Solving Linear Programming Problems: The Simplex Method

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The Simplex Method

- Overview: Core Idea of the Simplex Method
  - Location of the optimal solution
    - ✓ The optimal solution of a linear programming lies at the one of the corner-point feasible (CPF) solutions in the feasible region.
  - Movement strategy
    - ✓ The Simplex method starts from one basic feasible solution (BFS) and moves along an edge to an adjacent corner point where the objective function improves.
    - ✓ This adjacency holds when two corner points share a common boundary formed by the intersection of constraints
  - Iterative improvement
    - ✓ At each step, the entering variable and leaving variable are determined, and a pivot operation is performed to move to a new BFS (CPF solution).
    - ✓ When no further improvement is possible, the optimal solution is reached.

The Simplex Method

- Overview: Core Idea of the Simplex Method
  - Why do we only need to check the corner points?
  - Because the objective function is linear and the feasible region is a convex polyhedron, increasing the value in one direction will eventually hit a boundary and reach a maximum or minimum at a corner point.
    - ✓ Therefore, it is unnecessary to search every interior point; comparing the corner points alone is sufficient to find the global optimum.

The Simplex Method

- The Prototype Example
  - The corner-points of the feasible region are
    - ✓ (0, 0), (0, 6), (2, 6), (4, 3), (4, 0)
  - If we start at (0, 0), we can move to (4, 0) by increasing $x1$, or to (0, 6) by increasing $x2$.
  - The Simplex method changes only one variable at a time, moving to an adjacent corner point, and chooses the direction that increases the value of Z.
  - For example, moving from (0, 6) to (2, 6) increases Z from 30 to 36.

Maximize $Z = 3x1 + 5x2$
(s.t.)
$x1 \leq 4$
$x2 \leq 6$
$3x1 + 2x2 \leq 18$
$x1, x2 \geq 0$

The Simplex Method

- Need for Algebraic Formulation
  - The graphical method is intuitive, but it cannot be used to solve problems visually once there are more than three variables.
  - Instead, expressing an LP in matrix/algebraic form makes it possible to apply systematic computational methods.

The Simplex Method

- Algebraic Formulation: Slack Variable
  - Definition.
    - ✓ A non-negative variable needed to convert a "≤" constraint into an equality.
  - It represents the amount of unused resources.
  - Example
    - ✓ $x_1 + x_2 \leq 100 \quad \Rightarrow \quad x_1 + x_2 + s_1 = 100, \;\; s_1 \geq 0$
    - ✓ If the maximum production of two products is 100 units, then s_1 represents the shortfall from 100 (i.e., the unused production capacity).

The Simplex Method

- Algebraic Formulation: Basic and Non-Basic Variables
  - Basic Variables (BVs)
    - ✓ The m variables chosen to satisfy the m equations.
    - ✓ These variables directly determine the solution to the constraints.
  - Non-Basic Variables (NBVs)
      - The remaining variables, which are set to 0.
    - ✓ Basic solution
      - A solution obtained by setting the non-basic variables to 0 and solving the remaining m equations.
    - ✓ Basic Feasible Solution (BFS)
      - A basic solution in which all variables satisfy $x_j \geq 0$
  - Relationships between BFS and CPF solution
    - ✓ One-to-one correspondence: BFSs correspond exactly to CPF solutions.
    - ✓ Geometrically, they are the intersections of constraints; algebraically, they are the solutions obtained by solving with m variables.

The Simplex Method

- Slack Variables and the Initial BFS in the Simplex Method
  - In the initial BFS, all slack variables are set as the basic variables.
  - A decrease in their values indicates that resources are being consumed.
- Characteristics
  - ✓ Always $s \geq 0$
  - ✓ The value represents the remaining amount of the corresponding resource.
  - ✓ If the value is 0, it means that resource is fully used.
  - Example
    - ✓ x1 + x2 + s1 = 4
    - ✓ 2x1 + x2 + s2 = 6
    - ✓ ➔ Initial BFS: x1 = 0, x2 = 0, s1= 4, s2 = 6

The Simplex Method

- Pivot Operations – Determining the Entering and Leaving Variables
  - Meaning of pivot
    - ✓ A pivot is the operation of removing one variable from the current set of BVs and bringing in another variable to move to a new BFS.
    - ✓ Geometrically, this corresponds to moving along an edge from one corner point of the feasible region to an adjacent corner point.
  - How to select entering variable
    - ✓ Compare the objective function coefficients with the current contributions
    - ✓ In a maximization problem, choose the variable with the largest positive coefficient (the variable with the greatest potential improvement).
  - How to select leaving variable (Minimum Ratio Test)
    - ✓ When the entering variable increases by 1 unit, determine the maximum allowable increase without violating any constraints.
    - ✓ For each constraint, compute (current RHS value / coefficient of the entering variable), considering only positive coefficients.
    - ✓ The variable corresponding to the smallest ratio becomes the leaving variable (the first one to drop to zero).

The Simplex Method

- Pivot Operations – Determining the Entering and Leaving Variables
  - Pivot procedure
    - ✓ Select the pivot element: the position at the intersection of the entering variable’s column and the leaving variable’s row.
    - ✓ Divide the pivot row by the pivot element so that it becomes 1.
    - ✓ Adjust the other rows to make all other entries in the pivot column equal to 0.
    - ✓ Apply the same adjustments to the objective function row (Z-row).
  - Intuitive understanding
    - ✓ Entering: represents using a new resource or starting a new activity.
    - ✓ Leaving: represents depleting a resource or stopping another activity.
    - ✓ This process always keeps the number of BVs unchanged while moving from one solution to another.