Chapter 3. Determinants

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1. *3.1. Introduction to Determinants*

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3.1. Introduction to Determinants

Definition

- For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1i}\, det\, A_{1i}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, ..., a_{1n}$ are from the first row of A. In symbols,

$$
\begin{aligned}
detA &= \underline{a_{11} detA_{11} - a_{12} detA_{12} + \cdots + (-1)^{1+n} a^{1n} detA_{1n}} \\
&= \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j}\, det(A_{1j})
\end{aligned}
$$

co-factor expansion using 1^{st} row

- **Example 1.** Compute the determinant of

Other possible extension.
→ Expansion using 1^{st} column.
→ Expansion using 3^{rd} column.
⋮

Expansion using 1st row

$$A = \begin{bmatrix} \boxed{1} & \boxed{5} & \boxed{0} \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\underbrace{|A|}_{=\det A} = \overset{(-1)^{1+1}}{\oplus} 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} \overset{(-1)^{2+1}}{\ominus} 5 \cdot \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} \overset{(-1)^{3+1}}{\oplus} 0 \cdot \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$
$$= 1 \cdot (4 \cdot 0 - (-2)(-1)) - 5(2 \cdot 0 - (-1) \cdot 0) + 0 \cdot (2 \cdot (-2) - 4 \cdot 0)$$
$$= -2$$

- **Solution:** Compute

how to conduct sanity check? → get $|A|$ different way & check the same

$$\begin{aligned} detA &= a_{11}detA_{11} - a_{12}detA_{12} + a_{13}detA_{13} \\ &= 1 \cdot det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \\ &= 1(0-2) - 5(0-0) + 0(-4-0) = -2 \end{aligned}$$

- Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets, i.e. $det\ A = |A|$
- Thus the calculation in Example 1 can be written as

$$det\ A = 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \cdot \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \cdots = -2$$

- To state the next theorem, it is convenient to write the definition of $det\,A$ in a slightly different form. Given $A = [a_{ij}]$, the $(i, j)-$**cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} det\,A_{ij} \qquad (4)$$

- Then

$$det\,A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

- This formula is called a **cofactor expansion across the first row** of A.

- **Theorem 1:** The determinant of an $n \times n$matrix A can be computed by a cofactor across any row or down any column.
  - The expansion across the i-th row using the cofactors in (4) is

$$det\,A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

  - The cofactor expansion down the j th column is

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

- **Example 2.** Use a cofactor expansion across the third row to compute $det\ A$, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution:** Compute

$$\begin{aligned} detA &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= (-1)^{3+1}a_{31}detA_{31} + (-1)^{3+2}a_{32}detA_{32} + (-1)^{3+3}a_{33}detA_{33} \\ &= 0 \cdot \begin{bmatrix} 5 & 0 \\ 4 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + 0 \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \\ &= 0 + 2(-1) + 0 = -2 \end{aligned}$$

- **Theorem 2:** If A is a triangular matrix, then $det\ A$ is the product of the entries on the main diagonal of A.

e.g.,

$$\begin{vmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 3 \end{vmatrix} = 5 \cdot \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} = 5(6-0) - 0(0-0) + 2(0-0) = 30$$

$$= 5 \times 2 \times 3$$

expansion using 1st column

$$\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} = 1$$

Suggested Exercises

- 3.1.4
- 3.1.10

3.2. Properties of Determinants

- The Theorem 3 below answers to a question "How does an elementary row operation affect determinant?"

- **Theorem 3:** Let A be a square matrix
  a) (*Replacement*) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
  b) (*Interchange*) If two rows of A are interchanged to produce B, then $det\ B = (-1) \cdot det\ A$.
  c) (*Scaling*) If one row of A is multiplied by k to produce B, then $det\ B = k \cdot det\ A$.

Scaling $\Rightarrow k \cdot \begin{vmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{vmatrix} = \begin{vmatrix} - R_1 - \\ - k \cdot R_2 - \\ - R_3 - \end{vmatrix}$

- **Example 1** Compute $det\ A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$
- **Solution:**
  - The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

$$detA = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \overset{R_2 \leftarrow R_2 + 2R_1}{=} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} \overset{R_3 \leftarrow R_3 + R_1}{=} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

  - An interchange of rows 2 and 3 reverses the sign of the determinant, so

$R_2 \Leftrightarrow R_3$

$$det\ A = \underset{\because \text{Interchange}}{-}\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

- **Theorem 4:** A square matrix A is invertible if and only if $det\ A \neq 0$.

- **Example 3.** Compute det A, where $A = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

- **Solution**
  - Add 2 times row 1 to row 3 ($R3 \leftarrow R3 + 2R1$) to obtain

$$detA = det \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

$R_3 \leftarrow R_3 - R_2$ ...

$$\begin{vmatrix} x & x & x & x \\ x & x & x & x \\ 0 & 0 & 0 & 0 \\ x & x & x & x \end{vmatrix}$$

co-factor expansion using 3rd row

→ det = 0

if zero column or row, then det is zero

  - because the second and third rows of the matrix are equal.

Column Operations

$|A| = |A^T|$

- **Theorem 5:** If A is a matrix, then $det A^T = det\, A$.

using co-factor expansion!

- **Proof**
  - The theorem is obvious for $n = 1$.
  - Suppose the theorem is true for $k \times k$ determinants and let $n = k + 1$. Then the cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T, because the cofactors involve $k \times k$ determinants. Hence the cofactor expansion of $det\, A$ along the first row equals the cofactor expansion of $det\, A^T$ down the first column. That is, A and A^T have equal determinants.
  - Thus the theorem is true for $n = 1$, and the truth of the theorem for one value of k implies its truth for the next value of $k + 1$. By the principle of *mathematical induction*, the theorem is true for all $n \geq 1$.

Determinants and Matrix Products

- **Theorem 6:** If A and B are matrices, then $det\ AB = (det\ A)(det\ B)$
- **Example 5** Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.
- **Solution**
  - $$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

    and

    $$det\ AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$
  - On the other hand, since $det\ A = 9$ and $det\ B = 5$,

    $$(det\ A)(det\ B) = 9 \cdot 5 = 45 = det\ AB$$

unit lower triangular matrix

e.g., $\begin{bmatrix} 1 & & & \\ x & 1 & & \\ x & x & 1 & \\ x & x & x & 1 \end{bmatrix}$

$A = LU$

$|A| = \underbrace{|L|}_{=1} \cdot |U| = |U|$

Suggested Exercises

- 3.2.5
- 3.2.9

3.3. Cramer's Rule, Volume, and Linear Transformations

- how to use determinants to solve a system of linear equations.
- how computer solve $Ax=b$

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