Final

2020.

#1. Let $\mathbf{u}_1 = (1\ 1\ 1)$, $\mathbf{u}_2 = (-1\ 0\ 1)$, $\mathbf{y} = (3\ 5\ 10)$, and $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$.

(a) Find a unit vector in the direction of the vector $\mathbf{y}$.[10pt]

(b) Find a vector $\hat{\mathbf{y}}$, which is the orthogonal projection of $\mathbf{y}$ onto W.[10pt]

(c) Explain the relationship between $\mathbf{y}-\hat{\mathbf{y}}$ and $\mathbf{u}_1$. If possible, express the relationship in a mathematical expression.[10pt]

a) unit vector of $y = \frac{y}{\|y\|} = \frac{1}{\sqrt{134}}(3\ 5\ 10)$

$\|y\| = \sqrt{3^2+5^2+10^2} = \sqrt{134}$

b) $\hat{y} = \frac{u_1 \cdot y}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot y}{u_2 \cdot u_2} u_2 = \frac{3+5+10}{1^2+1^2+1^2} u_1 + \frac{-3+10}{1+0+1} u_2$

$= \frac{18}{3} u_1 + \frac{+7}{2} u_2 = \left(6-\frac{7}{2}\quad 6+0\quad 6+\frac{7}{2}\right) = \left(5/2\quad 6\quad \frac{19}{2}\right)$

c) $y - \hat{y} = (3\ 5\ 10) - (5/2\ 6\ 19/2) = \left(\frac{1}{2}\ -1\ \frac{1}{2}\right)$

$(y-\hat{y}) \cdot u_1 = 0$

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#2. From an experiment, four observations of two variables are collected: $(1,2), (2,6), (3,4)$, and $(4,8)$, answer the following.

(a) Set up a normal equation for a linear regression. [10pt]

(b) Solve the normal equation, and draw the regression line and the four points in a 2D plane. [10pt]

a) normal equation $A^TA\hat{x} = A^Tb$ linear regression $y = \alpha + \beta x$

$$\left.\begin{aligned} \alpha + \beta &= 2 \\ \alpha + 2\beta &= 6 \\ \alpha + 3\beta &= 4 \\ \alpha + 4\beta &= 8 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 8 \end{bmatrix}$$

$$A x = b$$

normal equation : $A^TA\hat{x} = A^Tb$.

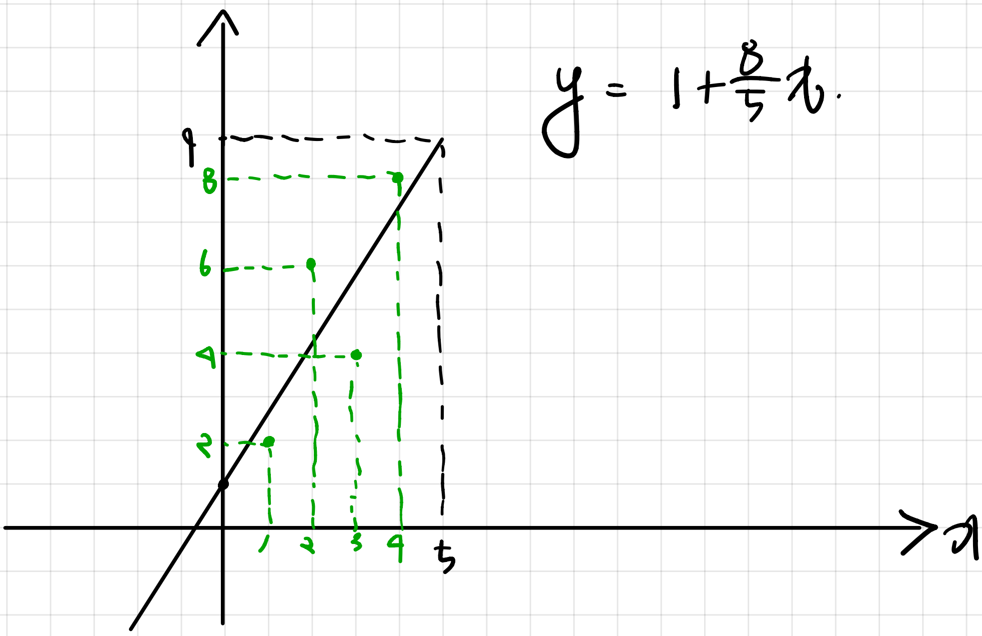
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \\ 8 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1+2+3+4 & 1+2^2+3^2+4^2 \\ 1+1+1+1 & 1+2+3+4 \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} 2+12+12+32 \\ 2+6+4+8 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 30 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} 58 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \frac{1}{100-120} \begin{bmatrix} 10 & -30 \\ -4 & 10 \end{bmatrix} \begin{bmatrix} 58 \\ 20 \end{bmatrix} = -\frac{1}{20} \begin{bmatrix} -20 \\ -32 \end{bmatrix} = \begin{bmatrix} 1 \\ 8/5 \end{bmatrix}$$



$$y = 1 + \frac{8}{5}x.$$

2020

#3. We have a matrix A of the following:

$$A = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix}$$

(a) Show that A is positive definite[1]. [10pt]

(b) Perform a Cholesky decomposition. [10pt]

a) $A\mathbb{x} = \lambda \mathbb{x} \quad \lambda \in \mathbb{R}. \quad (A - \lambda I)\mathbb{x} = 0 \quad \begin{bmatrix} 3-\lambda & 9 \\ 9 & 35-\lambda \end{bmatrix}$

$(-1)^{1+1}(3-\lambda)(35-\lambda) + (-1)^{2+1} \cdot 9 \cdot 9$

$(3-\lambda)(35-\lambda) - 81 = 105 - 38\lambda + \lambda^2 - 81 = \lambda^2 - 38\lambda + 24 \quad \lambda = \dfrac{38 \pm \sqrt{38^2 - 4 \cdot 24}}{2}$

λ is positive. A is positive definite.

b). $\begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix} \sim \begin{bmatrix} 3 & 9 \\ 0 & 8 \end{bmatrix} = U, \quad L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$R_2 \leftarrow R_2 - 3R_1$

$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 9 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix},$

$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 3\sqrt{3} & 2\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 3\sqrt{3} \\ 0 & 2\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix}$

$A = LL^T = \begin{bmatrix} \sqrt{3} & 0 \\ 3\sqrt{3} & 2\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 3\sqrt{3} \\ 0 & 2\sqrt{2} \end{bmatrix}$

2020

#4. Suppose $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{x} = [1 \;\; 2 \;\; 0]^t$ and $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} = [1 \;\; 1 \;\; -1]^t$. Is $\mathbf{x} = [3 \;\; 5 \;\; -1]^t$ a solution to $A\mathbf{x} = \mathbf{b}$ as well? Write True or False, and explain it. [10pt]

$$A\begin{bmatrix}1\\2\\0\end{bmatrix} = 0 \qquad A\begin{bmatrix}1\\1\\-1\end{bmatrix} = \mathbb{b} \qquad A\begin{bmatrix}3\\5\\-1\end{bmatrix} = \mathbb{b} \ ?$$

$$2\cdot A\begin{bmatrix}1\\2\\0\end{bmatrix} + A\begin{bmatrix}1\\1\\-1\end{bmatrix} = A\begin{bmatrix}2+1\\4+1\\0-1\end{bmatrix} = A\begin{bmatrix}3\\5\\-1\end{bmatrix} = \mathbb{b}$$

2020

#5. From an experiment, three observations of two variables are collected: $(1,6)$, $(2,5)$, and $(3,4)$, answer the following.

(a) Construct a sample covariance matrix. [10pt]

Using the sample covariance matrix above, perform a principal component analysis. In case you are unsure of your answer to the problem (a), then you may propose a legit 2×2 covariance matrix and proceed with it.

(b) Find the two principal components. [10pt]

(c) How much variance each principal component explain? [10pt]

(d) Any comment on your result regarding (c)? [Bonus 5pt]

a) $X = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$. mean, $M = \frac{1}{3}\begin{bmatrix} 6 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

mean deviation form, $B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

sample covariance matrix, $S = \frac{1}{N-1} BB^T = \frac{1}{2}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1+0+1 & -1+0-1 \\ -1+0-1 & 1+0+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

b) $\det(S - \lambda I) = 0$

$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 1 = 1 - 2\lambda + \lambda^2 - 1 = \lambda(\lambda - 2) = 0$

$\lambda = 0$ or 2.

i) $\lambda_1 = 0$

$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, $x_1 - x_2 = 0$, $x_1 = x_2$

eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ u_1, unit vector of $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

ii) $\lambda_2 = 2$

$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$ $-x_1 - x_2 = 0$, $x_1 = -x_2$

eigenvector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ u_2, unit vector of $v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

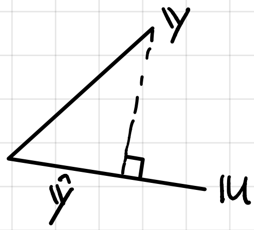
c) u_1 explain $\frac{0}{2+0}$, 0% of overall variation
u_2 explain $\frac{2}{2+0}$, 100% of overall variation

d) all observation exist in a single line
single vector can explain all variations,

2021

#1. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and the origin. [15pt]

$\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$



$\hat{\mathbf{y}} = \dfrac{\mathbf{u}\cdot\mathbf{y}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \dfrac{(-1)(1)+3\cdot 7}{(-1)^2+3^2}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

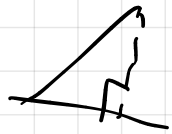
$= \dfrac{20}{10}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

2021

#2. The given set is a basis for a subspace W. Use the Gram–Schmidt process to produce an orthogonal basis for W. [15pt]

$$\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

$\mathbb{X}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ $\quad \mathbb{X}_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ $\quad \mathbb{V}_1, \mathbb{V}_2 \in \mathbb{R}^3.$



$\mathbb{V}_1 = \mathbb{X}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$

$$\mathbb{V}_2 = \mathbb{X}_2 - \frac{\mathbb{X}_1 \cdot \mathbb{X}_2}{\mathbb{X}_1 \cdot \mathbb{X}_1} \mathbb{X}_1 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{2 \cdot 4 + 5 \cdot 2}{2^2 + (-5)^2 + 1^2} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{15}{30} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 5/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

$$\mathbb{V}_1 \cdot \mathbb{V}_2 = 2 \times 3 + (-5)(3/2) + 1(3/2) = 6 - \frac{15}{2} + \frac{3}{2} = 0.$$

orthogonal basis for W is $\{\mathbb{V}_1, \mathbb{V}_2\}$

2021

#3. Construct the normal equations for $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

and find the solution $\hat{\mathbf{x}}$. [20pt]

normal equations = $A^TA\hat{x} = A^T b$

$$\begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2\cdot 2+(-2)(-2)+2\cdot 2 & 2+2\cdot 3 \\ 1\cdot 2+3\cdot 2 & 1+3\cdot 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2(-5)-2(8)+2\cdot 1 \\ -5+0+3 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-64+120}\begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix}\begin{bmatrix} -24 \\ -2 \end{bmatrix} = \frac{\pm 1}{56}\begin{bmatrix} -240+16 \\ 192-24 \end{bmatrix} = \begin{bmatrix} -224/56 \\ 168/56 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

2021

#4. We have the following matrix. $A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$.

(a) Show that A is positive definite. [10pt]

(b) Perform an orthogonal diagonalization. [15pt]

a) $\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = 0.$

$(6-\lambda)(9-\lambda) - 4 = 54 - 15\lambda + \lambda^2 - 4 = \lambda^2 - 15\lambda + 50 = (\lambda-10)(\lambda-5) = 0$

$\lambda = 10$ & 5 all eigenvalues are positive.

So, A is positive definite.

b) i) $\lambda = 10$

$(A-10I)x = 0$

$\left[\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right]$

$-4x_1 - 2x_2 = 0$

$-\frac{1}{2}x_2 = x_1$

x_2

$v_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

ii) $\lambda = 5$

$(A-5I) = 0$

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right]$

$x_1 - 2x_2 = 0$

$\begin{matrix} 2x_2 \\ x_2 \end{matrix} \to \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2.$

$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\frac{5}{4}}} v_1$

$= \frac{2}{\sqrt{5}} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$P = [u_1\ u_2]$, $P^T = P^{-1} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\because A$ is pd.

$A = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

$= \begin{bmatrix} -\frac{10}{\sqrt{5}} & \frac{10}{\sqrt{5}} \\ \frac{20}{\sqrt{5}} & \frac{5}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{10}{5} + \frac{20}{5} & -\frac{20}{5} + \frac{10}{5} \\ -\frac{20}{5} + \frac{10}{5} & \frac{40}{5} + \frac{5}{5} \end{bmatrix}$

$= \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$

2021

#5. From an experiment, three observations of two variables are collected: $(1,6)$, $(2,5)$, and $(3,4)$, answer the following.

(a) Construct a sample covariance matrix. [15pt]

Using the sample covariance matrix above, perform a principal component analysis. In case you are unsure of your answer to the problem (a), then you may propose a legit 2×2 covariance matrix and proceed with it.

(b) Find the two principal components. [10pt]

a) $X = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$ $M = \frac{1}{3}\begin{bmatrix} 6 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

$S = \frac{1}{3-1}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

b) $\det(S - \lambda I) = 0$

$(\lambda-1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda-2) = 0.$ $\lambda = 0, 2.$

i) $\lambda = 0$

$(S - 0I)x = 0$

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$

$x_1 - x_2 = 0$

$W_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$M_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

ii) $\lambda = 2$

$(S - 2I)x$

$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$

$-x_1 - x_2 = 0$

$W_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$U_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$