Exercises

Engineering Math
Linear Algebra and Its Application

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Chapter

I

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#1.1.5

eliminate coefficients of x_3 for 2 steps.

$$\begin{array}{cc}\left[\begin{array}{cccc|c} 1 & -4 & 5 & 0 & 7 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right] & \begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \end{array}\end{array}$$

1. $R2 \leftarrow R2 + 3R3$

$$\begin{array}{cc}\left[\begin{array}{ccccc} 1 & -4 & 5 & 0 & 7 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right] & \begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \end{array}\end{array}$$

2. $R1 \leftarrow R1 - 5R3$

$$\begin{array}{cc}\left[\begin{array}{ccccc} 1 & -4 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right] & \begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \end{array}\end{array}$$

3. $R1 \leftarrow R1 + 4R2$

$$\begin{array}{cc}\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 45 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right] & \begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \end{array}\end{array}$$

#1.1.6

$$\begin{array}{cc}\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{array}\right] & \begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \end{array}\end{array}$$

make the augmented matrix triangular form

$R4 \leftarrow R4 + 2R1$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{array}\right]$$

$R4 \leftarrow R4 - \frac{3}{2}R2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{array}\right]$$

$R4 \leftarrow R4 + R3$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

R4 means x_4 is a free variable. Thus, this linear system has infinite number of solution, so it is consistent.

or

No row $[0 \cdots 0\ b]$, $(b \neq 0)$ in the matrix. Thus, the linear system is consistent.

1.1.18

'three planes have at least one common point of intersection' means the three planes have at least one solution, so a linear system of the three planes is consistent then, they have common point of intersection.

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_2 - x_3 = 1 \\ x_1 + 3x_2 = 0 \end{cases}$$

augmented matrix of the linear system.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix}$$

$$\sim \begin{array}{cc} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} & \begin{matrix} R1 \\ R2 \\ R3 \end{matrix} \end{array}$$

R3 means $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -5$.

Thus, the solution of the linear system does not exist. Then, the common point of intersection also does not exist.

1.1.29

$$\begin{bmatrix} 0 & -2 & 5 \\ 1 & 4 & -7 \\ 3 & -1 & 6 \end{bmatrix} \sim \begin{array}{cc} \begin{bmatrix} 1 & 4 & -7 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{bmatrix} & \begin{matrix} R1 \\ R2 \\ R3 \end{matrix} \end{array}$$

interchange R1 & R2 in R1/R2 to be R2/R1.

$\boxed{R1 \Leftrightarrow R2}$

1.1.7

$$\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 7 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

inconsistent.

1.1.11

$$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$$

make sure be 1

$$\rightarrow \begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

inconsistent

#1.1.12

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

inconsistent

#1.1.19

$1 \times 6 - 3h \neq 0$ (non zero determinant)

$h \neq 2$

$$\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}$$

$6-3h \neq 0$ $h \neq 2$

#1.1.20

$1 \times 4 - (-2)h \neq 0$

$h \neq -2$ (X)

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & h & -3 \\ 0 & 4+2h & 0 \end{bmatrix}$$

$(4+2h) \times x_2 = 0.$

x_2 surely has solutions, so the system is consistent for any h

#1.1.21

$h - (-4) \times 3 \neq 0$

$h \neq -12$ (X)

$$\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & h+12 & 0 \end{bmatrix}$$

$(h+12)x_2$ surely has solutions, so the system is consistent for any h.

1.1.22

$-3h=5$, $h=-\frac{5}{3}$.

multiple of other $\Rightarrow$ consistent, ∞ sol'n

$$\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & h \\ 0 & 0 & 5+3h \end{bmatrix}$$

$5+3h=0$.

1.2.10

$$\begin{bmatrix} 1 & -2 & -1 & \vdots & 3 \\ 3 & -6 & -2 & \vdots & 2 \end{bmatrix} \begin{matrix} R1 \\ R2 \end{matrix}$$

$R2 \leftarrow R2 - 3R1$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

$x_3 = -7$

$x_1 - 2x_2 + 7 = 3$

$3x_1 - 6x_2 - 2 \times (-7) = 2$

$\rightarrow$ $x_1 - 2x_2 = -4$ — ①

$3x_1 - 6x_2 = -12$ — ② = ① × 3

general solution :

$x_3 = -7$, $x_1 - 2x_2 = -4$

#1.2.1

reduced echelon form : a, b
echelon form : d
not echelon form : c

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

#1.2.2

reduced echelon form : a
echelon form : b, d
not echelon form : c

$$a = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$d = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

#1.2.3

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

#1.2.4

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

#1.2.5

nonzero 2x2
echelon form

$$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$$

#1.2.6

3x2

$$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \blacksquare & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

1.2.7

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$x_3 = 3$

$x_1 + 3x_2 = -5$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

1.2.8

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & -1 & 0 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

$$\begin{cases} x_1 = -9 \\ x_2 = 4 \\ x_3 \text{ is free.} \end{cases}$$

#1.2.9

$$\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 7 & -6 \\ 0 & 1 & -6 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -5 & 4 \\ 0 & 1 & -6 & 5 \end{bmatrix}$$

$x_1 - 5x_3 = 4$
$x_2 - 6x_3 = 5$

$$\begin{cases} x_1 = 4 + 5x_3 \\ x_2 = 5 + 6x_3 \\ x_3 \text{ is free} \end{cases}$$

#1.2.11

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4/3 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - \frac{4}{3}x_2 + \frac{2}{3}x_3 = 0$

$$\begin{cases} x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

#1.2.12

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & -4 & 8 & 12 \\ 0 & 0 & 1 & -2 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 7x_2 + 6x_4 = 5$$
$$x_3 - 2x_4 = -3$$

$$\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_2 \text{ is free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ is free} \end{cases}$$

#1.2.13

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 3x_2 + 5x_5 = 2$$
$$x_2 - 4x_5 = 1$$
$$x_4 + 9x_5 = 4$$

$x_1 = 3x_2 - 5x_5 + 2$
$x_2 = 4x_5 + 1$
x_3 is free
$x_4 = 4 - 9x_5$
x_5 is free

1.2.14

$$\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 7 & 0 & 0 & -9 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 7x_3 = -9$

$x_2 - 6x_3 - 3x_4 = 2$

$x_5 = 0$

$$\begin{cases} x_1 = -9 - 7x_3 \\ x_2 = 6x_3 + 3x_4 + 2 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$$

1.2.17

$$\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & h \\ 0 & 0 & 7-2h \end{bmatrix}$$

$7 - 2h = 0 \qquad h = \frac{7}{2}$

1.2.18

$$\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{bmatrix}$$

$h+15 \neq 0 \qquad h \neq -15.$

1.2.19

$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$$

a) no solution

$8-4h=0, \quad k-8 \neq 0$

$h=2 \qquad k \neq 8$

b) unique solution

$8-4h \neq 0 \quad k-8$ is free

$h \neq 2$, k is free

c) many solution

$8-4h=0, \quad k-8=0$

$h=2, \ k=8$

1.2.20

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}$$

a) no solution

$$\left.\begin{matrix} h-9=0 \\ k-6\neq 0 \end{matrix}\right\} \quad \begin{matrix} h=9 \\ k\neq 6 \end{matrix}$$

b) unique solution

$$\left.\begin{matrix} h-9\neq 0 \\ k-6 \text{ is free} \end{matrix}\right\} \quad \begin{matrix} h\neq 9 \\ k \text{ is free} \end{matrix}$$

c) many solution

$$\left.\begin{matrix} h-9=0 \\ k-6=0 \end{matrix}\right\} \quad \begin{matrix} h=9 \\ k=6 \end{matrix}$$

1.2.23

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & b_3 \end{bmatrix} \cdots \text{echelon form}$$

three pivot columns. no zero row vector.

→ consistent!

#1.2.24

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \vdots & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & \vdots & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & \vdots & a_{35} \end{bmatrix}$$ ... echelon form.

$a_{35} \neq 0$.

$a_{31} \cdots a_{34}$ should be zero.

$0 = a_{35} \neq 0$ is not accepted logically

So the system is inconsistent.

#1.2.25.

$n \times m$ coefficient matrix.

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}$$

echelon form.

n pivot columns.

no zero row vector.

consistent.

#1.2.26

$$\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

echelon form.

3×3

$a \neq 0$, $b \neq 0$, $c \neq 0$.

no zero vector in row.
$\rightarrow$ consistent
no free variable.
$\rightarrow$ unique.

#1.2.27

every column is a pivot column.

#1.2.28

number of pivot columns.
number of rows and columns.
number of zero row vector in the matrix.

#1.3.14

Does $A\mathbf{x}=\mathbf{b}$ have solution?

$$(A|\mathbf{b}) = \left[\begin{array}{ccc:c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{array}\right]$$

the matrix has a solution because it has no free variables.

#1.3.18

h should make $\mathbf{y}$ exist in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{y}$, $(a_1, a_2 \in \mathbb{R})$

$$\left[\begin{array}{cc:c} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right]$$

if the matrix above is consistent, then $\mathbf{y}$ is in the plane by $\mathbf{v}_1$ & $\mathbf{v}_2$.

$\mathbf{v}_1$ and $\mathbf{v}_2$ should be linearly independent to span plane.

$$\sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix}$$

$a_2 = -5$, $2a_2 = -3+2h = -5\times 2 = -10$

$h = -\frac{7}{2}$

$a_1 - 3(-5) = -\frac{7}{2}$

$a_1 = -\frac{37}{2}$ $a_2 = -5$

When $h = -\frac{7}{2}$, the matrix has a solution.

or

$$\sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 7+2h \end{bmatrix}$$

To make the matrix has a solution

$0 = 7+2h$ is satisfied.

Thus, $h = -\frac{7}{2}$.

#1.4.11

$A\mathbf{x}=\mathbf{b}$

$$(A|\mathbf{b}) = \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution of matrix $(\mathbf{x}) = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$

#1.4.22

$$A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (a,b,c \in \mathbb{R})$$

Say $\mathbf{b}$ is any arbitrary 3-dimensional vector.
if $A\mathbf{x}=\mathbf{b}$ is consistent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$.

$$\begin{bmatrix} 0 & 0 & 4 & a \\ 0 & -3 & -1 & b \\ -2 & 8 & -5 & c \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & -5 & c \\ 0 & -3 & -1 & b \\ 0 & 0 & 4 & a \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 8 & -5 & c \\ 0 & -3 & -1 & b \\ 0 & 0 & 1 & a/4 \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & 0 & c+\frac{5a}{4} \\ 0 & -3 & 0 & b+a/4 \\ 0 & 0 & 1 & a/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 0 & 0 & c+\frac{5}{4}a+\frac{8}{3}(b+a/4) \\ 0 & -3 & 0 & b+a/4 \\ 0 & 0 & 1 & a/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2}\left(c+\frac{5}{4}a+\frac{8}{3}b+\frac{2}{3}a\right) \\ 0 & 1 & 0 & -\frac{1}{3}(b+a/4) \\ 0 & 0 & 1 & a/4 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\left(c+\frac{8}{3}b+\frac{23}{12}a\right) \\ -\frac{1}{3}(b+a/4) \\ a/4 \end{bmatrix}$$

$A\mathbf{x}=b$ has a unique solution because $(a,b,c \in \mathbb{R})$, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

or

A has a pivot position in every row.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$

#1.5.14

$$\mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_4 \\ 8+x_4 \\ 2-5x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_4 \\ x_4 \\ -5x_4 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

$$\mathbb{X} = x_4\mathbb{V} + \mathbb{P}, \quad \mathbb{V} = \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix}, \quad \mathbb{P} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix}$$

"the line through $\mathbb{P}$ in $\mathbb{R}^4$"

#1.5.18

$x_1 - 3x_2 + 5x_3 = 0$ — ①

$x_1 - 3x_2 + 5x_3 = 4$ — ②

Are 2 free variables needed?

1.8.1

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$T(u) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$

1.8.9

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (x_1, x_2, x_3, x_4 \in \mathbb{R})$$

$$\begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

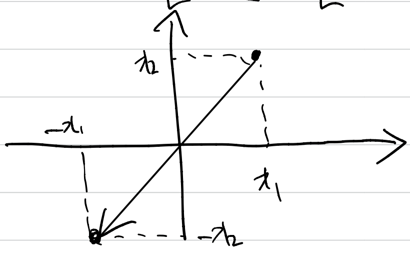
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -9 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 and x_4 is free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

#1.8.13

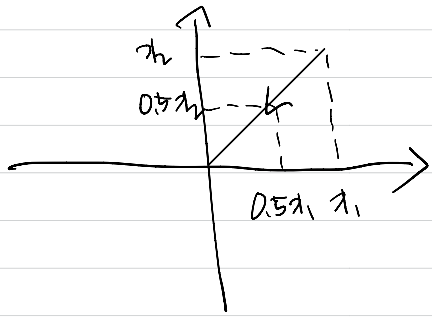
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$



reflection

#1.8.14

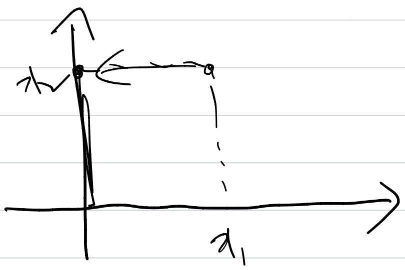
$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5x_1 \\ 0.5x_2 \end{bmatrix}$$



contraction

#1.8.15

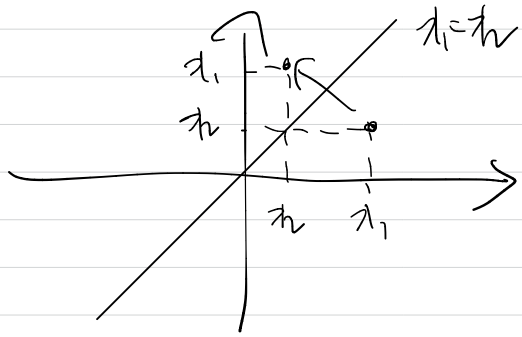
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$



projection onto x_2-axis

#1.8.16

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$



reflection

Chapter

Ⅱ

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Chapter 2.1 #27.

$$u = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \qquad v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$u^T = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} \qquad v^T = \begin{bmatrix} a & b & c \end{bmatrix}$$

$$u^T v = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -2a+3b-4c$$

$$v^T u = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = -2a+3b-4c$$

$$u v^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$$

$$v u^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -2a & 3a & -4 \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$$

Chapter 2.2 #9

a) True

b) False, $(AB)^{-1} = B^{-1}A^{-1}$

c) False, $\det(A) = ad-bc$

d) True

e) True

Chapter 2.2 #17

$AB = BC \rightarrow ABB^{-1} = BCB^{-1}$

$\rightarrow A = BCB^{-1}$

Chapter 2.2 #18

$A = PBP^{-1} \rightarrow AP = PBP^{-1}P = PB \rightarrow P^{-1}AP = B$

Chapter 2.2 #31

$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $[A \mid I_3] \sim [I_3 \mid A^{-1}]$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 10 & 3 & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 10 & 3 & 1 & 0 \\ 0 & 0 & 30 & 7 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 10 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7/30 & 1/10 & 1/30 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/15 & -\frac{1}{5} & -\frac{1}{15} \\ 0 & 1 & 0 & 2/3 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 7/30 & 1/10 & 1/30 \end{array}\right] = [I_3 \mid A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 8/15 & -1/5 & -1/15 \\ 2/3 & 0 & -1/3 \\ 7/30 & 1/10 & 1/30 \end{bmatrix}$$

Chapter 2.3 #11

a) True

b) True

c) False.

d) True

e) True.

Chapter 2.5 #7.

$Ax = b$

$\rightarrow \begin{cases} Ly = b \\ Ux = y \end{cases}$

$$\begin{bmatrix} 2 & 5 \\ -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 \\ 0 & 7/2 \end{bmatrix} = U$$

E_1: $R2 \leftarrow R2 + \frac{3}{2}R1$

$$\begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 7/2 \end{bmatrix}$$

$E_1 A = U$

$L = E_1^{-1}$

$$[E_1 | I_2] \sim [I_2 | E_1^{-1}]$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3/2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ -3/2 & 1 \end{bmatrix}$$

Chapter 2.5 #9

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix} = U$$

E_1: $R2 \leftarrow R2 + R1$, $R3 \leftarrow R3 - 3R1$ $\qquad$ E_2: $R3 \leftarrow R3 - \frac{2}{3}R2$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix}$$

Chapter 2.5 #11

$$A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 5 \end{bmatrix} = U$$

$R2 \leftarrow R2 - 2R_1$

$R_3 \leftarrow R_3 + \frac{1}{3}R_1$

$R3 \leftarrow R3 - R2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -\frac{1}{3} & 1 & 1 \end{bmatrix}$$

Chapter 2.8 #11

$$A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & -5 \\ 0 & 2 & 4 & -8 \\ 0 & -4 & -8 & 16 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & -5 \\ 0 & 2 & 4 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3×4

2 pivots, Col A spans $\mathbb{R}^2$ $\quad 2 = q$

Nul A spans $\mathbb{R}^{4-2}$ $\quad 4-2 = 2 = p$

Chapter 2.8 #12

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 9 & 15 \\ 0 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4×3

2 pivots Col A spans $\mathbb{R}^2$ $\quad q = 2$

Nul A spans $\mathbb{R}^{3-2} = \mathbb{R}^1$ $\quad p = 1$

Chapter 2 Supplementary Exercise

#1

a) True

$A : m \times n \qquad A^T : n \times m$

$B : m \times n \qquad B^T : n \times m$

b) False

$A : m \times n$
$B : n \times l \qquad \because \quad B \,\&\, C$ have l columns
$C : m \times l$

c) True.

d) False, if all columns in B are zero vectors?

e) False

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} = 0$$

$(a+c)w + (b+d)y = 0$
$(a+c)x + (b+d)z = 0$
if $(a+c) = 0, \quad y = 0, \quad z = 0$
$(b+d) \neq 0, \quad w \neq 0, \quad x \neq 0$

$$\begin{bmatrix} -c & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -cw & -cx \\ cw & cx \end{bmatrix} = 0$$

if $c = 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

f) False $\quad A^2 - AB + BA - B^2$

g) True

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

interchange & scaling $\rightarrow n$

replacement $\quad R_k \leftarrow xR_k + yR_{k-1} \rightarrow n+1$

h) True

$$\begin{bmatrix} 1 & & \\ & 1 & \\ 2 & & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_1 \leftarrow R_1 + 2R_3$$

i) True

j) False.

k) True

l) False.

m) False

n) True

$$A^{-1}AB = A^{-1}BA$$
$$B = A^{-1}BA$$
$$BA^{-1} = A^{-1}BAA^{-1}$$
$$= A^{-1}B$$

o) True

p) True

$$\begin{array}{cccc} a & b & c & 1 \\ & e & f & 0 \\ & & j & 0 \end{array}$$

#2

$$C^{-1} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$$

$$C = \frac{1}{4\times7-5\times6} \begin{bmatrix} 7 & -5 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} -7/2 & +5/2 \\ +3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -7/2 & +5/2 \\ +3 & -2 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} -14+15 & -\frac{35}{2}+\frac{35}{2} \\ 12-12 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = CC^{-1}$$

#3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = A^2 \qquad = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$I^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = I^2 - A^3$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = A^3$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \qquad A^3 = 0$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbb{O}$$

#4

$$A=\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}$$

$$I-A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}$$

$$[I-A|I]\sim[I|(I-A)^{-1}]$$

$$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\-1&1&0&0&1&0\\0&-1&1&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&-1&1&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&0&1&1&1&1\end{array}\right]$$

$$(I-A)^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

5

$$AI = IA$$

$$(A-I)^2 = A^2 - AI - IA + I^2 = A^2 - 2A + I = 0 \qquad A - I = 0$$

$$A = I.$$

$$\begin{cases} I = 3I - 2I. \\ I = 4I - 3I. \end{cases}$$

6

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad B^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$-BA = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

7

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 5 \\ 1 & 5 \\ 3 & 4 \end{bmatrix} \qquad A^{-1}B = [x_1, x_2], \qquad AA^{-1}B = B$$

1) $Ax_1 = \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}$, 2) $Ax_2 = \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$

1) $$\begin{bmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{bmatrix} x_1 = \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 3 & 8 & -3 \\ 2 & 4 & 11 & 1 \\ 1 & 2 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 8 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & -1 & -3 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 8 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 1 & 15 \end{bmatrix}$$

$x_1 = 10$

$x_2 = 9$

$x_3 = -5$

$$X_1 = \begin{bmatrix} 10 \\ 9 \\ -5 \end{bmatrix}$$

2) $$\begin{bmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{bmatrix} X_2 = \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}. \qquad \begin{bmatrix} 1 & 3 & 8 & 5 \\ 2 & 4 & 11 & 5 \\ 1 & 2 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 8 & 5 \\ 0 & -2 & -5 & -5 \\ 0 & -1 & -3 & -1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -1 \\ 10 \\ -3 \end{bmatrix} \qquad \begin{matrix} x_1 = 59 \\ x_2 = 10 \\ x_3 = -3 \end{matrix} \qquad \sim \begin{bmatrix} 1 & 3 & 8 & 5 \\ 0 & -2 & -5 & -5 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} 10 & -1 \\ 9 & 10 \\ -5 & -3 \end{bmatrix}$$

$AA^{-1}B = B,$

$$\begin{bmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 10 & -1 \\ 9 & 10 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 1 & 5 \\ 3 & 4 \end{bmatrix}$$

$10+27-40 \qquad -1+30-24$

$20+36-55 \qquad -2+40-33$

$10+18-25 \qquad -1+20-15$

#8

$$A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(2x2)

$$A = \begin{bmatrix} — a_1 — \\ — a_2 — \end{bmatrix} \qquad A \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \frac{1}{1} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

#9

$$AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}, \quad B^{-1} = \frac{1}{7-6} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

$$ABB^{-1} = A, \quad \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 5-8 & -15+28 \\ -2-6 & 6+21 \end{bmatrix} = \begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}$$

#10

A is invertible, then A^T is also invertible.

product of $(n \times n)$ invertible matrices is invertible

$A \to (n \times n)$, $A^T \to (n \times n)$

So, A^TA is invertible
$(n \times n)$

$$(A^TA)^{-1}(A^T) = A^{-1}(A^T)^{-1}(A^T) = A^{-1}I = A^{-1}$$

#11

$$\begin{bmatrix} - X_1 - \\ - X_2 - \\ \vdots \\ - X_n - \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$X_k = [\lambda_k^0 \ \ \lambda_k^1 \ \ \lambda_k^2 \ \cdots \ \lambda_k^{n-1}]$ in $\mathbb{R}^n$

$(1 \le k \le n)$

a.

$$\underset{(1\times n)\,(n\times 1)}{X_k \cdot C} = \underset{(1\times 1)}{C_0 \lambda_k^0 + C_1 \lambda_k^1 + C_2 \lambda_k^2 + \cdots + C_{n-1} \lambda_k^{n-1}} = y_k = p(\lambda_k)$$

b. ?

c. ?

#12 ?

#13

a) $P^2 = uu^T \cdot uu^T = u \cdot (u^T u) \cdot u^T = u \cdot 1 \cdot u^T = uu^T = P$

b) $P^T = (uu^T)^T = (u^T)^T u^T = uu^T = P$

c) $Q^2 = (I-2P)(I-2P) = I^2 - 2IP - 2PI + 4P^2 = I^2 - 4P + 4P^2 = I^2 - 4P + 4P = I^2 = I$

#14

$$P = uu^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q = I - 2P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

$$Px = \underset{3\times 3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}} \qquad Qx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$$

#15

$C \sim B$, Row reduced echelon form of both matrices, B & C is same.
because row reduced echelon form is unique.
E_1, E_2, E_3 is represent elementary row operations (EROs)
& ERO is $\boxed{\text{reversible}}$

$$B \sim E_1B \sim E_2E_1B \sim E_3E_2E_1B = C$$

#16

#17

#18