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Chapter 5. **Eigenvalues and Eigenvectors**

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5.1. Eigenvectors and Eigenvalues

Eigenvectors and Eigenvalues

• Definition

- An **eigenvector** of an $n \times n$ matrix A is a nonzero vector \mathbf{x} such that $A\mathbf{x} = \lambda \mathbf{x}$ for some scalar λ .
- A scalar λ is called an **eigenvalue** of A if there is a **nontrivial solution x** of $A\mathbf{x} = \lambda \mathbf{x}$, such an x is called an **eigenvector corresponding to λ** .

• Remark

- λ is an eigenvalue of an $n \times n$ matrix A if and only if the following equation has **nontrivial solution**.

$$(A - \lambda I)\mathbf{x} = 0 \quad (3)$$

- The set of all solutions of (3) is just the null space of the matrix $A - \lambda I$.
- So this set is a subspace of \mathbb{R}^n and is called the **eigenspace** of A corresponding to λ .
- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ .

In order for $(A - \lambda I)\mathbf{x} = 0$ has nontrivial solution \rightarrow space spanned by eigenvectors.

$\Leftrightarrow A$ is not invertible.

$\Leftrightarrow \det(A - \lambda I) = 0$ a.k.a. characteristic equation.

- **Example 3:** Show that 7 is an eigenvalue of matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.
- **Solution:** (proof that 7 is eigenvalue)
 - The scalar 7 is an eigenvalue of A if and only if the equation has a nontrivial solution.

$$A\mathbf{x} = 7\mathbf{x} \quad (1)$$

- But (1) is equivalent to $A\mathbf{x} - 7\mathbf{x} = 0$ or

$$(A - 7I)\mathbf{x} = 0 \quad (2)$$

- To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \boxed{\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}}$$

(A - 7I) $\mathbf{x} = 0$ has non-trivial solution!

- The columns of $A - 7I$ are obviously linearly dependent, so (2) has nontrivial solutions.

• **Solution:** (finding its corresponding eigenvector) $A\mathbf{x} = 7\mathbf{x}$

- To find the corresponding eigenvectors, use row operations:

$$\left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

- The general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda=7$, its corresponding eigenvectors are $[1], [^1], [^2], [^3], [^4], \dots$
- Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$.
- (If you multiply a constant to an eigenvector, it is again an eigenvector.)

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

① eigen-value (λ such that $A\mathbf{x} = \lambda\mathbf{x}$)

$$A - \lambda I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}$$

characteristic equation

$$\det(A - \lambda I) = (1-\lambda)(2-\lambda) - (6)(5) = 2 - 3\lambda + \lambda^2 - 30 = \underbrace{\lambda^2 - 3\lambda - 28}_{= 0} = (\lambda - 7)(\lambda + 4) = 0$$

$$\begin{cases} \lambda_1 = 7 \\ \lambda_2 = -4 \end{cases}$$

② eigen vectors (\mathbf{x} such that $A\mathbf{x} = \lambda\mathbf{x}$)

i) For $\lambda_1 = 7$ non-zero

$$A - 7I = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

$\downarrow 1 \quad 2$

$(A - 7I)\mathbf{x} = 0$ has augmented matrix.

$$\left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 5 & -5 & 0 \end{array} \right]$$

$$\therefore \mathbf{x}_1 = \text{d}_{\text{lin}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \dots$$

ii) For $\lambda_2 = -4$

$$(A + 4I) = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$$

$\downarrow 1 \quad 2$

$(A + 4I)\mathbf{x} = 0$ has augmented matrix,

$$\left[\begin{array}{cc|c} 5 & 6 & 0 \\ 5 & 6 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\therefore \mathbf{x}_2 = \text{d}_{\text{lin}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \dots$$

(3) Eigen vector matrix.

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 5 \end{bmatrix}$$

in this case, P is invertible. $P^{-1} = \frac{1}{5+6} \begin{bmatrix} 5 & 6 \\ -1 & 1 \end{bmatrix}$
however, P is not always invertible.

Eigen value matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix} = D$$

diagonal

(4) Diagonalization "A = [1 6; 1 5]".

$$A = PDP^{-1}, \text{ when } P \text{ is eigen-vector matrix } P = \begin{bmatrix} 1 & 6 \\ 1 & 5 \end{bmatrix} \quad P^{-1} = \frac{1}{11} \begin{bmatrix} 5 & 6 \\ -1 & 1 \end{bmatrix}$$

D is eigen-value matrix $D = \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix} \frac{1}{11} \begin{bmatrix} 5 & 6 \\ -1 & 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 7 & 24 \\ 7 & -20 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -1 & 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 11 & 66 \\ 55 & 11 \end{bmatrix}$$

equivalently, $A = PDP^{-1} \Leftrightarrow AP = PD$

check this!

$$\begin{bmatrix} 1 & 6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$$

$A = PDP^{-1}$. \leftrightarrow A & D are similar.

\leftrightarrow A & D have same eigen values.

A to D is similar transformation.

$$\underbrace{A}_{\text{original}} = \underbrace{P}_{\text{eigen vector matrix}} \cdot \underbrace{D}_{\text{eigen value matrix}} \cdot \underbrace{P^{-1}}_{\text{inverse of eigen vector matrix.}}$$

$A = LU$... LU factorization.

possible if A is non singular.

$A = PDP^{-1}$... Eigen factorization

possible if eigen vectors are linearly independent

P^{-1} can be found.

- Example 4: Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is λ_1 . Find a basis for the corresponding eigenspace.

Solution:

- Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathbf{x} = 0$.

$$\left[\begin{array}{cccc} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array} \right] \sim \left[\begin{array}{cccc} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

- At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A - 2I)\mathbf{x} = 0$ has free variables.

- The general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, x_2 and x_3 free.

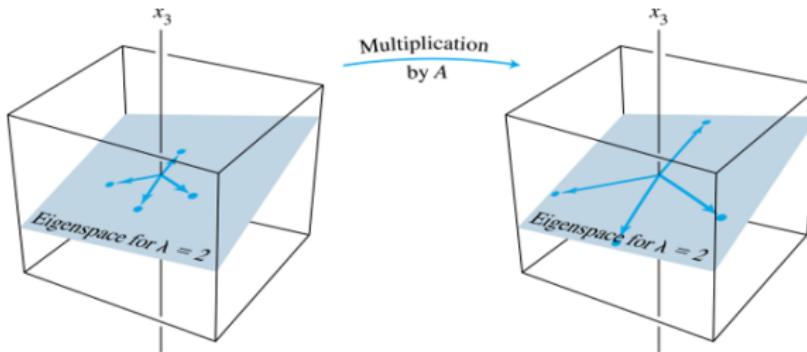
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

λ_1 corresponds to x_1 & x_2

• (*Solution continued*)

- The eigenspace, shown in the following figure, is a two-dimensional subspace of \mathbb{R}^3 . A basis is
composing vectors are linearly independent

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$



A acts as a dilation on the eigenspace.

- **Theorem 1:** The eigenvalues of a triangular matrix are the entries on its main diagonal.

• Proof:

- For simplicity, consider the 3×3 case. If A is upper triangular, the $A - \lambda I$ has the form

$$A - 2I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

- The scalar λ is an eigenvalue of A if and only if the equation $(A - \lambda I)\mathbf{x} = 0$ has a nontrivial solution, that is, if and only if the equation has a free variable.
- Because of the zero entries in $A - \lambda I$, it is easy to see that $(A - \lambda I)\mathbf{x} = 0$ has a free variable if and only if at least one of the entries on the diagonal of $A - \lambda I$ is zero.
- This happens if and only if λ equals one of the entries a_{11}, a_{22}, a_{33} in A .
- Sim: “The equation $((A - \lambda I)\mathbf{x} = 0)$ has a nontrivial solution if and only if $|A - \lambda I| = 0$.
- Sim: This equation is also called as characteristic polynomial.

- **Theorem 2:** If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A , then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.
 $(r \leq n)$ *distinct eigenvalue \Leftrightarrow eigenvectors are linearly independent.*

- **Proof:**

- Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly dependent. Since \mathbf{v}_1 is nonzero, Theorem 7 in Section 1.7 says that one of the vectors in the set is a linear combination of the preceding vectors.
- Let p be the least index such that \mathbf{v}_{p+1} is a linear combination of the preceding (linearly independent) vectors.
- Then, there exist scalars c_1, \dots, c_p such that

$$c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p = \mathbf{v}_{p+1} \quad (5)$$

- Multiplying both sides of (5) by A and using the fact that $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$ for each k , we obtain $c_1 A\mathbf{v}_1 + \cdots + c_p A\mathbf{v}_p = A\mathbf{v}_{p+1}$

$$c_1 \lambda_1 \mathbf{v}_1 + \cdots + c_p \lambda_p \mathbf{v}_p = \lambda_{p+1} \mathbf{v}_{p+1} \quad (6)$$

- Multiplying both sides of (5) by λ_{p+1} and subtracting the result from (6), we have

$$c_1 (\lambda_1 - \lambda_{p+1}) \mathbf{v}_1 + \cdots + c_p (\lambda_p - \lambda_{p+1}) \mathbf{v}_p = 0 \quad (7)$$

- Since $\mathbf{v}_1, \dots, \mathbf{v}_r$ is linearly independent, the weights in (6) are all zero.

• (*Proof continued*)

- But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct. Hence, $c_i = 0$ for $i = 1, \dots, p$.
- But then (4) says that $\mathbf{v}_{p+1} = 0$, which is impossible.
- Hence $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent.
- If A is an $n \times n$ matrix, then (8) is a recursive description of a sequence $\{x_k\}$ in \mathbb{R}^n .

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad (k = 0, 1, 2\dots) \quad (8)$$

- A **solution** of (8) is an explicit description of $\{x_k\}$ whose formula for each x_k does not depend directly on A or on the preceding terms in the sequence other than the initial term \mathbf{x}_0 .
- The simplest way to build a solution of (8) is to take an eigenvector \mathbf{x}_0 and its corresponding eigenvalue λ and let

$$\mathbf{x}_k = \lambda^k \mathbf{x}_0 \quad (k = 1, 2\dots) \quad (9)$$

- This sequence is a solution because

$$A\mathbf{x}_k = A(\lambda^k \mathbf{x}_0) = \lambda^k (A\mathbf{x}_0) = \lambda^k (\lambda \mathbf{x}_0) = \lambda^{k+1} \mathbf{x}_0 = \mathbf{x}_{k+1}$$

□

Suggested Exercises

- 5.1.3
- 5.1.6
- 5.1.15

5.2. The Characteristic Equation

Determinants (review of)

- Let A be an $n \times n$ matrix, let U be any echelon form obtained from A by row replacements and row interchanges (without scaling), and let r be the number of such row interchanges.
- Then the **determinant** of A , written as $\det A$, is $(-1)^r$ times the product of the diagonal entries u_{11}, \dots, u_{nn} in U .
- If A is invertible, then u_{11}, \dots, u_{nn} are all pivots (because $A \sim I_n$ and the u_{ii} have not been scaled to 1's).
- Otherwise, at least u_{nn} is zero, and the product u_{11}, \dots, u_{nn} is zero.
- Thus,

$$\det A = \begin{cases} (-1)^r \cdot (\text{product of pivots in } U), & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is not invertible} \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- Example 1: Compute $\det A$ for $A =$

Solution:

- The following row reduction uses one row interchange:

$$A \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = U_1$$

- So $\det A$ equals $(-1)^1(1)(-2)(-1) = -2$.
- The following alternative row reduction avoids the row interchange and produces a different echelon form.
- The last step adds $-1/3$ times row 2 to row 3:

$$A \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & 0 & 1/3 \end{bmatrix} = U_2$$

- This time $\det A$ is $(-1)^0(1)(-6)(1/3) = -2$, the same as before.

Theorems

- **The invertible matrix theorem (continued):** Let A be an $n \times n$ matrix. Then A is invertible if and only if:
 - s. The number 0 is not an eigenvalue of A . $\Leftrightarrow 0$ is not eigen value of A .
 - t. The determinant of A is not zero.
- $\Rightarrow |A| \neq 0, |A - \lambda I| \neq 0$
- **Theorem 3: (Properties of Determinants)** Let A and B be $n \times n$ matrices.
 - a. A is invertible if and only if $\det A \neq 0$.
 - b. $\det AB = (\det A)(\det B)$.
 - c. $\det A^T = \det A$.
 - d. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A .
 - e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.
- Remark
 - Theorem 3(a) shows how to determine when a matrix of the form $A - \lambda I$ is not invertible.

The characteristic equation

- The scalar equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** (or, **characteristic polynomial**) of A .
- A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$\det(A - \lambda I) = 0$
how to find eigen values.

Characteristic equation is n^{th} order polynomial equation
if A is $n \times n$ matrix.

- Example 3: Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Solution:

- Form $A - \lambda I$, and use Theorem 3(d):

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 5 - \lambda & -2 & 6 & -1 \\ 0 & 3 - \lambda & -8 & 0 \\ 0 & 0 & 5 - \lambda & 4 \\ 0 & 0 & 0 & 1 - \lambda \end{bmatrix} \\ &= (5 - \lambda)(3 - \lambda)(5 - \lambda)(1 - \lambda) \end{aligned}$$

- The characteristic equation is

$$(5 - \lambda)^2(3 - \lambda)(1 - \lambda) = 0 \text{ or } (\lambda - 5)^2(\lambda - 3)(\lambda - 1) = 0$$

- Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

preferred description of λ

$\lambda_1 = 5$ ↗ multiplicity root! (2x2)
 $\lambda_2 = 3$
 $\lambda_3 = 1$
 $\lambda_4 = 1$ ↗ multiplicity root!

• Remark

- If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the **characteristic polynomial** of A .
- The eigenvalue 5 in Example 3 is said to have *multiplicity* 2 because $(\lambda - 5)$ occurs two times as a factor of the characteristic polynomial.
- In general, the **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Similarity

- If A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.
- Writing Q for P^{-1} , we have $Q^{-1}BQ = A$.
- So B is also similar to A , and we say simply that A and B are similar.
- Changing A into $\underline{P^{-1}AP}$ is called a **similarity transformation**.
 $= B$

- **Theorem 4:** If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

- **Proof:**

- If $B = P^{-1}AP$ then,

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$
- Using the multiplicative property in Theorem 3(b), we compute

$$\det(B - \lambda I) = \det[P^{-1}(A - \lambda I)P] = \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P)$$
- Since $\det(P^{-1}) \cdot \det(P) = \det(P^{-1}P) = \det I = 1$, we see from the previous equation that $\det(B - \lambda I) = \det(A - \lambda I)$.

• Warnings:

- The matrices are not similar even though they have the same eigenvalues.

Similar \Rightarrow same eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

- Similarity is not the same as row equivalence. (If A is row equivalent to B , then $B = EA$ for some invertible matrix E). Row operations on a matrix usually change its eigenvalues.

Suggested Exercises

- 5.2.3

5.3. Diagonalization

Diagonalization (its benefit and definition)

- **Example 2:** Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k , given that $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$
 - **Solution:**
 - The standard formula for the inverse of a 2×2 matrix yields $P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$
 - Then, by associativity of matrix multiplication,

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) = PD \underbrace{(P^{-1}P)}_I DP^{-1} = PDDP^{-1} \\ &= PD^2P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \end{aligned}$$

- Again,

$$A^3 = (PDP^{-1})A^2 = \underbrace{(PD P^{-1})P}_I D^2 P^{-1} = PDD^2 P^{-1} = PD^3 P^{-1}$$

• (*solution continued*)

- In general, for $k \geq 1$,

$$\begin{aligned}A^k &= PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\&= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}\end{aligned}$$

The diagonalization theorem

- **Definition:** A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D .
- **Theorem 5 (diagonalization theorem):**
 - a. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
 - b. In other words, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A .
 - c. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P .
- **Remark:** In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of \mathbb{R}^n . We call such a basis an **eigenvector basis** of \mathbb{R}^n .

• **Proof** (only if part of *a* & *b*; and the statement *c*)

- First, observe that if P is any $n \times n$ matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$, and if D is any diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$, then

$$AP = A[\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n] = [A\mathbf{v}_1 \ A\mathbf{v}_2 \ \cdots \ A\mathbf{v}_n] \quad (1)$$

while

$$PD = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1 \mathbf{v}_1 \ \lambda_2 \mathbf{v}_2 \ \cdots \ \lambda_n \mathbf{v}_n] \quad (2)$$

- Now suppose A is diagonalizable and $A = PDP^{-1}$. Then right-multiplying this relation by P , we have $AP = PD$.
- In this case, equations (1) and (2) imply that

$$[A\mathbf{v}_1 \ A\mathbf{v}_2 \ \cdots \ A\mathbf{v}_n] = [\lambda_1 \mathbf{v}_1 \ \lambda_2 \mathbf{v}_2 \ \cdots \ \lambda_n \mathbf{v}_n] \quad (3)$$

- Equating columns, we find that

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2, \dots, A\mathbf{v}_n = \lambda_n \mathbf{v}_n \quad (4)$$

- (proof continued)

- Since P is invertible, its columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ must be linearly independent.
- Also, since these columns are nonzero, the equations in (4) show that $\lambda_1, \dots, \lambda_n$ are eigenvalues and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are corresponding eigenvectors.
- This argument proves the ‘only if’ parts of the first and second statements, along with the third statement, of the theorem.

- Proof (if part of *a* & *b*)

- Finally, given any n eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, use them to construct the columns of P and use corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ to construct D .
- By equations (1)–(3), $AP = PD$.
- This is true without any condition on the eigenvectors.
- If, in fact, the eigenvectors are linearly independent, then P is invertible (by the Invertible Matrix Theorem), and $AP = PD$ implies that $A = PDP^{-1}$. \square

Diagonalizing Matrices

- **Example 3:** Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

- **Solution:** There are four steps to implement the description in Theorem 5.
 - **Step 1. Find the eigenvalues of A.**
 - Here, the characteristic equation turns out to involve a cubic polynomial that can be factored: $0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$
 - The eigenvalues are $\lambda = 1$ and $\lambda = -2$. $\lambda_1=1$ $\lambda_2=-2$ $\lambda_3=-2$
 - **Step 2. Find three linearly independent eigenvectors of A.**
 - Three vectors are needed because A is a 3×3 matrix.
 - This is a critical step. If it fails, then Theorem 5 says that A cannot be diagonalized.

- (solution continued)

- Basis for $\lambda = 1$: $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_1 = 1$

- Basis for $\lambda = -2$: $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = -2$ $\lambda_3 = -2$

- You can check that $\{v_1, v_2, v_3\}$ is a linearly independent set.

- Step 3. Construct P from the vectors in step 2.**

- The order of the vectors is unimportant. Using the order chosen in step 2, form

$$P = [v_1 \ v_2 \ v_3] = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- Step 4. Construct D from the corresponding eigenvalues.**

- In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P .

- Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to

$$\lambda = -2: D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$A = PDP^{-1}$ is called diagonalization

• (solution continued - *sanity check*)

- To avoid computing P^{-1} , simply verify that $AP = PD$.
- Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

- **Theorem 6:** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

- **Proof:**

- Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A .
- Then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent, by Theorem 2 in Section 5.1.
- Hence A is diagonalizable, by Theorem 5.

n distinct eigen values

\Leftrightarrow n linearly independent eigen vectors

$\Leftrightarrow P$ is invertible

$\Leftrightarrow A$ is diagonalizable.

Materices whose eigenvalues are not distinct

• Remarks

- It is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable.
- The 3×3 matrix in Example 3 is diagonalizable even though it has only two distinct eigenvalues.
- If an $n \times n$ matrix A has n distinct eigenvalues, with corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and if $P = [\mathbf{v}_1, \dots, \mathbf{v}_n]$, then P is automatically invertible because its columns are linearly independent, by Theorem 2.
- When A is diagonalizable but has fewer than n distinct eigenvalues, it is still possible to build P in a way that makes P automatically invertible, as the next theorem shows.

Suggested Exercises

- 5.3.12
- 5.3.14
- 5.3.21

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