Linear Programming

↳ planning

↳ all functions in the model are linear (1st order)

⇒ simplex method

* Example Problem

company { prod #1 (\$3) : 1 hour @ F_1 , 3 hours @ F_3 ⇒ { # prod 1 : x_1 } decision variable

prod #2 (\$5) : 2 hours @ F_2 , 2 hours @ F_3 ⇒ # prod 2 : x_2

↓

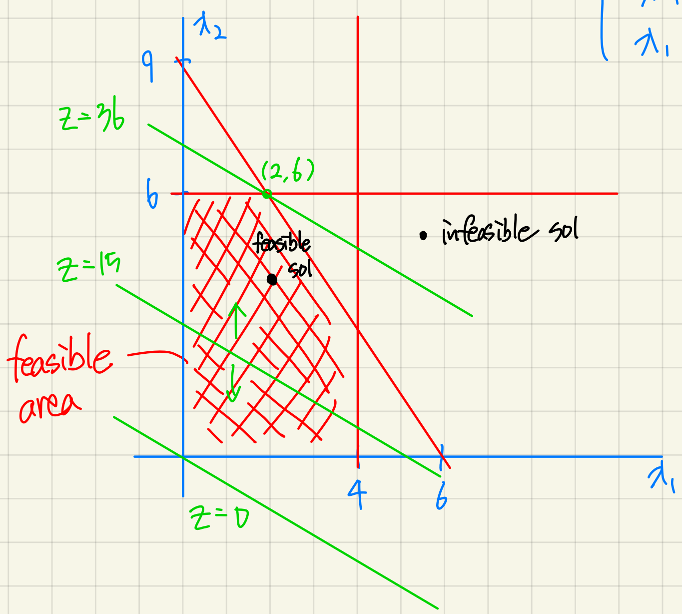
3 Factories

- F_1 4 hours
- F_2 12 hours
- F_3 18 hours

$z = \underbrace{\max(3x_1 + 5x_2)}_{\text{objective function}}$ with constraints

$$\begin{cases} x_1 \le 4 \\ 2x_2 \le 12 \\ 3x_1 + 2x_2 \le 18 \end{cases}$$ functional constraints / structural

$x_1, x_2 \ge 0$. } non-negativity constraints.



Cases
1. one optimal sol
2. multiple optimal sol
3. infeasible
4. unbounded

} No optimal sol

General LP problems

a_{ij} ?

m resources → b_i $(i = 1 \cdots m)$ ⇒ $\sum_j a_{ij} x_j \le b_i$

n activities → x_j $(j = 1 \cdots n)$

obj $Z = \sum_j c_j x_j$, c_j = contribution of unit activity j to obj

$\text{Max } Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

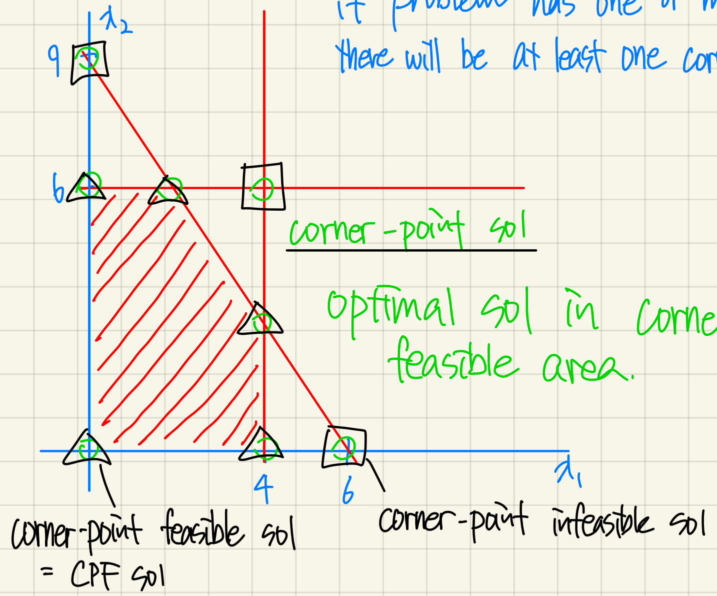
subject to

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \end{cases}$$

$x_1, x_2, \cdots x_n \ge 0$

other form
- Min obj
- = constraints
- ≥ constraints
- x_j can be negative.

if problem has one or more optimal sol,
there will be at least one corner-point optimal sol.



optimal sol in corner-point of feasible area.

* LP assumption.
- proportionality
- additivity
- divisibility : x_1 & x_2 can be real number
- certainty

* Description.

a patient with malicious tumor

two radiations can be used together

hurt both cells

(malicious & normal)

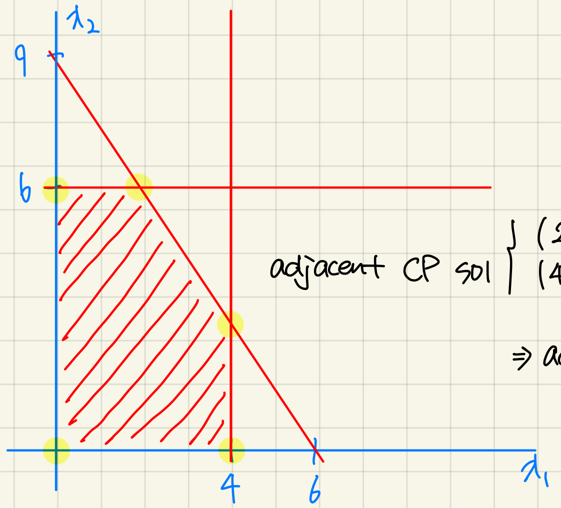
how to mix radiation will change the effect of treatment

	$x_1 \ge 0$ rad ①	$x_2 \ge 0$ rad ②		
normal	4	5		
major organs	3	1	≤ 27	$3x_1 + x_2 \le 27$
whole tumor	5	5	$= 60$	$5x_1 + 5x_2 = 60$
core tumor	6	4	≥ 60	$6x_1 + 4x_2 \ge 60$

Minimize effect on normal cell $Z = 4x_1 + 5x_2$

Max $Z = 3x_1 + 5x_2$

(s.t)
$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1,\ x_2 \ge 0$$



adjacent CP sol $\begin{cases} (2,6) & 3x_1+2x_2=18 \ \& \ 2x_2=12 \\ (4,3) & 3x_1+2x_2=18 \ \& \ x_1=4 \end{cases}$

⇒ adjacent CP sol share n-1 constraints (as equal)
have 1 different constraints (as equal)

n variables → CP sol will satisfy n constraints as equality

$x_1=2$ $S_1=2$ (remaining hours of F_1)

$$\Rightarrow \left.\begin{aligned} x_1 \quad\ \boxed{+S_1} &= 4 \\ 2x_2 \quad\ \boxed{+S_2} &= 12 \\ 3x_1+2x_2 \quad\ \boxed{+S_3} &= 18 \end{aligned}\right\} \text{augmented form}$$

slack var ≥ 0 (∵ $x_1, x_2 \ge 0$)

$(x_1, x_2) \to (x_1, x_2, S_1, S_2, S_3)$

$(2,6) \to (2,6,2,\boxed{0},\boxed{0})$
$(0,6) \to (\boxed{0},6,4,\boxed{0},6)$
$(4,3) \to (4,3,0,6,\boxed{0})$
$(0,0) \to (0,0,4,12,18)$

adjacent basic sol
⇒ share (n-1) NBVs $=0$
have 1 different NBVs & BVs $\neq 0$

x_2: entering variable (fix $x_1=0$)

$S_1 = 4$
leaving variable ⋯ $S_2 = 12-2x_2 \ge 0$ $x_2 \to 6$ ($12/2$)
$S_3 = 18-2x_2 \ge 0$ $x_2 \to 9$ ($18/2$)
} $x_2 \to 6$ minimum ratio test (MRT)

next solution $(0,6,4,0,6)$ ⇒ check optimal sol or not.

BVs: S_1, x_2, S_3
NBVs: x_1, S_2 $=0$

$$\Rightarrow \begin{aligned} x_1 \quad + S_1 \qquad\qquad &= 4 \\ x_2 \quad + \tfrac{1}{2}S_2 \qquad &= 6 \\ 3x_1 \qquad - S_2 + S_3 &= 6 \end{aligned}$$

② $(0,6,4,0,6)$
⇒ $Z=30$

$$\begin{aligned} Z &= 3x_1 + 5x_2 \\ -30 &= \quad -5x_2 - \tfrac{5}{2}S_2 \\ \hline Z-30 &= 3x_1 - \tfrac{5}{2}S_2 \end{aligned}$$

⇒ $Z = \underbrace{3x_1 - \tfrac{5}{2}S_2}_{\text{NBVs}=0} + 30$ ③ ? ⇒ Max $Z = 30$. ①

original		augmented
CP solution	⇒	basic solution
CPF solution	⇒	basic feasible solution (BFS)

$(x_1, x_2) = (0,0)$
⇒ $S_1 = 4$, $S_2 = 12$. $S_3 = 18$

Non-basic variables (NBVs) = (# all vars − # structural constraints) [m+n, m]
Basic variables (BVs) = # structural constraints (m)

x_1: entering variable
$S_1 = 4 - x_1$
$x_2 + \tfrac{1}{2}S_2 = 6$
leaving var ⋯ $S_3 = 6 - 3x_1$ ⋯ $x_1 \to 2$
MRT

$$\left.\begin{aligned} x_1 \quad + S_1 \qquad\qquad &= 4 \\ x_2 \quad + \tfrac{1}{2}S_2 \qquad &= 6 \\ x_1 \quad - \tfrac{1}{3}S_1 \quad + \tfrac{1}{3}S_3 &= 2 \end{aligned}\right] \to S_1 + \tfrac{1}{3}S_2 - \tfrac{1}{3}S_3 = 2$$

BFS = $(2,6,2,0,0)$

$$\begin{aligned} Z &= 3x_1 - \tfrac{5}{2}S_2 + 30 \\ 6 &= 3x_1 + S_2 + S_3 \\ \hline Z-6 &= -\tfrac{3}{2}S_2 - S_3 + 30 \end{aligned}$$

⇒ $Z = \underbrace{-\tfrac{3}{2}S_2 - S_3}_{\text{NBVs}=0} + 36$