Quiz 1

Engineering Math.

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2021 #1

(a) $(1,1,0)$ $(1,2,3)$ $(0,0,0)$

$a_1(1,1,0) + a_2(1,2,3) + a_3(0,0,0) = (0,0,0)$

$a_1=0,\ a_2=0,\ a_3=1$

(a), linearly dependent.

(b) $(2,3,0)$ $(0,2,-1)$ $(4,4,-1)$

$a_1(2,3,0) + a_2(0,2,-1) + a_3(4,4,-1) = (0,0,0)$

$2a_1 + 4a_3 = 0 \quad \therefore a_1 = -2a_3$
$3a_1 + 2a_2 + 4a_3 = 0$
$-a_2 - a_3 = 0 \quad \therefore a_2 = -a_3$

$-6a_3 - 2_3 + 4a_3 = 0$
$a_1 = a_2 = a_3 = 0$
(b), linearly independent

2021 #2

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -3 & -5 \\ 0 & -2 & -4 \end{bmatrix}$$

2021 #3

$n \times n$ matrix A

$A\mathbf{x} = \mathbf{b}$ has unique sol'n

(b) non-zero determinant

(d) non-singular

$\boxed{\text{(f) linearly dependent}}$ X

(g) invertible

(i) A^{-1} exists

(e) linearly independent

2021 #4

$$x - y = 3$$
$$2x + 3y = 7$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

$$|A| = (1 \times 3) - (-1 \times 2) = 5$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 16 \\ 1 \end{bmatrix} = \begin{bmatrix} 16/5 \\ 1/5 \end{bmatrix}$$

2021 #5

a)

$$A' = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

$$= \begin{bmatrix} 2a & 2b & 2c & 2d \\ -a+e & -b+f & -c+g & -d+h \\ 3i & 3j & 3k & 3l \\ 2e+m & 2f+n & 2g+o & 2h+p \end{bmatrix}$$

b) $(R1) = (a, b, c, d)$
$(R2) = (e, f, g, h)$
$(R3) = (i, j, k, l)$
$(R4) = (m, n, o, p)$

3. $3 \times (R3)$
4. $2 \times (R2) + (R4)$

2021 #6

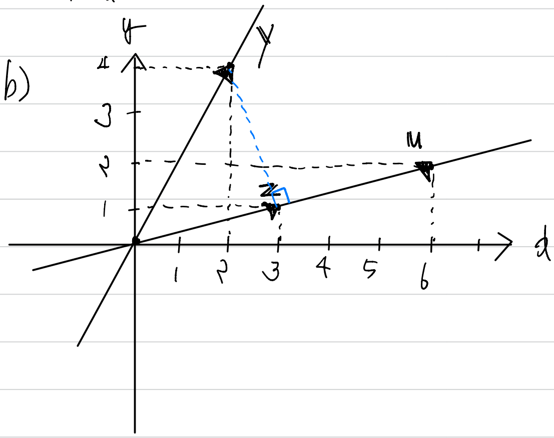
a) $y = [2, 4]$ $\quad u = [6, 2]$

$y \cdot u = 2 \times 6 + 4 \times 2 = 18$
$u \cdot u = 6 \times 6 + 2 \times 2 = 36$

y projected into u → $\perp$

$$\frac{y \cdot u}{u \cdot u} u = \frac{1}{2} u = [3, 1] = z$$

b)



2020 #1

determinant

$\det(A) = |A| = ad - bc$

$\det(5A) = |5A| = 25(ad-bc)$

$5\det(A) = 5(ad-bc) \neq \det(5A) = 25(ad-bc)$

False.

2020 #2

$a_1(4,1,0) + a_2(0,2,-1) + a_3(3,2,0) = (0,0,0)$

$4a_1 + 3a_3 = 0$ $\quad$ $4a_1 + 3a_3 = 0$

$a_1 + 2a_2 + 2a_3 = 0$ $\quad$ $4a_1 + 8a_3 = 0$

$-a_2 = 0$

$a_2 = 0 \quad a_3 = 0 \quad a_1 = 0$

$a_1 = a_2 = a_3 = 0$

column vectors are independent

$\det(A) \neq 0$

False!

2020 #3

$(5-\lambda)(3-\lambda) - (-3 \times -1) = 0$

$15 - 8\lambda + \lambda^2 = 3$

$\lambda^2 - 8\lambda + 12 = 0 \qquad (\lambda-2)(\lambda-6) = 0$

$\lambda = 2$ or 6

2020 #4

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 6 \\ 2 & 6 & 6 \end{bmatrix}$$

2020 #5 = 2021 #5

2020 #6

$Ax = b$ $\exists!$ sol'n

A has non-zero determinant

A is non-singular

A's column vectors are independent.

A is invertible

2020 #7

$v_1 = (0,1,-1)$
$v_2 = (1,0,1)$
$v_3 = (-1,1,0)$

$\left.\begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}\right\}$ basis of 3D?

$(x,y,z) = a_1 v_1 + a_2 v_2 + a_3 v_3$

$+a_2 - a_3 = x$
$a_1 + a_3 = y$
$-a_1 + a_2 = z$

$a_2 + a_3 = y + z$
$a_2 - a_3 = x$
$2a_2 = x+y+z$

$a_1 = \frac{(x+y-z)}{2}$

$a_2 = \frac{(x+y+z)}{2}$

$a_3 = \frac{(-x+y+z)}{2}$

real numbers.

2020 #8

$$\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$A x = b$

$$\frac{1}{20-6}\begin{bmatrix} 4 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 6 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{6}{14} \\ -\frac{1}{14} \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 6/14 \\ -1/14 \end{bmatrix}$$

$$= \begin{bmatrix} 30/14 - 2/14 \\ 18/14 - 4/14 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2020 #9

2020 #10 = 2021 #6

a) $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = A\mathbf{x}$

b) $A(A\mathbf{x}) = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 0 \end{bmatrix}$

c) $A^n\mathbf{x} = \begin{bmatrix} 2^{n+1} \\ 2^{n+1}-1 \\ 0 \end{bmatrix}$

4	8	16
3	7	15

Rc.

#1. Show that the set of the following vectors are linearly indepdent. [15pt]

$$(2,3,0),\ (0,2,-1),\ (4,8,-1)$$

if the vectors are independent, $a_1 = a_2 = a_3$ should be satisfied

$a_1(2,3,0) + a_2(0,2,-1) + a_3(4,8,-1) = (0,0,0)$

$2a_1 + 0a_2 + 4a_3 = 0$ ① $\cdots$ $a_1 = -2a_3$ ④

$3a_1 + 2a_2 + 8a_3 = 0$ ② $\cdots$ ②→$3(-2a_3) + 2(-a_3) + 8a_3 = 0$.

$0a_1 - a_2 - a_3 = 0$ ③ $\cdots$ $a_2 = -a_3$ ⑤

$a_1 = -2a_3$

$a_2 = -a_3$

($a_3 = 0$, $a_1 = a_2 = a_3 = 0$. vectors are independent.)

$a_1 = a_2 = a_3$

→ $-2a_3 = -a_3 = +a_3$

Only $a_3 = 0$ can validate the equation

So, $a_3 = 0$.

$a_1 = a_2 = a_3 = 0$

the vectors above are Independent

15

#1. Show that the set of the following vectors are linearly dependent. [15pt]

$$(2,3,0),\ (0,2,-1),\ (4,8,-1)$$

If k vectors $v_1, \dots, v_k$ are linearly independent with real numbers $a_1, \dots, a_k$,
the sol'n of $a_1 v_1 + \dots + a_k v_k = 0$ is only $a_1 = a_2 = \dots = a_k = 0$

not linearly independent means linearly dependent.

$$a_1 \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + a_3 \begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

if the sol'n is $a_1 = a_2 = a_3 = 0$, the vectors are linearly independent,
otherwise the vectors are linearly dependent.

$2a_1 + 4a_3 = 0 \quad \cdots ①$
$3a_1 + 2a_2 + 8a_3 = 0 \quad \cdots ②$
$-a_2 - a_3 = 0 \quad \cdots ③$

$a_1 = -2a_3 \ (\because ①)$
$a_2 = -a_3 \ (\because ③)$

$② = 3(-2a_3) + 2(-a_3) + 8a_3 = 0 \times a_3 = 0 \quad \cdots ④$

a_3 has infinite solution to satisfy ④
a_1 & a_2 is decided by a_3. then the vectors are not linearly independent.

if $a_3 = 1$, $a_1 = -2$ and $a_2 = -1$

$$\left.\begin{array}{l} 2 \times (-2) + 4 \times 1 = 0 \\ 3 \times (-2) + 2 \times (-1) + 8(1) = 0 \\ -(-1) - 1 = 0 \end{array}\right\} \ a_3 \text{ has other solution except } 0.$$

#2. Complete the following theorem. [20pt]

For a $n \times n$ matrix A, the followings are all equivalents.

- (invertibility) The matrix A is invertible.
- (determinant) A has non-zero determinant
- (solution of $A\mathbf{x} = \mathbf{b}$) unique solution
- (singularity) non singular matrix A
- (column vectors) column vectors are independent

20

Re

#3. Prove the following statement. [15pt]

- For a 2×2 matrix A, if its column vectors are independent, then its row vectors are independent.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad a_{11} \cdot a_{22} - a_{12} a_{21} \neq 0$$

· column vectors are independent $(a_{11}, a_{21}), (a_{12}, a_{22})$

$$a(a_{11}, a_{21}) + b(a_{12}, a_{22}) = (0,0)$$
$$a \cdot a_{11} + b a_{12} = 0$$
$$a \cdot a_{21} + b a_{22} = 0$$
$$a = b = 0$$

row vectors are independent $(a_{11}, a_{12}), (a_{21}, a_{22})$

$$c(a_{11}, a_{12}) + d(a_{21}, a_{22}) = (0,0)$$

is this true? or your assumption? or your claim?

$$c a_{11} + d a_{21} = 0$$
$$c a_{12} + d a_{22} = 0$$

Seem to apply 귀류법 (proof by contradiction).

if (c & $d \neq 0$)

$$c a_{12} + d\, a_{21} \cdot \frac{a_{12}}{a_{11}} = 0$$

(why?)

$$c a_{12} + d a_{22} = 0$$

$$a_{21} \cdot \frac{a_{12}}{a_{11}} = a_{22}$$

what if $a_{11} = 0$?

$$a_{21} \cdot a_{12} = a_{11} \cdot a_{22} \quad \text{(false)} \quad \because a_{11} a_{22} \neq a_{12} a_{21}$$

non-zero determinant.

therefore, $c = d = 0$.

row vectors are independent.

3 by 21102092 Lee Jeong-yun.

Q. Prove the following statement.
"For a 2x2 matrix A, if its column vectors are independent, then its row vectors are independent."

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$x_1 = (a, c)$ $\quad x_2 = (b, c)$

x_1 and x_2 is linearly independent.

So, A has non-zero determinant.

consider, $A' = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

if $y_1 = (a, b)$ and $y_2 = (c, d)$ are linearly independent, row vectors in A are linearly independent.

y_1 & y_2 are linearly independent, then $|A'| \neq 0$.

$|A| = ad - bc \neq 0$ ($\because$ x_1 & x_2 are linearly independent.)

$|A'| = ad - bc \neq 0$

A' has non-zero determinant.

so column vectors in A' are linearly independent.

$y_1 = (a, b)$ $\quad y_2 = (c, d)$

row vectors in A

$z_1 = (a, b)$ $\quad z_2 = (c, d)$

$y_1 = z_1$ & $y_2 = z_2$

y_1 & y_2 $\rightarrow$ linearly independent

z_1 & z_2 $\rightarrow$ linearly independent

#3 by Prof.

Q. Prove the following statement.

"For a 2x2 matrix A, if its column vectors are independent, then its row vectors are independent." q

p

Alt 1

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - bc \neq 0$$

$$B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$|B| = ad - bc \neq 0$$

column vectors in B linearly independent,
so row vectors in A linearly independent.

$n \times n$ matrix

column vectors linearly independent
$\Leftrightarrow$ row vectors linearly independent

Alt 2

$p \to q$: $\sim q \to \sim p$ (contraposition)

row vectors are linearly dependent,
then the column vectors are linearly dependent.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

linearly dependent $\Rightarrow$ constant multiple of other

$$[a, b] = m[c, d]$$

$$A = \begin{bmatrix} mc & md \\ c & d \end{bmatrix}$$

if $d \neq 0$, $\begin{bmatrix} mc \\ c \end{bmatrix} = \frac{c}{d} \begin{bmatrix} md \\ d \end{bmatrix}$

linearly dependent

if $d = 0$, $A = \begin{bmatrix} mc & 0 \\ c & 0 \end{bmatrix}$

zero vector $\to$ linearly dependent.

#4. Write the matrix formular for the following system of linear equation. Find the inverse of the coefficient matrix. Find the solution to the system of linear equation in vector form. [15pt]

$$\begin{aligned} 2x + 3y &= 13 \\ 4x + 2y &= 14 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \end{bmatrix}$$

$Ax = b$

$$A^{-1} = -\frac{1}{8}\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix}$$

$$AA^{-1} = I = -\frac{1}{8}\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

important to check correctness

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ 14 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} 26 - 42 \\ -52 + 28 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -16 \\ -24 \end{bmatrix} = \begin{bmatrix} +2 \\ +3 \end{bmatrix}$$

$x = +2, \quad y = +3$

$2(+2) + 3(3) = 13$

$2(4) + 2(3) = 14.$

15

Re.

#5. Suppose that $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a basis of 3-dimensional vector space. Carefully show that $\{\mathbf{x}, \mathbf{x}-\mathbf{y}, \mathbf{x}+\mathbf{y}-\mathbf{z}\}$ is a basis of 3-dimensional vector space as well. [15pt]

x, y, z should be 3-demensional vector.

addition between vector should be conducted with same demension.

also, the result of addition is still the same demension.

So, $x-y$ and $x+y-z$ are 3-demensional vectors

therefore, $\{x, x-y, x+y-z\}$ is basis of 3-dimensional vector space.

cf) n-dimensional vector space

→ n-dimension vectors in the set

The number of the element in the set is n.

X

0

#5 by 21102052 Lee Jeong-yun

Q. Suppose that $\{x, y, z\}$ is a basis of 3-dimensional vector space. Carefully show that $\{x, x-y, x+y-z\}$ is a basis of 3-dimensional vector space as well

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Since $\{x, y, z\}$ is a basis of 3D vector space, any 3D vector v can be expressed by

$$v = a_1 x + a_2 y + a_3 z \qquad (a_1, a_2, a_3 \in \mathbb{R})$$

if $\{x, x-y, x+y-z\}$ is a basis of 3D vector space, then

$$v = b_1 x + b_2 (x-y) + b_3 (x+y-z)$$

should be satisfied.

In other words, v should be expres by a linear combination of x, $x-y$, $x+y-z$.

$$\begin{aligned} v &= a_1 x + a_2 y + a_3 z \\ &= (b_1+b_2+b_3)x + (-b_2+b_3)y + (-b_3)z \end{aligned}$$

$a_1 = b_1+b_2+b_3$

$a_2 = -b_2+b_3$

$a_3 = -b_3$

then

$b_3 = -a_3$

$b_2 = -a_2-a_3$

$b_1 = a_1 - (-a_2-a_3) - (-a_3) = a_1+a_2+2a_3$

v can be expressed by

$$v = (a_1+a_2+2a_3)x + (-a_2-a_3)(x-y) + (-a_3)(x+y-z)$$

Since $(a_1, a_2, a_3 \in \mathbb{R})$,

$(a_1+a_2+2a_3, -a_2-a_3, -a_3 \in \mathbb{R})$

Thus, above expression any 3D vector v can be expressed as a linear combination of $\{x, x-y, x+y-z\}$, i.e., $\{x, x-y, x+y-z\}$ is a basis of 3D vector space.