ITM426, Quiz 3, 2020 Fall

Solution and Grading

- ITM 426 Engineering Mathematics 2020 F
- Nov 20, 2020
- Duration: 90 minutes
- 5 Questions
- Weighting of 25 % or 30 % depending on other quiz scores

- Name: ____________________
- Student ID: ____________________
- E-mail: ______________ @seoultech.ac.kr

- Write legibly.
- In on-line exam, start every problem in a new page.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

1	10
2	10
3	10
4	20
5	20
6	10
7	10
Total	90

#1. Compute the determinant of the following matrix. [10pt]

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$$

Difficulty: Easy
Amount of work: 15 %
Suggested answer:

First expand along either the second row or the second column. Using the second row,

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} = (-1)^{2+3} \cdot 2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

Now expand along the second column to find:

$$(-1)^{2+3} \cdot 2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = -2\left((-1)^{2+2} \cdot 3 \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} \right)$$

Now expand along either the first column or third row. The first column is used below.

$$-2\left((-1)^{2+2} \cdot 3 \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} \right) = -6\left((-1)^{1+1} \cdot 4 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + (-1)^{2+1} \cdot 5 \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix} \right) = (-6)(4(1) - 5(1)) = 6$$

#2. Assuming $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$, find the determinant of the following. `[10pt]`

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$$

Difficulty: Easy
Amount of work: 10 %
Suggested answer:

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(7) = 14$$

#3. Disprove the following statement by an example. `[10pt]`

The set H of all points in $\mathbb{R}^2$ of the form $(3s, 2+5s)$ is a vector space.

Difficulty: Easy
Amount of work: 10 %
Suggested answer: Let $s = 2$, then a vector $(6, 12) \in H$, but its scalar multiple $2(6, 12) = (12, 24)$ is not a form of $(3s, 2+5s)$. That is, $2(6, 12) \notin H$ Thus, H is not a vector space.

#4. In the following, the matrix A is row equivalent to B. [20pt]

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

(a) Write $rank\, A$ and $dim\, Nul\, A$.

(b) Find bases for $Col\, A$.

(c) Find bases for $Row\, A$.

(d) Find bases for $Nul\, A$.

Difficulty: Easy
Amount of work: 25 %
Suggested answer:

The matrix B is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three pivot rows, so the dimension of Row A is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A: $\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$. A basis for Row A is the pivot rows of B: $\{(1,-3,0,5,-7),(0,0,2,-3,8),(0,0,0,0,5)\}$. To find a basis for Nul A row reduce to reduced echelon form: $A \sim \begin{bmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The solution to $A\mathbf{x} = \mathbf{0}$ in terms of free variables is

$x_1 = 3x_2 - 5x_4$, $x_3 = (3/2)x_4$, $x_5 = 0$, with x_2 and x_4 free. Thus a basis for Nul A is
$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

#5. It is known that the eigenvalues of the following matrix is 3 and 1.

$$A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$$

(a) Find all eigenvectors corresponding to each eigenvalue. [10pt]

(b) Perform diagonalization. Sanity check is highly recommended. `[10pt]`

Difficulty: Medium
Amount of work: 20 %
Suggested answer:

The eigenvalues of A are given to be 3 and 1.

For $\lambda = 3$: $A-3I=\begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-3I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right\}$

For $\lambda = 1$: $A-I=\begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3=\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}=\begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then set $D=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

#6. Find a nonzero 2×2 matrix that is invertible but not diagonalizable. `[10pt]`

Difficulty: Medium-Hard
Amount of work: 10 %
Suggested answer:

For a 2×2 matrix A to be invertible, its eigenvalues must be nonzero. A first attempt at a construction might be something such as $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, whose eigenvalues are 2 and 4. Unfortunately, a 2×2 matrix with two distinct eigenvalues is diagonalizable (Theorem 6). So, adjust the construction to $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, which works. In fact, any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ has the desired properties when a and b are nonzero. The eigenspace for the eigenvalue a is one-dimensional, as a simple calculation shows, and there is no other eigenvalue to produce a second eigenvector.

#7. Find a nonzero 2×2 matrix that is diagonalizable but not invertible. `[10pt]`

Difficulty: Medium-Hard
Amount of work: 10 %
Suggested answer:

Any 2×2 matrix with two distinct eigenvalues is diagonalizable, by Theorem 6. If one of those eigenvalues is zero, then the matrix will not be invertible. Any matrix of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has the desired properties when a and b are nonzero. The number a must be nonzero to make the matrix diagonalizable; b must be nonzero to make the matrix not diagonal. Other solutions are $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$.

Write your name before detaching this page. Your Name: ______________________