the first and last parts of equation (13) by $\frac{8}{3}$ and solve for $8\mathbf{q}_2$.

c. Use equation (8) to substitute for $8\mathbf{q}_3$ and then apply part (a).

15. a. From equation (11), $\mathbf{y}'(1) = .5\mathbf{x}'(.5) = \mathbf{z}'(0)$.

b. Observe that $\mathbf{y}'(1) = 3(\mathbf{q}_3 - \mathbf{q}_2)$. This follows from equation (9), with $\mathbf{y}(t)$ and its control points in place of $\mathbf{x}(t)$ and its control points. Similarly, for $\mathbf{z}(t)$ and its control points, $\mathbf{z}'(0) = 3(\mathbf{r}_1 - \mathbf{r}_0)$. By part (a),

$3(\mathbf{q}_3 - \mathbf{q}_2) = 3(\mathbf{r}_1 - \mathbf{r}_0)$. Replace $\mathbf{r}_0$ by $\mathbf{q}_3$, and obtain $\mathbf{q}_3 - \mathbf{q}_2 = \mathbf{r}_1 - \mathbf{q}_3$, and hence $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$.

c. Set $\mathbf{q}_0 = \mathbf{p}_0$ and $\mathbf{r}_3 = \mathbf{p}_3$. Compute $\mathbf{q}_1 = (\mathbf{p}_0 + \mathbf{p}_1)/2$ and $\mathbf{r}_2 = (\mathbf{p}_2 + \mathbf{p}_3)/2$. Compute $\mathbf{m} = (\mathbf{p}_1 + \mathbf{p}_2)/2$. Compute $\mathbf{q}_2 = (\mathbf{q}_1 + \mathbf{m})/2$ and $\mathbf{r}_1 = (\mathbf{m} + \mathbf{r}_2)/2$. Compute $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$ and set $\mathbf{r}_0 = \mathbf{q}_3$.

17. a. $\mathbf{r}_0 = \mathbf{p}_0, \mathbf{r}_1 = \dfrac{\mathbf{p}_0 + 2\mathbf{p}_1}{3}, \mathbf{r}_2 = \dfrac{2\mathbf{p}_1 + \mathbf{p}_2}{3}, \mathbf{r}_3 = \mathbf{p}_2$

b. *Hint:* Write the standard formula (7) in this section, with $\mathbf{r}_i$ in place of $\mathbf{p}_i$ for $i = 0, \ldots, 3$, and then replace $\mathbf{r}_0$ and $\mathbf{r}_3$ by $\mathbf{p}_0$ and $\mathbf{p}_2$, respectively:

$$\begin{aligned}\mathbf{x}(t) = {} & (1 - 3t + 3t^2 - t^3)\mathbf{p}_0 \\ & + (3t - 6t^2 + 3t^3)\mathbf{r}_1 \\ & + (3t^2 - 3t^3)\mathbf{r}_2 + t^3\mathbf{p}_2\end{aligned} \qquad (iii)$$

Use the formulas for $\mathbf{r}_1$ and $\mathbf{r}_2$ from part (a) to examine the second and third terms in this expression for $\mathbf{x}(t)$.