Chapter 7. Symmetric Matrices and Quadratic Forms

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only possible

$PDP^{-1} = PDP^T$

advanced ver

1 7.1. Diagonalization of Symmetric Matrices

$$\begin{bmatrix} 1 & 3 & \boxed{4} \\ 3 & 1 & -2 \\ \boxed{4} & -2 & 1 \end{bmatrix}$$

7.1. Diagonalization of Symmetric Matrices

Symmetric Matrix

- A **symmetric matrix** is a matrix $\overset{m\times n}{A}$ such that $\underset{n\times m}{A^T} = \underset{m\times n}{A}$.
- Such a matrix is necessarily square. $n = m$ ∴ square matrix
- Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal.
- **Theorem 1:** If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.
- **Theorem 2:** An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric matrix.

- **Example 3:**: Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, whose characteristic equation is $-\lambda^3 + 12\lambda^2 - 21\lambda - 98 = \underline{-(\lambda - 7)^2(\lambda + 2)} = 0$

  $\lambda_1 = 7 \quad \lambda_2 = 7 \quad \lambda_3 = -2$

- **Solution:**
  - The usual calculations produce bases for the eigenspaces:

    $$\lambda = 7 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}; \ \lambda = -2 : \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

  - Although $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, they are not orthogonal. The projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$ is $\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$. The component of $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$ is

    $$\mathbf{z}_2 = \mathbf{v}_2 - \overset{\hat{V}_2}{\boxed{\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1}} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

    $V_1 = Z_1$

  - Then $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an orthogonal set in the eigenspace for $\lambda = 7$
  - (Note that $\mathbf{z}_2$ is linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, so $\mathbf{z}_2$ is in the eigenspace.)

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \qquad A - \lambda I = \begin{array}{c} 11 \\ 21 \\ 31 \end{array} \begin{bmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$$

$$(-1)^{1+1}(3-\lambda)\begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} \; (-1)^{2+1} \, -2 \begin{vmatrix} -2 & 4 \\ 2 & 3-\lambda \end{vmatrix} \; (-1)^{3+1} +4 \begin{vmatrix} -2 & 4 \\ 6-\lambda & 2 \end{vmatrix}$$

$$(3-\lambda)\{(6-\lambda)(3-\lambda)-4\} + 2\{-2(3-\lambda) - 2\times 4\} + 4\{-4-(6-\lambda)4\}$$

$$(3-\lambda)(18-9\lambda+\lambda^2-4) + 2(-6+2\lambda-8) + 4(-4-24+4\lambda)$$

$$(3-\lambda)(\lambda^2-9\lambda+14) + 2(2\lambda-14) + 4(4\lambda-28)$$

$$-(\lambda-3)(\lambda-7)(\lambda-2) + 4(\lambda-7) + 16(\lambda-7)$$

$$-(\lambda-7)\{+(\lambda-3)(\lambda-2) - 4 - 16\} = -(\lambda-7)\{\lambda^2-5\lambda-14\}$$

$$= -(\lambda-7)(\lambda-7)(\lambda+2) = -(\lambda-7)^2(\lambda+2)$$

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$\lambda_1 = 7$
$\lambda_2 = 7$
$\lambda_3 = -2$

$A - \lambda I = 0.$

$$\begin{bmatrix} -4 & -2 & 4 & 0 \\ -2 & -1 & 2 & 0 \\ 4 & 2 & -4 & 0 \end{bmatrix}$$

$-4x_1 - 2x_2 + 4x_3 = 0.$

x_2, x_3 is free.

$\frac{-2x_2 + 4x_3}{4} = -\frac{1}{2}x_2 + x_3$

$$\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3.$$

$$\begin{bmatrix} 5 & -2 & 4 & 0 \\ -2 & 8 & 2 & 0 \\ 4 & 2 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & -2 & 4 & 0 \\ -2 & 8 & 2 & 0 \\ 0 & 18 & 9 & 0 \end{bmatrix}$$

$5 \quad -2 \quad 4 \quad 0$

$$\begin{array}{r|cccc} & -2 & 8 & 2 & 0 \\ \frac{2}{5} & 2 & -\frac{4}{5} & \frac{8}{5} & 0 \\ \hline & 0 & \frac{36}{5} & \frac{18}{5} & \end{array}$$

$0 \quad \frac{36}{5} \quad \frac{18}{5}$

$\cancel{0 \quad 18 \quad 9}$

$5x_1 - 2x_2 + 4x_3 = 0$

$2x_2 + x_3 = 0$

$5x_1 + 5x_3 = 0$

$x_1 = -x_3$

$-x_3$

$-\frac{1}{2}x_3$

x_3

$$\begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} x_3$$

- (*solution continued:*)
  - Since the eigenspace is two-dimensional (with basis $\mathbf{v}_1, \mathbf{v}_2$), the orthogonal set $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an *orthogonal basis* for the eigenspace, by the Basis Theorem.
  - Normalize $\mathbf{v}_1$ and $\mathbf{z}_2$ to obtain the following orthonormal basis for the eigenspace for $\lambda = 7$:

$V_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\|V_1\| = \sqrt{2}$

$Z_2 = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$ $\|Z_2\| = \sqrt{\frac{1}{16}+1+\frac{1}{16}} = \frac{3\sqrt{2}}{4}$

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

  - An orthonormal basis for the eigenspace for $\lambda = -2$ is

$$\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

  - By Theorem 1, $\mathbf{u}_3$ is orthogonal to the other eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Hence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.
  - Let $P = [\overbrace{\underbrace{\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3}_{\text{orthonormal}}}^{\text{eigen vectors}}] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

$$A = PDP^{-1} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix} \begin{bmatrix} 7 & & \\ & 7 & \\ & & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix} \begin{bmatrix} 7 & & \\ & 7 & \\ & & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{18} & 4/\sqrt{18} & 1/\sqrt{18} \\ -2/3 & -1/3 & 2/3 \end{bmatrix} \qquad P^{-1} = P^T$$

Definition If $A = PDP^{-1}$ where P is orthonormal and D is diagonal, then A is orthogonally (orthonormally) diagonalizable

Remark orthogonal diagonalization $\Rightarrow$ $P^{-1} = P^T$

Suggested Excercises

- 7.5.1
- 7.5.3
- 7.6 (p.450)

$$A = PDP^{-1} = PDP^{T} = P\begin{bmatrix}7 & & \\ & 7 & \\ & & -2\end{bmatrix}P^{T} = P\begin{bmatrix}\sqrt{7} & & \\ & \sqrt{7} & \\ & & \sqrt{-2}\end{bmatrix}\begin{bmatrix}\sqrt{7} & & \\ & \sqrt{7} & \\ & & \sqrt{-2}\end{bmatrix}^{T}P^{T}$$

$$= P(\sqrt{D}\sqrt{D}^{T})P^{T} = (P\sqrt{D})(P\sqrt{D})^{T}$$

If A is orthogonal diagonalizable, then $A = (P\sqrt{D})(P\sqrt{D})^{T}$

$$A^{T} = A = PDP^{-1}$$

$$= A^{T} = (PDP^{-1})^{T} = (P^{-1})^{T}D^{T}P^{T}$$

$D = D^{T}$ ($\because$ diagonal)

$P = (P^{-1})^{T}$ $\quad P^{-1} = P^{T}$

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