Transportation & Assignment Problems

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Introduction

- Transportation Problem
  - A problem concerning "how to transport goods in an optimal way"
  - The main application areas of transportation problems are not necessarily limited to transportation itself.
- Assignment Problem
  - A problem concerning "how to assign people (assignee) to one of several tasks"
  - This is a special case of the transportation problem.
- Characteristics of transportation and assignment problems
  - Both problems belong to LP.
  - They typically involve a very large number of constraints and variables, which makes the computational burden of the Simplex Method increase significantly.
  - However, due to their structural nature, most of the constraint coefficients a_{ij} are 0, and the relatively small number of nonzero coefficients appear with a special pattern.
  - By exploiting these structural characteristics, efficient solution algorithms have been developed.

Transportation Problems

Transportation Problem

- Prototype Example
  - We want to transport cans produced at three canneries to four warehouses.
    - ✓ Canneries: (1) Bellingham, (2) Eugene, and (3) Albert Lea
    - ✓ Warehouses: (1) Sacramento, (2) Salt Lake City, (3) Rapid City, and (4) Albuquerque
  - The primary expense involved is the shipping cost.
  - For decision-making, we have forecasted production at each cannery and forecasted demand at each warehouse for the upcoming season.

Shipping cost		Warehouse				Output
		1	2	3	4	
Cannery	1	464	513	654	867	75
	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	300

    - ✓ The objective is to determine a shipping plan for each (cannery-warehouse) pair that minimizes total transportation cost.

Transportation Problem

- LP Formulation
  - Minimize Total Shipping Cost Z
    - ✓ Z = 464 x11 + 513 x12 + 654 x13 + 867 x14 + 352 x21 + 416 x22 + 690 x23 + 791 x24 + 995 x31 + 682 x32 + 388 x33 + 685 x34
    - ✓ Cannery constraints
      - x11 + x12 + x13 + x14 = 75
      - x21 + x22 + x23 + x24 = 125
      - x31 + x32 + x33 + x34 = 100
    - ✓ Warehouse constraints
      - x11 + x21 + x31 = 80
      - x12 + x22 + x32 = 65
      - x13 + x23 + x33 = 70
      - x14 + x24 + x34 = 85
    - ✓ x_{ij}: quantity shipped from cannery i to warehouse j, where $i \in \{1, 2, 3\}$, $j \in \{1, 2, 3, 4\}$.

Transportation Problem

- Assumptions about Supply and Demand in Transportation Problem
  - Each source has a fixed supply s_i, which must be distributed among several destinations.
  - Each destination has a fixed demand d_j, which must be satisfied by shipments from multiple sources.
  - There should be no surplus or shortage in the overall system, i.e.,
    - ✓ $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$
  - Assumption about unit shipping cost
    - ✓ The shipping cost from a source to a destination is proportional to the amount shipped (the proportionality assumption in LP).
    - ✓ Therefore, the shipping cost from source i to destination j is expressed as: $c_{ij} \times x_{ij}$
    - ✓ Here, c_{ij} is unit cost of shipping one unit from source i to destination j and x_{ij} is a quantity shipped along the route $i \to j$.

Transportation Problem

- Information for the Transportation Model
  - Objective: Minimize the total shipping cost $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$
    - ✓ Constraints
      - $\sum_{j=1}^{n} x_{ij} = s_i \ (\forall i = 1, \dots, m)$
      - $\sum_{i=1}^{m} x_{ij} = d_j \ (\forall j = 1, \dots, n)$
      - $x_{ij} \geq 0 \ \forall i, j$
  - Parameter table
    - ✓ (1) supply at each source, (2) demand at each destination, (3) unit shipping cost for all pairs

		Destination				Supply
		1	2	⋯	n	
Sources	1	c_{11}	c_{12}	⋯	c_{1n}	s_1
	2	c_{21}	c_{22}	⋯	c_{2n}	s_2
	⋮	⋮	⋮		⋮	⋮
	m	c_{m1}	c_{m2}	⋯	c_{mn}	s_m
Demand		d_1	d_2	⋯	d_n	$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$

Transportation Problem

- Integer Solution Property of the Transportation Problem
  - If all supply values (s_i) and demand values (d_j) are integers, then every BFS, including the optimal solution, will also have all BVs as integers.

Transportation Problem

- Example with a Dummy Destination
  - Machine Production Scheduling (over four months)
    - ✓ x_j: The number of machines to be produced in month j

Month	Scheduled Installations	Maximum Production	Unit Cost of Production	Unit Cost of Storage
1	10	25	1.08	0.015
2	15	35	1.11	0.015
3	25	30	1.10	0.015
4	20	10	1.13	

Transportation Problem

- Example with a Dummy Destination
  - Machine production scheduling (4 months)
    - ✓ Source i: Machine production in month i
    - ✓ Destination j: Machine installation in month j
    - ✓ x_{ij}: The number of machines produced in month i to be installed in month j
      - Constraint: $x_{ij} = 0$ if $i > j$ (cannot produce later for an earlier installation)
    - ✓ c_{ij}: x_{ij}
      - If $i = j$, unit production cost in month j. If $i < j$, unit production + storage cost until month j
    - ✓ s_i: Given as an upper bound, not a fixed supply → introduce a dummy destination to absorb unused production ($c_{ij} = 0$ for dummy)
      - $x_{11} + x_{12} + x_{13} + x_{14} \leq 25$
      - $x_{21} + x_{22} + x_{23} + x_{24} \leq 35$
      - $x_{31} + x_{32} + x_{33} + x_{34} \leq 30$
      - $x_{41} + x_{42} + x_{43} + x_{44} \leq 10$
    - ✓ d_j: scheduled number of machine installation in month j

		Destination					Supply
		1	2	3	4	5(D)	
Source	1	1.080	1.095	1.110	1.125	0	25
	2	M	1.110	1.125	1.140	0	35
	3	M	M	1.100	1.115	0	30
	4	M	M	M	1.130	0	10
Demand		10	15	25	20	30	100

Transportation Problem

- Example with a Dummy Source
  - There are three plants (A, B, C) and four cities (1, 2, 3, 4)
  - The supply capacities of the plants are 50, 60, 50, and the demands of the cities are 50, 70, 30, 60.
    - ✓ Thus, total supply is 160 total demand is 210, so supply is insufficient.
  - Assume the penalty (loss) cost for unsatisfied demand is the same regardless of the city.
  - The objective is to minimize total cost in distributing the available supply to the demands.
  - As indicated in the parameter table, shipping from plant C to city 4 is assumed to be impossible.

		Destination				Supply
		1	2	3	4	
Source	A	16	13	22	17	50
	B	14	13	19	15	60
	C	19	20	23	M	50
Demand		50	70	30	60	

Transportation Problem

- Example with a Dummy Source
  - We assume that the penalty cost for unsatisfied demand is the same across all cities.
    - ✓ Therefore, we do not need to consider which city's demand is left unsatisfied, and only the shipping cost matters in this example.
  - To balance total supply and demand, we introduce a dummy source.
    - ✓ Shipments from the dummy source to a city do not represent actual deliveries but instead indicate the shortfall of demand at that city.
    - ✓ If necessary, different penalty costs can be assigned to the dummy source-city pairs to reflect differences in the loss of unsatisfied demand across cities.

		Destination				Supply
		1	2	3	4	
Source	A	16	13	22	17	50
	B	14	13	19	15	60
	C	19	20	23	M	50
	D	M	M	M	M	50
Demand		50	70	30	60	210

Transportation Simplex Method

- (General) Simplex Method
  - Simplex tableau (with artificial variables introduced)
    - ✓ z_i and z_{m+j} are artificial variables.

	BV	Z	⋯	x_{ij}	⋯	z_i	⋯	z_{m+j}	⋯	RHS
	Z	-1		c_{ij}		M		M		0
	⋮	⋮								
	z_i	0		1		1				s_i
	⋮	⋮								
	z_{m+j}	0		1				1		d_j
	⋮	⋮								
(at any iteration)	Z	-1		$c_{ij} - u_i - v_j$		$M - u_i$		$M - v_j$		$-\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j$

Transportation Simplex Method

- (General) Simplex Method
  - Simplex tableau (with artificial variables introduced)
    - ✓ For a NBV x_{ij}
      - u_i: the multiple of row i that has been subtracted from row 0 to form the current simplex tableau.
      - v_j: the multiple of row $m+j$ that has been subtracted from row 0 to form the current simplex tableau.
    - ✓ For a BV x_{ij}: $c_{ij} - u_i - v_j = 0$

BV	Z	⋯	x_{ij}	⋯	z_i	⋯	z_{m+j}	⋯	RHS
Z	-1		c_{ij}		M		M		0
⋮	⋮								
z_i	0		1		1				s_i
⋮	⋮								
z_{m+j}	0		1				1		d_j
⋮	⋮								
(at any iteration) Z	-1		$c_{ij} - u_i - v_j$		$M - u_i$		$M - v_j$		$-\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j$

Transportation Simplex Method

- (General) Simplex Method
  - Initialization
    - ✓ Objective: To obtain an initial BFS that serves as the starting point for the Simplex method
    - ✓ In a general LP, the number of BVs is m+n, but in the transportation problem, the number of BVs is only m+n-1.
      - This is because, under the balance condition (total supply = total demand), the final equation is automatically satisfied.
    - ✓ To construct an initial BFS, select m+n-1 variables and assign values so that each chosen variable satisfies one of the constraints.

Transportation Simplex Method

- General Initialization
  - 1. From the current row/column under consideration, select an entering BV according to a chosen rule.
  - 2. Assign as much as possible so that either the remaining supply of that row or the remaining demand of that column is exhausted.
    - ✓ Consume the smaller of the two (remaining supply or demand).
  - 3. Remove the row/column that has been exhausted from further consideration (if both are exhausted simultaneously, remove the row first.)
  - 4. When only one row or column remains, allocate the remaining feasible assignments, select all associated variables, and terminate the initialization process.

Transportation Simplex Method

- Transportation Simplex Method
  - Northwest Corner Method
    - ✓ Start with x_{11}.
    - ✓ If the current source i still has remaining supply, then proceed to select $x_{i,j+1}$.
    - ✓ Otherwise, move down and select $x_{i+1,j}$.

Transportation Simplex Method

- Transportation Simplex Method
  - Vogel's Approximation Method
    - ✓ For each row and each column, calculate the difference between the smallest unit shipping cost c_{ij} and the next smallest c_{ij}.
      - If they are the same, the difference is taken as 0.
    - ✓ Select the row or column with the largest difference.
      - Within that row or column, choose the variable x_{ij} corresponding to the smallest c_{ij}.
      - If there is a tie (multiple choices with the same value), select the one randomly.

Transportation Simplex Method

- Transportation Simplex Method
  - 러셀의 근사법(Russel's Approximation Method)
    - ✓ 남은 source i에 대해 $\bar{u}_i$ (남아있는 가장 큰 단위비용 c_{ij})
    - ✓ 남은 destination j에 대해 $\bar{v}_j$ (남아있는 가장 큰 단위비용 c_{ij})
    - ✓ 아직 선택되지 않은 x_{ij}에 대해 $\Delta_{ij} = c_{ij} - \bar{u}_i - \bar{v}_j$ 계산
    - ✓ 가장 작은 음수 Δ_{ij} 선택(같은 값은 임의 선택)

Transportation Simplex Method

- Transportation Simplex Method
  - Comparison of three initialization alternatives
    - ✓ Northwest Corner Method
      - Fast and simple
      - Does not consider the shipping cost c_{ij}.
      - Likely to produce an initial BFS that is far from optimal, which may increase the number of iterations needed later.
    - ✓ Vogel's Approximation Method
      - Generally requires fewer iterations than the Northwest Corner Method.
      - Still relatively easy to apply by hand.
    - ✓ Russel's Approximation Method
      - Tends to find even better initial BFS compared to Vogel's method.
      - More complex to compute manually, but well-suited for computer-based implementation.

Transportation Simplex Method

- Transportation Simplex Method
  - Optimality Test
    - ✓ A BFS is optimal if and only if for every NBV x_{ij}, $c_{ij} - u_i - v_j \geq 0$ holds.
      - To perform this test, derive the values of u_i and v_j for the current BFS and compute $c_{ij} - u_i - v_j$.
      - To determine u_i, v_j, use the equality $c_{ij} - u_i - v_j = 0$ that must holds for every BV x_{ij}.
    - ✓ Since there are $m + n - 1$ BVs and $m + n$ unknowns (m values of u_i and n values of v_j), one of them can be assigned arbitrarily.
      - A convenient choice is to set $u_i = 0$ for the row with the most allocations.
  - Example: Performing optimality test starting from Russel's initial solution:

Transportation Simplex Method

- Transportation Simplex Method
  - Iterations
    - ✓ 1. Determining the entering BV: For an NBV x_{ij}, $c_{ij} - u_i - v_j$ indicates the change in the objective function Z if x_{ij} increases.
      - To reduce the total cost Z, choose the x_{ij} with the most negative value $c_{ij} - u_i - v_j$.
    - ✓ 2. Determining the leaving BV: Increasing the chosen NBV triggers a chain reaction due to the constraints.
      - Since this chain reaction occurs by the same amount throughout, the leaving BV is the donor cell x_{ij} with the smallest allocation.
    - ✓ 3. Find the new BFS
      - Update allocations by adding and subtracting the leaving BV's value along the chain (recipient/donor cells).
      - The total cost change is $\Delta Z = (\text{value of the leaving BV}) \times (c_{ij} - u_i - v_j \text{ of the entering BV})$

Transportation Simplex Method

- Transportation Simplex Method
  - Initialization: Use one of the initialization methods such as the Northwest Corner Method, Vogel's Approximation method, Russel's Approximation method.
  - Optimality Test:
    - ✓ Set $u_i = 0$ for the row with the largest number of allocations.
    - ✓ For each BV x_{ij}, use the condition $c_{ij} - u_i - v_j = 0$ to recursively determine the values of u_i, v_j.
    - ✓ For each NBV x_{ij}, if $c_{ij} - u_i - v_j \geq 0$, the solution is optimal. If any negative values exist, proceed to iterations.
  - Iterations:
    - ✓ Determine the entering BV: Choose the NBV x_{ij} with the most negative value of $c_{ij} - u_i - v_j$.
    - ✓ Determine the leaving BV: Increasing the entering BV triggers a chain reaction. Identify the donor cells in this cycle, and select the BV with the smallest allocation as the leaving BV.
    - ✓ Compute the new BFS: Adjust allocations by adding/subtracting the leaving BV's value across recipient/donor cells.

Transportation Simplex Method

- Transportation Simplex Method
  - Special characteristics of the example
    - ✓ 1. Since the BV $x_{31} = 0$, the initial BFS is degeneracy. In this example, x_{31} increases in a later iterations.
    - ✓ 2. Both x_{21}, x_{34} simultaneously reach 0; therefore, one of them is arbitrarily chosen as the leaving BV, while the other becomes degeneracy.
    - ✓ 3. If $c_{ij} - u_i - v_j \geq 0$, the solution is optimal. However, due to (2), two alternative optimal solutions occur.

Transportation Simplex Method

- Transportation Simplex Method
  - Solve the example to the optimal point:

Assignment Problems

Assignment Problem

- Overview of the Assignment Problem
  - An LP problem where assignees are assigned to tasks.
  - Characteristics
    - ✓ 1. The number of assignees equals the number of tasks. (n)
    - ✓ 2. Each assignee must be assigned to exactly one task.
    - ✓ 3. Each task must be performed by exactly one assignee.
    - ✓ 4. The cost c_{ij} is given for assignee i performing task j.
    - ✓ 5. The objective is to minimize the total cost.

Assignment Problem

- Prototype Example
  - There are three machines (1, 2, 3) and four locations (1, 2, 3, 4) available for installation.
    - ✓ Constraint: Machine 2 cannot be installed at Location 2.

		Location			
		1	2	3	4
Machine	1	13	16	12	11
	2	15	-	13	20
	3	5	7	10	6

Assignment Problem

- Prototype Example
  - There are three machines (1, 2, 3) and four possible locations (1, 2, 3, 4).
    - ✓ However, machine 2 cannot be installed at location 2. This restriction is modeled using the Big-M method (assigning a very large penalty cost).
  - Just as in the transportation problem, the balance condition requires total supply = total demand, in the assignment problem (a special case of the transportation problem) the number of assignees and the number of tasks must be equal.

		Location			
		1	2	3	4
Machine	1	13	16	12	11
	2	15	M	13	20
	3	5	7	10	6
	4(D)	0	0	0	0

Assignment Problem

- Mathematical Model
  - $x_{ij} = 1$ if assignee i performs task j; otherwise, $x_{ij} = 0$.
    - ✓ With the balance assumption (same number of assignees and tasks), $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, n\}$
  - Objective: Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$
    - ✓ Constraints:
      - $\sum_{j=1}^{n} x_{ij} = 1, \quad \forall i \in \{1, 2, \dots, n\}$
      - $\sum_{i=1}^{n} x_{ij} = 1, \quad \forall j \in \{1, 2, \dots, n\}$
      - $x_{ij} \geq 0$ (Binary variables) $\forall i, j \in \{1, 2, \dots, n\}$
  - Relation to the transportation model (special case)
    - ✓ Interpretable as a transportation problem with # sources = # destinations = n, where $s_i = 1$ for all $i = 1, \dots, n$ & $d_j = 1$ for all $j = 1, \dots, n$.

Assignment Problem

- Solution Methods for the Assignment Problem
  - General Simplex Method (without explicit integer constraints)
  - Transportation Simplex Method
- Applying the Transportation Simplex Method
  - The number of BVs is $m + n - 1$ → in the assignment, this becomes $2n - 1$.
    - ✓ However, in the assignment problem, exactly n variables take the value 1, and the remaining $n - 1$ BVs are degenerate (value 0).
  - Therefore, the transportation simplex method does not fully exploit the special structure of the assignment problem.

Assignment Problem

- Approach via Transportation/Assignment Models
  - We need to produce four products (1,2, 3, 4) using three plants (1, 2, 3).
    - ✓ Constraint: plant 2 cannot produce product 3.

Unit production cost		Product				Supply
		1	2	3	4	
Source	1	41	27	28	24	75
	2	40	29	-	23	75
	3	37	30	27	21	45
Demand		20	30	30	40	

Assignment Problem

- Approach via Transportation/Assignment Models
  - We need to produce four products (1,2, 3, 4) using three plants (1, 2, 3).
    - ✓ Constraint: plant 2 cannot produce product 3.

Unit production cost		Product				Supply
		1	2	3	4	
Source	1	41	27	28	24	75
	2	40	29	-	23	75
	3	37	30	27	21	45
Demand		20	30	30	40	

    - ✓ If the production of a single product can be split across multiple plants, model it as a transportation problem.
      - Optimal solution in transportation formulation: $x_{12} = 30, x_{13} = 30, x_{15} = 15, x_{24} = 15, x_{25} = 60, x_{31} = 20, x_{34} = 25$ → $Z = 3260$

Assignment Problem

- Approach via Transportation/Assignment Models
  - We need to produce four products (1,2, 3, 4) using three plants (1, 2, 3).
    - ✓ Constraint: plant 2 cannot produce product 3.

단위 생산 비용		Product				
		1	2	3	4	5(D)
Assignee	1a	820	810	840	960	0
	1b	820	810	840	960	0
	2a	800	870	M	920	0
	2b	800	870	M	920	0
	3	740	900	810	840	M

- ✓ If a product's production cannot be split across multiple plants → Assignment problem
  - Optimal solution: $x_{12} = 1, x_{13} = 1, x_{21} = 1, x_{34} = 1$ → $Z = 3290$

Algorithm for Assignment Problem

- Role of Equivalent Cost Tables
  - An <u>equivalent cost table</u> is a transformed version of the original cost table where all final entries are either 0 or positive.
    - ✓ every assignment can be made only at positions where the cost value is 0.
  - Key points
    - ✓ How it works
      - By adding or subtracting constants to entire rows or columns, we can transform the table.
      - These transformations do not change the actual optimal assignment or the total cost. They only simplify the problem.
    - ✓ Interpretation
      - The 0's in the equivalent cost table represent feasible positions for assignments.
      - As long as the assignment follows the one-to-one rule (each person → one task, each task → one person), the solution obtained is equivalent to the original problem.

Algorithm for Assignment Problem

- Equivalent Cost Table
  - Example.

	1	2	3	4
1	13	16	12	11
2	15	M	13	20
3	5	7	10	6
4(D)	0	0	0	0

	1	2	3	4
1	2	5	1	0
2	15	M	13	20
3	5	7	10	6
4(D)	0	0	0	0

	1	2	3	4
1	2	5	1	0
2	2	M	0	7
3	0	2	5	1
4(D)	0	0	0	0

✓ In the original table, if you sum the costs at the assigned positions, you get $Z = 11 + 13 + 5 + 0 = 29$

Algorithm for Assignment Problem

- Equivalent Cost Table
  - Example.
    - ✓ Since every row already contains a zero (except the last row), subtract the minimum of each column from that column.

	1	2	3	4	5(D)
1a	820	810	840	960	0
1b	820	810	840	960	0
2a	800	870	M	920	0
2b	800	870	M	920	0
3	740	900	810	840	M

	1	2	3	4	5(D)
1a	80	0	30	120	0
1b	80	0	30	120	0
2a	60	60	M	80	0
2b	60	60	M	80	0
3	0	90	0	0	M

    - ✓ Even after transformation, a complete assignment using only the zero positions is still not possible.변형 후에도 여전히 0의 위치만을 이용한 온전한 할당해 도출 불가능

Algorithm for Assignment Problem

- Equivalent Cost Table: Generating Additional Zeros
  - Example.
    - ✓ Draw lines across rows/columns to cover all zeros with the minimum number of lines.

	1	2	3	4	5(D)
1a	820	810	840	960	0
1b	820	810	840	960	0
2a	800	870	M	920	0
2b	800	870	M	920	0
3	740	900	810	840	M

→

	1	2	3	4	5(D)
1a	80	0	30	120	0
1b	80	0	30	120	0
2a	60	60	M	80	0
2b	60	60	M	80	0
3	0	90	0	0	M

→

	1	2	3	4	5(D)
1a	50	0	0	90	0
1b	50	0	0	90	0
2a	30	60	M	50	0
2b	30	60	M	50	0
3	0	120	0	0	M

    - ✓ Smallest remaining value =30 (choose one from the two belows)
      - Method 1. Subtract 30 from all entries, then restore the positions of zeros, and add 30 to each line (adding 60 at intersections).
      - Method 2. Subtract 30 from the entries not covered by any line, add 30 to the entries at the intersections of lines, and leave the other covered entries unchanged.

Algorithm for Assignment Problem

- Equivalent Cost Table: Generating Additional Zeros
  - Example.
    - ✓ Once an assignment using only zero positions becomes possible, derive the optimal assignment solution:
      - $Z = 810 + 840 + 800 + 0 + 840 = 3290$

	1	2	3	4	5(D)
1a	50	0	0	90	0
1b	50	0	0	90	0
2a	30	60	M	50	0
2b	30	60	M	50	0
3	0	120	0	0	M

→

	1	2	3	4	5(D)
1a	50	0	0	90	30
1b	50	0	0	90	30
2a	0	30	M	20	0
2b	0	30	M	20	0
3	0	120	0	0	M

Algorithm for Assignment Problem

- Hungarian Algorithm
  - 1. Subtract the minimum value in each row (row reduction)
  - 2. Subtract the minimum value in each column (column reduction)
  - 3. Check for optimality: if the minimum number of lines needed to cover all zeros equals the number of rows (or columns) n, then an optimal solution can be found.
  - 4. If the number of lines is less than n: (a) Subtract the smallest uncovered value from all uncovered elements, (b) Add this value to the elements at the intersections of the lines, (c) Leave all other elements unchanged.
  - 5. Repeat steps 3 and 4 until an optimal solution found.
  - 6. Make the assignment using only zero positions (start with rows/columns that have exactly one zero).

Q&A