ITM426, Quiz 2, 2021 Fall

Solution and Grading

- ITM 426 Engineering Mathematics 2021 F
- Oct 29, 2021
- Duration: 90 minutes
- Weights: 25% or 30% depending on other quiz scores
- 5 Questions

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- Write legibly.
- In on-line exam, start every problem in a new page.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

1	24
2	16
3	20
4	20
5	20
Total	100

#1. Mark True or False. No justification is necessary. `[Each 6pt]`

- In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations. (`TRUE` / `FALSE`)
- If one row in an echelon form of an augmented matrix is [0 0 0 5 0], then the associated linear system is inconsistent. (`TRUE` / `FALSE`)
- Let A be a 3×2 matrix. The equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. (`TRUE` / `FALSE`)
- If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution to $A\mathbf{x} = \mathbf{b}$ is unique. (`TRUE` / `FALSE`)

Difficulty: Hard
Amount of work: 24 %
Suggested answer:

- False. See Chapter 1, Theorem 1.
- False. The row shown corresponds to the equation $5x_4 = 0$, which does not by itself lead to a contradiction. So the system might be consistent or it might be inconsistent.
- True. A 3×2 matrix has three rows and two columns. With only two columns, A can have at most two pivot columns, and so A has at most two pivot positions, which is not enough to fill all three rows. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. Generally, if A is an m×n matrix with $m > n$, then A can have at most n pivot positions, which is not enough to fill all m rows. Thus, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$.
- True
  (Geometric argument using Theorem 6.) Since $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  (Proof using free variables.) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

#2. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5 + 4x_3$, $x_2 = -2 - 7x_3$, with x_3 free. Use vectors to describe this set as a line in $\mathbb{R}^3$. [16pt]

Difficulty: Easy
Amount of work: 16 %
Suggested answer:

To write the general solution in parametric vector form, pull out the constant terms that do not involve the free variable:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5+4x_3 \\ -2-7x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -7x_3 \\ x_3 \end{bmatrix} = \underset{\substack{\uparrow \\ \mathbf{p}}}{\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}} + x_3 \underset{\substack{\uparrow \\ \mathbf{q}}}{\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}} = \mathbf{p} + x_3\mathbf{q}.$$

Geometrically, the solution set is the line through $\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ in the direction of $\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$.

#3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$.

- Prove that f is a linear transformation when $b = 0$. [10pt]
- Find a property of a linear transformation when $b = 0$. [10pt]

Difficulty: Medium
Amount of work: 20 %
Suggested answer:

a. When $b = 0$, $f(x) = mx$. In this case, for all x, y in $\mathbb{R}$ and all scalars c and d,
$f(cx + dy) = m(cx + dy) = mcx + mdy = c(mx) + d(my) = c \cdot f(x) + d \cdot f(y)$
This shows that f is linear.

b. When $f(x) = mx + b$, with b nonzero, $f(0) = m(0) + b = b \neq 0$. This shows that f is not linear, because every linear transformation maps the zero vector in its domain into the zero vector in the codomain. (In this case, both zero vectors are just the number 0.) Another argument, for instance, would be to calculate $f(2x) = m(2x) + b$ and $2f(x) = 2mx + 2b$. If b is nonzero, then $f(2x)$ is not equal to $2f(x)$ and so f is not a linear transformation.

#4. Find an LU factorization of the following matrix `[20pt]`

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$$

Difficulty: Medium
Amount of work: 20 %
Suggested answer:

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix} = U$$

$$\begin{bmatrix} 3 \\ -3 \\ 9 \end{bmatrix} \qquad \begin{bmatrix} -3 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} -8 \end{bmatrix}$$

$$\div 3 \qquad \div -3 \qquad \div -8$$

$$\begin{bmatrix} 1 & & \\ -1 & 1 & \\ 3 & 2/3 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix}$$

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#5. Find an inverse of the following matrix [20pt]

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

Difficulty: Medium
Amount of work: 20 %
Suggested answer:

$$[A \quad I] = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}. \quad A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$$

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