# Introduction to Linear Programming

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# Linear Programming

- Linear Programming (LP)
  - Linear: Both the objective function and the constraints are expressed as linear combinations.
  - Programming: Here, "programming" means planning – solving problems of how to make plans to achieve a goal
- What is LP?
  - An optimization tool have been widely used in real industries to reduce costs and improve efficiency since the 1950s
  - Applications include production planning, logistics and transportation, investment portfolios, policy resource allocation, among many others.
- What problems does linear programming solve?
  - Allocating "limited resources" to "competing activities" in the most optimal way
    - ✓ Resources: budget, machines/equipment, time, manpower, raw materials, storage space, etc.
    - ✓ Activities: product manufacturing, marketing campaigns, route assignments, investment decisions, etc.
  - Once the activity levels are decided, the corresponding resource usage and outcomes (profit/cost) are determined.
  - The goal is to find the best possible option among all feasible choices.

# Linear Programming

- Steps in LP Modeling
  - The general process of converting a problem description into a LP model is
    - ✓ 1. Understand the problem
      - Identify the background, objectives, available resources, and constraints.
    - ✓ 2. Set the objective
      - Decide what to maximize or minimize (e.g., profit, cost, time).
    - ✓ 3. Identify the constraints
      - Consider resource limitations, technical restrictions, and policy requirements.
    - ✓ 4. Define the decision variables
      - Represent the “decisions to be made” as variables.
    - ✓ 5. Formulate the mathematical model
      - Express the objective function and constraints in terms of these variables.

# Linear Programming

- Prototype Example
  - Products: doors, windows
  - Facilities: Plant 1, Plant 2, Plant 3
  - Given/Assumptions
    - ✓ Each plant has a fixed number of available hours per week
    - ✓ Making a door or a window requires different plants and time.
  - Goal: Maximize total profit
    - ✓ Information about profit: \$300 per door, \$500 per door

| Factory | Time/door | Time/window | Available hours |
|---|---|---|---|
| 1 | 1 | 0 | 4 |
| 2 | 0 | 2 | 12 |
| 3 | 3 | 2 | 18 |

# Linear Programming

Profit: \$300 per door, \$500 per window

| Factory | Time/door | Time/window | Available hours |
|---|---|---|---|
| 1 | 1 | 0 | 4 |
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- Prototype Example
  - Definition of decision variables
    - ✓ x1: weekly production of doors
    - ✓ x2: weekly production of windows
  - Objective function:

$$\max Z = 3\,x1 + 5\,x2 \text{ (unit: \$100)}$$

  - Constraints (convert the available hours in each plant into conditions)
    - ✓ Plant 1: $1\,x1 + 0\,x2 \le 4$
    - ✓ Plant 2: $0\,x1 + 2\,x2 \le 12$
    - ✓ Plant 3: $3\,x1 + 2\,x2 \le 18$
      - These three constraints are called functional (or structural) constraints
    - ✓ Non-negativity constraints: $x1, x2 \ge 0$

# Linear Programming

- Standard Form of LP
  - In LP, the standard form is designed so that algorithms (e.g., the Simplex method) work properly.
  - 1. The objective function is “Maximize”
  - 2. All constraints are in the form of “≤”
  - 3. All variables have non-negativity constraints

# Linear Programming

- Standard Form of LP – Generalization
  - Setting
    - ✓ $m$ resources → the limitation of the $i$-th resource $b_i$ (i=1, ⋯, m)
      - The amount of resource $i$ consumed by 1 unit of activity $j$: $a_{ij}$
    - ✓ $n$ activities → the quantity of the $j$-th activity $x_j$ (j=1, ⋯, n)
      - Contribution of 1 unit of activity j to total performance Z: $c_j$
    - ✓ Total performance measure: Z
  - Formulation
    - ✓ Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
    - ✓ (subject to)
      - $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$
      - ⋯
      - $a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$
      - $x_1, x_2, \ldots, x_n \geq 0$

# Linear Programming

- Other Forms
  - Minimization objective function
    - ✓ Minimize Z = c1x1 + c2x2 + ⋯ + cnxn
  - Greater-than-or-equal-to constraints
    - ✓ ai1x1 + ai2x2 + ⋯ ainxn ≥ bi
  - Equality constraints
    - ✓ ai1x1 + ai2x2 + ⋯ ainxn = bi
  - Variables without non-negativity restrictions

- Although we introduced management science as allocating limited resources among competing activities, not all problems necessarily fall into this framework.

# Linear Programming

- Assumptions of LP
  - Since an LP model is a simplified mathematical representation of real-world problems, several assumptions are made.
  - 1. Proportionality
    - ✓ Changes in variable values cause proportional changes in the objective function and constraints. (e.g., if producing 1 unit of a product requires 5 kg of material, then producing 2 units requires 10kg.
    - ✓ Violation in real-world: bulk purchase discounts, economies of scale
  - 2. Additivity
    - ✓ The total resource usage or total profit is assumed to be the simple sum of each variable’s contribution.
    - ✓ Violation in real-world: when there is interaction between variables (synergy or interference).
      - When one plant produce two types of products, the efficiency decreases.

# Linear Programming

- Assumptions of LP
  - 3. Divisibility
    - ✓ Variables are assumed to take continuous values (no integer restrictions).
      - For example, we assume that producing 2.5 units of a product is possible.
    - ✓ Violation in real-world: a person cannot work in half-unit increments
      - → requires Integer Programming
  - 4. Certainty
    - ✓ Coefficients in objective function or constraints, and resource availability are assumed to be known with certainty.
    - ✓ Violation in real-world: fluctuations in raw material prices, variations in production time
      - → Requires Stochastic Programming

# Graphical Solution

- Graphical Method
  - The graphical method finds solutions visually when there are two variables.
  - It is useful for intuitively understanding the basic structure of LP and the nature of the optimal solution.
- Ket Concepts of the Graphical Method
  - Graphing the constraints
    - ✓ Each constraint is a line, and the feasible side of the line is identified.
      - Example: $x1 + x2 \leq 4$ corresponds to the line $x1 + x2 = 4$ and the region below the line satisfies the constraint.
  - Feasible region
    - ✓ The region that satisfies all constraints simultaneously.
    - ✓ The solution of the LP problem always lies within this region.
    - ✓ The region forms a convex polygon.

# Graphical Solution

- Key Concepts of the Graphical Method
  - Objective function line (Iso-profit or Iso-cost line)
    - ✓ The line representing all points with the same value of the objective function Z = c1x1 + c2x2
    - ✓ By moving this line in a parallel manner, the last point where it touches the feasible region is the optimal solution.
  - Property of the optimal solution
    - ✓ In LP, the optimal solution always occurs at one of the corner-points of the feasible region.
    - ✓ This property forms the foundation of the Simplex method.

# Graphical Solution

- Prototype Example
  - Determine the mix of two products that maximizes profit under limited resources.
    - ✓ x1: the number of doors produced
    - ✓ x2: the number of windows produced
  - Objective function: Maximize $Z = 3x_1 + 5x_2$
  - Constraints
    - ✓ $x_1 \le 4$
    - ✓ $x_2 \le 6$
    - ✓ $3x_1 + 2x_2 \le 18$
    - ✓ Non-negativity constraints: $x_1, x_2 \ge 0$

# Graphical Solution

- Procedure of the Graphical Method in Prototype Example
  - 1. Draw the constraints
    - ✓ x1 = 4
    - ✓ x2 = 6
    - ✓ 3x1 + 2x2 = 18
  - 2. Find the feasible region
    - ✓ Shade the area that satisfies each constraint
    - ✓ The region that satisfies all constraints simultaneously is the feasible region.
  - 3. Move the objective function line
    - ✓ From Z = 3 x1 + 5 x2, choose an arbitrary value and draw the line.
    - ✓ Move this line in parallel until it reaches the boundary of the feasible region (the last contact point is the optimal solution)
  - 4. Calculate the corner points
    - ✓ (0, 0), (0, 6), (4, 0), (4, 3), (2, 6)
  - 5. Evaluate the objective function at each corner point
    - ✓ (0, 0) = 0, (0, 6) = 30, (4, 0) = 12, (4, 3) = 27, **(2, 6) = 36(최대값)**

# Linear Programming

- Special Cases: Infeasible, Multiple Optimal Solutions, Unbounded
  - Infeasible solution
    - ✓ Occurs when there is no point that satisfies all constraints simultaneously
    - ✓ Diagnosis: in the graph, no overlapping shaded region exists; in modeling, conflicting (contradictory) constraints may be present.
    - ✓ Response: review or relax constraints, check data for errors, or adjust the objective
  - Multiple optimal solutions
    - ✓ Occurs when the objective function line is parallel to one edge of the feasible region.
    - ✓ Diagnosis: if two different corner points give the same optimal value, then every points on the line segment between them is also optimal.
    - ✓ When there are multiple optimal solutions, they act as alternatives each other, so increase flexibility.
    - ✓ Caution: in sensitivity analysis, allowable increase/decrease values and slack may be subtle.
  - Unbounded solution
    - ✓ Occurs when the feasible region is open in one direction, and that direction aligns with the improvement direction of the objective function.
    - ✓ Diagnosis: in the graph, moving the objective function line in the improvement direction never reaches a boundary.
    - ✓ Response: in practice, an unbounded solution usually indicates that some constraint is missing from the model.

# Linear Programming

- Additional Example: Radiation Therapy Design
  - Background: Designing a treatment plan using two types of radiations beams for a patient’s tumor.
    - ✓ Deliver a sufficient high does of radiation to kill malignant tumor cells.
    - ✓ Minimize damage to healthy tissue cells.
    - ✓ Satisfy allowable radiation does limits for major organs.
  - Decision variables: amounts of the two types of radiation beams x1, x2
  - Objective function: minimize the average radiation delivered to healthy tissue
  - Constraints
    - ✓ Average radiation to critical organs ≤ 27
    - ✓ Average radiation to the entire tumor = 60
    - ✓ Radiation to the tumor center ≥ 60

| Target | Radiation 1 | Radiation 2 |
|---|---|---|
| Healthy tissue | 4 | 5 |
| Critical organs | 3 | 1 |
| Entire tumor | 5 | 5 |
| Tumor center | 6 | 4 |

# Linear Programming

- Additional Example: Radiation Therapy Design
  - Objective function: Minimize the average radiation delivered to healthy tissue
  - Constraints
    - ✓ Average radiation to critical organs ≤ 27
    - ✓ Average radiation to the entire tumor = 60
    - ✓ Radiation to the tumor center ≥ 60
  - Mathematical formulation
    - ✓ Minimize Z = 4x1 + 5x2
    - ✓ (subject to)
      - 3x1 + x2 ≤ 27
      - 5x1 + 5x2 = 60
      - 6x1 + 4x2 ≥ 60

| Target | Radiation 1 | Radiation 2 |
|---|---|---|
| Healthy tissue | 4 | 5 |
| Critical organs | 3 | 1 |
| Entire tumor | 5 | 5 |
| Tumor center | 6 | 4 |