Lecture 6. Discrete Time Markov Chain (1)

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Markov Chain - Motivation

- I drink a bottle of soda everyday. I drink either Coke or Pepsi everyday. When I choose what to drink for today, I only consider what I drank yesterday.

- Suppose I drank Coke yesterday, then chance of drinking Coke again today is 0.7. (What is chance of drinking Pepsi today then?)

- Suppose I drank Pepsi yesterday, then chance of drinking Pepsi again today is 0.5. (What is chance of drinking Coke today then?)

- In a tabular form?

- How would you describe this situation in diagram?

- How would you represent this situation to mathematical form?

- Suppose I do this for an year. Which brand of soda I will drink more? (Pepsi or Coke?)
- If I drink Coke today, what is the probability that I will be drinking Pepsi on three days later?

- If I do this for 10 years (3650 days) from now, then how many days I will drink Pepsi?
- Suppose Pepsi is \$1 and Coke is \$1.5. How much on average I spend on soda in a month?
- Suppose that there are a billion customers (who has the same type of consumption pattern) like me in the world. You are working for Pepsi's marketing department and wish to boost Pepsi $\rightarrow$ Pepsi probability from 0.5 to 0.6. On average, how much revenue will be additionally generated by this marketing effort for a day?

Definitions

Definition - Stochastic Process

- Stochastic Process: Time + Random
    - Discrete Time Stochastic Process
        - Discrete Time + Random
        - $\{X_n : n \geq 0,\ n \in \mathbb{N}\}$
    - Continuous Time Stochastic Process
        - Continuous Time + Random
        - $\{X_t : t \geq 0,\ t \in \mathbb{R}^+\}$

Definitions - State and State space

- State: value of X_n.
- State space (S): a set of all possible values that X_n can take.

Definitions

Definitions - Markov Property

- The nearest future only depends on the present.
- For a discrete time stochastic process $\{X_n : n \geq 0,\ n \in \mathbb{N}\}$,
  - X_{n+1} depends only on the state of X_n.
  - X_{n+1} is function of X_n and some randomness.
  - $\mathbb{P}(X_{n+1} = j | X_0 = i_0, X_1 = i_1, ..., X_n = i) = \mathbb{P}(X_{n+1} = j | X_n = i)$

Definition - DTMC

- A discrete time stochastic process with Markov Property is said to be a DTMC.

Definition - Transition probability (matrix)

- **Transition probability** p_{ij} is the probability that governs "transition" within the state space.
  - $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_n = j | X_{n-1} = i) = \mathbb{P}(X_1 = j | X_0 = i)$
- **Transition probability matrix P** is collection of p_{ij}
  - $\mathbf{P} = [p_{ij}]$

Definition - Initial Distribution

- The information of where the chain starts at time 0.
- $a_0 :=$ distribution of X_0 as a row vector.

Fomularize Coke & Pepsi MC

- State Space

- Transition probability Matrix

- Transition Diagram

- Initial Distribution

1-step transition

- Suppose $\mathbb{P}(X_0 = c) = 0.6$ and $\mathbb{P}(X_0 = p) = 0.4$, then what is $\mathbb{P}(X_1 = c)$?

Math Review 14 - Probabilities - properties (Revisited)

- $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$, „ if $E_1 \cap E_2 = \emptyset$.
- $\mathbb{P}(E) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c)$ „ $\because (E \cap F) \cap (E \cap F^c) = \emptyset$
- If $F_1 \cap F_2 \cap \cdots \cap F_n = \emptyset$ and $\mathbb{P}(F_1 \cup \cdots \cup F_n) = 1$, then $\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \cap F_i)$

Math Review 15 - Bayes' rule (Revisited)

Definition - Bayes' rule

- Suppose $F_1, \ldots, F_n$ are mutually exclusive events
  - $F_i \cap F_j = \emptyset$, or, $\mathbb{P}(F_i \cap F_j) = 0$, for any $i \neq j$
  - $\mathbb{P}(F_1 \cup \cdots \cup F_n) = 1$
- In other words, exactly one of the events $F_1, \ldots, F_n$ will occur.
- Then, the following holds.

$$
\begin{aligned}
\mathbb{P}(E) &= \mathbb{P}(E \cap F_1) + \cdots + \mathbb{P}(E \cap F_n) \\
&= \sum_{i=1}^{n} \mathbb{P}(E \cap F_i) \\
&= \sum_{i=1}^{n} \mathbb{P}(E | F_i)\mathbb{P}(F_i)
\end{aligned}
$$

2-step transition

- Suppose $\mathbb{P}(X_0 = c) = 0.6$ and $\mathbb{P}(X_0 = p) = 0.4$, then what is $\mathbb{P}(X_2 = c)$?

Transitions in DTMC

2-step transition

- Suppose $X_0 = p$, then what is $\mathbb{P}(X_2 = p)$?
- (Or, equivalently, what is $\mathbb{P}(X_2 = p | X_0 = p)$)?

2-step transition

- Suppose $X_0 = p$, then what is $\mathbb{P}(X_2 = p)$?
- (Or, equivalently, what is $\mathbb{P}(X_2 = p | X_0 = p)$)?

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"If I only had an hour to chop down a tree, I would spend the first 45 minutes
sharpening my axe. - A. Lincoln"
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