Lecture 6. Discrete Time Markov Chain (1)

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Markov Chain - Motivation

discrete time {State Space}

- I drink a bottle of soda everyday. I drink either Coke or Pepsi everyday. When I choose what to drink for today, I only consider what I drank yesterday. Markov property

- Suppose I drank Coke yesterday, then chance of drinking Coke again today is 0.7. (What is chance of drinking Pepsi today then?)

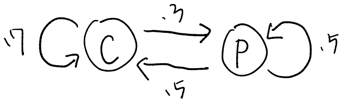
- Suppose I drank Pepsi yesterday, then chance of drinking Pepsi again today is 0.5. (What is chance of drinking Coke today then?)

- In a tabular form?

Yesterday \ Today	C	P
C	.7	.3
P	.5	.5

- How would you describe this situation in $\boxed{\text{diagram}}$?

DTMC transition diagram.



- How would you represent this situation to mathematical form?

matrix A

$$
\begin{array}{cc}
 & \begin{array}{cc} \text{to} & \\ c & p \end{array} \\
\text{from} \begin{array}{c} c \rightarrow \\ p \rightarrow \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}
\end{array}
$$

DTMC transition matrix

1) $|S| \times |S|$ matrix

2) all elements $\in [0,1]$

3) row sum $= 1$ ... Stochastic matrix

$S = \{c, p\}$

$|S| = \#$ elem in S

$$\mathbb{P}(X_{n+1} = p \mid X_n = c) = 0.3 = A_{cp}$$

- Suppose I do this for an year. Which brand of soda I will drink more? (Pepsi or Coke?) Coke
- If I drink Coke today, what is the probability that I will be drinking Pepsi on three days later?

0	1	2	3
C	C	C	
	P	P	P

.7 .7 .3

.7 .3 .5

.3 .5 .3

.3 .5 .5

$$\mathbb{P}[X_3 = P \mid X_0 = C]$$

- If I do this for 10 years (3650 days) from now, then how many days I will drink Pepsi?
- Suppose Pepsi is \$1 and Coke is \$1.5. How much on average I spend on soda in a month?
- Suppose that there are a billion customers (who has the same type of consumption pattern) like me in the world. You are working for Pepsi's marketing department and wish to boost Pepsi $\rightarrow$ Pepsi probability from 0.5 to 0.6. On average, how much revenue will be additionally generated by this marketing effort for a day?

Definitions

Newsvendor
time 0 : make decision
time 1 : demand is realized
revenue is determined
1 period : stochastic model

Definition - Stochastic Process

- Stochastic Process: Time + Random
  - Discrete Time Stochastic Process
    - Discrete Time + Random
    - $\{X_n : n \geq 0,\ n \in \mathbb{N}\} = \{X_0, X_1, \cdots,\}$ index ⇒ time
  - Continuous Time Stochastic Process
    - Continuous Time + Random
    - $\{X_t : t \geq 0,\ t \in \mathbb{R}^+\} = \{X_0 \cdots X_{\sqrt{2}} \cdots X_3 \cdots\}$

Definitions - State and State space

- State: value of X_n.
- State space (S): a set of all possible values that X_n can take.

Definitions

Definitions - Markov Property

- The nearest future only depends on the present.
- For a discrete time stochastic process $\{X_n : n \geq 0,\ n \in \mathbb{N}\}$,
  - X_{n+1} depends only on the state of X_n.
  - X_{n+1} is function of X_n and some randomness. $X_{n+1} = f(X_n, \text{random})$
  - $\mathbb{P}(X_{n+1} = j | \cancel{X_0 = i_0, X_1 = i_1, \ldots}, \boxed{X_n = i}) = \mathbb{P}(X_{n+1} = j | X_n = i)$

outdated info
irrelevant history

most recent history

$= \mathbb{P}(X_3 = j \mid X_2 = i)$

$= \mathbb{P}(X_4 = j \mid X_3 = i)$

Definition - DTMC

- A discrete time stochastic process with Markov Property is said to be a DTMC.

Definition - Transition probability (matrix)

- **Transition probability** p_{ij} is the probability that governs "transition" within the state space.
  - (1 step) $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_n = j | X_{n-1} = i) = \mathbb{P}(X_1 = j | X_0 = i)$
- **Transition probability matrix P** is collection of p_{ij}
  - $\mathbf{P} = [p_{ij}]$

    $P \in \mathbb{R}^{|S| \times |S|}$, $|S|$ is # elem in set S

Definition - Initial Distribution

- The information of where the chain starts at time 0.
- $a_0 :=$ distribution of X_0 as a row vector.

$X_0 = \text{coke} \Leftrightarrow \mathbb{P}(X_0 = c) = 1, \ \mathbb{P}(X_0 = p) = 0 \Leftrightarrow a_0 = (1 \quad 0)$

time index

$\mathbb{P}(X_2 = c) = 0.6, \ \mathbb{P}(X_2 = p) = 0.4 \Leftrightarrow a_2 = (0.6 \quad 0.4)$

Fomularize Coke & Pepsi MC

- State Space $S = \{\underline{c, p}\}$

- Transition probability Matrix $P = \begin{array}{cc} \begin{bmatrix} .7 & .3 \\ .5 & .5 \end{bmatrix} & \begin{array}{c} c \\ p \end{array} \end{array}$

- Transition Diagram
$$\begin{tikzcd} C \arrow[loop left, ".7"] \arrow[r, shift left, ".3"] & P \arrow[l, shift left, ".5"] \arrow[loop right, ".5"] \end{tikzcd}$$

- Initial Distribution
$$a_0 = (\underbrace{.6}_{\mathbb{P}(X_0=c)}, \overbrace{.4}^{\mathbb{P}(X_0=p)})$$

1-step transition

- Suppose $\mathbb{P}(X_0 = c) = 0.6$ and $\mathbb{P}(X_0 = p) = 0.4$, then what is $\mathbb{P}(X_1 = c)$?

$X_0 \to X_1$

$$\left.\begin{array}{l} c \to c : \underline{.6} \times .7 = .42 \\ p \to c : \underline{.4} \times .5 = .20 \end{array}\right\} \Sigma = .62 = \mathbb{P}(X_1 = c)$$

$\mathbb{P}(X_1 = c) = \mathbb{P}(X_1 = c, X_0 = c) + \mathbb{P}(X_1 = c, X_0 = p)$ cf) $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$

$= \mathbb{P}(X_1 = c \mid X_0 = c)\mathbb{P}(X_0 = c) + \mathbb{P}(X_1 = c \mid X_0 = p)\mathbb{P}(X_0 = p)$

$= 0.7 \times \underline{0.6} + 0.5 \times \underline{0.4} = (0.6 \;\; 0.4)\begin{pmatrix} 0.7 \\ 0.5 \end{pmatrix}$

$= 0.62$

$a_0 = (.6, .4) \quad P = \begin{bmatrix} .7 & .3 \\ .5 & .5 \end{bmatrix}$

$a_0 \cdot P = (.6 \;\; .4)\begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} = (.62 \;\; .38) = (\mathbb{P}(X_1 = c), \; \mathbb{P}(X_1 = p)) = a_1$

$\Rightarrow a_1 = a_0 P^1$

$a_2 = a_1 P = (a_0 P) P = a_0 P^2$

$a_3 = a_2 P = (a_0 P^2) P = a_0 P^3$

$\vdots$

$a_n = a_0 P^n = a_i P^{n-i}$

Math Review 14 - Probabilities - properties (Revisited)

- $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$, „ if $E_1 \cap E_2 = \emptyset$.
- $\mathbb{P}(E) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c)$ „ $\because (E \cap F) \cap (E \cap F^c) = \emptyset$
- If $F_1 \cap F_2 \cap \cdots \cap F_n = \emptyset$ and $\mathbb{P}(F_1 \cup \cdots \cup F_n) = 1$, then $\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \cap F_i)$

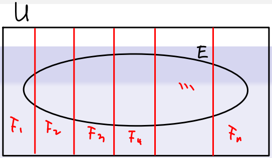
$A = (\quad)$, then A^k

$A = A \Lambda P^{-1}$

$\Rightarrow A^k = P \Lambda^k P^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & .7 \end{pmatrix}^k \xrightarrow{\infty} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^k \xrightarrow{\infty} \begin{pmatrix} 1 & 0 \\ 0 & \infty \end{pmatrix}$

Math Review 15 - Bayes' rule (Revisited)



Definition - Bayes' rule

- Suppose $F_1, \ldots, F_n$ are mutually exclusive events
  - $F_i \cap F_j = \emptyset$, or, $\mathbb{P}(F_i \cap F_j) = 0$, for any $i \neq j$
  - $\mathbb{P}(F_1 \cup \cdots \cup F_n) = 1$
- In other words, exactly one of the events $F_1, \ldots, F_n$ will occur.

mathmatical partition.

- Then, the following holds.

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(E \cap F_1) + \cdots + \mathbb{P}(E \cap F_n) \\ &= \sum_{i=1}^{n} \mathbb{P}(E \cap F_i) \\ &= \sum_{i=1}^{n} \mathbb{P}(E|F_i)\mathbb{P}(F_i)\end{aligned}$$

2-step transition

- Suppose $\mathbb{P}(X_0 = c) = 0.6$ and $\mathbb{P}(X_0 = p) = 0.4$, then what is $\mathbb{P}(X_2 = c)$?

$$\mathbb{P}(X_2=c) = \mathbb{P}(X_2=c,\ X_1=c) + \mathbb{P}(X_2=c,\ X_1=p)$$

$$= \mathbb{P}(X_2=c \mid X_1=c)\,\mathbb{P}(X_1=c) + \mathbb{P}(X_2=c \mid X_1=p)\,\mathbb{P}(X_1=p)$$

$$= 0.7 \times 0.62 + 0.5 \times 0.38$$

$$a_0 = (0.6 \quad 0.4) \qquad a_1 P = a_0 P^2 = \big(\mathbb{P}(X_2=c) \quad \mathbb{P}(X_2=p)\big)$$

$$a_1 = (0.62 \quad 0.38)$$

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{pmatrix}$$

Transitions in DTMC

$$a_0 \xrightarrow{\times P} a_1 \xrightarrow{\times P} a_2 \xrightarrow{\times P} \cdots \xrightarrow{\times P} a_n$$

$$a_n = a_0 P^n$$

$$a_{n+1} = a_n P$$

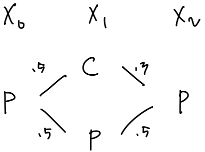
$$\begin{aligned} a_{n+k} &= a_n P^k \\ &= a_k P^n \\ &= a_{n+1} P^{k-1} \end{aligned}$$

2-step transition

- Suppose $X_0 = p$, then what is $\mathbb{P}(X_2 = p)$?
- (Or, equivalently, what is $\mathbb{P}(X_2 = p | X_0 = p)$)?

$$X_0 = p \rightarrow \begin{matrix} \mathbb{P}(X_1 = c) = 0.5 \\ \mathbb{P}(X_1 = p) = 0.5 \end{matrix} \rightarrow \begin{aligned} \mathbb{P}(X_2 = p) &= \mathbb{P}(X_2 = p \mid X_1 = c)\, \mathbb{P}(X_1 = c) \\ &+ \mathbb{P}(X_2 = p \mid X_1 = p)\, \mathbb{P}(X_1 = p) \end{aligned}$$

$$a_2 = a_0 P^2 = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{pmatrix}^2 = \begin{pmatrix} 0.6 & \underbrace{0.4}_{\mathbb{P}(X_2 = p)} \end{pmatrix}$$



$.5 \times .3 = .15$

$.5 \times .5 = .25$

$\mathbb{P}(X_2 = p \mid X_0 = p)$

$= .15 + .25 = .4$

2-step transition

- Suppose $X_0 = p$, then what is $\mathbb{P}(X_2 = p)$?
- (Or, equivalently, what is $\mathbb{P}(X_2 = p | X_0 = p)$)?

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"If I only had an hour to chop down a tree, I would spend the first 45 minutes
sharpening my axe. - A. Lincoln"
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