Quiz 3

2020

#1. Compute the determinant of the following matrix. [10pt]

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$$

$$\det\left(\begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}\right) = (-1)^{2+3} \cdot 2 \cdot \det\left(\begin{bmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{bmatrix}\right)$$

$$= (-1)^{2+3} \cdot 2 \cdot (-1)^{2+2} \cdot 3 \cdot \left\{ \det\begin{bmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} \right\}$$

$$= -6 \cdot \left\{ (-1)^{1+1} \cdot 4 \cdot \det\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} + (-1)^{1+2} \cdot 3 \cdot \det\begin{pmatrix} 5 & 3 \\ 0 & 2 \end{pmatrix} + (-1)^{1+3} \cdot (-5) \det\begin{pmatrix} 5 & 2 \\ 0 & -1 \end{pmatrix} \right\}$$

$$= -6 \left\{ 4 \times (4-3) - 3(10-0) - 5(-5-0) \right\}$$

$$= -6 \left\{ -1 \right\} = 6$$

2020

#2. Assuming $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$, find the determinant of the following. [10pt]

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \underset{2R_2}{\sim} \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix} \underset{R_2+R_1}{\sim} \begin{bmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{bmatrix} = B$$

$\det(A) = \lambda,$ $\quad \det B = 2\lambda$

2020

#3. Disprove the following statement by an example. [10pt]

The set H of all points in $\mathbb{R}^2$ of the form $(3s, 2+5s)$ is a vector space.

$s=2$, $(6, 12) = v_1$

$v_2 = 2v_2 = (12, 24)$, $s=4$, $3s=12$, $2+5s=22$. No!

2020

#4. In the following, the matrix A is row equivalent to B.[20pt]

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

(a) Write $rank\, A$ and $dim\, Nul\, A$. $\quad$ rank $A = 3$ $\quad$ dim Nul $A = 2$

(b) Find bases for $Col\, A$.

(c) Find bases for $Row\, A$.

(d) Find bases for $Nul\, A$.

b. $\begin{bmatrix} 1 & 4 & 9 \\ -2 & -6 & -10 \\ -3 & -6 & -3 \\ 3 & 4 & 0 \end{bmatrix}$

c. $[(1,-3,0,5,-7)\ (0,0,2,-3,8)\ (0,0,0,0,5)]$

d. $x_1 - 3x_2 + 5x_4 - 7x_5 = 0$

$2x_3 - 3x_4 + 8x_5 = 0$

$5x_5 = 0$

x_2, x_4 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_4 \\ x_2 \\ \frac{3}{2}x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} x_4$$

$$\begin{bmatrix} 3 & -5 \\ 1 & 0 \\ 0 & 3/2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

2020

#5. It is known that the eigenvalues of the following matrix is 3 and 1.

$$A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$$

(a) Find all eigenvectors corresponding to each eigenvalue. [10pt]

(b) Perform diagonalization. Sanity check is highly recommended. [10pt]

i) $\lambda = 3$

$(A-3I)x = 0$ $\quad (A-3I) = \begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix} \sim \begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_2 + 4x_3 = 0.$ $\quad \begin{bmatrix} -x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} x_3$

eigenvectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

ii) $\lambda = 1$

$(A-I)x = 0$ $\quad (A-I) = \begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & 2 & 2 \\ -2 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ -2 & -2 & -6 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

$x_1 + x_2 + 3x_3 = 0$

$x_2 + x_3 = 0$

x_3 free.

$\begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} x_3$

eigenvectors $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underline{AP = PD}$

$$\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix} \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -12 & -2 \\ 3 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -12 & -2 \\ 3 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}$$

2020

#6. Find a nonzero 2×2 matrix that is invertible but not diagonalizable. [10pt]

$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad ad-bc \neq 0$

$\begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix}$ $\quad \lambda \neq 0$

$(a-\lambda)(d-\lambda) - bc$

$ad - (a+d)\lambda + \lambda^2 - bc$

$\underbrace{(ad-bc)}_{\neq 0} - (a+d)\lambda + \underbrace{\lambda^2}_{\neq 0}$

#7. Find a nonzero 2×2 matrix that is diagonalizable but not invertible. [10pt]

#1. Answer the following questions. [Each 10pt]

- If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is 5×7 matrix, what is the rank of A?

$Ax=0$, 3 free variables.

$3 + \underbrace{\# \text{ of pivot}}_{=4} = 7.$

rank $A = 4$.

- Construct a 4×3 matrix with rank 1.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

20

124 8
-1 3 6 7
0 5 10 15

1 2 4 8
2 4 8

0 5 10 15
0 5 10 20
0 0 0 -5

#2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why or why not? [10pt]

$\{v_1, v_2, v_3\} = A$

$Ax = w$ $\quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 3 & 6 \end{bmatrix} x = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$

$$[A|w] = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 5 & 10 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ \boxed{0 \; 0 \; 0 \; -5} \end{bmatrix},$$

$Ax = w$ inconsistent.

Subspace is closed under vector sum.
however, w cannot be made by linear combination of set of vectors v_1, v_2 and v_3.

So w is not in subspace spanned by A

10

15
9
135

36
3
108

#3. Assume that A is row equivalent to B. Find bases for $Nul\,A$ and $Col\,A$.[Each 10pt]

$$A=\begin{bmatrix}1 & 2 & -5 & 0 & -1\\ 2 & 5 & -8 & 4 & 3\\ -3 & -9 & 9 & -7 & -2\\ 3 & 10 & -7 & 11 & 7\end{bmatrix},\ B=\begin{bmatrix}1 & 2 & -5 & 0 & -1\\ 0 & 1 & 2 & 4 & 5\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

1 2 -5 0 -1
0 -2 -4 -8 -10

$$Col\,A=\begin{bmatrix}1 & 2 & 0\\ 2 & 5 & 4\\ -3 & -9 & -7\\ 3 & 10 & 11\end{bmatrix}$$

1 0 -9 -8 -11
0 0 0 8 -16

0 1 2 4 5
0 0 0 -4 8

$$Ax=0 \quad [A|0]\sim\begin{bmatrix}1 & 2 & -5 & 0 & -1\\ 0 & 1 & 2 & 4 & 5\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 0 & -9 & -8 & -11\\ 0 & 1 & 2 & 4 & 5\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & -9 & 0 & -27\\ 0 & 1 & 2 & 0 & 13\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$x_1-9x_3-27x_5=0.$

$x_2+2x_3+13x_5=0$

$x_4-2x_5=0.$

$$\begin{bmatrix}9x_3+27x_5\\ -2x_3-13x_5\\ x_3\\ 2x_5\\ x_5\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}$$

$$Nul\,A=\begin{bmatrix}9 & 27\\ -2 & -13\\ 1 & 0\\ 0 & 2\\ 0 & 1\end{bmatrix}$$

20

#4. For the following matrix, one eigenvalue is 5 and one eigenvector is $(-2, 1, 2)$. Identify all eigenvalues and their corresponding eigenvector. Then, perform a diagonalization. [20pt]

$$A = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$$

5 $(A-\lambda I)x=0$ 3

$$[A-\lambda I \mid 0] = \begin{bmatrix} -7-\lambda & -16 & 4 & 0 \\ 6 & 13-\lambda & -2 & 0 \\ 12 & 16 & 1-\lambda & 0 \end{bmatrix} \sim \begin{bmatrix} -7-\lambda & -16 & 4 & 0 \\ 6 & 13-\lambda & -2 & 0 \\ 5-\lambda & 0 & 5-\lambda & 0 \end{bmatrix} \sim \begin{bmatrix} 5-\lambda & 10-2\lambda & 0 & 0 \\ 6 & 13-\lambda & -2 & 0 \\ 5-\lambda & 0 & 5-\lambda & 0 \end{bmatrix}$$

$\lambda = 5$

$$[A-5I \mid 0] \sim \begin{bmatrix} 6 & 8 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$6x_1 + 8x_2 - 2x_3 = 0$

x_2, x_3 is free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}x_2 + \frac{1}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} x_3$$

$\lambda = 13.$

$$[A-13I \mid 0] \sim \begin{bmatrix} -8 & -16 & 0 \\ 6 & 0 & -2 \\ -8 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -4 \\ 0 & -2 & 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 = 0$

$x_1 = 0$

0

#5. Compute the determinants of the following matrix [15pt]

$$A = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 2 & -5 \\ -1 & -1 & -10 \\ \hline 0 & 1 & -15 \end{array}$$

$$\begin{array}{ccc} 1 & 2 & 5 \\ -1 & -2 & 5 \\ \hline 0 & 0 & 5 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & -5 \\ 0 & 0 & 10 \\ 0 & 1 & -15 \end{bmatrix}$$

$$\det(A) = (-1)^{1+1} \cdot 1 \det\left(\begin{bmatrix} 1 & 2 & -5 \\ 1 & 2 & 5 \\ -1 & -1 & -10 \end{bmatrix}\right)$$

$$\begin{array}{ccc} 1 & 2 & 5 \\ -1 & -2 & 5 \\ \hline 0 & 0 & 10 \end{array}$$

$$= (-1)^{2+3} \cdot 10 \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right) = -10 \cdot (1 \cdot 1 - 0) = -10$$

$$\begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\det(A) = 1 \times 1 \times 1 \times 10 \times (-1) = -10.$$

15

#6. Show that if A is both diagonalizable and invertible, then A^{-1} is also diagonalizable. [15pt]

~~A is square matrix.~~
~~$n \times n$ ($n \in \mathbb{R}$)~~

~~A has n linearly independent vectors.~~

~~the vectors are eigen vectors.~~
~~also linearly independent~~

invertible P
diagonal D

$A = PDP^{-1}$

Is D invertible?

$\rightarrow$ A is invertive
non-zero eigen value.
$\therefore$ D all entries : non-zero
$\therefore$ $\det(D) \neq 0$. invertible

$AP = PD$

$(AP)^{-1} = (PD)^{-1}$

$P^{-1}A^{-1} = D^{-1}P^{-1}$

$A^{-1} = PD^{-1}P^{-1}$

if D is invertible, then A^{-1} is diagonalizable.

D is diagonal matrix. and Its entries are non-zero. why?

so $\det(D) \neq 0$, so, D is invertible.

Thus A^{-1} is diagonalizable.

A is invertible
$\det(A) \neq 0$