Lecture 09. Poisson Process

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Math Review - Exponential distribution AGAIN?!

- pdf, cdf $\lambda e^{-\lambda x}$, $1-e^{-\lambda x}$ $(x \geq 0)$

- $\mathbb{E}X, Var(X), c_X, c_X^2$ $C_X = \frac{sd(X)}{\mathbb{E}X} = 1$
  $\frac{1}{\lambda}$ $\frac{1}{\lambda^2}$

- Memoryless property $\mathbb{P}(X > s+t \mid X > t) = \mathbb{P}(X > s)$ for all $s, t \geq 0$

- $X_1 \sim exp(\lambda_1), X_2 \sim exp(\lambda_2)$, and they are indep.
  $\Rightarrow \mathbb{P}(X_1 < X_2) = ?$ $\frac{\lambda_1}{\lambda_1+\lambda_2}$

  ex) X_1: lifetime of #1 bulb
  X_2: liftime of #2 bulb
  $\mathbb{P}(X_1 < X_2)$: prob. #1 broken before #2
  $min(X_1, X_2)$: time until one of bulbs broken

- [NEW] $X_1 \sim exp(\lambda_1), X_2 \sim exp(\lambda_2)$, and they are indep.
  $\Rightarrow min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$

Examples

- Two servers, A and B, with service times following $X_A \sim \exp(\lambda_A)$ and $X_B \sim \exp(\lambda_B)$. Their service times are independent.
  - The time for one customer finishes? $\min(X_A, X_B) \sim \exp(\lambda_A + \lambda_B)$
  - If $X_A \sim \exp(1/2)$, $X_B \sim \exp(1/2)$ then $\mathbb{E}[min(X_A, X_B)]$?

    $\lambda_A = \frac{1}{2}$, $\lambda_B = \frac{1}{2}$ $\quad \mathbb{E}\left[\exp\left(\frac{1}{2}+\frac{1}{2}\right)\right] = \frac{1}{\lambda} = \frac{1}{\frac{1}{2}+\frac{1}{2}} = 1$ minutes
  - If $X_A \sim \exp(1/2)$, $X_B \sim \exp(1/4)$ then $\mathbb{E}[min(X_A, X_B)]$?

    $\lambda_A = \frac{1}{2}$, $\lambda_B = \frac{1}{4}$ $\quad \mathbb{E}\left[\exp\left(\frac{1}{2}+\frac{1}{4}\right)\right] = \frac{1}{\lambda} = \frac{1}{\frac{1}{2}+\frac{1}{4}} = \frac{4}{3}$ minutes

- Two servers, A, B with $X_A \sim \exp(\lambda_A)$, $X_B \sim \exp(\lambda_B)$ serving John and Paul, respectively. Their service times are independent.
  - On average, how long does it take to clear the system? $\left(\because -\max(i, j) = \min(-i, -j) \right)$

$\mathbb{E}[\max(X_A, X_B)] = \mathbb{E}[-\min(-X_A, -X_B)] = -\mathbb{E}[\min(-X_A, -X_B)]$?

$X_A + X_B = \min(X_A, X_B) + \max(X_A, X_B)$

$\Rightarrow \mathbb{E}X_A + \mathbb{E}X_B = \mathbb{E}[\min(X_A, X_B)] + \mathbb{E}[\max(X_A, X_B)]$

$\Rightarrow \frac{1}{\lambda_A} + \frac{1}{\lambda_B} = \frac{1}{\lambda_A + \lambda_B} + \mathbb{E}[\max(X_A, X_B)]$

$\Rightarrow \mathbb{E}[\max(X_A, X_B)] = \frac{1}{\lambda_A} + \frac{1}{\lambda_B} - \frac{1}{\lambda_A + \lambda_B}$

- In a call center for Amazon, there are 100 incoming calls per minute on average. Answer the followings in most naive and plain but intuitive way.
  - $\mathbb{E}$[# of calls bet'n 9:00 and 9:01]? 100 $\mathbb{E}[N(1)] = 100$
  - $\mathbb{E}$[# of calls bet'n 9:05 and 9:08]? 300 $\mathbb{E}[N(8)-N(5)] = 300$
  - Suppose 150 calls bet'n 9:00 and 9:02, $\mathbb{E}$[# of calls bet'n 9:03 and 9:05]? 200

    $\mathbb{E}[N(5)-N(3) \mid N(2)=150] = \mathbb{E}[N(5)-N(3)] = 200$
  - Suppose 800 calls bet'n 9:00 and 9:10, $\mathbb{E}$[# of calls bet'n 9:05 and 9:10]? $800 \times \frac{5}{10} = 400$

    $\mathbb{E}[N(10)-N(5) \mid N(10)=800] = 400$
  - Suppose 900 calls bet'n 9:10 and 9:20, $\mathbb{E}$[# of calls bet'n 9:18 and 9:20]? $900 \times \frac{2}{10} = 180$

    $\mathbb{E}[N(20)-N(18) \mid N(20)-N(10)=900] = 180$
  - Suppose 850 calls bet'n 9:00 and 9:10, $\mathbb{E}$[# of calls bet'n 9:08 and 9:12]? $850 \times \frac{2}{10} + 100 \times 2 = 370$

$$
\begin{aligned}
&\mathbb{E}[N(12)-N(8) \mid N(10)=850] \\
&= \mathbb{E}[N(12)-N(10)+N(10)-N(8) \mid N(10)=850] \\
&= \mathbb{E}[N(12)-N(10) \mid N(10)=850] + \mathbb{E}[N(10)-N(8) \mid N(10)=850] \\
&= \mathbb{E}[N(12)-N(10)] + \mathbb{E}[N(10)-N(8) \mid N(10)=850] = 200+170 = 370
\end{aligned}
$$

counting process

- Let $t = 0$ at 9:00, $t = 1$ at 9:01,… and consider a function $N(t)$ that counts # of total calls since $t = 0$.
- Go back to previous slide and express in terms of $N(\cdot)$.
- Observations on $N(t)$ in common sense. (not PP yet!)
  - $N(0) = 0$
  - $N(t)$ is non-decreasing function
  - $N(t)$ has a nonnegative value.
  - For $t_1 \le t_2 \le t_3 \le t_4$, $N(t_2) - N(t_1)$ is indep. of $N(t_4) - N(t_3)$
  - For $t_1 \le t_2$, $\mathbb{E}[N(t_2) - N(t_1)] = \mathbb{E}[N(t_2 - t_1)]$

↳ non-overlapping intervals

Math Review - Poisson distribution

Definition - Poisson distribution

- A discrete random variable X is said to follow **Poisson distribution** with parameter λ, and write $X \sim Poi(\lambda)$, if pmf

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for } k = 0, 1, 2, ...$$

- What is cdf of $Poi(\lambda)$? $\mathbb{P}(X \leq k) = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!}$, for $k = 0, 1, 2, \cdots$
- $\mathbb{E}X = Var(X) = \lambda$.

Definition

Definition - Poisson process

- $N = \{N(t), t \geq 0\}$ is said to be **counting process** if $N(0) = 0$, and the integer-valued function $N(t)$ is non-decreasing.
- A counting process $N = \{N(t), t \geq 0\}$ is said to be a **Poisson process** with rate λ, and write $PP(\lambda)$ if
  1. $N(t_2) - N(t_1) \sim poi(\lambda(t_2 - t_1))$ for any $0 \leq t_1 \leq t_2$ ⇒ Stationary increment
  2. For $t_1 \leq t_2 \leq t_3 \leq t_4$, $N(t_2) - N(t_1)$ is indep. of $N(t_4) - N(t_3)$ ⇒ independent increment

- Above statements can be understood as:
  1. (Amount of increment follows Poisson dist. with parameter proportional to time length) [handwritten: λ above "parameter", $t_2 - t_1$ above "time length"]
  2. (Increments in non-overlapping intervals are independent)

Example for $0 \le t_1 \le t_2$, $N(t_2)-N(t_1) \sim Poi(\lambda(t_2-t_1))$

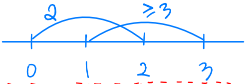
$$\frac{\lambda^k e^{-\lambda}}{k!}$$

- Suppose N is $PP(\lambda = 2/minutes)$
  1. The probability that there are 4 arrivals in the first 3 minutes.

$$\mathbb{P}[\underbrace{N(3)-N(0)}_{= Poi(2(3-0))}=4] = \mathbb{P}[X=4 \mid X \sim Poi(6)] = \frac{6^4 e^{-6}}{4!}$$

  2. The probability that two arrivals in [0,2] and at least 3 arrivals in [1,3].

$$\mathbb{P}[N(2)-N(0)=2,\ N(3)-N(1) \geq 3]$$



0	2	≥1
1	1	≥2
2	0	≥3

$$= \mathbb{P}[N(1)-N(0)=0,\ N(2)-N(1)=2,\ N(3)-N(2) \geq 1]$$

$\lambda=2$

$$+ \mathbb{P}[N(1)-N(0)=1,\ N(2)-N(1)=1,\ N(3)-N(2) \geq 2]$$

$$+ \mathbb{P}[N(1)-N(0)=2,\ N(2)-N(1)=0,\ N(3)-N(2) \geq 3]$$

$$= \frac{2^0 e^{-2}}{0!} \cdot \frac{2^2 e^{-2}}{2!} \cdot \left(1 - \frac{2^0 e^{-2}}{0!}\right)$$

$$+ \frac{2^1 e^{-2}}{1!} \cdot \frac{2^1 e^{-2}}{1!} \left(1 - \frac{2^0 e^{-2}}{0!} - \frac{2^1 e^{-2}}{1!}\right)$$

$$+ \frac{2^2 e^{-2}}{2!} \cdot \frac{2^0 e^{-2}}{0!} \left(1 - \frac{2^0 e^{-2}}{0!} - \frac{2^1 e^{-2}}{1!} - \frac{2^2 e^{-2}}{2!}\right)$$

non overlapping → independent.

blank

- Suppose N is $PP(\lambda = 2/minutes)$ made in counting domain
  1. The probability that there is no arrival in [0,4]?

     $\mathbb{P}(N(4)-N(0)=0) = \mathbb{P}[X=0 \mid X\sim Poi(2(4-0))] = \frac{8^0 e^{-8}}{0!}$ $\qquad \frac{\lambda^k e^{-\lambda}}{k!}$
  2. The probability that the first arrival takes at least 4 minutes? made in time domain

     $\mathbb{P}(N(4)-N(0)=0) = \frac{8^0 e^{-8}}{0!}$
  3. The probability that the first arrival takes at least t minutes?

     $\mathbb{P}(N(t)-N(0)=0) = \mathbb{P}(X=0 \mid X\sim Poi(2t)) = \frac{2t^0 e^{-2t}}{0!} = e^{-2t}$
  4. What is the distribution of T_1? $T_1 :=$ time to first arrival

     $\mathbb{P}(T_1 > t) = e^{-2t}$

     $\Rightarrow \mathbb{P}(T_1 \leq t) = 1-\mathbb{P}(T_1 > t) = 1-e^{-2t}$

     $\Rightarrow T_1 \sim exp(2)$

Theorem

- Time to first arrival in $PP(\lambda)$ follows $exp(\lambda)$

proof) let T_1 be the time to first arrival

$$\mathbb{P}(T_1 \leq t) = \mathbb{P}(N(t)-N(0) > 0) = 1 - \mathbb{P}(N(t)-N(0) = 0) = 1 - \mathbb{P}(X=0 \mid X \sim Poi(\lambda t))$$

$$= 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = 1 - e^{-\lambda t}$$

$\therefore T_1 \sim exp(\lambda)$

Merging PP

Example

- Consider the north confluence area of I75 and I85.
  - Suppose, for South bound, N_{I75S} is $PP(100/min)$ and N_{I85S} is $PP(200/min)$.
  - Then, at the north confluence point, we have $N_{I75S} + N_{I85S}$, which is $PP(300/min)$ traffic heading South.

Theorem

- Suppose $N_A = \{N_A(t), t \geq 0\}$ is $PP(\lambda_A)$ and $N_B = \{N_B(t), t \geq 0\}$ is $PP(\lambda_B)$, and they are independent. Then $N = \{N(t) = N_A(t) + N_B(t)\}$ is $PP(\lambda_A + \lambda_B)$

Thinning PP

Example

- Consider the north confluence area of I75 and I85.
  - Suppose, for North bound, we have $PP(400/min)$ heading North.
  - Each car heading north chooses I75N with probability 0.4 and I85N with probability 0.6.
  - Then, we have N_{I75N}, which is $PP(160/min)$ and N_{I85N}, which is $PP(240/min)$.

Theorem

- Suppose $N = \{N(t), t \geq 0\}$ is $PP(\lambda)$ and each arrival will choose subset A or B, with probability p and $q = 1 - p$, respectively. Then, $N_A = \{N_A(t), t \geq 0\}$ is $PP(p\lambda)$ and $N_B = \{N_B(t), t \geq 0\}$ is $PP(q\lambda)$

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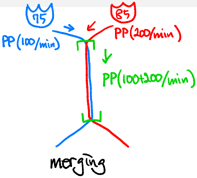
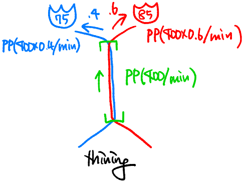


75
85
PP(100/min)
PP(200/min)
PP(100+200/min)
merging



75
.4
.6
85
PP(300×0.4/min)
PP(300×0.6/min)
PP(300/min)
thinning

Criticism and Remedy of Poisson Process

- Criticism: Arrival process may not be *time homogeneous* for systems such as…
  - Call center traffic in the morning vs afternoon.
  - Hospital emergency patients in Friday night vs Tuesday morning
  - Aircraft arrival rate in an airport during holiday vs today

- Remedy: Why don't we make λ to be *time-dependent*? $\lambda \to \lambda(t)$!
  - We call this process "Non-homogeneous Poisson process".
  - KEY function: $\lambda(t)$ as a function for arrival rate that changes over time.
  - We can still use very similar mathematical technique of (Stationary or Homogeneous) Poisson Process!

Review: Definitions of Poisson Process

Definition (ver 1)

- A counting process $N = \{N(t), t \geq 0\}$ is said to be a **Poisson process** with rate λ, and write $PP(\lambda)$ if
  1. $N(0) = 0$
  2. $N(t)$ has independent increment.
  3. $\mathbb{P}(N(t+s) - N(s) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ for $n = 0, 1, 2, ...$

Definition (ver 2)

- A counting process $N = \{N(t), t \geq 0\}$ is said to be a **Poisson process** with rate λ, and write $PP(\lambda)$ if
  1. $N(0) = 0$
  2. $N(t)$ has independent increment.
  3. $N(t+s) - N(s) \sim Poisson(\lambda t)$

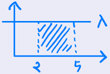
Definition of Non-homogeneous Poisson Process

Definition (ver 2 of PP)

- A counting process $N = \{N(t), t \geq 0\}$ is said to be a *Poisson process* with rate λ, and write $PP(\lambda)$ if
  1. $N(0) = 0$
  2. $N(t)$ has independent increment.
  3. $N(t+s) - N(s) \sim Poisson(\lambda t)$ $= \lambda(t+s) - \lambda(s) = \int_s^{s+t} \lambda \, du$

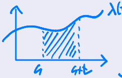
$N(3+2) - N(2) \sim Poi(3\lambda)$

$\therefore 3\lambda = 5\lambda - 2\lambda = \int_2^5 \lambda \, du$



Definition - Non-homogeneous Poisson process

- A counting process $N = \{N(t), t \geq 0\}$ is said to be a Non-homogeneous Poisson process with rate function $\lambda(t), t \geq 0$, and write $NHPP(\lambda(t))$ if
  1. $N(0) = 0$
  2. $N(t)$ has independent increment.
  3. $N(t+s) - N(s) \sim Poisson(\int_s^{t+s} \lambda(u) du)$



$\int_s^{s+t} \lambda(u) \, du$

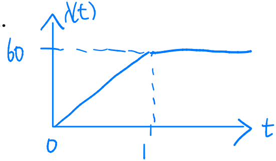
- Remark: PP is special case of NHPP where rate function of NHPP is a constant.

Example

Non-Stationary Poisson Process

- Assume that call arrival to a call center follows a NHPP.
  - The call center opens from 9 am to 5 pm.
  - During the first hour, the arrival rate increases linearly from 0 at 9 am to 60 calls per hour at 10 am.
  - After 10 am, the arrival rate is constant at 60 calls per hour.

(a) Plot the arrival rate function $\lambda(t)$ as a function of time t; indicate clearly the time unit used.



$$\lambda(t) = \begin{cases} 60t & (0 \le t \le 1) \\ 60 & (t > 1) \end{cases}$$

(b) Find the probability that exactly 5 calls have arrived by 9:10am.

$$\mathbb{P}\left[N\left(\tfrac{1}{6}\right) - N(0) = 5\right] = \frac{\left(\frac{5}{6}\right)^5 e^{-5/6}}{5!}$$

$$N\left(\tfrac{1}{6}\right) - N(0) \sim \mathrm{Poi}\left(\int_0^{1/6} \lambda(u)\,du = \int_0^{1/6} 60u\,du = 30u^2\Big|_0^{1/6} = \frac{5}{6}\right)$$

Example

(c) Find the probability that exactly k calls will arrive between 9:50 am to 10:10 am.

$$\mathbb{P}\left[N\left(\tfrac{7}{6}\right)-N\left(\tfrac{5}{6}\right)=k\right]=\mathbb{P}\left[\mathrm{Poi}\left(\int_{5/6}^{7/6}\lambda(u)\,du\right)=k\right]=\mathbb{P}\left[\mathrm{Poi}\left(\int_{\frac{5}{6}}^{1}\lambda(u)\,du+\int_{1}^{7/6}\lambda(u)\,du\right)=k\right]$$

$$=\mathbb{P}\left[\mathrm{Poi}\left(30u^2\Big]_{5/6}^{1}+60u\Big]_{1}^{7/6}\right)=k\right]=\mathbb{P}\left[\mathrm{Poi}\left(\tfrac{115}{6}\right)=k\right]=\frac{\left(\frac{115}{6}\right)^k e^{-(115/6)}}{k!}$$

Exercise 1

- Nortel in Canada operates a call center for customer service. Assume that each caller speaks either English or French, but not both. Suppose that the arrival for each type of calls follows a Poisson process. The arrival rates for English and French calls are 2 and 1 calls per minute, respectively. Assume that call arrivals for different types are independent.
  - *(a)* What is the probability that the 2nd English call will arrive after minute 5?
  - *(b)* Find the probability that, in first 2 minutes, there are at least 2 English calls and exactly 1 French call?
  - *(c)* What is the expected time for the 1st call that can be answered by a bilingual operator (that speaks both English and French) to arrive?
  - *(d)* Suppose that the call center is staffed by bilingual operators that speak both English and French. Let $N(t)$ be the total number of calls of both types that arrive during $(0, t]$. What kind of a process is $N(t)$? What is the mean and variance of $N(t)$?
  - *(e)* What is the probability that no calls arrive in a 10 minute interval?
  - *(f)* What is the probability that at least two calls arrive in a 10 minute interval?
  - *(g)* What is the probability that two calls arrive in the interval 0 to 10 minutes and three calls arrive in the interval from 5 to 15 minutes?
  - *(h)* Given that 4 calls arrived in 10 minutes, what is the probability that all of these calls arrived in the first 5 minutes?
  - *(i)* Find the probability that, in the first 2 minutes, there are at most 1 call, and, in the first 4 minutes, there are exactly 3 calls?

(Solution)

Let N_E denote the Poisson process corresponding to English calls, and let $N_F(\cdot)$ denote the Poisson process corresponding to French calls. Then, N_E is $PP(2/min)$ and N_F is $PP(1/min)$.

(a)

$$\begin{aligned} & \mathbb{P}[N_E(5) - N_E(0) < 2] = \mathbb{P}[Poi(2(5-0)) < 2] \\ = \;& \mathbb{P}[Poi(10) = 0] + \mathbb{P}[Poi(10) = 1] = e^{-10} + 10e^{-10} = 11e^{-10} \end{aligned}$$

(b)

$$\begin{aligned} & \mathbb{P}[N_E(2) - N_E(0) \geq 2, N_F(2) - N_F(0) = 1] \\ = \;& \mathbb{P}[N_E(2) - N_E(0) \geq 2]\mathbb{P}[N_F(2) - N_F(0) = 1] \\ = \;& \mathbb{P}[Poi(2(2-0)) \geq 2]\mathbb{P}[Poi(1(2-0)) = 1] \\ = \;& [1 - \mathbb{P}(Poi(4) = 0) - \mathbb{P}(Poi(4) = 1)][\mathbb{P}(Poi(2) = 0)] \\ = \;& (1 - e^{-4} - 4e^{-4})(2e^{-2}) \\ = \;& (1 - 5e^{-4})(2e^{-2}) \end{aligned}$$

(c) Note that [the arrival time of the first English call] is exponential distribution with parameter 2/min, and [the arrival time of the first French call] is exponential distribution with parameter 1/min. Hence [the arrival time of the first call] is minimum of the two times, which we now know is exponential with parameter 3/min (since the two processes are independent). Thus, the expected arrival time of the first call is just $1/3$ minutes.

(d) $N = N_E + N_F$, and N_E and N_F are independent, so N is $PP((1+2)/min)$ by merging principle. Thus, $N(t) - N(0) \sim Poi(\lambda(t-0)) = Poi(3t)$, and $\mathbb{E}[N(t)] = 3t$ and $Var(N(t)) = 3t$ because $N(0) = 0$ always by the definition of counting process.

(e) The probability that no calls arrive in a 10 minute interval is just $\mathbb{P}(N(10) - N(0) = 0) = \mathbb{P}[Poi(3(10-0)) = 0] = \frac{30^0 e^{-30}}{0!} e^{-30}$

(f)

$$\begin{aligned} & \mathbb{P}(N(10) - N(0) \geq 2) = \mathbb{P}[Poi(3(10-0)) \geq 2] \\ = \quad & 1 - \mathbb{P}[Poi(30) = 0] - \mathbb{P}[Poi(30) = 1] \\ = \quad & 1 - 1e^{-30} - 30e^{-30} = 1 - 31e^{-30} \end{aligned}$$

(g)

$$
\begin{aligned}
& \mathbb{P}(N(10)=2, N(15)-N(5)=3) \\
= \;& \mathbb{P}(N(5)-N(0)=2, N(10)-N(5)=0, N(15)-N(10)=3) \\
& +\mathbb{P}(N(5)-N(0)=1, N(10)-N(5)=1, N(15)-N(10)=2) \\
& +\mathbb{P}(N(5)-N(0)=0, N(10)-N(5)=2, N(15)-N(10)=1) \\
= \;& \mathbb{P}(N(5)-N(0)=2)\mathbb{P}(N(10)-N(5)=0)\mathbb{P}(N(15)-N(10)=3) \\
& +\mathbb{P}(N(5)-N(0)=1)\mathbb{P}(N(10)-N(5)=1)\mathbb{P}(N(15)-N(10)=2) \\
& +\mathbb{P}(N(5)-N(0)=0)\mathbb{P}(N(10)-N(5)=2)\mathbb{P}(N(15)-N(10)=1) \\
= \;& \mathbb{P}(Poi(15)=2)\mathbb{P}(Poi(15)=0)\mathbb{P}(Poi(15)=3) \\
& +\mathbb{P}(Poi(15)=1)\mathbb{P}(Poi(15)=1)\mathbb{P}(Poi(15)=2) \\
& +\mathbb{P}(Poi(15)=0)\mathbb{P}(Poi(15)=2)\mathbb{P}(Poi(15)=1) \\
= \;& (\frac{15^2}{2}e^{-15})(e^{-15})(\frac{15^3}{6}e^{-15}) + (15e^{-15})(15e^{-15})(\frac{15^2}{2}e^{-15}) \\
& +(e^{-15})(\frac{15^2}{2}e^{-15})(15e^{-15})
\end{aligned}
$$

(h) $(5/10)^4$

(i)

$$
\begin{aligned}
& \mathbb{P}(N(2) \leq 1, N(4) = 3) \\
= \ & \mathbb{P}(N(2) = 0, N(4) = 3) + \mathbb{P}(N(2) = 1, N(4) = 3) \\
= \ & \mathbb{P}(N(2) - N(0) = 0, N(4) - N(2) = 3) + \mathbb{P}(N(2) - N(0) = 1, N(4) - \\
= \ & \mathbb{P}(Poi(6) = 0)\mathbb{P}(Poi(6) = 3) + \mathbb{P}(Poi(6) = 1)\mathbb{P}(Poi(6) = 2) \\
= \ & e^{-6}(\frac{6^3}{3!}e^{-6}) + (6e^{-6})(\frac{6^2}{2}e^{-6}) \\
= \ & 6^2e^{-12} + 6^3e^{-12}/2
\end{aligned}
$$

Exercise 2

- Calls to a center follow a Poisson process with rate 120 calls per hour. Each call has probability 1/4 from a male customer. The call center opens at 9am each morning.
    - *(a)* What is the probability that there are no incoming calls in the first 2 minutes?
    - *(b)* What is the probability that there are exactly one calls in the first 2 minutes and exactly three calls from minute 1 to minute 4?
    - *(c)* What is the probability that during the first 2 minutes there are 1 call from male customer and 2 calls from female customers?
    - *(d)* What is the expected elapse of time from 9am until the second female customer arrives?
    - *(e)* What is the probability that the second call arrives after 9:06am?

(Solution)

(a) Let $N(t)$ be the total number of calls by minute t, then N is $PP(2/min)$.
$\mathbb{P}(N(2) = 0) = \mathbb{P}(Poi(4) = 0) = e^{-2\lambda} = e^{-4}$

(b)

$$
\begin{aligned}
&\mathbb{P}(N(2) = 1, N(4) - N(1) = 3) \\
=&\mathbb{P}(N(1) - N(0) = 1, N(2) - N(1) = 0, N(4) - N(2) = 3) \\
&+ \mathbb{P}(N(1) - N(0) = 0, N(2) - N(1) = 1, N(4) - N(2) = 2) \\
=&\lambda e^{-\lambda} \times e^{-\lambda} \times e^{-2\lambda} \frac{(2\lambda)^3}{6} + e^{-\lambda} \times \lambda e^{-\lambda} \times e^{-2\lambda} \frac{(2\lambda)^2}{2} \\
=&\frac{112}{3} e^{-8}
\end{aligned}
$$

(c) By thinning principle, we have that N_M is $PP(0.5/min)$ and N_F is $PP(1.5/min)$. $\mathbb{P}(N_M(2) = 1, N_F(2) = 2) = \mathbb{P}(Poi(1) = 1)\mathbb{P}(Poi(3) = 2) = e^{-1} \times \frac{9}{2}e^{-3} = \frac{9}{2}e^{-4}$

Alternative approach - Not using thinning principle. Let N be the number of calls received in the call center, regardless of the gender of the customer.
$\mathbb{P}(N(2) - N(0) = 3) = \mathbb{P}(Poi(2 \cdot (2-0)) = 3) = \mathbb{P}(Poi(4) = 3) = \frac{e^{-4} \cdot 4^3}{3!}$
Among the three calls, there is only one call from a male customer with a probability of 1/4. $\mathbb{P}(Poi(4) = 3) \times {}_3\mathrm{C}_1 \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 = \frac{\mathrm{e}^{-4} \cdot 4^3}{3!} \times 3 \times \frac{3^2}{4^3} = \frac{9}{2}\mathrm{e}^{-4}$

(d) Let T_i be the interarrival time between the $(i-1)$th female customer and the ith female customer, which follows exponential distribution with rate $\frac{3}{2}$:
$\mathbb{P}[T_1 + T_2] = \mathbb{E}[T_1] + \mathbb{E}[T_2] = 2 \times \frac{1}{1.5} = \frac{4}{3}$ minutes.

(e)

$$\begin{aligned} & \mathbb{P}(N(6) < 2) = \mathbb{P}(N(6) = 0) + \mathbb{P}(N(6) = 1) \\ = \; & \mathbb{P}(Poi(12) = 0) + \mathbb{P}(Poi(12) = 1) \\ = \; & e^{-12} + 12e^{-12} = 13e^{-12} \end{aligned}$$

Exercise 3

- Assume that call arrival to a call center follows a non-homogeneous Poisson process. The call center opens from 9am to 5pm. During the first hour, the arrival rate increases linearly from 0 at 9am to 60 calls per hour at 10am. After 10am, the arrival rate is constant at 60 calls per hour.
  - *(a)* Plot the arrival rate function $\lambda(t)$ as a function of time t; indicate clearly the time unit used.
  - *(b)* Find the probability that exactly 5 calls have arrived by 9:10am.
  - *(c)* What is the probability that the first call arrives after 9:20am?
  - *(d)* What is the probability that there are exactly one call between 11:00am and 11:05am and at least two calls between 11:03am and 11:06am?

(Solution)

(a) If we use time unites of minutes, then the rate function $\lambda(t) = \frac{1}{60}t$ for $0 \leq t \leq 60$, and $\lambda(t) = 1$ for $t > 60$

From now on, N is defined as $NHPP(\lambda(t))$ where $\lambda(t)$ is given at (a).

(b)

$$\mathbb{P}[N(10) - N(0) = 5] = \mathbb{P}[Poi(\int_0^{10} \lambda(t)dt) = 5]$$

$$= \mathbb{P}[Poi(\int_0^{10} \frac{1}{60}tdt) = 5] = \mathbb{P}[Poi(\frac{5}{6}) = 5] = \frac{(\frac{5}{6})^5}{5!}e^{-\frac{5}{6}}$$

(c)

$$\mathbb{P}[N(20) - N(0) = 0] = \mathbb{P}[Poi(\int_0^{20} \lambda(t)dt) = 0]$$

$$= \mathbb{P}[Poi(\int_0^{20} \frac{1}{60}tdt) = 0] = \mathbb{P}[Poi(\frac{10}{3}) = 0] = e^{-\frac{10}{3}}$$

(d)

$$
\begin{aligned}
& \mathbb{P}(N(125)-N(120)=1, N(126)-N(123)\geq 2) \\
= & \mathbb{P}(N(123)-N(120)=0, N(125)-N(123)=1, N(126)-N(125)\geq 1) \\
& +\mathbb{P}(N(123)-N(120)=1, N(125)-N(123)=0, N(126)-N(125)\geq 2) \\
= & \mathbb{P}(N(123)-N(120)=0, N(125)-N(123)=1, N(126)-N(125)\geq 1) \\
& +\mathbb{P}(N(123)-N(120)=1, N(125)-N(123)=0, N(126)-N(125)\geq 2)
\end{aligned}
$$

From above, $N(123)-N(120)\sim Poi(\int_{120}^{123} 1dt)=Poi(3)$, and similarly, we have $N(125)-N(123))\sim Poi(2)$ and $N(126)-N(125))\sim Poi(1)$. Hence, we can write

$$
\begin{aligned}
= & \mathbb{P}(Poi(3)=0)\mathbb{P}(Poi(2)=1)\mathbb{P}(Poi(1)\geq 1) \\
& +\mathbb{P}(Poi(3)=1)\mathbb{P}(Poi(2)=0)\mathbb{P}(Poi(1)\geq 2) \\
= & e^{-3}\cdot 2e^{-2}\cdot(1-e^{-1})+3e^{-3}\cdot e^{-2}\cdot(1-2e^{-1}) \\
= & 5e^{-5}-8e^{-6}
\end{aligned}
$$

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"Success isn't permanent, and failure isn't fatal. - Mike Ditka"
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