ITM426, Final Exam, 2020 Fall

Solution and Grading

- ITM 426 Engineering Mathematics 2020 F
- Dec 11, 2020
- Duration: 120 minutes
- 5 Questions
- Weighting of 30 %

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- Write legibly.
- In on-line exam, start every problem in a new page.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

1	30
2	20
3	20
4	10
5	30
Total	110

#1. Let $\mathbf{u}_1 = (1\ \ 1\ \ 1)$, $\mathbf{u}_2 = (-1\ \ 0\ \ 1)$, $\mathbf{y} = (3\ \ 5\ \ 10)$, and $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$.

(a) Find a unit vector in the direction of the vector $\mathbf{y}$. `[10pt]`

(b) Find a vector $\hat{\mathbf{y}}$, which is the orthogonal projection of $\mathbf{y}$ onto W. `[10pt]`

(c) Explain the relationship between $\mathbf{y} - \hat{\mathbf{y}}$ and $\mathbf{u}_1$. If possible, express the relationship in a mathematical expression. `[10pt]`

Difficulty: Easy
Amount of work: 15 %
Suggested answer:
(a) $\frac{1}{\sqrt{3^2+5^2+10^2}}(3\ \ 5\ \ 10)$
(b) $\hat{\mathbf{y}} = (3\ \ 5\ \ 10) - \frac{18}{3}(1\ \ 1\ \ 1) - \frac{7}{2}(-1\ \ 0\ \ 1) = \frac{1}{2}(5\ \ 12\ \ 19)$
(c) $(\mathbf{y} - \hat{\mathbf{y}}) \circ \mathbf{u}_1 = 0$

#2. From an experiment, four observations of two variables are collected: $(1,2),(2,6),(3,4)$, and $(4,8)$, answer the following.

(a) Set up a normal equation for a linear regression. [10pt]

(b) Solve the normal equation, and draw the regression line and the four points in a 2D plane. [10pt]

Difficulty: Medium
Amount of work: 25 %
Suggested answer:
Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 8 \end{bmatrix}$, then normal equation is $A^T A\mathbf{x} = A^T\mathbf{b}$. It leads to $\mathbf{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 8/5 \end{bmatrix}$

#3. We have a matrix A of the following:

$$A = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix}$$

(a) Show that A is positive definite[1]. `[10pt]`

(b) Perform a Cholesky decomposition. `[10pt]`

Difficulty: Medium
Amount of work: 20 %
Suggested answer:
(a) A characteristic equation is $\lambda^2 - 38\lambda + 24 = 0$, and the two roots are both positive numbers. This proves the pd.
(b) $A = LL^t$, where $L = \begin{bmatrix} \sqrt{3} & 0 \\ 3\sqrt{3} & 2\sqrt{2} \end{bmatrix}$

[1] Following may or may not help: a quadratic equation of $ax^2 + bx + c = 0$ has a solution $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

#4. Suppose $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{x} = [1 \quad 2 \quad 0]^t$ and $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} = [1 \quad 1 \quad -1]^t$. Is $\mathbf{x} = [3 \quad 5 \quad -1]^t$ a solution to $A\mathbf{x} = \mathbf{b}$ as well? Write True or False, and explain it. `[10pt]`

Difficulty: Easy
Amount of work: 10 %
Suggested answer:

True. $A\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} = A \cdot 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + A\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{0} + \mathbf{b} = \mathbf{b}$.

#5. From an experiment, three observations of two variables are collected: $(1, 6), (2, 5)$, and $(3, 4)$, answer the following.

(a) Construct a sample covariance matrix. `[10pt]`

Using the sample covariance matrix above, perform a principal component analysis. In case you are unsure of your answer to the problem (a), then you may propose a legit 2×2 covariance matrix and proceed with it.

(b) Find the two principal components. `[10pt]`

(c) How much variance each principal component explain? `[10pt]`

(d) Any comment on your result regarding (c)? `[Bonus 5pt]`

Difficulty: Medium-Hard
Amount of work: 30 %
Suggested answer:
(a) $X = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}, M = \begin{bmatrix} 2 & 2 & 2 \\ 5 & 5 & 5 \end{bmatrix}, S = (X - M)(X - M)^t/(3 - 1) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.
(b)(c) eigenvalues: $2, 0$; eigenvectors: $(1 \ -1)$ and $(1 \ 1)$; principal components: $\frac{1}{\sqrt{2}}(1 \ -1)$ and $\frac{1}{\sqrt{2}}(1 \ 1)$. Each principal component explains 100% and 0%, respectively.
(d) All observations fall into a single line, thus a single vector (principal component) can explain all variations.

Write your name before detaching this page. Your Name: ____________________