#1. Mark True or False. No justification is necessary. [Each 5pt]

- Every matrix is row equivalent to a unique matrix in echelon form. (TRUE / FALSE)
- Any system of n linear equations in n variables has at most n solutions. (TRUE / FALSE)
- If a system of linear equations has no free variables, then it has a unique solution. (TRUE / FALSE)
- If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. (TRUE / FALSE)
- If matrix A is an $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions. (TRUE / FALSE)

$$\left[\begin{array}{cc|c} 2 & 1 & 9 \\ 0 & 0 & 1 \end{array}\right]$$

$x_1 + x_2 = 8$

$x_1 + x_2 = 9$

$$\begin{matrix} 1 & 1 & 8 \\ 1 & 1 & 9 \end{matrix}$$

20

#2. Find all solutions to the system of the homogeneous equations of A. [15pt]

$$A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$$

$Ax=0$

augmented matrix $[A|0]$

$$= \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, x_3 is free variable.

$x_4 = 0$.

$x_2 = -2x_3$

$x_1 = -3x_2 - 9x_3 = -3(-2x_3) - 9x_3 = \underline{-3x_3}$

solution, $x = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ $x_3 \in \mathbb{R}$. ✓

a line through origin, direction of $\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^4$

15

#3. Find an LU factorization of the following matrix [15pt]

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

$$\begin{array}{ccc} 3 & 7 & 5 \\ -3 & +3 & -6 \\ \hline 0 & 10 & -1 \end{array}$$

$$\begin{array}{ccc} 1 & -3 & 1 \\ -1 & +1 & -2 \\ \hline 0 & -2 & -1 \end{array}$$

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - \frac{1}{2}R_1 \\ R_3 \leftarrow R_3 - \frac{3}{2}R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} \quad R_3 \leftarrow R_3 + 5R_2$$

$$\begin{array}{ccc} 0 & 10 & -1 \\ 0 & -10 & -5 \\ \hline 0 & 0 & -6 \end{array}$$

$$\sim \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = U$$

flip the sign

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -5 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

$-3+10$

$6+-6$

15

#4. Find an inverse of the following matrix [15pt]

$$A = \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 1 \end{bmatrix}$$

$$[A \mid I_3] \sim [I_3 \mid A^{-1}]$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ -3 & 6 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & -9 & -11 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccc} 1 & -5 & -4 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & -5 & 0 & 13 & 12 & 4 \\ 0 & 3 & 0 & -12 & -11 & -4 \\ 0 & 0 & 1 & 3 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccc} 1 & 0 & 0 & -7 & -\frac{19}{3} & -\frac{8}{3} \\ 0 & 1 & 0 & -4 & -\frac{11}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & 3 & 3 & 1 \end{array}\right] = [I_3 \mid A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -7 & -\frac{19}{3} & -\frac{8}{3} \\ -4 & -11/3 & -4/3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} -7 & -19/3 & -8/3 \\ -4 & -11/3 & -4/3 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -7+20-12 & -\frac{19}{3}+\frac{55}{3}-12 & -\frac{8}{3}+\frac{20}{3}-4 \\ -12+12 & -11+12 & -4+4 \\ 21-24+3 & 19-22+3 & 8-8+1 \end{bmatrix}$$

$-7-20+12$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

15

#5. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible. [15pt]

if A is invertible, $(AB)^{-1}AB = I = B^{-1}(A^{-1}A)B = B^{-1}IB = I$

if A is not invertible, then $(AB)^{-1}$ cannot be determined.

All invertible matrix should satisfy $(CD)^{-1} = D^{-1}C^{-1}$, when CD, C, D are invertible matrices.

Because of $(AB)^{-1} = \underline{B^{-1}A^{-1}}$, to determine $(AB)^{-1}$, A^{-1} should be defined

So. A is invertible.

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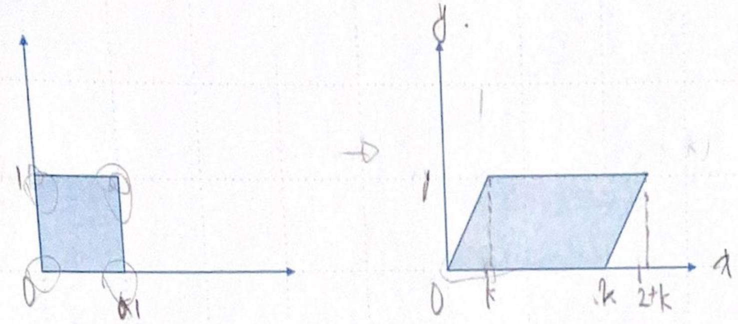
#6. The unit square on the left becomes the parellogram on the right by a linear transformation. What would be the standard matrix for this linear transformation? Justification is necessary. [15pt]



Standard matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $a, b, c, d \in \mathbb{R}$. $\boxed{k = \frac{1}{2}}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & k & 2+k \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{bmatrix}$$

x y

$a=2 \quad b=k$

$c=0 \quad d=1$

$$A = \begin{bmatrix} 2 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

15