Ch 1. Linear Equations (2/2)

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1.5. Solution Sets of Linear Systems

1.7. Linear Indepdence

1.8. Introduction to Linear Transformations

1.9. The Matrix of Linear Transformation

Suggested Exercises

1.5. Solution Sets of Linear Systems

- A system of linear equations is said to be *homogeneous* if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.
- Such a system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the *trivial solution*.
- There being a trivial solution, the important question for a homogeneous system is whether there exists a *nontrivial solution*, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$.

Homogeneous Linear Systems

Example 1. Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

Solution: Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[A\ \mathbf{0}]$ to echelon form.

$$\begin{pmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Since x_3 is a free variable, we know that there exists a nontrivial solution. That is, setting x_3 equal to any arbitrary number will still generate a legit solution.

- Continue the row reduction of $[A \ \mathbf{0}]$ to *reduced* echelon form:

$$\begin{pmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 - \frac{4}{3}x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

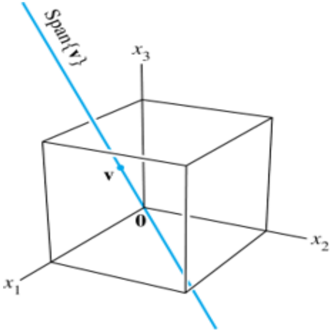
- Solving for basic variables, x_1, x_2 to obtain $x_1 = \frac{4}{3}x_3$, $x_2 = 0$, with x_3 free. As a vector, this solution can be written as

homogeneous sol'n
general sol'n

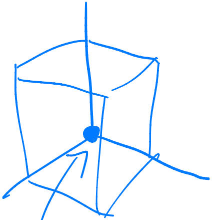
$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix} = x_3\mathbf{v}, \quad \text{where } \mathbf{v} = \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}$$

Solution set is Span $\left\{ \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix} \right\}$

- Here x_3 is factored out of the expression for the general solution vector.
- This shows that every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of $\mathbf{v}$.
- The trivial solution is obtained by choosing $x_3 = 0$.
- Geometrically, the solution set is a line through 0 in $\mathbb{R}^3$.



if only trivial sol'n



sol'n

Parametric vector form

- The equation of the form $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ $(s, t \in \mathbb{R})$ is called *a parametric vector equation* of the plane.
- In **Example 1**, the equation $\mathbf{x} = x_3\mathbf{v}$ (with x_3 free), or $\mathbf{x} = t\mathbf{v}$ (with $t \in \mathbb{R}$) is a parametric vector equation of a line.
- Whenever a solution set is described explicitly with vectors as in Example 1, we say that the solution is in *parametric vector form*.

1 vector → span a line

2 vector → span a plane ← u & v should be (linearly independent / not multiple of each other)

3 vector → span a 3D space

$\{ (1,1), (0,2)\ (2,5) \}$

span 2D space (plane)

min (3,2)

$\{ (1,1,1), (0,1,0) \}$

span 2D space (plane)

min(2,3)

$\{(1,1,1)\ (0,1,0)\ (1,2,3)\}$

min(3,3)

Span 3D space

# of linearly independent vectors : m

dimension of a vectors : n

span min (m, n) dimension space

Spanning set theorem

① There can be at most n linearly independent nD vectors.

Solutions of nonhomogenous systems $Ax = b$

- When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

Example 3. Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \text{ and } b = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$$

Solution Row operations on $[A\ b]$ produce

reduced echelon form-

$$[A\ b] \sim \left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right) \sim \begin{pmatrix} \boxed{1} & 0 & -\frac{4}{3} & -1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

row equivalent

- Thus, $x_1 = -1 + \frac{4}{3}x_3,\ \ x_2 = 2$, and x_3 is free.
- As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form

W in Ex1

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{pmatrix}}_{\text{sol'n set expressed in terms of } x_3} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \underbrace{x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}}_{\text{Sol'n set for homogeneous system}}$$

$$\overbrace{x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}}^{x_3 W} \text{ solves } A\mathbf{x} = 0 \ \ldots \text{ homo-sys}$$

$$\underbrace{x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{x_3 W + P} \text{ solves } A\mathbf{x} = \begin{pmatrix} 17 \\ -1 \\ -4 \end{pmatrix} \ldots \text{ non homo-sys}$$

- The equation $x = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$x = \mathbf{p} + t\mathbf{v} \ (t \in \mathbb{R})$$

describes the solution set in parametric vector form.
- The vector $\mathbf{p}$ itself is a solution to the nonhomogeneous system, by letting $t = 0$.
- The solution to nonhomogenous solution is composed of *a constant multiple of homogeneous solution* ($\mathbf{v}$) and *particular solution* ($\mathbf{p}$).

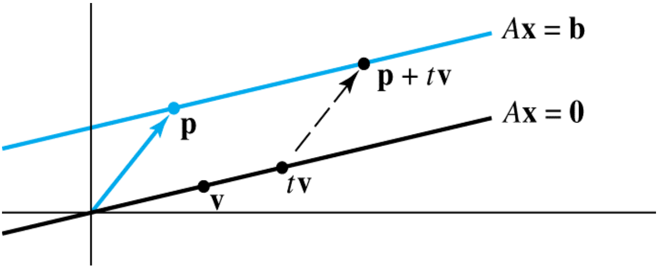
x_h : homogeneous ($\mathbf{v}$) : $Ax_h = \mathbb{0}$
x_p : particular sol'n ($\mathbf{p}$) : $Ax_p = \mathbb{b}$

$A(tx_h + x_p) = t \cdot Ax_h + Ax_p = t \cdot 0 + \mathbb{b} = \mathbb{b}$

x_h is determined by A.

$$x = \mathbf{p} + t\mathbf{v} \ (t \in \mathbb{R})$$

- *The solution is, therefore, the line through* $\mathbf{p}$ *parallel to* $\mathbf{v}$.
- The solution is, a line through $\mathbf{p}$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$.



- The relation between the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ shown in the figure above generalizes to any consistent equation $A\mathbf{x} = \mathbf{b}$, although the solution set will be larger than a line when there are several free variables.

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ *is consistent for some given* $\mathbf{b}$*, and let* $\mathbf{p}$ *be a solution. Then the solution set of* $A\mathbf{x} = \mathbf{b}$ *is the set of all vectors of the form* $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$*, where* v_h *is any solution of the homogeneous equation.*

- If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$ for the translation.
- (Translating means moving toward a specific direction).

어떻게 translate ?

Writing a solution set (of a consistent system) in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution **x** as a vector whose entries depend on the free variables, if any.
4. Decompose **x** into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Example ?

1.7. Linear Indepdence

1.7. Linear Indepdence

Definition (Linear independence)

An indexed set of vectors $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is said to be *linearly independent* if the vector equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \dots + x_p\mathbf{v_p} = \mathbf{0} \tag{1}$$

has only the trivial solution.

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = 0$$ ⋯ homogeneous system

Definition (Linear dependence)

An indexed set of vectors $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is said to be *linearly dependent* if the vector equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \dots + x_p\mathbf{v_p} = \mathbf{0} \tag{2}$$

has only the nontrivial solution. (some of x_i, $1 \le i \le p$, is not zero.)

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + ... + x_p\mathbf{v_p} = \mathbf{0}$$

- Adopting the perspective of matrix multiplication, it follows

$$\begin{pmatrix} | & | & & | \\ \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_p} \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_p \end{pmatrix} = \mathbf{0}$$

- Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$
- The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- *In order to check the linear dependence/independence of vectors, arrange them into column vectors of a matrix, and look for a homogeneous solution. If any nontrivial solution, they are linearly dependent, otherwise linearly dependent.*

independent ?

typo?

Sets of one vector

- A set containing only one vector – say, $\mathbf{v}$ – is linearly independent if and only if $\mathbf{v}$ is not the zero vector.
- This is because the vector equation $x_1\mathbf{v} = \mathbf{0}$ has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$.
- The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

Sets of two vectors

- A set of two vectors $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly dependent if at least one of the vectors is a multiple of the other.
- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Sets of two or more vectors

Theorem (Characterization of linearly dependent sets)

An indexed set $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v_1} \neq \mathbf{0}$, then some $\mathbf{v_j}$ (with $j > 1$) is a linear combination of the preceding vectors, $v_1, ..., v_{j-1}$.

Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v_1}, ..., \mathbf{v_p}\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array} \rightarrow \quad \begin{matrix} n=3 \\ p=5 \end{matrix}$$

$n < p$

⋮

columns are linearly dependent

- If $p > n$, the columns are linearly dependent.

Theorem

If a set $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Theorem

최대

There can be at most n $\boxed{\text{vectors}}$ where each vector is in $\mathbb{R}^n$.

linearly independent

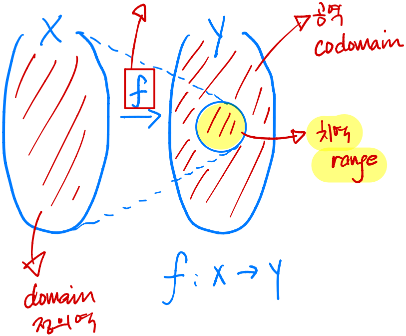
n차원 벡터들이
linearly independent 한
집합 되기 위한 최대 벡터 개수
= n

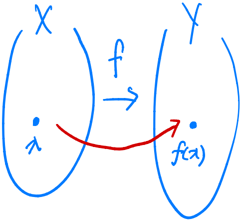
focusing on relationship between input & output

1.8. Introduction to Linear Transformations

function

function 함수
transformation 변환.

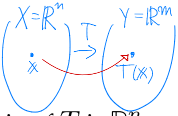




f(x) is image of x.

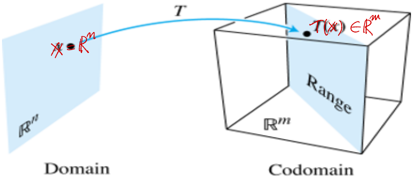
A few definitions

- A *transformation* (or *function* or *mapping*) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.



- The set $\mathbb{R}^n$ is called *domain*(정의역) of T
- The set $\mathbb{R}^m$ is called the *codomain*(공역) of T.
- The notation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For x in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the *image*(치역) of $\mathbf{x}$ (under the action of T).
- The set of all images $T(\mathbf{x})$ is called the range of T. See figure below.

* $A \subset B$: A is subset of B
$A \subsetneq B$: A is proper subset of B
(진부분집합)
자기자신 제외한 부분집합

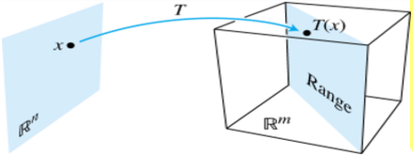


Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

Sometimes,

Range = codomain (equal)

Range $\subsetneq$ codomain (subset & not equal)



Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

$x \in \mathbb{R}^n \quad T(x) = Ax \in \mathbb{R}^m$

$$\underbrace{\boxed{A}}_{m \times n} \underbrace{\begin{bmatrix} x \end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} y \end{bmatrix}}_{m \times 1}$$

$T : X \to Y$

$T(x) = Ax$

- For each $\mathbf{x} \in \mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an matrix.
- For simplicity, we denote such a *matrix transformation* by $\mathbf{x} \mapsto A\mathbf{x}$ ("maps to").
- Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries.
- The range of T is the set of all linear combinations of the columns of A, because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.

linear combination
all the columns of A
weighted by each element of x

Example 1. Let

$$A = \underbrace{\begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}}_{3\times 2}, \quad \mathbf{u} = \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{2\times 1}, \quad \mathbf{b} = \underbrace{\begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}}_{3\times 1}, \quad \mathbf{c} = \underbrace{\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}}_{3\times 1}$$

and define a transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(x) = A\mathbf{x} = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{pmatrix}$$

- **a.** Find $T(\mathbf{u})$, the image of u under the transformation T.

$$T(\mathbf{u}) = \begin{bmatrix} 2-3(-1) \\ 3\times 2+5(-1) \\ -2+7(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

- **b.** Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

$x_2 = -\frac{1}{2}$

$A\mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

$\begin{cases} x_1 = \frac{3}{2} \\ x_2 = -\frac{1}{2} \end{cases}$

$\mathbf{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$

- **c.** Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

$\exists!$ sol'n $\iff$ # of pivots = # of columns.

$A = (n \times n)$ matrix

- **d.** Determine if $\mathbf{c}$ is in the range of the transformation T.

Is there any $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{c}$?

$A\mathbf{x} = \mathbf{c}$ has a solution?

$A\mathbf{x} = \mathbf{c}$ is a consistent system?

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

$\boxed{\nexists \text{ sol'n}}$

$\mathbf{c}$ is not in the range of the transformation T

b. approach augmented matrix

$$[A \mid b] = \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \begin{array}{l} R1 \\ R2 \leftarrow R2-3R1 \\ R3 \leftarrow R3+R1 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right] \begin{array}{l} R1 \\ R2 \leftarrow \frac{1}{14}R2 \\ R3 \leftarrow R3-\frac{2}{7}R2 \end{array}$$

2x2

2 columns
2 pivots

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

ignorable!

$x_1 = \frac{3}{2}$

$x_2 = -\frac{1}{2}$

c. approach augmented matrix

$$[A \mid c] = \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \begin{array}{l} R1 \\ R2 \leftarrow R2-3R1 \\ R3 \leftarrow R3+R1 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{array}\right] \begin{array}{l} R1 \\ R2 \leftarrow \frac{1}{14}R2 \\ R3 \leftarrow R3-\frac{2}{7}R2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 10 \end{array}\right]$$

inconsistent.

Sheer transformation

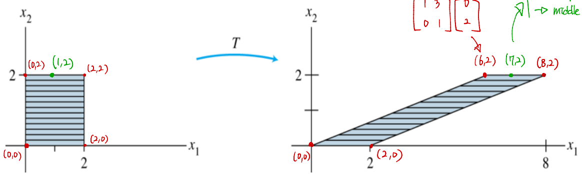
Example 3. Let $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a *sheer transformation*.

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

middle point of input
→ middle point of output

pf) $(1,2) = \frac{1}{2}((0,2)+(2,2))$

$A(1,2) = \frac{1}{2}A(0,2) + \frac{1}{2}A(2,2)$
$= \frac{1}{2}(6,2) + \frac{1}{2}(8,2)$
$= (7,2)$



- The key idea is to show that T maps line segments onto line segments and then to check that the *corners of the square* map onto the vertices of the parallelogram. → unit square will become parallelogram
- For instance, the image of the point For instance, the image of the point $\mathbf{u} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ is $T(\mathbf{u}) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

Linear transformation

$T(x) = Ax$ ⋯ linear function

Definition (Linear transformation)

A transformation (or mapping) T is linear if:

- $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T
- $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Linear transformations *preserve the operations of vector addition and scalar multiplication*.

$y = ax + b$ ⋯ affine function

$y = ax$ ⋯ linear function (no constant term)

대수학적

linear function in algebraic sense

$f(x+y) = f(x) + f(y)$ ⋯ independent on dimension of x

$f(cx) = cf(x)$

def. of linear function

$T(\underbrace{x-x}_{=0}) = T(x) - T(x) = 0$

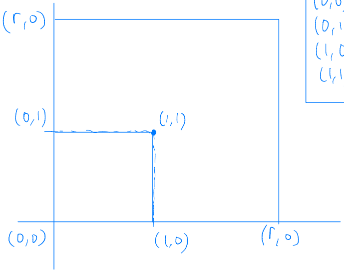
- It follows both $\boxed{T(\mathbf{0}) = \mathbf{0}}$ and $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.
- In engineering and physics, the following equation is referred to as a *superposition principle*.

$$T(c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p}) = c_1T(\mathbf{v_1}) + \cdots + c_pT(\mathbf{v_p})$$

- Given a scalar r, define $\boxed{T : \mathbb{R}^n \longrightarrow \mathbb{R}^n}$ by $\boxed{T(\mathbf{x}) = r\mathbf{x}}$. What is the corresponding matrix?

$$n\overbrace{\begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{bmatrix}}^{n}$$

- T is called a *contraction*(수축) when $0 \le r \le 1$ and a *dilation*(확대) when $r > 1$.



$$\begin{array}{ccc} (0,0) & & (0,0) \\ (0,1) & \xrightarrow{rx} & (0,r) \\ (1,0) & & (r,0) \\ (1,1) & & (r,r) \end{array}$$

1.9. The Matrix of Linear Transformation

The matrix of linear transformation

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Remark

- In fact, A is the $m \times n$ matrix whose j-th column is the vector $T(\mathbf{e_j})$ where $\mathbf{e_j}$ is the j-th column of identity matrix in $\mathbb{R}^n$. (or a j-th *unit vector*). Namely,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$A = \begin{pmatrix} | & & | \\ T(\mathbf{e_1}) & \cdots & T(\mathbf{e_n}) \\ | & & | \end{pmatrix}$$

unit matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$n \times n$

$$\begin{cases} T(e_1) = A \cdot e_1 = A_{\cdot 1} \\ T(e_2) = A \cdot e_2 = A_{\cdot 2} \\ \vdots \\ T(e_n) = A \cdot e_n = A_{\cdot n} \end{cases}$$

↳ you can determine A by checking what the linear transformation does to all unit vectors.

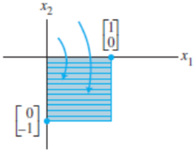
Remark

- This matrix A is called the *standard matrix for the linear transformation*.
- Every *linear transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as a *matrix transformation*, and vice versa.
- The term *linear transformation* focuses on a property of a mapping, while matrix transformation describes how such a mapping is implemented.

Geomatric linear transformation in $\mathbb{R}^2$

Reflections

TABLE 1 Reflections

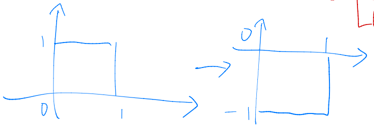
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{array}{c}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\end{array}\begin{array}{l}\lambda_1\\ \lambda_2\end{array}$

original

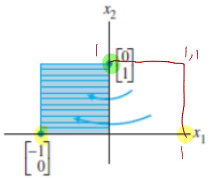
$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1 & 0 & 1\\ 0 & 0 & -1 & -1\end{bmatrix}$$

reflection



Reflection through the x_2-axis

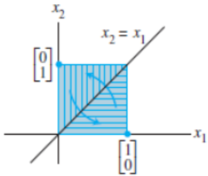


$$\begin{array}{cc} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{matrix} x_1 \\ x_2 \end{matrix} \end{array}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Reflection through the line $x_2 = x_1$

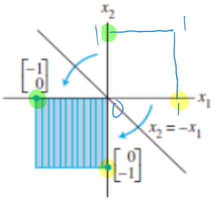


$$\begin{array}{cc} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{matrix} x_1 \\ x_2 \end{matrix} \end{array}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Reflection through the line $x_2 = -x_1$

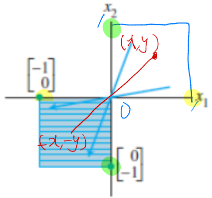


$$\begin{array}{c} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \end{array} \begin{array}{l} x_1 \\ x_2 \end{array}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

Reflection through the origin



$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{array}{l} x_1 \\ x_2 \end{array}$$

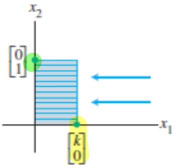
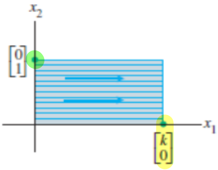
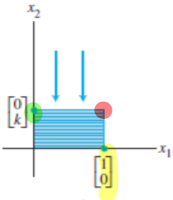
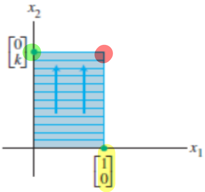
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

축약 확장

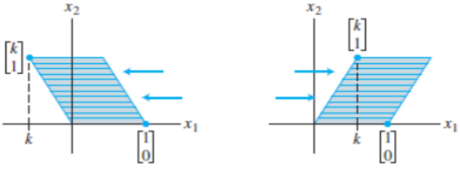
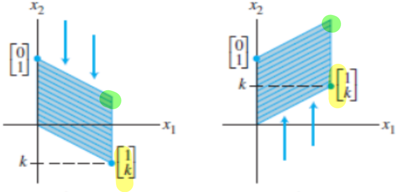
Contractions and expansions

TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$ $\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$ $= \begin{bmatrix}0 & k & 0 & k\\0 & 0 & 1 & 1\end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$ $\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$ $= \begin{bmatrix}0 & 1 & 0 & 1\\0 & 0 & k & k\end{bmatrix}$

Sheers

TABLE 3 Shears

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 $k<0$ $k>0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k<0$ $k>0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

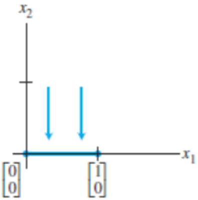
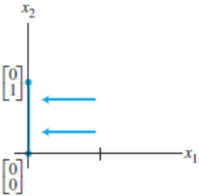
$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & k & k+1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & k & 1 & k+1 \end{bmatrix}$$

Projections

투사 reduction of dimension

TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Definition (onto)

codomain = range

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^n$ if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one $\mathbf{x} \in \mathbb{R}^n$.

codomain $\supset$ range

i) $=$ "onto"

ii) $\supsetneq$ "not onto"

Remark

- If *onto*, *codomain* is same set as *range*.
- If *not onto*, there is some $\mathbf{b} \in \mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution.

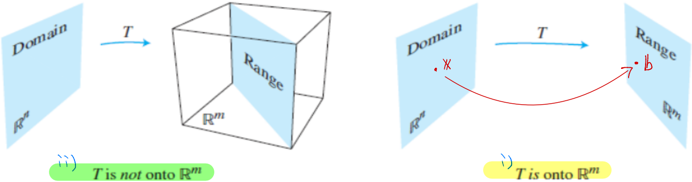


FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

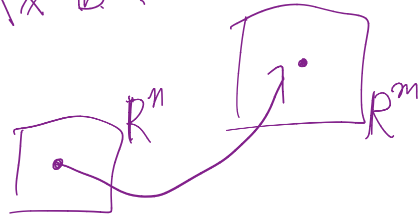
any point in $\mathbb{R}^m$ must correspond some point x in $\mathbb{R}^n$ $\rightarrow$ $\forall \mathbb{b}$, $\exists$ x (s.t.) such that $T(x) = \mathbb{b}$

Definition (one-to-one)

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b} \in \mathbb{R}^m$ is the image of **at most one** $\mathbf{x} \in \mathbb{R}^m$.

Why "at most" ?

$Ax = b$?



Example 4. Let T be the linear transformation whose standard matrix is

$$A = \begin{pmatrix} \boxed{1} & -4 & 8 & 1 \\ 0 & \boxed{2} & -1 & 3 \\ 0 & 0 & 0 & \boxed{5} \end{pmatrix}$$

linearly independent

$3^{\#}$ pivots

$= 3^{\#}$ linearly independent column vectors

$= 3^{\#}$ linearly independent row vectors.

" consistency of $A\mathbf{x}=\mathbf{b}$ "

- Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$?

$\Leftrightarrow$ For every $\mathbf{b} \in \mathbb{R}^3$, $\exists$ x such that $A\mathbf{x}=\mathbf{b}$

($A\mathbf{x}$ is linear combination of column vectors of A)

$\Leftrightarrow$ $\forall \mathbf{b} \in \mathbb{R}^3$ is it possible to express $\mathbf{b}$ as a linear combination of columns of A?

Since columns of A can express any arbitrary vectors in $\mathbb{R}^3$, T is onto.

- Is T a one-to-one mapping?

related to uniqueness of sol'n / # of free variable

The homogeneous system, $A\mathbf{x}=\mathbf{0}$ have non-zero sol'n.

3 basic variabls & 1 free variable

= not one-to-one

So, not one-to-one.

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is ***one-to-one*** *if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.* = { columns of A are linearly independent / no free variables

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then,

• independent $\begin{pmatrix} \text{column} \\ \text{row} \end{pmatrix}$ vectors • # of pivot or basic variables = m

- *T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.*
- *T is one-to-one if and only if the columns of A are linearly independent.*

Suggested Exercises

- 1.5.14, 1.5.18
- 1.8.1, 1.8.9, 1.8.13, 1.8.14, 1.8.15, 1.8.16

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"Man can learn nothing unless he proceeds from the known to the unknown. - Claude Bernard"
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