Sensitivity Analysis

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Simplex Method in Matrix Form

- Matrix Representation of the Problem
  - A general LP (in standard form) can be expressed using matrices as follows:
    - ✓ Maximize $Z = \boldsymbol{c}^T \boldsymbol{x}$
    - ✓ Subject to $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \;\; \boldsymbol{x} \ge 0$
    - ✓ where the dimensions of each matrix/vector are:
      - Objective coefficient vector $\boldsymbol{c} \in \mathbb{R}^{1\times n}$
      - Decision variable vector $\boldsymbol{x} \in \mathbb{R}^{n\times 1}$
      - Constraint coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$
      - RHS vector $\boldsymbol{b} \in \mathbb{R}^{m\times 1}$
    - ✓ If we add slack variables $\boldsymbol{x}_s$, the constraints become
      - $[\boldsymbol{A}|\boldsymbol{I}]\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{bmatrix} = \boldsymbol{b}, \;\; \boldsymbol{x} \ge 0, \boldsymbol{x}_s \in \mathbb{R}^{m\times 1} \ge 0$

Simplex Method in Matrix Form

- Algebraic Operations of the Simplex Method
  - Computation process using the Simplex tableau
    - ✓ Classification of variables in the Simplex tableau
      - Baisc Variables (BVs): variables that determine the current solution
      - Non-Basic Variables (NBVs): variables not involved in the current solution and set to 0
    - ✓ Pivoting: the process of moving from the current BFS by introducing a selected NBV into the basis (entering variable) and simultaneously removing one BV from the basis (leaving variable)
    - ✓ To do this, the selected pivot element is scaled to 1, and row operations are applied so that all other entries in the pivot column become 0.

Simplex Method in Matrix Form

- Algebraic Operations of the Simplex Method
  - Comparison with solving systems of equations ($\boldsymbol{Ax} = \boldsymbol{b}$)
    - ✓ Systems of equations:
      - $2x_1 + x_2 = 5$
      - $3x_1 + 2x_2 = 8$
    - ✓ Matrix form:
      - $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ → $\boldsymbol{A}^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$
      - $\boldsymbol{x} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
      - In this case, the left-hand side can be interpreted $\boldsymbol{I} \cdot \boldsymbol{x} = \boldsymbol{A}^{-1} \cdot \boldsymbol{b}$
  - Connection to the Simplex Method: The row operations in the Simplex Method are similar to the procedure of solving systems of equations, transforming the tableau step by step to maintain feasibility while finding the optimal solution.

Simplex Method in Matrix Form

- Matrix Computation of a Basic Solution
  - Distinguishing BVs and NBVs
    - ✓ In a given solution, the variable set $\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} \in \mathbb{R}^{m+n}$ can be divided into n NBVs and m BVs.
    - ✓ Focusing only on the BVs in the current solution, $\boldsymbol{B}\boldsymbol{x}_B = \boldsymbol{b}$
      - Here, $\boldsymbol{x}_B = \begin{bmatrix} x_{B1} \\ \vdots \\ x_{Bm} \end{bmatrix} \in \mathbb{R}^m$ is the vector of BVs. The order is determined by which BV corresponds to each constraint, so it may differ from the original variable order.
      - Likewise, $\boldsymbol{B} = \begin{bmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mm} \end{bmatrix} \in \mathbb{R}^{m \times m}$ is obtained from $[\boldsymbol{A}|\boldsymbol{I}]$ by removing the columns corresponding to NBVs.
    - ✓ If $\boldsymbol{B}^{-1}$ exists and $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$, then a BFS is obtained with all NBVs set to zero.
    - ✓ In this case, the objective function value is $Z = \boldsymbol{c}_B\boldsymbol{x}_B = \boldsymbol{c}_B\boldsymbol{B}^{-1}\boldsymbol{b}$

Simplex Method in Matrix Form

- Matrix Computation of a Basic Solution: Example
  - Max Z = 3x1 + 5x2
    - ✓ x1 + s1 = 4
    - ✓ 2x2 + s2 =12
    - ✓ 3x1 + 2x2 + s3 = 18
    - ✓ x1, x2, s1, s2, s3 ≥ 0
  - Matrix form:
    - ✓ $\boldsymbol{c} = [3, 5, 0, 0, 0] \in \mathbb{R}^{1\times5}$
    - ✓ $\boldsymbol{A} = \begin{bmatrix} 1, 0, 1, 0, 0 \\ 0, 2, 0, 1, 0 \\ 3, 2, 0, 0, 1 \end{bmatrix} \in \mathbb{R}^{3\times5}$
    - ✓ $\boldsymbol{b} = [4, 12, 18]^T \in \mathbb{R}^{3\times1}$
    - ✓ $\boldsymbol{x} = [x_1, x2]^T \in \mathbb{R}^{2\times1}, \quad \boldsymbol{x}_s = [s_1, s_2, s_3]^T \in \mathbb{R}^{3\times1}$

Simplex Method in Matrix Form

- Matrix Computation of a Basic Solution: Example
  - Computation of $\boldsymbol{x}_B, \boldsymbol{B}$ (및 $\boldsymbol{B}^{-1}$), $\boldsymbol{c}_B, Z$ per-iteration.

Simplex Method in Matrix Form

- Matrix Computation of a Basic Solution: Example
  - Simplex Tableau in Matrix Form

Iteration	BVs	Z	$\boldsymbol{x}$	$\boldsymbol{x}_s$	RHS
0	Z	1	$-\boldsymbol{c}$	$\boldsymbol{0}$	0
	$\boldsymbol{x}_B (= \boldsymbol{x}_s)$	0	$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Any iter i	Z	1	$\boldsymbol{c}_B \boldsymbol{B}^{-1} \boldsymbol{A} - \boldsymbol{c}$	$\boldsymbol{c}_B \boldsymbol{B}^{-1}$	$\boldsymbol{c}_B \boldsymbol{B}^{-1} \boldsymbol{b}$
	$\boldsymbol{x}_B$	0	$\boldsymbol{B}^{-1} \boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1} \boldsymbol{b}$

✓ $\boldsymbol{x}_B, \boldsymbol{B}, \boldsymbol{c}_B$ are determined by the BVs in each iteration (therefore, as the BVs change each iteration, they are extracted differently from iteration 0.)

Simplex Method in Matrix Form

- Summary: Simplex Method in Matrix Form
  - Initialization: After introducing slack variables (or else), derive $\boldsymbol{x}_B, \boldsymbol{c}_B, \boldsymbol{B}, \boldsymbol{B}^{-1}$
    - ✓ In standard form, $\boldsymbol{B} = \boldsymbol{I}$.
  - Optimality check: Examine $[\boldsymbol{c}_B \boldsymbol{B}^{-1} \boldsymbol{A} - \boldsymbol{c} \mid \boldsymbol{c}_B \boldsymbol{B}^{-1}]$
    - ✓ If any entry corresponding to an NBV is negative, further improvement is possible.
  - Iteration.
    - ✓ Determine entering variable: Choose the NBV with the most negative value in $[\boldsymbol{c}_B \boldsymbol{B}^{-1} \boldsymbol{A} - \boldsymbol{c} \mid \boldsymbol{c}_B \boldsymbol{B}^{-1}]$.
    - ✓ Determine leaving variable:
      - At a given iteration, the RHS is $\boldsymbol{B}^{-1}\boldsymbol{b}$
      - The coefficients of the entering variable's column depend on its type:
        - If it is an original variable, use the column from $\boldsymbol{B}^{-1}\boldsymbol{A}$ → Ratio test: $\boldsymbol{B}^{-1}\boldsymbol{b}/\boldsymbol{B}^{-1}\boldsymbol{A}$
        - If it is a slack variable, use the column from $\boldsymbol{B}^{-1}$ → Ratio test: $\boldsymbol{B}^{-1}\boldsymbol{b}/\boldsymbol{B}^{-1}$
    - ✓ Update to new BFS
      - Based on the entering/leaving variables, update the BV set, and then recompute $\boldsymbol{x}_B, \boldsymbol{c}_B, \boldsymbol{B}, \boldsymbol{B}^{-1}$.

Simplex Method in Matrix Form

- Insights from Simplex Method in Matrix Form
  - At a given iteration:
    - ✓ Row 0: $[-c, \mathbf{0}, 0] + \boldsymbol{c}_B \boldsymbol{B}^{-1}[\boldsymbol{A}, \boldsymbol{I}, \boldsymbol{b}]$ → Row 0 + $\boldsymbol{y}^*$ ×(other rows)
    - ✓ Other rows: $\boldsymbol{B}^{-1}[\boldsymbol{A}, \boldsymbol{I}, \boldsymbol{b}]$ → $\boldsymbol{S}^*$ ×(other rows)
  - Key elements can be expressed with new notation (Note: $\boldsymbol{z}$ and Z are distinct values)
    - ✓ $\boldsymbol{S}^* = \boldsymbol{B}^{-1}$
    - ✓ $\boldsymbol{A}^* = \boldsymbol{B}^{-1}\boldsymbol{A}$
    - ✓ $\boldsymbol{y}^* = \boldsymbol{c}_B \boldsymbol{B}^{-1}$
    - ✓ $\boldsymbol{z}^* = \boldsymbol{c}_B \boldsymbol{B}^{-1}\boldsymbol{A}$
    - ✓ $Z^* = \boldsymbol{c}_B \boldsymbol{B}^{-1}\boldsymbol{b}$
    - ✓ $\boldsymbol{b}^* = \boldsymbol{B}^{-1}\boldsymbol{b}$

반복	BVs	Z	$\boldsymbol{x}$	$\boldsymbol{x}_s$	RHS
0	Z	1	$-\boldsymbol{c}$	$\mathbf{0}$	0
	$\boldsymbol{x}_B(=\boldsymbol{x}_s)$	0	$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
⋮	⋮	⋮	⋮	⋮	⋮
Any iter i	Z	1	$\boldsymbol{c}_B\boldsymbol{B}^{-1}\boldsymbol{A}-\boldsymbol{c}$	$\boldsymbol{c}_B\boldsymbol{B}^{-1}$	$\boldsymbol{c}_B\boldsymbol{B}^{-1}\boldsymbol{b}$
	$\boldsymbol{x}_B$	0	$\boldsymbol{B}^{-1}\boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

반복	BVs	Z	$\boldsymbol{x}$	$\boldsymbol{x}_s$	RHS
Any iter i	Z	1	$\boldsymbol{z}^*-\boldsymbol{c}$	$\boldsymbol{y}^*$	$\boldsymbol{Z}^*$
	$\boldsymbol{x}_B$	0	$\boldsymbol{A}^*$	$\boldsymbol{B}^*$	$\boldsymbol{b}^*$

Sensitivity Analysis

- Sensitivity Analysis
  - Model parameters (a_{ij}, b_i, c_j, etc) are estimates based on predictions, and may change for various reasons.
    - ✓ It is necessary to study how changes in model parameters affect the optimal solution.
  - Results of sensitivity analysis
    - ✓ Identify sensitive parameters (those whose change alters the optimal solution)
    - ✓ Identify the allowable range of parameters that do not affect the optimal solution.
    - ✓ Determine the range within which the optimal BFS remains feasible.
  - Idea of sensitivity analysis
    - ✓ Sensitivity analysis does not mean re-solving the LP from scratch whenever parameters change.
    - ✓ Instead, when parameters change slightly, the already computed optimal Simplex tableau is used to estimate the effect of those changes.

Sensitivity Analysis

- Sensitivity Analysis
  - Types of parameter changes
    - ✓ Change in the right-hand side (RHS): $\boldsymbol{b} \rightarrow \bar{\boldsymbol{b}}$
    - ✓ Change in the objective function coefficients: $\boldsymbol{c} \rightarrow \bar{\boldsymbol{c}}$
    - ✓ Change in the constraint coefficients: $\boldsymbol{A} \rightarrow \bar{\boldsymbol{A}}$
  - To track how the optimal Simplex tableau changes under parameter variation, we make use of insights from the matrix form.

Iteration	BVs	Z	$\boldsymbol{x}$	$\boldsymbol{x}_s$	RHS
New init.	Z	1	$-\bar{\boldsymbol{c}}$	$\boldsymbol{0}$	0
	$\boldsymbol{x}_B(=\boldsymbol{x}_s)$	0	$\bar{\boldsymbol{A}}$	$\boldsymbol{I}$	$\bar{\boldsymbol{b}}$
⋮	⋮	⋮	⋮	⋮	⋮
Using original $\boldsymbol{B}^{-1}$	Z	1	$\boldsymbol{y}^*\bar{\boldsymbol{A}}-\bar{\boldsymbol{c}}$	$\boldsymbol{y}^*$	$\boldsymbol{y}^*\bar{\boldsymbol{b}}$
	$\boldsymbol{x}_B$	0	$\boldsymbol{S}^*\bar{\boldsymbol{A}}$	$\boldsymbol{S}^*$	$\boldsymbol{S}^*\bar{\boldsymbol{b}}$

Sensitivity Analysis

- Example
  - In a prototype example, consider the following changes:
    - ✓ $c_1 = 3 \rightarrow 4$
    - ✓ $a_{31} = 3 \rightarrow 2$
    - ✓ $b_2 = 12 \rightarrow 24$

Sensitivity Analysis

- Case 1. Changes in b_i
  - When the only changes in the final Simplex tableau are in the RHS
    - ✓ The rest of the tableau remains in the already transformed form.
    - ✓ In row 0, the NBVs are non-negative.
    - ✓ (Q) Are the values of the BVs still non-negative (i.e., is the solution still feasible)?
  - When $\boldsymbol{b} \rightarrow \overline{\boldsymbol{b}}$:
    - ✓ $Z^* = \boldsymbol{y}^* \overline{\boldsymbol{b}}$
    - ✓ $\boldsymbol{b}^* = \boldsymbol{S}^* \overline{\boldsymbol{b}}$

Sensitivity Analysis

- Case 1. Changes in b_i
  - Example. In the prototype example, let $b_2 = 12 \rightarrow 24$

Sensitivity Analysis

- Case 1. Changes in b_i
  - Allowable range of RHS
    - ✓ For $\Delta b_2 = 12$, the change alters the set of BVs and feasibility. → What is the allowable range (i.e., how much can it change while still keeping the same BVs and feasibility)?
    - ✓ When $\Delta b_2 = \theta$:
      - $b_1^* =$
      - $b_2^* =$
      - $b_3^* =$

Sensitivity Analysis

- Case 2. Changes in objective function coefficient c_j and constraint coefficients $\boldsymbol{A}_j$ of an NBV.
  - Allowable range of an NBV's objective function coefficient c_j (only c_j changes)
    - ✓ The current solution remains optimal if $z_j^* - \bar{c}_j \geq 0$
    - ✓ Since $z_j^* = \boldsymbol{y}^* \boldsymbol{A}_j$ is assumed unchanged, the allowable range is $c_j \leq \boldsymbol{y}^* \boldsymbol{A}_j$

    - ✓ Reduced cost: For NBV x_j, the reduced cost is $z_j^* - c_j$ which represents the unit cost of variable x_j.
      - To make x_j enter the basis, its unit cost must decrease by at least this amount.
      - Therefore, $z_j^* - c_j$ is the maximum allowable increase (Δc_j) that keeps the current solution optimal.

Sensitivity Analysis

- Case 2. Changes in objective function coefficient c_j and constraint coefficients $\boldsymbol{A}_j$ of an NBV.
  - Changes in the constraint coefficients $\boldsymbol{A}_j$
    - ✓ Only the corresponding column in the final Simplex tableau is affected.

Sensitivity Analysis

- Case 2b. Introduction of a new variable
  - Treated in a similar way as changes in the coefficients of an existing NBV.
  - The newly considered variable can be thought of as if it had already existed, with all its coefficients initially set to zero as an NBV.

Sensitivity Analysis

- Case 3. Changes in the coefficients of a BV
  - When a changes occurs for a BV x_j in the final Simplex tableau
    - ✓ First-step results using the changed values:
      - $z_j^* - \bar{c}_j = \boldsymbol{y}^* \overline{\boldsymbol{A}}_j - \bar{c}_j$
      - $A_j^* = \boldsymbol{S}^* \overline{\boldsymbol{A}}_j$
    - ✓ Using the first-step results, construct a modified final Simplex tableau.
    - ✓ Then, apply Gaussian elimination to transform it into the proper form.
    - ✓ After the transformation, check for optimality and feasibility.
  - Example. In the prototype example, suppose $c_2 = 5 \rightarrow \bar{c}_2 = 3$ & $\overline{\boldsymbol{A}}_2 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$

Sensitivity Analysis

- Case 3. Changes in the coefficients of a BV
  - Allowable range of the objective function coefficient c_j when x_j is a BV
    - ✓ In the original final Simplex tableau, for a BV we have $z_j^* - c_j = 0$
    - ✓ If the coefficient changes as $\bar{c}_j = c_j + \Delta c_j$, then $z_j^* - \bar{c}_j = z_j^* - c_j - \Delta c_j = -\Delta c_j$
    - ✓ Therefore, to eliminate the residual term $-\Delta c_j$ from the BV's objective function row, Gaussian elimination must be carried out.
    - ✓ After Gaussian elimination, use the condition that all NBV coefficients in the new row 0 must be non-negative to determine the allowable range.
  - Example. When $c_2 = 3 + \Delta c_2$

Sensitivity Analysis

- Case 4. Introduction of a New Constraint
  - A new constraint can only reduce the existing feasible region.
    - ✓ Therefore, if the new constraint is satisfied at the current optimal solution, the optimal solution is retained.
    - ✓ If the current optimal solution is no longer feasible due to the new constraint:
      - Add the constraint row to the final Simplex tableau (a row is added where a slack variable or artificial variable becomes a BV.)
      - Eliminate the non-zero coefficients for other BVs using Gaussian elimination, and then re-optimize.

Q&A