Quiz 2

2020 #1

#1. Find the solution set of the following system of linear equations by describing in parametric vector form. Also, give a geometric description of the solution set. [15pt]

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned} \quad \Rightarrow \quad A\mathbf{x} = \mathbf{b}$$

augmented matrix $[A \mid \mathbf{b}] = \begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, x_3 is a free variable.

$x_2 = -2x_3 + 1$

$x_1 = -3x_2 - x_3 + 1 = -3(-2x_3 + 1) - x_3 + 1 = 5x_3 - 2$

parametric form, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2+5x_3 \\ 1-2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

solution set, a line through $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, in direction of $\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

parallel to $A\mathbf{x} = 0$ solution set

2020 #2

#2. Find the inverse of these matrices. [20pt]

(a)

$$\begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [A \mid I_3] \sim [I_3 \mid A^{-1}]$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} -2 & -7 & -9 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -2 & -7 & -9 & 1 & 0 & 0 \\ 0 & -2 & -3 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -2 & -7 & -9 & 1 & 0 & 0 \\ 0 & -2 & -3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -2 & -7 & 0 & 10 & -9 & 36 \\ 0 & -2 & 0 & 4 & -2 & 12 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -6 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right]$$

$$AA^{-1} = I, \quad \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & -6 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & -6 \\ 1 & -1 & 4 \end{bmatrix}$$

2020 #2

(b)

$$\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad [A|I_4] \sim [I_4|A^{-1}]$$

$$[A|I_4] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_3 \end{array}$$

$$= [I_4|A^{-1}]$$

$$AA^{-1} = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

2020 #3

#3. Find an LU factorization of the following matrix.[15pt]

$$\begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

$$A = \underset{3\times 4}{\begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}} = \underset{3\times3}{L}\underset{3\times4}{U}$$

$$\begin{array}{rrrr} 6 & -9 & 7 & -3 \\ -6 & 12 & -12 & 6 \end{array}$$

$L^{-1}A = U$, $A \sim U$

$$\begin{array}{rrr} -6 & 10 & -1 \\ 6 & -10 & 6 \\ \hline 0 & 0 & +5 \end{array}$$

$$A \sim \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix} \begin{array}{l} R_2 \leftarrow R_2 - 3R_1 \\ R_3 \leftarrow R_3 + \frac{1}{2}R_1 \end{array}$$

$$\sim \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} R_3 \leftarrow R_3 + 2R_2 \quad = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

2020 #4

#4. If the following statement is True, then provide a mathematical proof or explain it further. If False, then provide a counter-example. [10pt]

> For a $m \times n$ matrix A, if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then the number of solution to $A\mathbf{x} = \mathbf{b}$ is infinite.

False, counter example, $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$

$A\mathbf{x} = \mathbf{0}$, $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 & -4 & 0 \\ 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

x_2 is free.

$x_1 = 2x_2$ $\therefore$ $A\mathbf{x} = \mathbf{0}$ has nontrivial solrn

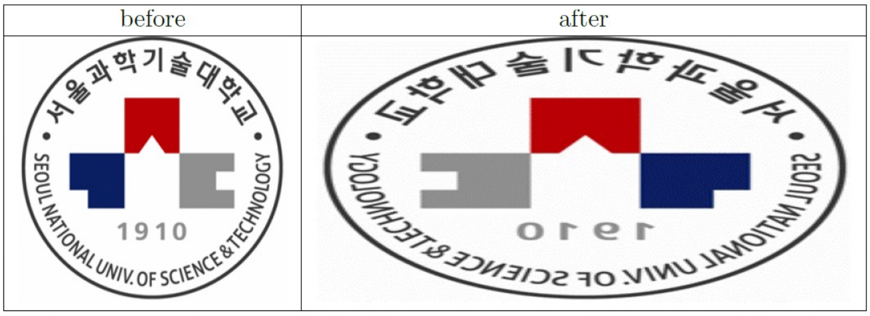
$A\mathbf{x} = \mathbf{b}$, $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$[A|\mathbf{b}]$ $\begin{bmatrix} 2 & -4 & 2 \\ 1 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 \\ 0 & 0 & 2 \end{bmatrix}$, $\because 0x_1 + 0x_2 = 2$

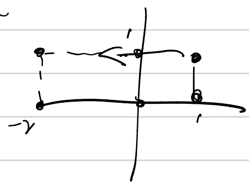
$A\mathbf{x} = \mathbf{b}$ is inconsistent.

2020 #5

#5. After a linear transformation, SNUTech's amblem has been transformed. Specifically, the emblem is horizontally reversed and its horizontal length is doubled. Suggest a standard matrix for this linear transformation and explain it. [15pt]



x- axis reverse
horizonal length doubled



$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{array}{c} x \\ y \end{array} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Standard matrix : $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

2021 #1

#1. Mark True or False. No justification is necessary. [Each 6pt]

- In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations. (TRUE / FALSE) unique row reduced echelon form!
- If one row in an echelon form of an augmented matrix is [0 0 0 5 0], then the associated linear system is inconsistent. (TRUE / FALSE)
- Let A be a 3×2 matrix. The equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. (TRUE / FALSE)
- If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution to $A\mathbf{x} = \mathbf{b}$ is unique. (TRUE / FALSE)

▷ 3x2 matrix, at most two pivot columns.
not enough for span $\mathbb{R}^3$

▷ $A\mathbf{x}=\mathbf{b}$ has a solution.
a unique solution!
every column in A is pivot column.
→ $A\mathbf{x}=\mathbf{0}$ has only trivial solution!

2021 #V

#2. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5 + 4x_3$, $x_2 = -2 - 7x_3$, with x_3 free. Use vectors to describe this set as a line in $\mathbb{R}^3$. [16pt]

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5+4x_3 \\ -2-7x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} = p + x_3 q$$

a line through $\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ and its direction is $\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$

2021 #3

#3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$.

- Prove that f is a linear transformation when $b = 0$. [10pt]
- Find a property of a linear transformation when $b = 0$. [10pt]

Linear transformation preserves vector addition & scalar multiplication. $u, v \in \mathbb{R}$, $c, d \in \mathbb{R}$

$f(u+v) = f(u)+f(v)$ | $m(u+v)+b = mu+b+mv+b = m(u+v)+2b$

$f(cv) = cf(v)$ | $mcv+b = c(mv+b) = mcv+bc$

To satisfy both expressions above, b should be 0, $(b=0)$

$f(0) = 0$

$f(cu+dv) = cf(u)+df(v)$

2021 #4.

#4. Find an LU factorization of the following matrix [20pt]

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix} \begin{matrix} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{bmatrix} R_3 \leftarrow R_3 - \frac{2}{3}R_2$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix} = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & -\frac{2}{3} & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix} = L$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$$

2021 #5

#5. Find an inverse of the following matrix [20pt]

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$[A \mid I] \sim [I \mid A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix} = [I \mid A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$$

$$AA^{-1} = I = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#1. Mark True or False. No justification is necessary. [Each 5pt]

- Every matrix is row equivalent to a unique matrix in echelon form. (TRUE / FALSE)
- Any system of n linear equations in n variables has at most n solutions. (TRUE / FALSE)
- If a system of linear equations has no free variables, then it has a unique solution. (TRUE / FALSE)
- If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. (TRUE / FALSE)
- If matrix A is an $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.
  (TRUE / FALSE)

$$\left[\begin{array}{cc|c} 2 & 1 & 9 \\ 0 & 0 & 1 \end{array}\right]$$

$x_1 + x_2 = 8$

$x_1 + x_2 = 9$

$$\begin{array}{ccc} 1 & 1 & 8 \\ 1 & 1 & 9 \end{array}$$

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#2. Find all solutions to the system of the homogeneous equations of A. [15pt]

$$A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$$

$Ax=0$

augmented matrix $[A|0]$

$$= \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, x_3 is free variable.

$x_4 = 0$.

$x_2 = -2x_3$

$x_1 = -3x_2 - 9x_3 = -3(-2x_3) - 9x_3 = \underline{-3x_3}$.

solution, $x = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ $x_3 \in \mathbb{R}$. ✓

a line through origin, direction of $\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^4$

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Sanity check!

#3. Find an LU factorization of the following matrix [15pt]

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

$$\begin{array}{ccc} 3 & 7 & 5 \\ -3 & +3 & -6 \\ \hline 0 & 10 & -1 \end{array}$$

$$\begin{array}{ccc} 1 & -3 & 1 \\ -1 & +1 & -2 \\ \hline 0 & -2 & -1 \end{array}$$

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - \frac{1}{2}R_1 \\ R_3 \leftarrow R_3 - \frac{3}{2}R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} \quad R_3 \leftarrow R_3 + 5R_2$$

$$\begin{array}{ccc} 0 & 10 & -1 \\ 0 & -10 & -5 \\ \hline 0 & 0 & -6 \end{array}$$

$$\sim \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = U$$

flip the sign

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -5 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

$-3+10$

$6+-6$

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#4. Find an inverse of the following matrix [15pt]

$$A = \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 1 \end{bmatrix}$$

$$[A \mid I_3] \sim [I_3 \mid A^{-1}]$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ -3 & 6 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & -9 & -11 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -5 & 0 & 13 & 12 & 4 \\ 0 & 3 & 0 & -12 & -11 & -4 \\ 0 & 0 & 1 & 3 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & -\frac{19}{3} & -\frac{8}{3} \\ 0 & 1 & 0 & -4 & -\frac{11}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & 3 & 3 & 1 \end{array}\right] = [I_3 \mid A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -7 & -\frac{19}{3} & -\frac{8}{3} \\ -4 & -11/3 & -4/3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} -7 & -19/3 & -8/3 \\ -4 & -11/3 & -4/3 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -7+20-12 & -\frac{19}{3}+\frac{55}{3}-12 & -\frac{8}{3}+\frac{20}{3}-4 \\ -12+12 & -11+12 & -4+4 \\ 21-24+3 & 19-22+3 & 8-8+1 \end{bmatrix}$$

$-7-20+12$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

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#5. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible. [15pt]

if A is invertible, $(AB)^{-1}AB = I = B^{-1}(A^{-1}A)B = B^{-1}IB = I$

if A is not invertible, then $(AB)^{-1}$ cannot be determined.

All invertible matrix should satisfy $(CD)^{-1} = D^{-1}C^{-1}$, when CD, C, D are invertible matrices.

Because of $(AB)^{-1} = B^{-1}A^{-1}$, to determine $(AB)^{-1}$, A^{-1} should be defined

So, A is invertible.

No use (A^{-1}) because A^{-1} is not confirmed.

pf 1)

$C = AB$

$CC^{-1} = I$

$BB^{-1} = I$

consider, CB^{-1}, CB^{-1} is invertible, $\because$ C & B is invertible.

$CB^{-1} = (AB)B^{-1} = A$. So, A is invertible.

pf 2)

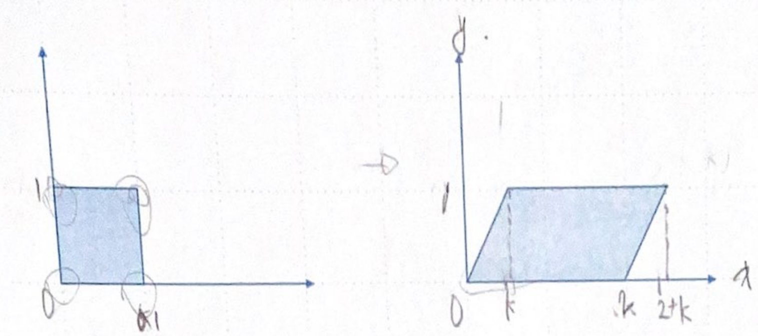
$\underbrace{(CB^{-1})}_{\text{invertible}} X = I$

$X(CB^{-1}) = I$

$X = (\overset{=A}{CB^{-1}})^{-1}$

$\Rightarrow$ inverse of A.

#6. The unit square on the left becomes the parellogram on the right by a linear transformation. What would be the standard matrix for this linear transformation? Justification is necessary. [15pt]



Standard matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $a, b, c, d \in \mathbb{R}$. $\boxed{k = \frac{1}{2}}$

$$\underbrace{\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}}_{A} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & k & 2+k \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{bmatrix}$$

$= I_2$

$a = 2 \quad b = k$

$c = 0 \quad d = 1$

$$A = \begin{bmatrix} 2 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

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