Lecture 12. CTMC-Exercises

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Exercise 1

- Consider a CTMC $X = \{X(t), t \geq 0\}$ on $S = \{A, B, C\}$ with a rate matrix G given by

$$G = \begin{pmatrix} -12 & 4 & 8 \\ 5 & -6 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

(a) Draw the rate diagram

(b) Use a computer software like to directly compute the transition probability matrix $\mathbf{P}(t)$ at $t = 0.20$ minutes.

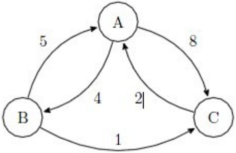
(c) Do the previous part for $t = 1.0$ minute.

(d) Using the results from parts (b) and (c), but without using a software package or calculator, find $\mathbb{P}[X(1.2) = C | X(0) = A]$ and $\mathbb{P}[X(2) = A | X(1) = B]$.

(e) Do part (b) for t = 5 minutes. What phenomenon have you observed?

(Solution)

(a)



(b)

$$\mathbf{P}(0.2) = \begin{bmatrix} .2299 & .1805 & .5896 \\ .2383 & .3989 & .3628 \\ .1411 & .0508 & .8081 \end{bmatrix}$$

(c)

$$\mathbf{P}(1.0) = \begin{bmatrix} .1673 & .1131 & .7196 \\ .1688 & .1176 & .7137 \\ .1662 & .1097 & .7241 \end{bmatrix}$$

(d)

$$\mathbf{P}(1.2) = \mathbf{P}(1.0)\mathbf{P}(0.2) = \begin{bmatrix} .1669 & .1119 & .7212 \\ .1675 & .1136 & .7189 \\ .1665 & .1105 & .7230 \end{bmatrix}$$

Hence, $\mathbb{P}(X(1.2) = C | X(0) = A) = 0.7212$. Using the results from part (c), $\mathbb{P}(X(2.0) = A | X(1.0) = B) = 0.1688$.

(e)

$$\mathbf{P}(5.0) = \begin{bmatrix} .1667 & .1111 & .7222 \\ .1667 & .1111 & .7222 \\ .1667 & .1111 & .7222 \end{bmatrix}$$

The Markov Chain has reached to steady-state by time 5. Each row is same as the stationary distribution.

(For your reference, the following is screenshot for MATLAB results for above.)

```
>> G=[-12 4 8;
      5 -6 1;
      2 0 -2];
>> expm(0.2*G)

ans =

    0.2299    0.1805    0.5896
    0.2383    0.3989    0.3628
    0.1411    0.0508    0.8081

>> expm(G)

ans =

    0.1673    0.1131    0.7196
    0.1688    0.1176    0.7137
    0.1662    0.1097    0.7241

>> expm(5*G)

ans =

    0.1667    0.1111    0.7222
    0.1667    0.1111    0.7222
    0.1667    0.1111    0.7222

>>
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Exercise 2

- Let $X = \{X(t) : t \geq 0\}$ be a continuous time Markov chain with state space $\{1, 2, 3\}$ with a generator matrix G of the following.

$$G = \begin{pmatrix} -4 & ? & 1 \\ 1 & ? & 2 \\ 1 & ? & -2 \end{pmatrix}$$

Using a software, one can compute e^G as following:

$$e^G = \begin{pmatrix} 0.20539 & 0.36211 & 0.43250 \\ 0.19865 & 0.35727 & 0.44407 \\ 0.19865 & 0.33896 & 0.46239 \end{pmatrix}$$

(a) Fill in entries in G.

(b) Find $\mathbb{P}[X(2) = 3 | X(1) = 2]$.

(c) Find $\mathbb{P}[X(1) = 2, X(3) = 3 | X(0) = 2]$

(d) Find the stationary distribution of the Markov chain.

(e) Find $\mathbb{P}[X(200) = 3 | X(1) = 2]$

(Solution)

(a) Since each row of G should sum up to 0,

$$G = \begin{bmatrix} -4 & 3 & 1 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{bmatrix}$$

(b) Let $\mathbf{P}(t)$ be the transition probability matrix. (i.e., $\mathbf{P}_{ij}(t) = \mathbb{P}(X_t = j|X_0 = i)$
Then we have $\mathbf{P}(t) = e^{tG}$. This implies
$\mathbb{P}(X(2) = 3|X(1) = 2) = \mathbf{P}(1)_{23} = expm(G)_{23} = 0.44407$

(c) By markov property, we obtain

$$\begin{aligned} \mathbb{P}(X(1) = 2, X(3) = 3|X(0) = 2] &= \mathbf{P}(1)_{22}\mathbf{P}(2)_{23} \\ &= [e^{G}]_{22} \cdot [e^{2G}]_{23} \\ &= [e^{G}]_{22} \cdot [e^{G} \times e^{G}]_{23} \\ &= 0.35727 \cdot 0.4499 = 0.1607 \end{aligned}$$

(d) Solving $(\pi_1 \ \pi_2 \ \pi_3)\, G = (0 \ 0 \ 0)$ and $\pi_1 + \pi_2 + \pi_3 = 1$, we get stationary distribution $\pi = (.2 \ .35 \ .45)$.

(e) $\mathbb{P}(X(200) = 3 | X(1) = 2) = \pi_3 = .45$

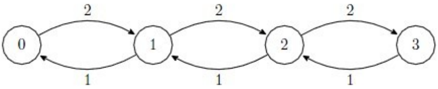
Exercise 3

- Consider a call center that is staffed by K agents with three phone lines. Call arrivals follow a Poisson process with rate 2 per minute. An arrival call that finds all lines busy is lost. Call processing times are exponentially distributed with mean 1 minute. An arrival call that finds both agents busy will wait in the third phone line until beginning service.
- *(a)* Find the throughput and average waiting time when K = 1.
- *(b)* Find the throughput and average waiting time when K = 2.
- *(c)* Find the throughput and average waiting time when K = 3.

(Solution)

For all the sub-problem below, the state space will always be $\{0, 1, 2, 3\}$ indicating the number of calls in the system.

(a) The transition diagram is as the following.



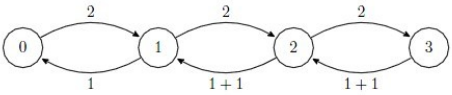
Solve the stationary distribution using "cuts", we get

$$\pi = (1/15, 2/15, 4/15, 8/15)$$

So the throughput is $\lambda_{eff} = 2(1 - \pi_3) = 14/15$. To find the average waiting time, we use Little's law. Note that $L_q = 1\pi_2 + 2\pi_3 = 20/15$, so

$$W = \frac{L_q}{\lambda_{eff}} = 10/7$$

(b) The transition diagram is as the following.



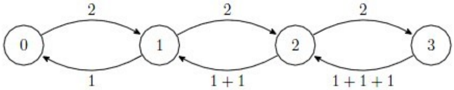
Solve the stationary distribution using "cuts", we get

$$\pi = (1/7, 2/7, 2/7, 2/7)$$

. So the throughput is $\lambda_{eff} = 2(1 - \pi_3) = 10/7$. To find the average waiting time, we use Little's law. Note that $L_q = 1\pi_3 = 2/7$, so

$$W_q = \frac{L_q}{\lambda_{eff}} = 1/5.$$

(c) The transition diagram is as the following.



Solve the stationary distribution using "cuts", we get

$$\pi = (3/19, 6/19, 6/19, 4/19)$$

So the throughput is $\lambda_{eff} = 2(1 - \pi_3) = 30/19$. To find the average waiting time, we use Little's law. Note that $L_q = 0$, so

$$W_q = \frac{L_q}{\lambda_{eff}} = 0.$$

Exercise 4

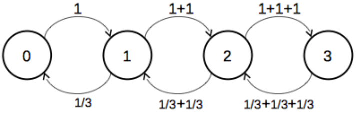
- A production system has three machines working in parallel. The up times of a machine is assumed to be iid, exponentially distributed with mean 3 hours. When a machine is down, its repair times are iid, exponentially distributed with mean 1 hour. There are two repairmen, working at the same speed.

(a) Using a CTMC to find the long-run fraction of time that all machines are up and running; you need to specify the meaning of each state; draw the rate diagram; spell out necessary equations that are used in your calculations.

(b) When a machine is up and running, it processes customer orders at rate 4 orders per minute. Assume the customer orders are backlogged in the next month. What is the average production rate per minute of the production system?

(Solution)

(a) Let $X(t)$ be the number of machines that are up at time t. Then the state space is $\{0, 1, 2, 3\}$. The rate diagram is as follows.



We can write the following cut equations for the stationary distribution π:

$$\begin{aligned} 2\pi_0 &= (1/3)\pi_1 \\ 2\pi_1 &= (2/3)\pi_2 \\ \pi_2 &= \pi_3 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

Solving these equations, we have $\pi = (1/43, 6/43, 18/43, 18/43)$. Thus, the long run fraction of time that all machines are up is $18/43$.

(b) $0\pi_0 + 4\pi_1 + 8\pi_2 + 12\pi_3 = 24/43 + 144/43 + 216/43 = 384/43$ jobs per minute.

Exercise 5

- Customers arrive at a two-server system according to a Poisson process with rate $\lambda = 10$ per hour. An arrival finding server 1 (John) free will begin his service with server 1. An arrival finding server 1 busy and server 2 (Mary) free will join server 2. An arrival finding both servers busy goes away. Once a customer is served by either server, he departs the system. The service times at both servers are exponential random variables. Assume that the service rate ofthe first server is 6 per hour and the service rate of the second sever is 4 per hour.

(a) Describe a continuous time Markov chain to model the system and give the rate transition diagram. (First, start with identifying state space)

(b) Find the stationary distribution of the continuous time Markov chain.

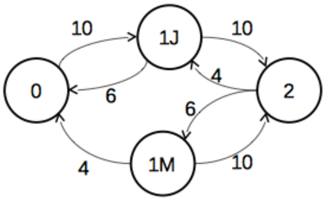
(c) What is the long-run fraction of time that server i is busy for $i = 1, 2$, respectively?

(Solution)

(a) This process can be modelled as follows: here our state space is $S = \{0, 1J, 1M, 2\}$ where: 0 = no jobs; $1J$ = only one job, at first server; $1M$ = only one job, at second server; 2 = two jobs, one at each server. The generator is as follows

$$G = \begin{array}{c} 0 \\ 1\text{J} \\ 1\text{M} \\ 2 \end{array} \begin{bmatrix} -10 & 10 & 0 & 0 \\ 6 & -16 & 0 & 10 \\ 4 & 0 & -14 & 10 \\ 0 & 4 & 6 & -10 \end{bmatrix}$$

and the transition diagram is the following



(b) To find the stationary distribution, we need to again solve the system of balance and normalizing equations. From the generator, we see that

$$\begin{aligned} 10\pi_0 &= 6\pi_{1\mathrm{J}} + 4\pi_{1\mathrm{M}} \\ 16\pi_{1\mathrm{J}} &= 10\pi_0 + 4\pi_2 \\ 14\pi_{1\mathrm{M}} &= 6\pi_2 \\ 10\pi_2 &= 10\pi_{1\mathrm{J}} + 10\pi_{1\mathrm{M}} \\ 1 &= \pi_0 + \pi_{1\mathrm{J}} + \pi_2 + \pi_{1\mathrm{M}} \end{aligned}$$

Solving this, we that $\pi_0 = 18/88$, $\pi_{1\mathrm{J}} = 20/88$, $\pi_2 = 35/88$, and $\pi_{1\mathrm{M}} = 15/88$.

(c) The long-run fraction of time that server 1 is busy is $\pi_{1\mathrm{J}} + \pi_2 = 55/88 = 62.5\%$. The long-run fraction of time that server 2 is busy is $\pi_2 + \pi_{1\mathrm{M}} = 50/88 \approx 56.8\%$.

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"It's not that I'm so smart, it's just that I stay with problems longer. - A. Einstein"
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