#1.1.5

eliminate coefficients of x_3 for 2 steps.

$$\begin{array}{c}\left[\begin{array}{cccc|c} 1 & -4 & 5 & 0 & 7 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]\end{array}\begin{array}{l} R1 \\ R2 \\ R3 \\ R4 \end{array}$$

1. $R2 \leftarrow R2 + 3R3$

$$\left[\begin{array}{ccccc} 1 & -4 & 5 & 0 & 7 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]\begin{array}{l} R1 \\ R2 \\ R3 \\ R4 \end{array}$$

2. $R1 \leftarrow R1 - 5R3$

$$\left[\begin{array}{ccccc} 1 & -4 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]\begin{array}{l} R1 \\ R2 \\ R3 \\ R4 \end{array}$$

3. $R1 \leftarrow R1 + 4R2$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 45 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]\begin{array}{l} R1 \\ R2 \\ R3 \\ R4 \end{array}$$

#1.1.16

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{array}\right]\begin{array}{l} R1 \\ R2 \\ R3 \\ R4 \end{array}$$

make the augmented matrix triangular form

$R4 \leftarrow R4 + 2R1$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{array}\right]$$

$R4 \leftarrow R4 - \frac{3}{2} R2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{array}\right]$$

$R4 \leftarrow R4 + R3$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(R4 means x_4 is a free variable. Thus, this linear system has infinite number of solution, so it is consistent.

or

(No row $[0 \cdots 0\, b]$, $(b \neq 0)$ in the matrix. Thus, the linear system is consistent.

#1.1.18

'three planes have at least one common point of intersection' means the three planes have at least one solution, so a linear system of the three planes is consistent then, they have common point of intersection.

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_2 - x_3 = 1 \\ x_1 + 3x_2 = 0 \end{cases}$$

augmented matrix of the linear system.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \end{array} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

R3 means $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -5$.

Thus, the solution of the linear system does not exist. Then, the common point of intersection also does not exist.

#1.1.29

$$\begin{bmatrix} 0 & -2 & 5 \\ 1 & 4 & -7 \\ 3 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -7 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{bmatrix} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

interchange R1 & R2 in R1/R2 to be R2/R1.

$\boxed{R1 \Leftrightarrow R2}$

#1.2.10

$$\begin{array}{c} \left[\begin{array}{ccc:c} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{array}\right] \end{array} \begin{array}{l} R1 \\ R2 \end{array}$$

$R2 \leftarrow R2 - 3R1$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

$x_3 = -7$

$x_1 - 2x_2 + 7 = 3$

$3x_1 - 6x_2 - 2 \times (-7) = 2$

$\rightarrow$ $x_1 - 2x_2 = -4$ $\quad - ①$

$3x_1 - 6x_2 = -12$ $\quad - ② = ① \times 3$

general solution :

$x_3 = -7$, $\quad x_1 - 2x_2 = -4$

#1.3.14

Does $A\mathbf{x}=\mathbf{b}$ have solution?

$$(A|\mathbf{b}) = \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{array}\right]$$

the matrix has a solution because it has no free variables.

#1.3.18

h should make $\mathbf{y}$ exist in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{y}$, $(a_1, a_2 \in \mathbb{R})$

$$\left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right]$$

if the matrix above is consistent, then $\mathbf{y}$ is in the plane by $\mathbf{v}_1$ & $\mathbf{v}_2$.

$\mathbf{v}_1$ and $\mathbf{v}_2$ should be linearly independent to span plane.

$$\sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix}$$

$a_2 = -5$, $2a_2 = -3+2h = -5\times 2 = -10$

$h = -\frac{7}{2}$

$a_1 - 3(-5) = -\frac{7}{2}$

$a_1 = -\frac{37}{2}$ $a_2 = -5$

When $h = -\frac{7}{2}$, the matrix has a solution.

or

$$\sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 7+2h \end{bmatrix}$$

To make the matrix has a solution

$0 = 7+2h$ is satisfied.

Thus, $h = -\frac{7}{2}$.

#1.4.11

$A\mathbf{x}=\mathbf{b}$

$$(A|\mathbf{b}) = \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution of matrix $(\mathbf{x}) = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$

#1.4.22

$$A = \begin{bmatrix} \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (a,b,c \in \mathbb{R})$$

Say $\mathbf{b}$ is any arbitrary 3-dimensional vector.
if $A\mathbf{x}=\mathbf{b}$ is consistent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$.

$$\begin{bmatrix} 0 & 0 & 4 & a \\ 0 & -3 & -1 & b \\ -2 & 8 & -5 & c \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & -5 & c \\ 0 & -3 & -1 & b \\ 0 & 0 & 4 & a \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 8 & -5 & c \\ 0 & -3 & -1 & b \\ 0 & 0 & 1 & a/4 \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & 0 & c+\frac{5a}{4} \\ 0 & -3 & 0 & b+a/4 \\ 0 & 0 & 1 & a/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 0 & 0 & c+\frac{5}{4}a+\frac{8}{3}(b+a/4) \\ 0 & -3 & 0 & b+a/4 \\ 0 & 0 & 1 & a/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2}\left(c+\frac{5}{4}a+\frac{8}{3}b+\frac{2}{3}a\right) \\ 0 & 1 & 0 & -\frac{1}{3}(b+a/4) \\ 0 & 0 & 1 & a/4 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\left(c+\frac{8}{3}b+\frac{23}{12}a\right) \\ -\frac{1}{3}(b+a/4) \\ a/4 \end{bmatrix}$$

$A\mathbf{x}=b$ has a unique solution because $(a,b,c \in \mathbb{R})$, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

or

A has a pivot position in every row.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$

#1.5.14

$$\mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_4 \\ 8+x_4 \\ 2-5x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_4 \\ x_4 \\ -5x_4 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

$\mathbb{X} = x_4\mathbb{V} + \mathbb{P}$, $\mathbb{V} = \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix}$, $\mathbb{P} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix}$

"the line through $\mathbb{P}$ in $\mathbb{R}^4$"

#1.5.18

$x_1 - 3x_2 + 5x_3 = 0$ — ①

$x_1 - 3x_2 + 5x_3 = 4$ — ②

Are 2 free variables needed?

~~#1.8.1~~

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$T(u) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$

~~#1.8.9~~

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (x_1, x_2, x_3, x_4 \in \mathbb{R})$$

$$\begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

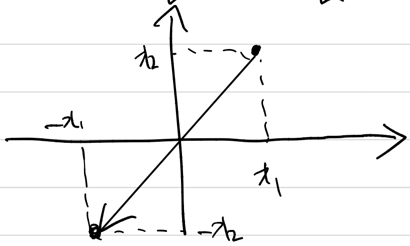
$$\begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & -9 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 and x_4 is free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

#1.8.13

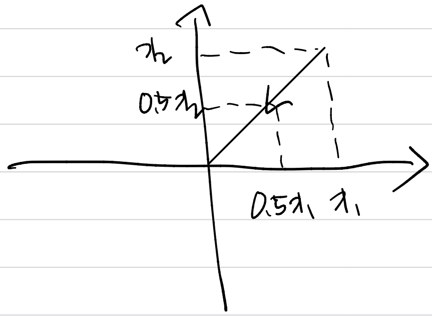
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$



reflection

#1.8.14

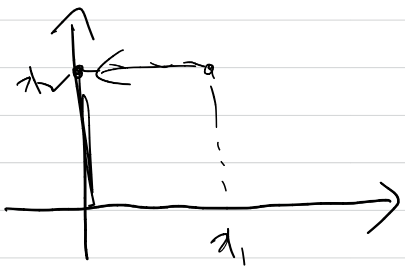
$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5x_1 \\ 0.5x_2 \end{bmatrix}$$



contraction

#1.8.15

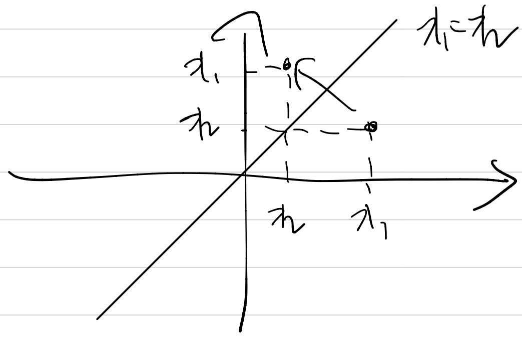
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$



projection onto x_2-axis

#1.8.16

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$



reflection