ITM426, Final Exam, 2021 Fall

Solution and Grading

- ITM 426 Engineering Mathematics 2021 F
- Dec 17, 2021
- Duration: 90 minutes
- Weights: 30%
- 5 Questions

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- Write legibly.
- In on-line exam, start every problem in a new page.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

1	15
2	15
3	20
4	25
5	25
Total	100

#1. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and the origin. `[15pt]`

Difficulty: Easy
Amount of work: 15%
Suggested answer:

Let $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. The orthogonal projection of $\mathbf{y}$ onto the line through $\mathbf{u}$ and the origin is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$, and this vector is $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = -\frac{2}{5} \mathbf{u} = \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix}$.

Partial credit: If the student knows the formula exactly, then +5 pts.

#2. The given set is a basis for a subspace W. Use the Gram–Schmidt process to produce an orthogonal basis for W.`[15pt]`

$$\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

Difficulty: Medium
Amount of work: 15%
Suggested answer: Set $\mathbf{v_1} = \mathbf{x_1}$ and compute that $\mathbf{v_2} = \mathbf{x_2} - \frac{\mathbf{x_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \mathbf{x_2} - \frac{1}{2} \mathbf{v_1} = (3\ 3/2\ 3/2)^t$. Thus, an orthogonal basis for W is $[(0\ 4\ 2)^t, (3\ 3/2\ 3/2)^t]$. Another correct answer is $(-6/7\ -30/7\ -3/7)^t$ and $(4\ -1\ 2)^t$ in case the second vector was fixed.
Partial credit: If the student knows the formula exactly, then +5 pts. If a mistake occurs that could have been fixed with sanity check, the penalty is 10pts (mistake for 5pts and not conducting sanity check for 5pts).

#3. Construct the normal equations for $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

and find the solution $\hat{\mathbf{x}}$. [20pt]

Difficulty: Medium
Amount of work: 20%
Suggested answer:

To find the normal equations and to find $\hat{\mathbf{x}}$, compute

$$A^T A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}; \; A^T\mathbf{b} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}.$$

a. The normal equations are $(A^T A)\mathbf{x} = A^T\mathbf{b}$: $\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$.

b. Compute $\hat{\mathrm{x}} = (A^T A)^{-1} A^T\mathbf{b} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}^{-1} \begin{bmatrix} -24 \\ -2 \end{bmatrix} = \frac{1}{56} \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix} \begin{bmatrix} -24 \\ -2 \end{bmatrix} = \frac{1}{56} \begin{bmatrix} -224 \\ 168 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

Partial credit: Constructing normal equation counts 10pts and solving the equation counts 10pts. If a mistake occurs that could have been fixed with sanity check, the penalty is 10pts (making a mistake for 5pts and not conducting sanity check for 5pts).

#4. We have the following matrix. $A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$.

(a) Show that A is positive definite. [10pt]

(b) Perform an orthogonal diagonalization. [15pt]

Difficulty: Easy-Medium
Amount of work: 25%
Suggested answer:
For (a), the eigenvalues are 5 and 10, thus positive definite. For (b),

Let $A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$. Then the characteristic polynomial of A is $(6-\lambda)(9-\lambda)-4 = \lambda^2 - 15\lambda + 50$ $= (\lambda-5)(\lambda-10)$, so the eigenvalues of A are 5 and 10. For $\lambda = 5$, one computes that a basis for the eigenspace is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. For $\lambda = 10$, one computes that a basis for the eigenspace is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

Partial credit: For (b), if the matrix is not normalized, then the penalty is 10pts (not normalizing for 5pts and this error could have been fixed by conducting sanity check for 5pts).

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#5. From an experiment, three observations of two variables are collected: $(1,6),(2,5)$, and $(3,4)$, answer the following.

(a) Construct a sample covariance matrix. `[15pt]`

Using the sample covariance matrix above, perform a principal component analysis. In case you are unsure of your answer to the problem (a), then you may propose a legit 2×2 covariance matrix and proceed with it.

(b) Find the two principal components. `[10pt]`

Difficulty: Medium-Hard
Amount of work: 25%
Suggested answer:
(a) $X = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}, M = \begin{bmatrix} 2 & 2 & 2 \\ 5 & 5 & 5 \end{bmatrix}, S = (X-M)(X-M)^t/(3-1) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.
(b) eigenvalues: $2, 0$; eigenvectors: $(1 \ -1)$ and $(1 \ 1)$; principal components: $\frac{1}{\sqrt{2}}(1 \ -1)$ and $\frac{1}{\sqrt{2}}(1 \ 1)$.
Partial credit: For (b), unnormalized PC presentation loses 5pts. Since PC is for directional information and not for length information, it must be normalized.

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Write your name before detaching this page. Your Name: ______________________