Solving Linear Programming Problems: The Simplex Method

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The Simplex Method

- Overview: Core Idea of the Simplex Method
  - Location of the optimal solution
    - ✓ The optimal solution of a linear programming lies at the one of the corner-point feasible (CPF) solutions in the feasible region.
  - Movement strategy
    - ✓ The Simplex method starts from one basic feasible solution (BFS) and moves along an edge to an adjacent corner point where the objective function improves.
    - ✓ This adjacency holds when two corner points share a common boundary formed by the intersection of constraints
  - Iterative improvement
    - ✓ At each step, the entering variable and leaving variable are determined, and a pivot operation is performed to move to a new BFS (CPF solution).
    - ✓ When no further improvement is possible, the optimal solution is reached.

The Simplex Method

- Overview: Core Idea of the Simplex Method
  - Why do we only need to check the corner points?
  - Because the objective function is linear and the feasible region is a convex polyhedron, increasing the value in one direction will eventually hit a boundary and reach a maximum or minimum at a corner point.
    - ✓ Therefore, it is unnecessary to search every interior point; comparing the corner points alone is sufficient to find the global optimum.

The Simplex Method

Maximize $Z = 3x1 + 5x2$
(s.t.)
$x1 \leq 4$
$x2 \leq 6$
$3x1 + 2x2 \leq 18$
$x1, x2 \geq 0$

- The Prototype Example
  - The corner-points of the feasible region are
    - ✓ (0, 0), (0, 6), (2, 6), (4, 3), (4, 0)
  - If we start at (0, 0), we can move to (4, 0) by increasing x1, or to (0, 6) by increasing x2.
  - The Simplex method changes only one variable at a time, moving to an adjacent corner point, and chooses the direction that increases the value of Z.
  - For example, moving from (0, 6) to (2, 6) increases Z from 30 to 36.

The Simplex Method

- Need for Algebraic Formulation
  - The graphical method is intuitive, but it cannot be used to solve problems visually once there are more than three variables.
  - Instead, expressing an LP in matrix/algebraic form makes it possible to apply systematic computational methods.

The Simplex Method

- Algebraic Formulation: Slack Variable
  - Definition.
    - ✓ A non-negative variable needed to convert a "≤" constraint into an equality.
  - It represents the amount of unused resources.
  - Example
    - ✓ $x_1 + x_2 \leq 100 \quad \Rightarrow \quad x_1 + x_2 + s_1 = 100, \;\; s_1 \geq 0$
    - ✓ If the maximum production of two products is 100 units, then s_1 represents the shortfall from 100 (i.e., the unused production capacity).

The Simplex Method

- Algebraic Formulation: Basic and Non-Basic Variables
  - Basic Variables (BVs)
    - ✓ The m variables chosen to satisfy the m equations.
    - ✓ These variables directly determine the solution to the constraints.
  - Non-Basic Variables (NBVs)
      - The remaining variables, which are set to 0.
    - ✓ Basic solution
      - A solution obtained by setting the non-basic variables to 0 and solving the remaining m equations.
    - ✓ Basic Feasible Solution (BFS)
      - A basic solution in which all variables satisfy $x_j \geq 0$
  - Relationships between BFS and CPF solution
    - ✓ One-to-one correspondence: BFSs correspond exactly to CPF solutions.
    - ✓ Geometrically, they are the intersections of constraints; algebraically, they are the solutions obtained by solving with m variables.

The Simplex Method

- Slack Variables and the Initial BFS in the Simplex Method
  - In the initial BFS, all slack variables are set as the basic variables.
  - A decrease in their values indicates that resources are being consumed.
- Characteristics
    - ✓ Always $s \geq 0$
    - ✓ The value represents the remaining amount of the corresponding resource.
    - ✓ If the value is 0, it means that resource is fully used.
  - Example
    - ✓ x1 + x2 + s1 = 4
    - ✓ 2x1 + x2 + s2 = 6
    - ✓ ➔ Initial BFS: x1 = 0, x2 = 0, s1= 4, s2 = 6

The Simplex Method

- Pivot Operations – Determining the Entering and Leaving Variables
  - Meaning of pivot
    - ✓ A pivot is the operation of removing one variable from the current set of BVs and bringing in another variable to move to a new BFS.
    - ✓ Geometrically, this corresponds to moving along an edge from one corner point of the feasible region to an adjacent corner point.
  - How to select entering variable
    - ✓ Compare the objective function coefficients with the current contributions
    - ✓ In a maximization problem, choose the variable with the largest positive coefficient (the variable with the greatest potential improvement).
  - How to select leaving variable (Minimum Ratio Test)
    - ✓ When the entering variable increases by 1 unit, determine the maximum allowable increase without violating any constraints.
    - ✓ For each constraint, compute (current RHS value / coefficient of the entering variable), considering only positive coefficients.
    - ✓ The variable corresponding to the smallest ratio becomes the leaving variable (the first one to drop to zero).

The Simplex Method

- Pivot Operations – Determining the Entering and Leaving Variables
  - Pivot procedure
    - ✓ Select the pivot element: the position at the intersection of the entering variable's column and the leaving variable's row.
    - ✓ Divide the pivot row by the pivot element so that it becomes 1.
    - ✓ Adjust the other rows to make all other entries in the pivot column equal to 0.
    - ✓ Apply the same adjustments to the objective function row (Z-row).
  - Intuitive understanding
    - ✓ Entering: represents using a new resource or starting a new activity.
    - ✓ Leaving: represents depleting a resource or stopping another activity.
    - ✓ This process always keeps the number of BVs unchanged while moving from one solution to another.

The Simplex Method

- Construction of the Simplex Tableau
  - Purpose of tableau
    - ✓ Provides a tabular format to easily view Simplex calculations and update them iteratively.
    - ✓ Allows checking all variable coefficients, the right-hand side (RHS), and the objective function value in one place.
    - ✓ Enables row operations to easily exchange entering and leaving variables.
  - Basic structure of the tableau
    - ✓ Rows: constraints + the objective function row (Z-row)
    - ✓ Column: all variables (decision variables + slack variables) and the RHS
    - ✓ Basic Variables: the list of basic variables in the current BFS
    - ✓ $z_j - c_j$ row: reduced cost of each variable

The Simplex Method

- Construction of the Simplex Tableau
  - Building the initial tableau
    - ✓ After converting the LP into standard form, add slack variables.
    - ✓ Set the slack variables as the basic variables (BVs), the other variables as non-basic variables (NBVs with values = 0), and fill in the RHS.
    - ✓ Objective function row calculation
      - $Z_j = \sum_i (c_B \times a_{ij})$
      - Fill in the reduced costs using $Z_j - c_j$
  - Using the tableau in iterations
    - ✓ Select the entering variable: the variable with the smallest negative value in the $z_j - c_j$ row.
    - ✓ Select the leaving variable: by the Minimum Ratio Test
    - ✓ Perform the pivot: update the tableau to calculate the new BFS.

The Simplex Method

- Example

Maximize Z = 3x1 + 5x2
(subject to)
x1 ≤ 4, 2x2 ≤ 12, 3x1 + 2x2 ≤ 18, x1, x2 ≥ 0

1. Conversion to standard form
(adding slack variables s1, s2, s3)
x1 + s1 = 4
2x2 + s2 = 12
3x1 + 2x2 + s3 = 18

2. Set the initial BFS
BVs: s1 = 4, s2 = 12, s3 = 18
NBVs: x1 = 0, x2 = 0
Initial Z = 0

3. Initial Tableau (Entering variable: x_3)

BVs	Z	x_1	x_2	s_1	s_2	s_3	RHS
Z	1	-3	-5	0	0	0	0
s_1	0	1	0	1	0	0	4
s_2	0	0	2	0	1	0	12
s_3	0	3	2	0	0	1	18

4. Select leaving variable (Minimum Ratio Test)
s1 row: 4 / 0 → ∞ (exclude if a coefficient is zero)
s2 row: 12 / 2 = 6
s3 row: 18 / 2 = 9
➔ Select s_2 as the leaving variable since the minimum ratio is 6

5. Pivot operations
Pivot element = (s2 row, x2 column) = 2
Divide s2 row by 2 to make pivot element as 1
Perform row operations to make the other entries in that column equal to 0.
A new tableau is then created, and the BFS is updated.

The Simplex Method

- Optimality & Termination – When to Stop, What is Optimal
  - Optimality test rule
    - ✓ For the Max problem(standard form, $x \geq 0$)
      - Check the reduced cost row (usually $z_j - c_j$ row) in the tableau.
      - If $z_j - c_j \geq 0$ for all NBVs, no further improvement is possible → the solution is optimal
    - ✓ Sometimes the row $c_j - z_j$ is used instead; in that case, if all the values ≤ 0, it is optimal.
      - In either notation, the key idea is: stop when no coefficient indicates potential improvement.
  - Unboundedness test
    - ✓ If, in the entering variable's column, there are no positive coefficients (so the Minimum Ratio Test cannot be applied)
      → the objective function can be improved indefinitely. → Unbounded
  - Degeneracy & Cycling
    - ✓ If the minimum ratio in the Minimum Ratio Test is 0, degeneracy occurs (the basic solution changes but Z does not improve).
    - ✓ Rarely, cycling can occur → resolved using rules such as Bland's rule.

The Simplex Method

- Optimality & Termination – Example

3. Initial Tableau							
BVs	Z	x_1	x_2	s_1	s_2	s_3	RHS
Z	1	-3	-5	0	0	0	0
s_1	0	1	0	1	0	0	4
s_2	0	0	2	0	1	0	12
s_3	0	3	2	0	0	1	18

Entering: x_2 (the most negative in Z-row) Leaving: Ratio test → s_2 (12/2=6, 18/2=9 → Min = 6) Pivot element: 2 (s_2 row, x_2 column)							
BVs	Z	x_1	x_2	s_1	s_2	s_3	RHS
Z	1	-3	0	0	2.5	0	30
s_1	0	1	0	1	0	0	4
x_2	0	0	1	0	0.5	0	6
s_3	0	3	0	0	-1	1	6

The Simplex Method

- Optimality & Termination – Example

Entering: x_1 (the most negative -3 in Z-row) Leaving: Ratio test → s_3 (6/3=2, 4/1=4 → Min = 2) Pivot element: 3 (s_3 row, x_1 column)							
BVs	Z	x_1	x_2	s_1	s_2	s_3	RHS
Z	1	0	0	0	1.5	1	36
s_1	0	0	0	1	1/3	-1/3	2
x_2	0	0	1	0	0.5	0	6
x_1	0	1	0	0	-1/3	1/3	2

Optimality test

- No negative coefficient in Z-row(z_j - c_j) → optimal
- Optimal solution is x1=2, x2=6
- Optimal objective value Z = 36

The Simplex Method

- Two Phase Simplex Method
  - Background
    - ✓ The Simplex method requires an initial BFS.
    - ✓ However, when there are ≥ constraints or equality constraints, adding only slack variables may not provide an initial BFS.
    - ✓ To handle this, artificial variables are introduced, and the problem is solved in two phases.
  - Procedure overview
    - ✓ Phase 1: Solve an auxiliary problem that minimizes W = a1 + a2 + ⋯
      - Goal: drive all artificial variables to zero, thereby finding a BFS within the feasible region of the original problem.
      - If $W_{min} > 0$, the original problem is infeasible.
    - ✓ Phase 2: Use the BFS obtained from Phase 1 as the starting point, then continue the Simplex method with the original objective function.
  - Advantages
    - ✓ Can handle all types of constraints within the same framework.
    - ✓ Provides a unified procedure that addresses both feasibility and optimality.

The Simplex Method

- Two Phase Simplex Method
  - Construction of the tableau
    - ✓ Convert the original problem into standard form.
      - For ≤ constraints: add slack variables.
      - For ≥ constraints: add surplus variables and artificial variables.
      - For = constraints: add artificial variables
    - ✓ Construct the initial tableau so that the artificial variables become the BVs.

  - Phase 1. Minimize W = a1 + a2 + ⋯
      - Assign $c_j = 1$ for artificial variables, and $c_j = 0$ for all other variables.
      - When we transform it into the maximization problem (Max –W), then $c_j = -1$ for artificial variables.
      - Compute $z_j - c_j$ and perform pivots.

The Simplex Method

- Two Phase Simplex Method
  - Example

Maximize $Z = 3x1 + 2x2$
(s.t.) $x1 + x2 \geq 4$, $x1, x2 \geq 0$

➡

Maximize $Z = 3x1 + 2x2$
(s.t.)
$x1 + x2 - s1 + a1 = 4$
$x1, x2 \geq 0$

✓ Phase 1. Minimize $W = a1$
✓ Phase 2. After removing the artificial variables, Maximize $Z = 3x1 + 2x2$

The Simplex Method

- Two Phase Simplex Method
  - Example

Maximize Z = 3x1 + 2x2
(s.t.) x1 + x2 ≥ 4, x1, x2 ≥ 0

➡

Maximize Z = 3x1 + 2x2
(s.t.)
x1 + x2 - s1 + a1 = 4
x1, x2 ≥ 0

Initial tableau in Phase 1						
BVs	W	x_1	x_2	s_1	a_1	RHS
W	-1	0	0	0	1	0
a_1	0	1	1	-1	1	4

✓ In maximizing -W, select the variable with the most negative $z_j - c_j$, such as x1 or x2.

✓ Phase 1 Pivot
- x1 (Entering variable), a2 (Leaving variable)
- After the pivot, a1 is removed and x1 becomes a BV. → A feasible BFS is obtained.

The Simplex Method

- Big-M Method
  - Purpose
    - ✓ Designed as an alternative to the Two-Phase Method, allowing the procedure to be carried out in a single run.
    - ✓ Artificial variables are added to create an initial BFS, but a large penalty M is assigned to them in the objective function from the start.
  - Procedure
    - ✓ When converting to standard form, for constraints requiring artificial variables:
      - Assign a coefficient of M(-M) for artificial variables in a minimization(maximization) problem.
    - ✓ During the simplex process:
      - Both the tendency to reduce artificial variables (due to the penalty) and the tendency to optimize the original objective function are reflected simultaneously.
    - ✓ M is treated as a large positive constant (calculations are carried out symbolically with M)

The Simplex Method

- Big-M Method
  - Example

Maximize $Z = 3x1 + 2x2$
(s.t.) $x1 + x2 \geq 4$

➡

Maximize $Z = 3x1 + 2x2 - Ma1$
(s.t.) $x1 + x2 - s1 + a1 = 4$

✓ Initial BFS: $a1 = 4$

✓ As the Simplex method proceeds, the artificial variable has a large negative coefficient, so it is forced to leave the basis first.

The Simplex Method

- Big-M Method
  - Advantages
    - ✓ Performs the calculation in a single run without dividing into Phase 1 and 2.
  - Disadvantages
    - ✓ Relies on the assumption that M is extremely large → can cause numerical instability.
    - ✓ Theoretically correct, but in practice with floating-point computations, the Two-Phase Method is more stable.

The Simplex Method

- Special Cases
  - Multiple Optimal Solutions
    - ✓ Condition: At the optimal solution, there exists an NBV with $z_j - c_j = 0$
    - ✓ Meaning: Moving to another adjacent BFS still yields the same objective function value.
    - ✓ Geometrically, the optimal solution is not a single CP but lies along an entire edge.
    - ✓ In this case, alternative optima can be found by introducing such variables as additional BVs.

  - Infeasibility
    - ✓ Condition: No feasible solution exists in the feasible region, even after checking all BFS.
    - ✓ Cause: The constraints are mutually contradictory.
    - ✓ In the Simplex method, if the optimal value of artificial variables in Phase 1 is nonzero. → The problem is declared infeasible.

The Simplex Method

- Special Cases
  - Unboundedness
    - ✓ Condition: In the entering variable's column, there are no positive coefficients (for a maximization problem) → the Minimum Ratio Test cannot be applied.
    - ✓ Meaning: The objective function can grow without bound.
    - ✓ Geometrically, the feasible region extends infinitely, and the direction of objective function improvement is not blocked by any constraint.
  - Degeneracy
    - ✓ Condition: The minimum ratio in the Minimum Ratio Test is 0 → even if the BFS changes, the objective value remains the same.
    - ✓ Effect: The same BFS may repeat even after multiple pivots (cycling may occur).
    - ✓ How to solve: apply variable selection rules such as Bland's rule

The Simplex Method

- Special Cases
  - Example of multiple optimal solutions
    - ✓ Maximize Z = x1 + x2
    - ✓ (s.t.) x1 + x2 ≤ 4, x1, x2 ≥ 0
      - Add Slack variable s1 → x1 + x2 + s1 = 4
    - ✓ Pivot 1: x1 (entering variable), s1 (leaving variable) → x1 = 4
    - ✓ After Pivot 1:
      - Optimality test shows no further improvement, but the NBV x2 has z_j-c_j = 0 → alternative optimal solution exists.
    - ✓ Therefore,
      - The solution set (x1, x2) = (4-t, t), 0 ≤ t ≤ 4 are all optimal solutions with Z=4

Initial tableau

BVs	Z	x_1	x_2	s_1	RHS
Z	1	-1	-1	0	0
s_1	0	1	1	1	4

After pivot 1

BVs	Z	x_1	x_2	s_1	RHS
Z	1	0	0	1	4
x_1	0	1	1	1	4

The Simplex Method

- Special Cases
  - Example of unboundedness
    - ✓ Maximize Z = x1 + x2
    - ✓ (s.t.) x1 – x2 ≥ 0, x1, x2 ≥ 0
      - Converting the ≥ constraint: – x1 + x2 ≤ 0
      - Add Slack variable s1 → - x1 + x2 + s1 = 0
    - ✓ Choose the entering variable
      - If x1 is selected, the coefficient is negative → Minimum Ratio Test not possible.
      - If x2 is selected, the coefficient is positive, but RHS is 0 → allowable increase is 0
    - ✓ Therefore, even though there exists a column with zj – cj < 0 (improvement possible), there are no positive coefficients in that column → Unbounded.
    - ✓ In this problem, if we increase x1 = x2 = t by the same amount, then Z = 2t and the constraint remains satisfied for all t → ∞. Hence, the solution is unbounded.

Initial tableau					
BVs	Z	x_1	x_2	s_1	RHS
Z	1	-1	-1	0	0
s_1	0	-1	1	1	0

The Simplex Method

- Special Cases
  - Example of Infeasibility
    - ✓ Maximize Z = x1
    - ✓ (s.t.) x1 ≥ 1, x1 ≤ 0.5, x1 ≥ 0
      - The first and second constraints contradict each other → Infeasible.

The Simplex Method

- Special Cases
  - Example of Degeneracy
    - ✓ Maximize Z = x1 + x2
    - ✓ (s.t.) x1 ≤ 2, x2 ≤ 0, x1 + x2 ≤ 2, x1, x2 ≥ 0
      - → x1 + s1 = 2
      - → x2 + s2 = 0
      - → x1 + x2 + s3 = 2
    - ✓ Initial BFS:
      - x1 = 0, x2 = 0, s1 = 2, s2 = 0, s3 = 2
      - One of the BVs has RHS=0 (here, s2 = 0)
      - In such a case, the Minimum Ration Test often yields a minimum value of 0 → After pivoting, Z does not increase, and the Simplex method may keep cycling around the same BFS.

Q&A