L07. E1. 남은 재고 < 3 ⇒ 재고량 6으로 재주문. 수요 1 : 1/6 , 2 : 3/6 , 3 : 2/6

a) X_n = Stock of beginning of day n. $X_0 = 5$.

a-1) Markov chain? Yes. Stock of tomorrow is depending on only today's stock.

a-2) State space : $\{3,4,5,6\}$, initial distribution $a_0 = \{0,0,1,0\}$

a-3) transition matrix :

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1/6 & 0 & 0 & 5/6 \\ 3/6 & 1/6 & 0 & 2/6 \\ 2/6 & 3/6 & 1/6 & 0 \end{pmatrix}$$

b) Y_n : amount of stock at the end of day n. $Y_0 = 2$.

b-1) Markov chain? Yes.

b-2) State Space : $\{0,1,2,3,4,5\}$, initial distribution $a_0 = \{0,0,1,0,0,0\}$

b-3) transition matrix :

$0\text{-}6 < \begin{matrix}5\\4\\3\end{matrix}$ $\quad 3 < \begin{matrix}2\\1\\0\end{matrix}$

$1\text{-}6 < \begin{matrix}5\\4\\3\end{matrix}$ $\quad 4 < \begin{matrix}3\\2\\1\end{matrix}$

$2\text{-}6 < \begin{matrix}5\\4\\3\end{matrix}$ $\quad 5 < \begin{matrix}4\\3\\2\end{matrix}$

$$\begin{pmatrix} & & & 2/6 & 3/6 & 1/6 \\ & & & 2/6 & 3/6 & 1/6 \\ & & & 2/6 & 3/6 & 1/6 \\ 2/6 & 3/6 & 1/6 & & & \\ & 2/6 & 3/6 & 1/6 & & \\ & & 2/6 & 3/6 & 1/6 & \end{pmatrix}$$

c) a-3 = P_X, $P_X^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1/6 & 0 & 0 & 5/6 \\ 3/6 & 1/6 & 0 & 2/6 \\ 2/6 & 3/6 & 1/6 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1/6 & 0 & 0 & 5/6 \\ 3/6 & 1/6 & 0 & 2/6 \\ 2/6 & 3/6 & 1/6 & 0 \end{pmatrix} = \begin{pmatrix} 2/6 & 3/6 & 1/6 & 0 \\ 10/36 & 15/36 & 5/36 & 6/36 \\ 5/36 & \frac{6}{36} & \frac{2}{36} & 23/36 \\ 6/36 & 1/36 & 0 & 29/36 \end{pmatrix}$

d) Diff of a) & b). pros/cons.

a) & b) both DTMC. a) has less states than b). More concise state space make easier analysis

e) $P(X_2 = 6 \mid X_0 = 5) = 23/36$

$a_2 = a_0 P_X^2 = \{0,0,1,0\} \begin{pmatrix} 2/6 & 3/6 & 1/6 & 0 \\ 10/36 & 15/36 & 5/36 & 6/36 \\ 5/36 & \frac{6}{36} & \frac{2}{36} & 23/36 \\ 6/36 & 1/36 & 0 & 29/36 \end{pmatrix} = (5/36, 6/36, 2/36, 23/36)$

f) $a_0 = \{0,0,1/2,1/2\}$, $P(X_2 = 6) = 52/72$

$a_0 P_X^2 = \{0,0,1/2,1/2\} \begin{pmatrix} 2/6 & 3/6 & 1/6 & 0 \\ 10/36 & 15/36 & 5/36 & 6/36 \\ 5/36 & \frac{6}{36} & \frac{2}{36} & 23/36 \\ 6/36 & 1/36 & 0 & 29/36 \end{pmatrix} = (11/72, 7/72, 2/72, 52/72)$

$\sum_{i=3}^{6} P(X_2 = 6 \mid X_0 = i) P(X_0 = i) = 0 \times 0 + 6/36 \times 0 + 23/36 \times 1/2 + 29/36 \times 1/2 = 52/72.$ (Bayes' rule)

L07. E2 Stock 1 selling price : $\begin{array}{c} 10 \\ 20 \end{array}\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}$, Stock 2 selling price : $\begin{array}{c} 10 \\ 25 \end{array}\begin{pmatrix} .9 & .1 \\ .15 & .85 \end{pmatrix}$

X_n $\qquad\qquad\qquad\qquad$ Y_n

a) transition matrix for $\{X_n : n \geq 0\}$: $\begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix}$ ⇒ irreducible

b) transition matrix for $\{Y_n : n \geq 0\}$: $\begin{pmatrix} .9 & .1 \\ .15 & .85 \end{pmatrix}$ ⇒ irreducible

c) Stationary dist of X_n : $\pi_x = \pi_x P_x \Rightarrow (\pi_{x_1}, \pi_{x_2}) = (\pi_{x_1}, \pi_{x_2}) \begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix} \Rightarrow \begin{array}{l} \pi_{x_1} = .8\pi_{x_1} + .1\pi_{x_2} \\ \pi_{x_2} = .2\pi_{x_1} + .9\pi_{x_2} \end{array} \Rightarrow .2\pi_{x_1} = .1\pi_{x_2}$

$\therefore \pi_x = (1/3, 2/3)$

d) Stationary dist of Y_n : $\pi_y = \pi_y P_y \Rightarrow (\pi_{y_1}, \pi_{y_2}) = (\pi_{y_1}, \pi_{y_2}) \begin{pmatrix} .9 & .1 \\ .15 & .85 \end{pmatrix} \Rightarrow \begin{array}{l} \pi_{y_1} = .9\pi_{y_1} + .15\pi_{y_2} \\ \pi_{y_2} = .1\pi_{y_1} + .85\pi_{y_2} \end{array} \Rightarrow .1\pi_{y_1} = .15\pi_{y_2}$

$\therefore \pi_y = (3/5, 2/5)$

e) 300 shares of Stock 1 : $10 \times 1/3 \times 300 + 20 \times 2/3 \times 300 = 1000 + 4000 = 5000$
Stock 2 : $10 \times 3/5 \times 300 + 25 \times 2/5 \times 300 = 4800$

$\therefore$ select stock 1.

L07 E3. 무사고 확률 0.98. X_n : num posted on the morning after n full days. $X_0=0$.

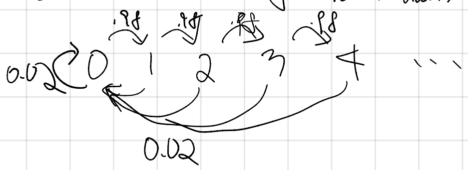
a) $X_n \Rightarrow$ markov chain? yes. depending on current state. & random.

State space : $\{0,1,2,\dots\}$. initial distribution = $\{1,0,0,\cdots\}$

$$P_{ij} = \begin{cases} 0.98 & (j=i+1) \quad \cdots \text{ i일 연속 무사고. 내일도 사고 X} \\ 0.02 & (j=0) \quad \cdots \text{ i일 연속 무사고. 내일 사고 발생.} \\ 0 & (\text{otherwise}) \end{cases}$$

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots & n+1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ n \end{matrix} & \begin{pmatrix} .02 & .98 & & & & & \\ .02 & 0 & .98 & & & & \\ .02 & 0 & 0 & .98 & & & \\ .02 & 0 & 0 & 0 & .98 & & \\ & & & & & \ddots & \\ .02 & 0 & 0 & 0 & 0 & \cdots & 0.98 \end{pmatrix} \end{array}$$

b) irreducible. every state is accessible.



c) aperiodic. $P_{00}^{n} > 0$, $n=1,2,3,4,\dots$ gcd of $n=1$

d) stationary distribution $\pi = \pi P$. $(\pi_0, \pi_1, \pi_2, \cdots) = (\pi_0, \pi_1, \pi_2, \cdots) \begin{pmatrix} .02 & .98 & & \\ .02 & 0 & .98 & \\ .02 & 0 & 0 & .98 \\ \vdots & & & \ddots \end{pmatrix}$

$= (.02(\pi_0+\pi_1+\pi_2+\cdots), .98\pi_0, .98\pi_1, .98\pi_2, \cdots)$

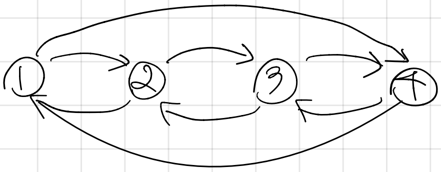
$\Rightarrow \pi_1 = 0.98\pi_0$, $\pi_2 = 0.98\pi_1$ $\cdots$ $\pi_k = (0.98)^k \pi_0$. $\sum_{k=0}^{\infty} (0.98)^k \pi_0 = \frac{\pi_0}{1-0.98} = 1$. $\pi_0 = 0.02$

$\Rightarrow \pi_k = (0.98)^k \cdot (0.02)$

$\pi = (\pi_k = (0.98)^k \times 0.02, \ (k=0,1,2,\cdots))$

e) irredcible. stationary dist exists.

LOM E4. $P=\begin{pmatrix} 0 & .5 & 0 & .5 \\ .6 & 0 & .4 & 0 \\ 0 & .7 & 0 & .3 \\ .8 & 0 & .2 & 0 \end{pmatrix}$



State space $= \{1,2,3,4\}$

a) irreducible. period is class property. period of 1 : $\gcd(2,4,6,\cdots)=2$. periodic.

b) $\pi=(\pi_1,\pi_2,\pi_3,\pi_4)=(33/96,\ 27/96,\ 15/96,\ 21/96) \Rightarrow$ stationary?

Stationary distribution = State space S, transition prob matrix P, $\pi_i \geq 0$ for all $i\in S$, $\sum_{i\in S}\pi_i=1$, $\pi=\pi P$

$$\pi=\pi P \Rightarrow (33/96,\ 27/96,\ 15/96,\ 21/96) = (33/96,\ 27/96,\ 15/96,\ 21/96)\begin{pmatrix} 0 & .5 & 0 & .5 \\ .6 & 0 & .4 & 0 \\ 0 & .7 & 0 & .3 \\ .8 & 0 & .2 & 0 \end{pmatrix}$$

$$\Rightarrow \left(\frac{27}{96}\times\frac{6}{10}+\frac{21}{96}\times\frac{8}{10}=\frac{33}{96},\ \frac{33}{96}\times\frac{5}{10}+\frac{15}{96}\times\frac{7}{10}=\frac{27}{96},\ \frac{27}{96}\times\frac{4}{10}+\frac{21}{96}\times\frac{2}{10}=\frac{15}{96},\ \frac{33}{96}\times\frac{5}{10}+\frac{15}{96}\times\frac{3}{10}=\frac{21}{96}\right)$$

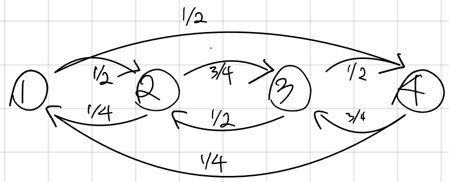
$\therefore$ stationary distribution

c) $P_{11}^{101}=0.$ ($\because$ period $=2$) $\qquad \pi_1=\dfrac{P_{11}^{100}+P_{11}^{101}}{2}=\dfrac{33}{96}$

$P_{11}^{100}=2\times 33/96$

L07 E5. $S=\{1,2,3,4\}$ $P=\begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix}$

a) transition diagram.



b) $\mathbb{P}(X_2=4 \mid X_0=2) = \frac{1}{4}\times\frac{1}{2} + \frac{3}{4}\times\frac{1}{2} = \frac{1}{2}$

c) $\mathbb{P}(X_2=2, X_4=4, X_5=1 \mid X_0=2) = \mathbb{P}(X_5=1 \mid X_4=4, X_2=2, X_0=2)\,\mathbb{P}(X_4=4 \mid X_2=2, X_0=2)\,\mathbb{P}(X_2=2 \mid X_0=2)$

$= \mathbb{P}(X_5=1 \mid X_4=4) \times \mathbb{P}(X_4=4 \mid X_2=2) \times \mathbb{P}(X_2=2 \mid X_0=2) = \mathbb{P}(X_5=1 \mid X_4=4) \times \mathbb{P}(X_2=4 \mid X_0=2) \times \mathbb{P}(X_2=2 \mid X_0=2)$

$= (1/4)\times(1/2)\times\left(\frac{1}{8}+\frac{3}{8}\right) = 1/16.$

d) period $=2$

e) $\pi = \{1/8, 1/4, 3/8, 1/4\}$, π is unique? DTMC is irreducible. So, Stationary dist. is unique.

$1/8 + 1/4 + 3/8 + 1/4 = 1$

$\pi = \pi P \Rightarrow (1/8\ \ 1/4\ \ 3/8\ \ 1/4)\begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix} = (1/8\ \ 1/4\ \ 3/8\ \ 1/4)$

f) $\lim_{n\to\infty} P_{14}^{n} = 1/4$? periodic DTMC. No convergence $P^{2n-1} \neq P^{2n}$

$\lim_{n\to\infty} \frac{P_{14}^{n} + P_{14}^{n+1}}{2} = 1/4.$

L07 E6

d	10	20	30
$P(D=d)$	.2	.5	.3

inventory Fri $\leq 10 \Rightarrow$ order up to 30 until Mon.

State space : $\{20, 30\}$

$$\begin{tikzcd} 20 \arrow[r, bend left, "1"] & 30 \arrow[l, bend left, ".2"] \arrow[loop right, ".8"] \end{tikzcd}$$

irreducible
aperiodic

a) $P(X_2=10 \mid X_0=20)=0.$

b) $P(X_2=20 \mid X_0=20)= 0.2$

$$P=\begin{pmatrix} 0 & 1 \\ .2 & .8 \end{pmatrix}, \quad a_0=(1,0), \quad a_2=a_0P^2=(1,0)\left\{\begin{pmatrix} 0 & 1 \\ .2 & .8 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ .2 & .8 \end{pmatrix}\right\}=(1\,0)\begin{pmatrix} .2 & .8 \\ .16 & .84 \end{pmatrix}=(.2 \;\; .8)$$

c) $P(X_{100}=10 \mid X_0=20)=0.$ out of state space.

d) $P(X_{100}=20 \mid X_0=20) \cong 1/6$

$$\pi=\pi P, \quad (\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ .2 & .8 \end{pmatrix} \Rightarrow \begin{matrix} \pi_1=0.2\pi_2 \\ \pi_1+\pi_2=1 \end{matrix} \Rightarrow \pi_1=\frac{1}{6} \quad \pi_2=\frac{5}{6} \qquad \pi=(1/6, 5/6), \quad P^{\infty}=\begin{pmatrix} 1/6 & 5/6 \\ 1/6 & 5/6 \end{pmatrix} \cong P^{100}$$

e) item price = \$200. item cost = \$100. Stock cost = \$20. order? fixed cost = \$500.

$$200 \times (\text{sales}) - 100 \times \underbrace{(30-\text{inventory})}_{=\text{order}} - 20(\text{inventory}) - 500$$

$X=20 \Rightarrow$ $D=10$ (0.2) order 20 inventory $=10 \Rightarrow 2000-2000-200-500 = -700$
(1/6) $D=20$ (0.8) order 30 inventory $=0 \Rightarrow 4000-3000-0-500 = 500$
$\therefore 0.2 \times (-700) + 0.8(500) = 260$

$X=30 \Rightarrow$ $D=10$ (0.2) order 0 inven 20 $\Rightarrow 2000-0-400-0-0 = 1600$
(5/6) $D=20$ (0.5) order 20 inven 10 $\Rightarrow 4000-2000-200-500 = 1300$
$D=30$ (0.3) order 30 inven 0 $\Rightarrow 6000-3000-0-500 = 2500$
$\therefore 0.2 \times 1100 + 0.5 \times 1300 + 0.3 \times 2500 = 1720$

Total $= 1/6 \times 260 + 5/6 \times 1720$

L07 E7 MC 아님을 증명하기.

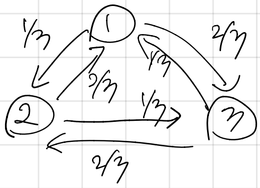
MC ⇒ 과거에 상관없이 오늘의 상태가 같다면 내일의 확률 동일.

!MC ⇒ 오늘의 상태가 같아도 어제의 상태에 따라 내일의 확률이 달라짐.

X_{n-1}의 정보가 X_{n+1} 예측에 영향 준다

$$\mathbb{P}(X_{n+1}=R \mid X_n=S, X_{n-1}=S) \neq \mathbb{P}(X_{n+1}=R \mid X_n=S, X_{n-1}=R)$$

L08. E1. $P = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$



i) irreducible? yes.

class = {1, 2, 3}.

ii) Stationary dist? unique?

irreducible. ⇒ unique

$\pi = \pi P \Rightarrow (\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3) \begin{pmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$

$$\left.\begin{aligned} \pi_1 &= 2/3\pi_2 + 1/3\pi_3 \\ \pi_2 &= 1/3\pi_1 + 2/3\pi_3 \\ \pi_3 &= 2/3\pi_1 + 1/3\pi_2 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}\right\}$$

$\pi_2 = \frac{1}{3}\left(\frac{2}{3}\pi_2 + \frac{1}{3}\pi_3\right) + \frac{2}{3}\pi_3 \Rightarrow \pi_2 = \pi_3$

$\pi_3 - \frac{1}{3}\pi_2 = \frac{2}{3}\pi_1 \Rightarrow \pi_2 = \pi_1$

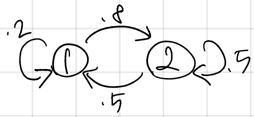
$\pi_1 = \pi_2 = \pi_3 = 1/3$ $\qquad \pi = \{1/3, 1/3, 1/3\}$

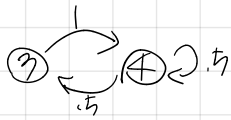
iii) periodic? No. Aperiodic.

iv) period of each state : gcd(2, 3, 4, ⋯) = 1

v) $P^{100} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$ finite. irreducible. aperiodic.

L08. E2. $P = \begin{pmatrix} .2 & .8 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & .5 & .5 \end{pmatrix}$





i) irreducible? No. 2 classes : $\{1,2\}$. $\{3,4\}$

ii) Stationary dist? unique?

$$\pi = \pi P \Rightarrow (\pi_1\ \pi_2\ \pi_3\ \pi_4) = (\pi_1\ \pi_2\ \pi_3\ \pi_4) \begin{pmatrix} .2 & .8 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & .5 & .5 \end{pmatrix}$$

$\pi_1 = .2\pi_1 + .5\pi_2$, $\pi_2 = .8\pi_1 + .5\pi_2$

$\Rightarrow \pi_1 \times 0.8 = \pi_2 \times 0.5 \Rightarrow \pi_1 = \frac{5}{13}\alpha$, $\pi_2 = \frac{8}{13}\alpha$

$\pi_3 = 0.5\pi_4$

$\Rightarrow \pi_3 = \frac{1}{3}\beta$, $\pi_4 = \frac{2}{3}\beta$

$\Rightarrow \alpha + \beta = 1$, $\pi = \left(\frac{5}{13}\alpha, \frac{8}{13}\alpha, \frac{1}{3}\beta, \frac{2}{3}\beta\right)$

MC is reducible, not unique stat. dist.

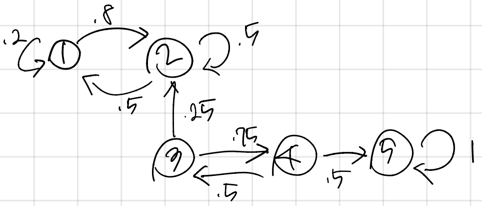
iii) periodic? $\{1,2\} \Rightarrow$ aperiodic
$\{3,4\} \Rightarrow$ aperiodic

iv) period of each state $\{1,2\} \Rightarrow 1$
$\{3,4\} \Rightarrow 1$

v) $P^{100} = \begin{pmatrix} 5/13 & 8/13 & & \\ 5/13 & 8/13 & & \\ & & 1/3 & 2/3 \\ & & 1/3 & 2/3 \end{pmatrix}$ each class $\Rightarrow$ irreducible. aperiodic, finite.

L08. E7.

$$P=\begin{pmatrix} .2 & .8 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 \\ 0 & .25 & 0 & .75 & 0 \\ 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



i) irreducible? No. 3 classes: $\{1,2\}$, $\{3,4\}$, $\{5\}$

$$\begin{array}{ccc} \{1,2\} & \{3,4\} & \{5\} \\ 1 & 2 & 1 \\ A & T & A \end{array}$$

ii) Stationary dist? unique? Not unique. ($\because$ reducible)

$$\pi=\pi P \Rightarrow (\pi_1\ \pi_2\ \pi_3\ \pi_4\ \pi_5) = (\pi_1\ \pi_2\ \pi_3\ \pi_4\ \pi_5)\begin{pmatrix} .2 & .8 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 \\ 0 & .25 & 0 & .75 & 0 \\ 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} \pi_1 &= 0.2\pi_1+0.5\pi_2 \\ \pi_2 &= 0.8\pi_1+0.5\pi_2+0.25\pi_3 \\ \pi_3 &= 0.5\pi_4 \\ \pi_4 &= 0.75\pi_3 \\ \pi_5 &= 0.5\pi_4+\pi_5 . \end{aligned}\right\} \begin{aligned} &\pi_4=0.\ \pi_3=0. \\ &0.8\pi_1=0.5\pi_2 \end{aligned}$$

$\pi_1+\pi_2+\pi_3+\pi_4+\pi_5=1$

$\pi=\left(\frac{5}{13}p,\ \frac{8}{13}p,\ 0,\ 0,\ 1-p\right)$

iii) periodic? yes. $\exists$ periodic states

iv) period of each state $\{1,2\}\Rightarrow 1$ $\{3,4\}\Rightarrow 2$ $\{5\}\Rightarrow 1$

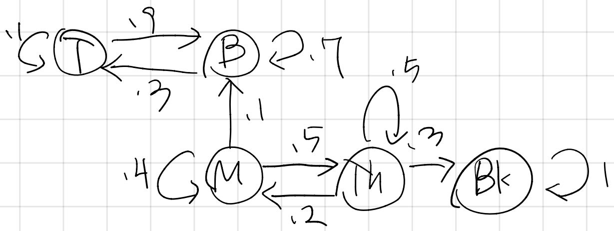
v) $$P^{100}=\begin{pmatrix} 5/13 & 8/13 & 0 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 & 0 \\ f_{3,\{1,2\}} & & 0 & 0 & f_{3,5} \\ f_{4,\{1,2\}} & & 0 & 0 & f_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\pi_1\ \pi_2)\begin{pmatrix} .2 & .8 \\ .5 & .5 \end{pmatrix}=(5/13\ \ 8/13)$$

$$\left.\begin{aligned} f_{3,\{1,2\}} &= \mathbb{P}(X_{100}=\{1,2\}\mid X_0=3) = 1/4+3/4 f_{4,\{1,2\}} \\ f_{4,\{1,2\}} &= 0.5 f_{3,\{1,2\}} \end{aligned}\right\} \begin{aligned} &f_{3,\{1,2\}}=\tfrac{1}{4}+\tfrac{3}{4}\times\tfrac{1}{2}f_{3,\{1,2\}} \\ &\Rightarrow f_{3,\{1,2\}}=0.4,\ f_{4,\{1,2\}}=0.2 \end{aligned}$$

$$=\begin{pmatrix} 5/13 & 8/13 & 0 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 & 0 \\ \frac{5}{13}\left(\frac{2}{5}\right) & \frac{8}{13}\left(\frac{2}{5}\right) & 0 & 0 & .6 \\ \frac{5}{13}\left(\frac{1}{5}\right) & \frac{8}{13}\left(\frac{1}{5}\right) & 0 & 0 & .8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

L08 E4.



class: {T, B} {M, Th} {Bk} ⇒ reducible
period 1 1 1 ⇒ aperiodic
R/T/A A T A

transition prob. matrix
$$P = \begin{array}{c} T \\ B \\ M \\ Th \\ Bk \end{array}\begin{pmatrix} .1 & .9 & & & \\ .3 & .7 & & & \\ & .1 & .4 & .5 & \\ & & .2 & .5 & .3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{359} = \begin{array}{c} T \\ B \\ M \\ Th \\ Bk \end{array}\begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ f_{M,\{T,B\}} & & 0 & 0 & f_{M,Bk} \\ f_{Th,\{T,B\}} & & 0 & 0 & f_{Th,Bk} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{array}{c} T \\ B \\ M \\ Th \\ Bk \end{array}\begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ \frac{1}{16} & \frac{3}{16} & 0 & 0 & 3/4 \\ \frac{1}{40} & \frac{3}{40} & 0 & 0 & 9/10 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\pi_1\ \pi_2)\begin{pmatrix} .1 & .9 \\ .3 & .7 \end{pmatrix} = (\pi_1\ \pi_2)$$

$\Rightarrow \pi_1 = 0.1\pi_1 + 0.3\pi_2$

$\pi_1 = \frac{1}{4}$, $\pi_2 = \frac{3}{4}$

$$\underbrace{f_{M,\{T,B\}}}_{=\alpha} = 0.1 + 0.4 f_{M,\{T,B\}} + 0.5 f_{Th,\{T,B\}}$$

$$\underbrace{f_{Th,\{T,B\}}}_{\beta} = 0.2 f_{M,\{T,B\}} + 0.5 f_{Th,\{T,B\}}$$

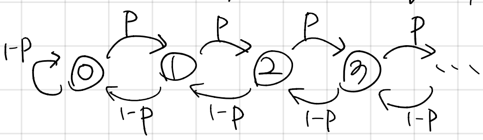
$\Rightarrow \beta - 0.5\beta = 0.2\alpha \Rightarrow \frac{2}{5}\alpha = \beta$

$\alpha = 0.1 + 0.4\alpha + 0.5 \times \frac{2}{5}\alpha \Rightarrow 0.4\alpha = 0.1$

$\alpha = \frac{1}{4}$ $\beta = \frac{1}{10}$

$\mathbb{P}(X_{360} = T \mid X_0 = Th) = 1/40$, $P^{360} \simeq P^{359}$

L08. E5 . $S=\{0,1,2,\cdots\}$ $p=1/4$ balance equation, stationary distribution



State	inflow	outflow		
0	$(1-p)\pi_0+(1-p)\pi_1$	π_0	$\Rightarrow (1-p)\pi_1 = p\pi_0$,	$\pi_1=\frac{1}{3}\pi_0$
1	$p\pi_0+(1-p)\pi_2$	π_1	$\Rightarrow \frac{3}{4}\pi_2=\frac{1}{12}\pi_0$,	$\pi_2=\frac{1}{9}\pi_0$
2	$p\pi_1+(1-p)\pi_3$	π_2	$\Rightarrow \frac{3}{4}\pi_3=\frac{1}{36}\pi_0$,	$\pi_3=\frac{1}{27}\pi_0$
$\vdots$	$\vdots$	$\vdots$		$\vdots$

$$\sum_{i=0}^{\infty}\pi_i = \pi_0+\frac{1}{3}\pi_0+\frac{1}{9}\pi_0+\cdots$$

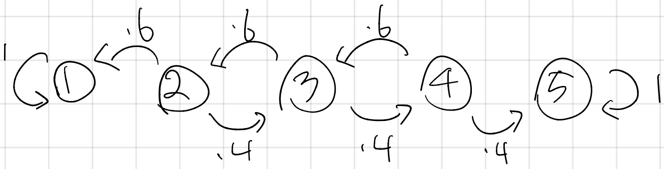
$$=\frac{1}{1-\frac{1}{3}}\pi_0=1, \quad \pi_0=\frac{2}{3}$$

$$\pi_i=\frac{2}{3}\left(\frac{1}{3}\right)^i, \quad \pi=\left\{\frac{2}{3},\frac{2}{9},\frac{2}{27},\cdots\right\}$$

Log Eb

L08. E7

$$P=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 & 0 \\ 0 & .6 & 0 & .4 & 0 \\ 0 & 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



class =	{1}	{2, 3, 4}	{5}
period	1	2	1
R/T/A	A	T	A

$$P^{100}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ f_{2,1} & 0 & 0 & 0 & f_{2,5} \\ f_{3,1} & 0 & 0 & 0 & f_{3,5} \\ f_{4,1} & 0 & 0 & 0 & f_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 57/65 & 0 & 0 & 0 & 8/65 \\ 9/13 & 0 & 0 & 0 & 4/13 \\ 27/65 & 0 & 0 & 0 & 38/65 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$f_{2,1} = 0.6 + 0.4 \times f_{3,1}$

$f_{3,1} = 0.6 \times f_{2,1} + 0.4 \times f_{4,1}$

$f_{4,1} = 0.6 \times f_{3,1}$

$f_{3,1} = x \Rightarrow f_{4,1} = 0.6x \Rightarrow x = 0.6 f_{2,1} + 0.24x$, $\quad 0.76x = 0.6 f_{2,1} \Rightarrow f_{2,1} = \frac{19}{15}x$

$\frac{19}{15}x = 0.6 + \frac{6}{15}x$, $\quad x = \frac{3}{5} \times \frac{15}{13} = \frac{9}{13}$

$f_{3,1} = \frac{9}{13}$, $\quad f_{2,1} = \frac{19}{15} \times \frac{9}{13} = \frac{57}{65}$ $\quad f_{4,1} = \frac{27}{65}$

LO9. E1. Eng : N_E is PP(2/min) Fra : N_F is PP(1/min), Eng & Fra independent.

a) $\mathbb{P}[N_E(5)-N_E(0)<2] = \mathbb{P}[X<2 \mid X\sim Poi(2\times5)] = \frac{10^0e^{-10}}{0!}+\frac{10^1e^{-10}}{1!} = 11e^{-10}$.

b) $\mathbb{P}[N_E(2)-N_E(0)\geq 2,\ N_F(2)-N_F(0)=1] = (1-\mathbb{P}[N_E(2)-N_E(0)<2])\times(\mathbb{P}[N_F(2)-N_F(0)=1])$
$= (1-\mathbb{P}[X<2 \mid X\sim Poi(2\times2)])\times\mathbb{P}[Y=1 \mid Y\sim Poi(1\times2)] = \left(1-\left(\frac{4^0e^{-4}}{0!}+\frac{4^1e^{-4}}{1!}\right)\right)\times\left(\frac{2^1e^{-2}}{1!}\right) = 2e^{-2}(1-5e^{-4})$

c) $X_E = T_1$ of Eng $\sim exp(2)$ $X_F = T_1$ of Fra $\sim exp(1)$
T_1 of any lang $\sim \min(X_E, X_F) \sim exp(2+1)$
$\mathbb{E}[\min(X_E,X_F)] = \mathbb{E}[exp(3)] = 1/3$ min

☆d) $N = N_E+N_F$. N is PP(3).
$N(t)-N(0)$ is PP$((2+1)t)$, mean $=3t$ std $=3t$

e) $\mathbb{P}[N(10)-N(0)=0] = \mathbb{P}[X=0 \mid X\sim Poi(30)] = \frac{30^0e^{-30}}{0!} = e^{-30}$

f) $\mathbb{P}[N(10)-N(0)\geq2] = 1-\mathbb{P}[N(10)-N(0)<2] = 1-\mathbb{P}[X<2 \mid X\sim Poi(30)] = 1-\left(\frac{30^0e^{-30}}{0!}+\frac{30^1e^{-30}}{1!}\right) = 1-31e^{-30}$

g) $\mathbb{P}[N(10)-N(0)=2,\ N(15)-N(5)=3] = \mathbb{P}[N(5)-N(0)=2]\,\mathbb{P}[N(10)-N(5)=0]\,\mathbb{P}[N(15)-N(10)=3]$
$+ \mathbb{P}[N(5)-N(0)=1]\,\mathbb{P}[N(10)-N(5)=1]\,\mathbb{P}[N(15)-N(10)=2]$
$+ \mathbb{P}[N(5)-N(0)=0]\,\mathbb{P}[N(10)-N(5)=2]\,\mathbb{P}[N(15)-N(10)=1]$
$= \frac{15^2e^{-15}}{2!}\cdot\frac{15^0e^{-15}}{0!}\cdot\frac{15^3e^{-15}}{3!}+\frac{15^1e^{-15}}{1!}\cdot\frac{15^1e^{-15}}{1!}\cdot\frac{15^2e^{-15}}{2!}+\frac{15^0e^{-15}}{0!}\cdot\frac{15^2e^{-15}}{2!}\cdot\frac{15^1e^{-15}}{1!}$

0	5	10	15 (min)
	2	0	3 (calls)
	1	1	2
	0	2	1

☆h) $\mathbb{P}[N(5)-N(0)=4 \mid N(10)-N(0)=4] = \frac{\mathbb{P}[(N(5)-N(0)=4)\cap(N(10)-N(0)=4)]}{\mathbb{P}[N(10)-N(0)=4]} = \frac{\mathbb{P}[N(5)-N(0)=4,\ N(10)-N(5)=0]}{\mathbb{P}[N(10)-N(0)=4]}$
$= \left(\frac{15^4e^{-15}}{4!}\times\frac{15^0e^{-15}}{0!}\right)\Big/\frac{30^4e^{-30}}{4!} = \frac{15^4e^{-30}}{4!}\cdot\frac{4!}{30^4e^{-30}} = \left(\frac{1}{2}\right)^4$

i) $\mathbb{P}[N(2)-N(0)\leq1,\ N(4)-N(0)=3] = \mathbb{P}[N(2)-N(0)=0]\,\mathbb{P}[N(4)-N(2)=3]+\mathbb{P}[N(2)-N(0)=1]\,\mathbb{P}[N(4)-N(2)=2]$
$= \frac{6^0e^{-6}}{0!}\times\frac{6^3e^{-6}}{3!}+\frac{6^1e^{-6}}{1!}\times\frac{6^2e^{-6}}{2!} = e^{-12}(6^2+3\times6^2) = 144e^{-12}$

0	2	4 (min)
	0	3 (calls)
	1	2

L09. E2 N is $PP(120/hour) \Rightarrow N_M$ is $PP(30/hour)$ & N_F is $PP(90/hour)$
N is $PP(2/min) \Rightarrow N_M$ is $PP(0.5/min)$ N_F is $PP(1.5/min)$

a) $\mathbb{P}[N(2)-N(0)=0] = \mathbb{P}[X=0 \mid X\sim Poi(2\times 2)] = \frac{4^0 e^{-4}}{0!} = e^{-4}$

b) $\mathbb{P}[N(2)-N(0)=1,\ N(4)-N(1)=3] = \mathbb{P}[N(1)-N(0)=0]\,\mathbb{P}[N(2)-N(1)=1]\,\mathbb{P}[N(4)-N(2)=2]$
$+\ \mathbb{P}[N(1)-N(0)=1]\,\mathbb{P}[N(2)-N(1)=0]\,\mathbb{P}[N(4)-N(2)=3]$

0	1	2	4
0	1	2	
1	0	3	

$= \frac{2^0e^{-2}}{0!}\cdot\frac{2^1e^{-2}}{1!}\cdot\frac{4^2e^{-4}}{2!} + \frac{2^1e^{-2}}{1!}\cdot\frac{2^0e^{-2}}{0!}\cdot\frac{4^3e^{-4}}{3!}$
$= \frac{112}{3}e^{-8}$

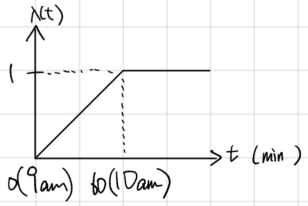
c) $\mathbb{P}[N_M(2)-N_M(0)=1,\ N_F(2)-N_F(0)=2] = \mathbb{P}[M=1 \mid M\sim Poi(\frac{1}{2}\times 2)]\,\mathbb{P}[F=2 \mid F\sim Poi(\frac{3}{2}\times 2)] = \frac{1^1e^{-1}}{1!}\cdot\frac{3^2e^{-3}}{2!} = \frac{9}{2}e^{-4}$

☆ d) $T_i \sim exp(\frac{3}{2})$ (interarrival $i-1^{th} \sim i^{th}$ customer)
$\mathbb{E}[T_1+T_2] = \mathbb{E}[T_1]+\mathbb{E}[T_2] = \frac{1}{1.5}+\frac{1}{1.5} = \frac{4}{3}$ min

☆ e) $\mathbb{P}[N(6)-N(0)<2] = \mathbb{P}[Poi(2\times 6)<2] = \frac{12^0e^{-12}}{0!} + \frac{12^1e^{-12}}{1!} = 13e^{-12}$

L09. E3. N is NHPP($\lambda(t)$)

a)



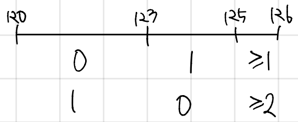
$$\lambda(t) = \begin{cases} \frac{1}{60} t & (0 \leq t \leq 60) \\ 60 & (60 < t \leq 60 \times 8) \\ 0 & (o.w) \end{cases}$$

b) $\mathbb{P}[N(10)-N(0)=5] = \mathbb{P}[Poi(\int_0^{10} \lambda(u)du) = 5] = \mathbb{P}[Poi(\frac{5}{6}) = 5] = \frac{(\frac{5}{6})^5 e^{-\frac{5}{6}}}{5!}$

c) $\mathbb{P}[N(20)-N(0)<1] = \mathbb{P}[Poi(\int_0^{20} \lambda(u)du) < 1] = \mathbb{P}[Poi(\frac{400}{120} = \frac{10}{3}) < 1] = \frac{(\frac{10}{3})^0 e^{-\frac{10}{3}}}{0!} = e^{-10/3}$

d) $\mathbb{P}[N(125)-N(120)=1,\ N(126)-N(123) \geq 2] = \mathbb{P}[Poi(1 \times 3)=0]\,\mathbb{P}[Poi(1 \times 2)=1]\,\mathbb{P}[Poi(1 \times 1) \geq 1]$

$+ \mathbb{P}[Poi(1 \times 3)=1]\,\mathbb{P}[Poi(1 \times 2)=0]\,\mathbb{P}[Poi(1 \times 1) \geq 2]$



$= \frac{3^0 e^{-3}}{0!} \cdot \frac{2^1 e^{-2}}{1!} \cdot \left(1 - \frac{1^0 e^{-1}}{0!}\right) + \frac{3^1 e^{-3}}{1!} \cdot \frac{2^0 e^{-2}}{0!} \cdot \left(1 - \frac{1^0 e^{-1}}{0!} - \frac{1^1 e^{-1}}{1!}\right)$

$= (1-e^{-1}) \cdot 2e^{-5} + (1-2e^{-1}) 3e^{-5} = 5e^{-5} - 8e^{-6}$

L10 E1.

$$P=\begin{array}{c}1\\2\\3\\4\\5\\6\\7\end{array}\begin{pmatrix} .2 & .8 & & & & & \\ .5 & .5 & & & & & \\ .5 & & 0 & & & .5 & \\ & & .8 & 0 & .2 & & \\ & 1 & & & & & \\ & & & & & .2 & .8 \\ & & & & & .5 & .5 \end{pmatrix}$$

class	$\{1,2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6,7\}$
period	1	1	1	1	1
R/T/A	A	T	T	T	A

$$(\pi_1\ \pi_2)\begin{pmatrix} .2 & .8 \\ .5 & .5 \end{pmatrix}=(\pi_1\ \pi_2) \Rightarrow \pi_1=0.2\pi_1+0.5\pi_2. \Rightarrow 0.8\pi_1=0.5\pi_2 \qquad \frac{5}{13}\ \ \frac{8}{13}$$
$$\pi_1+\pi_2=1$$

$$(\pi_6\ \pi_7)\begin{pmatrix} .2 & .8 \\ .5 & .5 \end{pmatrix}=(\pi_6\ \pi_7) \Rightarrow \pi_6=\frac{5}{13},\ \pi_7=\frac{8}{13} \quad (\because \pi_6+\pi_7=1)$$

$$P^{(100)}=\begin{pmatrix} & & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \\ \multicolumn{2}{c}{f_{3,\{1,2\}}} & 0 & 0 & 0 & \multicolumn{2}{c}{f_{3,\{6,7\}}} \\ \multicolumn{2}{c}{f_{4,\{1,2\}}} & 0 & 0 & 0 & \multicolumn{2}{c}{f_{4,\{6,7\}}} \\ \multicolumn{2}{c}{f_{5,\{1,2\}}} & 0 & 0 & 0 & \multicolumn{2}{c}{f_{5,\{6,7\}}} \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & & \end{pmatrix}$$

$$=\begin{pmatrix} 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(5/13) & \frac{1}{2}(8/13) & 0 & 0 & 0 & \frac{1}{2}(5/13) & \frac{1}{2}(8/13) \\ \frac{3}{5}(5/13) & \frac{3}{5}(8/13) & 0 & 0 & 0 & \frac{2}{5}(5/13) & \frac{2}{5}(8/13) \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \end{pmatrix}$$

$f_{3,\{1,2\}}=0.5$

$f_{4,\{1,2\}}=0.8f_{3,\{1,2\}}+0.2f_{5,\{1,2\}}=0.6$

$f_{5,\{1,2\}}=1$

L10. E3

$Poi(\lambda)$ pdf $\mathbb{P}(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$, $P(X \leq k) = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!}$

$X \sim Poi(\lambda)$, $\mathbb{E}[X] = \sum_{x=0}^{\infty} x P(X=x) = \sum_{x=1}^{\infty} x P(X=x) = \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=1}^{\infty} \lambda \cdot \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$

Stochastic Matrix : Sum of row = 1.

Doubly Stochastic Matrix : Sum of column = 1 & stochastic matrix.

Finite DTMC $n \times n$ Doubly Stochastic matrix $\Rightarrow \pi_i = 1/n \quad i \in S$

$X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2) \Rightarrow \min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$ X_1 & X_2 indep.

$Y = \min(X_1, X_2)$, $\left(\mathbb{P}(X_i > t) = e^{-\lambda_i t}\right) \Rightarrow \mathbb{P}(Y > t) = \mathbb{P}(\min(X_1, X_2) > t) = \mathbb{P}[X_1 > t \ \& \ X_2 > t)$

$= \mathbb{P}(X_1 > t)\,\mathbb{P}(X_2 > t) = e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} = e^{-(\lambda_1 + \lambda_2)t} = \mathbb{P}(Z > t)$

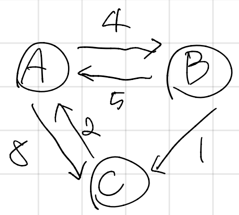
$Z \sim exp(\lambda_1 + \lambda_2)$

$\therefore Y \sim exp(\lambda_1 + \lambda_2)$

L12 E1

CTMC $X = \{X(t), t \geq 0\}$, $S = \{A, B, C\}$, $G = \begin{bmatrix} -12 & 4 & 8 \\ 5 & -6 & 1 \\ 2 & 0 & -2 \end{bmatrix}$

a) rate diagram.



* markov chain ⇒ time interval matters.

L12. E2.

$X = \{X(t), t \geq 0\}$ is CTMC, $S = \{1,2,3\}$, $G = \begin{pmatrix} -4 & ? & 1 \\ 1 & ? & 2 \\ 1 & ? & -2 \end{pmatrix}$, $e^G = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

a) $G = \begin{pmatrix} -4 & 3 & 1 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{pmatrix}$, $(P(t) = e^{tG}.)$

b) $\mathbb{P}[X(2)=3 \mid X(1)=2] = P(2-1)_{23}$

c) $\mathbb{P}[X(1)=2, X(3)=3 \mid X(0)=2] = \mathbb{P}[X(3)=3 \mid X(1)=2, X(0)=2]\,\mathbb{P}[X(1)=2 \mid X(0)=2]$
$= \mathbb{P}[X(3)=3 \mid X(1)=2]\,\mathbb{P}[X(1)=2 \mid X(0)=2]$
$= P(3-1)_{23} \times P(1-0)_{22}$

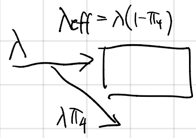
d) Stat. dist. π, $\pi G = 0$. $(\pi_1\ \pi_2\ \pi_3)\begin{pmatrix} -4 & 3 & 1 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{pmatrix} = (0\ 0\ 0) \Rightarrow$

$-4\pi_1 + \pi_2 + \pi_3 = 0$
$3\pi_1 - 3\pi_2 + \pi_3 = 0$
$\pi_1 + 2\pi_2 - 2\pi_3 = 0$
$\pi_1 + \pi_2 + \pi_3 = 1$

$\Rightarrow 4\pi_1 = \pi_2 + \pi_3$
$1 - \pi_1 = \pi_2 + \pi_3$
$\Rightarrow \pi_1 = \frac{1}{5}$, $\pi_3 = \frac{9}{20}$, $\pi_2 = \frac{7}{20}$
$\pi = \left(\frac{4}{20}\ \frac{7}{20}\ \frac{9}{20}\right)$

e) $\mathbb{P}[X(200)=3 \mid X(1)=2] = P(199)_{23} \cong 9/20$

L12. E3. M/M/K/3 Arrival is PP(2/min) Service ~ exp(1)



a) M/M/1/3, $S = \{0,1,2,3\}$ $⓪ \underset{1}{\overset{2}{\rightleftarrows}} ① \underset{1}{\overset{2}{\rightleftarrows}} ② \underset{1}{\overset{2}{\rightleftarrows}} ③$

$$G = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$(\pi_0\ \pi_1\ \pi_2\ \pi_3)\begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} = (0\ 0\ 0\ 0)$$

$-2\pi_0 + \pi_1 = 0 \Rightarrow \pi_1 = 2\pi_0$
$2\pi_0 - 3\pi_1 + \pi_2 = 0 \Rightarrow \pi_2 = 4\pi_0$
$2\pi_1 - 3\pi_2 + \pi_3 = 0$
$2\pi_2 - \pi_3 = 0 \Rightarrow \pi_3 = 2\pi_2$

$\sum \pi_i = \pi_0(1+2+4+8) = 1$
$\pi_0 = \frac{1}{15}$, $\pi_i = \frac{2^i}{15}$, $\pi = (1/15,\ 2/15,\ 4/15,\ 8/15)$

$TH = \lambda_{eff} = 2(1-\frac{8}{15}) = 14/15$.

$L_q = 0\cdot\pi_0 + 0\cdot\pi_1 + 1\cdot\pi_2 + 2\cdot\pi_3 = 1\times\frac{4}{15} + 2\times\frac{8}{15} = 20/15 = 4/3$. customers.
$W_q = L_q/\lambda_{eff} = 4/3 \times \frac{15}{14} = 10/7$ min

b) M/M/2/3. $S = \{0,1,2,3\}$ $⓪ \underset{1}{\overset{2}{\rightleftarrows}} ① \underset{1+1}{\overset{2}{\rightleftarrows}} ② \underset{1+1}{\overset{2}{\rightleftarrows}} ③$
($\because \min(X_A, X_B)$)

cutting method	L→R	=	R→L
0 → 1,2,3	$2\pi_0$	=	π_1
0,1 → 2,3	$2\pi_1$	=	$2\pi_2$
0,1,2 → 3	$2\pi_2$	=	$2\pi_3$

$\pi_0 = \alpha$
$\alpha + \frac{1}{2}\alpha + \frac{1}{2}\alpha + \frac{1}{2}\alpha = 1$
$\pi = \{1/7\ \ 2/7\ \ 2/7\ \ 2/7\}$

$TH = \lambda_{eff} = \lambda(1-\pi_3) = 2\times\frac{5}{7} = \frac{10}{7}$

$L_q = 0\cdot\pi_0 + 0\cdot\pi_1 + 0\cdot\pi_2 + 1\cdot\pi_3 = \frac{2}{7}$ customers.
$W_q = L_q/\lambda_{eff} = 2/7 \times \frac{7}{10} = 1/5$ min

c) M/M/3/3. $S = \{0,1,2,3\}$ $⓪ \underset{1}{\overset{2}{\rightleftarrows}} ① \underset{1+1}{\overset{2}{\rightleftarrows}} ② \underset{1+1+1}{\overset{2}{\rightleftarrows}} ③$

cutting method	L→R	=	R→L
0 → 1,2,3	$2\pi_0$	=	π_1
0,1 → 2,3	$2\pi_1$	=	$2\pi_2$
0,1,2 → 3	$2\pi_2$	=	$3\pi_3$

$\pi_1 = 2\pi_0$
$\pi_2 = \pi_1 = 2\pi_0$
$\pi_3 = \frac{2}{3}\pi_2 = \frac{4}{3}\pi_0$

$\pi_0 + 2\pi_0 + 2\pi_0 + \frac{4}{3}\pi_0 = 1$
$\pi_0 = \left(\frac{3+6+6+4}{3}\right)^{-1} = \frac{3}{19}$.
$\pi = (3/19\ \ 6/19\ \ 6/19\ \ 4/19)$

$TH = \lambda(1-\frac{4}{19}) = 2\times\frac{15}{19} = \frac{30}{19}$

$L_q = 0\cdot\pi_0 + 0\cdot\pi_1 + 0\pi_2 + 0\cdot\pi_3 = 0$
$W_q = 0$.

L12. E4 up time $\sim \exp(1/3)$ repair $\sim \exp(1)$

a) $X(t)$ is # machine that are up at time t. $S = \{0,1,2,3\}$

$$\begin{tikzcd}
0 \arrow[r, bend left, "1+1"] & 1 \arrow[l, bend left, "1/3"] \arrow[r, bend left, "1+1"] & 2 \arrow[l, bend left, "1/3+1/3"] \arrow[r, bend left, "1"] & 3 \arrow[l, bend left, "1/3+1/3+1/3"]
\end{tikzcd}$$

cutting method.	L→R	=	R→L
$0 \to 1,2,3$	$2\pi_0$	=	$\frac{1}{3}\pi_1$
$0,1 \to 2,3$	$2\pi_1$	=	$\frac{2}{3}\pi_2$
$0,1,2 \to 3$	$1\pi_2$	=	$\frac{3}{3}\pi_3$

$\pi_1 = 6\pi_0$
$\pi_2 = 3\pi_1 = 18\pi_0$
$\pi_3 = \pi_2 = 18\pi_0$

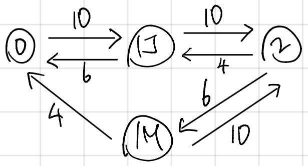
$\pi_0 = 1/43$
$\pi = \{1/43, 6/43, 18/43, 18/43\}$

long-run frac. of time that all machines are up & running $= \pi_3 = 18/43$

b) $4\times0\times\pi_0 + 4\times1\times\pi_1 + 4\times2\times\pi_2 + 4\times3\times\pi_3 = 4\left(6/43 + 36/43 + 54/43\right) = 384/43$

L12. E5. arrival is PP(10/h) 2 server $\begin{cases} J^{(1)} \sim exp(6/h) \\ M^{(2)} \sim exp(4/h) \end{cases}$

a) State space, $S = \{0, 1J, 1M, 2\}$ $X(t)$ is number of customer in the system.



$$G = \begin{Bmatrix} -10 & 10 & & \\ 6 & -16 & & 10 \\ 4 & & -14 & 10 \\ & 4 & 6 & -10 \end{Bmatrix}$$

b) $\pi G = 0$. $(\pi_0 \ \pi_{1J} \ \pi_{1M} \ \pi_2)\begin{pmatrix} -10 & 10 & & \\ 6 & -16 & & 10 \\ 4 & & -14 & 10 \\ & 4 & 6 & -10 \end{pmatrix} = (0\ 0\ 0\ 0) \Rightarrow$

$$\begin{cases} -10\pi_0 + 6\pi_{1J} + 4\pi_{1M} = 0 \\ 10\pi_0 - 16\pi_{1J} + 4\pi_2 = 0 \\ -14\pi_{1M} + 6\pi_2 = 0 \\ 10\pi_{1J} + 10\pi_{1M} - 10\pi_2 = 0 \\ \pi_0 + \pi_{1J} + \pi_{1M} + \pi_2 = 1 \end{cases}$$

$\pi_{1M} = \frac{3}{7}\pi_2$, $\pi_{1J} = \frac{4}{7}\pi_2$

$\pi_{1J} + \pi_{1M} = \pi_2$

$\pi_0 + 2\pi_2 = 1$

$\pi = [18/88, 20/88\ \ 15/88\ \ 35/88]$

c) Server 1, J busy $\Rightarrow \pi_{1J} + \pi_2 = 55/88$

Server 2, M busy $\Rightarrow \pi_{1M} + \pi_2 = 50/88$