Lecture 10. More on PP and DTMC

Sim, Min Kyu, Ph.D.
mksim@seoultech.ac.kr

서울과학기술대학교 데이터사이언스학과

BK21 FOUR 데이터사이언스와 비즈니스포텐셜

Minimum of exponential random variables

- Minimum of exponentially distributed random variables is again exponential random variable with parameter that is the sum of parameters of original exponential random variables.
- Thm. $X_1 \sim exp(\lambda_1), X_2 \sim exp(\lambda_2)$, X_1 and X_2 are indep.
  $\Rightarrow min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$

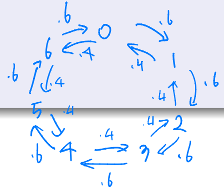
Doubly stochastic matrix

Definition - Doubly stochastic matrix

- A matrix is said to be *stochastic* if each row sums up to 1.
    - Every legit transition probability matrix in DTMC is *stochastic*.
- A stochastic matrix is said to be *doubly stochastic* if each column sums up to 1 as well.
    - Ex) The first example of periodic matrix.

Theorem

- n by n doubly stochastic matrix for finite states DTMC has stationary distribution $\pi_i = 1/n$ for every state $i \in S$.
    - Ex1) The above example.
    - Ex2) Ring structure DTMC.



Chapman-Kolmogrov Equation for DTMC

- n-step probability review
- We want to have (m+n)-step transition probability (matrix) using m-step and n-step transition matrix.
- $\mathbf{P}_{ij}^{n+m} = \sum_{k \in S} \mathbf{P}_{ik}^{n} \mathbf{P}_{kj}^{m}$
- Perspective of ''path''.
- pf) $\mathbf{P}_{ij}^{n+m} =$

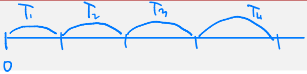
- Matrix Algebra point of view

-

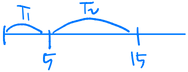
$$\mathbf{P} = \begin{array}{r} (C,C) \\ (NC,C) \\ (C,NC) \\ (NC,NC) \end{array} \begin{pmatrix} .2 & & .8 & \\ .4 & & .6 & \\ & .6 & & .4 \\ & .8 & & .2 \end{pmatrix}$$

Q1) $\mathbb{P}[Y_{n+2} = (NC,C) | Y_n = (C,C)]$=?

Interarrival times in PP



- In LN( ), p( ). We let T_1 :=*Time to first arrival* in $PP(\lambda)$, and showed $T_1 \sim exp(\lambda)$.
- Furthermore, for $n > 1$, let T_n denote the elapsed time between $(n-1)$st and the nth event. The sequence $\{T_n, n = 1, 2, ...\}$ is called *sequence of interarrival times.*
- ex) What does $T_1 = 5$ and $T_2 = 10$ mean?



- Theorem: $T_n \sim exp(\lambda)$, for all $n = 1, 2, ...$.
- Remark. The PP from any point on is independent of all that has previously occurred (by independent increments), and also has the same distribution as the original process (by stationary increments).
- Remark. In other words, the process has *no memory*, and hence exponential interarrival times are naturally expected.

blank

Extended applications of DTMC and PP

- We can model various things.
- Sports Analysis
- Gambling
- Financial Market
- and more…

- This lecture could help your project as well as the exam.

Gambler's ruin probability

18/38

1 0 1 2 3 4 5 6 7 8 1

20/38

absorbing state

1~7 : temporary

0,8 : permanant

- Suppose you have \$3(=x), and bet \$1 with winning probability $p = 18/38$ until your wealth becomes \$0(=a) or your wealth becomes \$8(=b). What is chance of you will leave Casino with \$8? $P^{\infty}_{38} = ?$

$$\mathbf{P} = \begin{array}{r} 0(lose) \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8(won) \end{array} \begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ .53 & \bullet & .47 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & .53 & \bullet & .47 & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & .53 & \bullet & .47 & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & .53 & \bullet & .47 & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & .53 & \bullet & .47 & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & .53 & \bullet & .47 & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & .53 & \bullet & .47 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 \end{pmatrix}$$

$P^{1}_{34} = 0.47$

- Result of $a = 0, b = 8, p = 18/38$

$$\mathbf{P}^{\infty} = \begin{array}{c} 0(lose) \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8(won) \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .08 \\ .82 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .18 \\ .72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .28 \\ .60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .40 \\ .48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .52 \\ .33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .67 \\ .18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .82 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$P^{\infty}_{38} = 0.28$

- Result of $a \overset{\text{lose}}{=} 0, b \overset{\text{win}}{=} 1000, p = 18/38, x \overset{\text{current}}{=} 100.$
  - What is the quantity for $\mathbf{P}^{\infty}_{100\$\to win}$?

6.2×10^{-42}

- Result of $a = 0, b = 1000, p = 19/38, x = 100.$
  - What is the quantity for $\mathbf{P}^{\infty}_{100\$\to win}$?

$X_0 = \mathbb{E}X_1 = \cdots = \mathbb{E}X_T$

0.1

$100 = X_0 = \mathbb{E}X_T = P_{win} \times 1000 + (1 - P_{win}) \times 0$

$P_{win} = 10\%$

- Result of $a = 0, b = 10 \times 100\$, p = 18/38, x = 1 \times 100$ \$ (bet 100\$ for each)
  - What is the quantity for $\mathbf{P}^{\infty}_{1\times 100\$\to win}$?

$0 \xleftarrow{20/38} 100 \xrightarrow{18/38} 200 \quad \cdots \quad 1000$

6%

- Appendix - MATLAB code for Gambler's ruin

```
>>
>>
>>
>>
>>
>> p=18/38;
q=1-p;
a=0; % lose
b=1000; % won
n=b-a+1;
P=diag(p*ones(n-1,1),1)+diag(q*ones(n-1,1),-1);
P(1,:)=[1' zeros(n-1,1)']';
P(n,:)=[zeros(n-1,1)' 1']';
P;
P50000=P^500000;
P50000(101,n)
```

blank

Squash

- Racket sports (court number 5 in CRC)
- Rules
  - Two players, three or five games.
  - Only the server scores points.
  - The server, on winning a rally, scores a point
  - The receiver, on winning a rally, becomes the server.
  - The player who scores nine points wins the game

- Rules (cont'd)
  - Suppose A and B are playing for the first set and $8 : \overline{7}$ now. (A's score is 8, B's score is 7, and B is serving)
  - Suppose B wins this play so that it becomes $8 : \overline{8}$.
  - Because A got to 8 first, A can decide either
    - *i)* This set ends at 9
    - *ii)* This set ends at 10
- Questions
  - Suppose the chance of A winning a play is 0.6, then should A choose i) or ii)?

- Suppose A decides "i) This set ends at 9".
- DTMC
  - Transition diagram and matrix

  - Classification of states

  - What is the chance of A winning this game?

- Suppose A decides “ii) This set ends at 10”.
- DTMC

$$
\mathbf{P} = \begin{array}{c} lose \\ 8:\overline{8} \\ \overline{8}:8 \\ 8:\overline{9} \\ \overline{8}:9 \\ 9:\overline{8} \\ \overline{9}:8 \\ 9:\overline{9} \\ \overline{9}:9 \\ win \end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{pmatrix}
$$

- What is the chance of A winning this game?

- What if the chance of A winning a rally is not 0.6, but for general p?

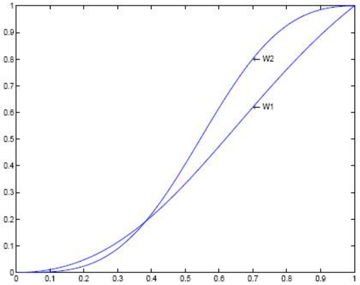


Figure 1: Probability of winning

- optimal decision
  - If $p \leq$, then choose i) ends at 9
  - Otherwise, choose ii) ends at 10

- Reference
  - Optimal Decision for the Squash Player
  - Jan Vecer, Columbia University, Department of Statistics
  - Journal of Chinese Statistical Association, 2004.
  - www.stat.columbia.edu/∼vecer/squash.ps

Stock price - binomial tree

- Let X_n be the closing price of the stock at nth day.
- Let $p = \mathbb{P}(X_{n+1} = x + 1 | X_n = x)$, and $1 - p = \mathbb{P}(X_{n+1} = x - 1 | X_n = x)$
- Consider future evolution, starting with $X_0 = 100$.

- Consider an European call option which matures at day 5 with exercise price 101.
- (If you possess one unit of the call option, then at the day 5, you have a right to buy the stock at 101 dollars.)
- If $X_5 = 103$, then you can buy the stock at 101 and sell at 103. In this case, you earn 2 dollar.
- If $X_5 = 99$, then you still can buy the stock at 101. But you would not do it because you can buy a stock at 99 dollars. (Possessing call option is the "right" not the "obligation")
- i.e., the payoff of a call option is $(X_5 - 101)^+$

- What is the expected payoff for the option, when $p = 0.6$?

blank

Soccer scoring by PP

- Sports like soccer can be modelled by Poisson process because of many trials with small probability to make goal.
- Suppose the team A has goal-ability of $PP(1\ per\ 30\ minutes)$, and the team B has goal-ability of $PP(1\ per\ 25\ minutes)$.
- Assume the two Poisson processes are independent. (very absurd assumption…)
- What are the chance for A to win/draw/lose?
- Let $X_A :=$ # of goals that A team makes during 90 minutes, and define X_B same way.
- By Poisson Process principle, $X_A \sim poi(3)$, $X_B \sim poi(90/25)$.
- $\mathbb{P}(X_A > X_B)$, $\mathbb{P}(X_A = X_B)$, $\mathbb{P}(X_A < X_B)$
- Why PP is not a good model for Soccer game scoring?

blank

$$\mathrm{Ber}(p) \xrightarrow{n\ \text{trial}} \mathrm{bin}(p,n) \xrightarrow{np\ \text{large}} N(\mu,\sigma^2)$$

$$\mathrm{bin}(p,n) \xrightarrow[p\ \text{small}]{n\ \text{large}} \mathrm{Poi}(\lambda) \xrightarrow[\text{inter event time}]{} \exp(\lambda)$$

$\exp(\lambda)$ $\begin{cases} \text{continuous} \\ \text{memoryless} \end{cases}$

$$\exp(\lambda) \longleftrightarrow \mathrm{geo}(p)$$

$\mathrm{geo}(p)$ $\begin{cases} \text{discrete} \\ \text{memoryless} \end{cases}$

X_n: n번째 시행에서 관찰한 눈의 개수

1회 1.2회 1.5회 2회 3회 6회

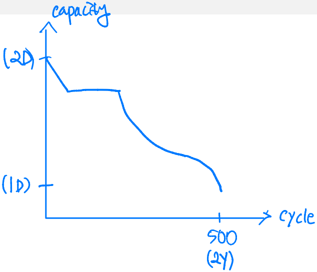
$$\begin{tikzcd} 0 \arrow[r, "1"] & 1 \arrow[r, "5/6"] \arrow[loop below, "1/6"] & 2 \arrow[r, "4/6"] \arrow[loop below, "2/6"] & 3 \arrow[r, "3/6"] \arrow[loop below, "3/6"] & 4 \arrow[r, "2/6"] \arrow[loop below, "4/6"] & 5 \arrow[r, "1/6"] \arrow[loop below, "5/6"] & 6 \arrow[loop below, "1"] \end{tikzcd}$$

"13.7회"

Previous Exercise

$$\mathbf{P} = \begin{pmatrix} .2 & .8 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 \\ 0 & .25 & 0 & .75 & 0 \\ 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

blank



Previous Exercise

- Suppose you are modelling your economic status using DTMC.
  - There are five possible economic status: Trillionaire (T), Billionaire (B), Millionaire (M), Thousandaire (Th), Bankrupt (Bk).
  - Your economic status changes every month.
  - Once you are bankrupt, there is no chance of coming back to other states.
  - Also, once you become a billionaire or a trillionaire, you will never be millionaire, thousandaire, or bankrupt.
  - Other chances of transitions are as follows. (Chances for transitions to itself are omitted.)

Transition	Prob.
T $\rightarrow$ B	0.9
B $\rightarrow$ T	0.3
M $\rightarrow$ B	0.1
M $\rightarrow$ Th	0.5
Th $\rightarrow$ M	0.2
Th $\rightarrow$ Bk	0.3

blank

Previous Exercise

- Nortel in Canada operates a call center for customer service.
  - Assume that each caller speaks either English or French, but not both.
  - Suppose that the arrival for each type of calls follows a Poisson process.
  - The arrival rates for English and French calls are 2 and 1 calls per minute, respectively.
  - Assume that call arrivals for different types are independent.

(a) What is the probability that the 2nd English call will arrive after minute 5?

(c) What is the expected time for the 1st call that can be answered by a bilingual operator (that speaks both English and French) to arrive?

(d) Suppose that the call center is staffed by bilingual operators that speak both English and French.

- Let $N(t)$ be the total number of calls of both types that arrive during $(0, t]$. What kind of a process is $N(t)$?
- What is the mean and variance of $N(t)$?

(g) What is the probability that two calls arrive in the interval 0 to 10 minutes and three calls arrive in the interval from 5 to 15 minutes?

blank

blank

Previous Exercise

- Calls to a center follow a Poisson process with rate 120 calls per hour.
  - Each all has probability 1/4 from a male customer.
  - The call center opens at 9am each morning.

(d) What is the expected elapse of time from 9am until the second female customer arrives?

blank

Previous Exercise

Assume that call arrival to a call center follows a NHPP. The call center opens from 9am to 5pm. During the first hour, the arrival rate increases linearly from 0 at 9am to 60 calls per hour at 10am. After 10am, the arrival rate is constant at 60 calls per hour.

(a) Plot the arrival rate function $\lambda(t)$ as a function of time t; indicate clearly the time unit used.

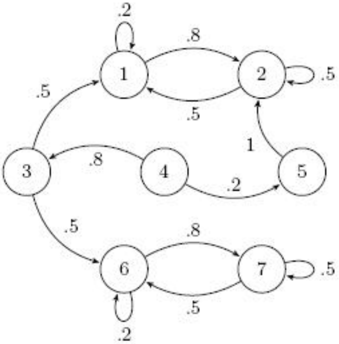
(b) Find the probability that exactly 5 calls have arrived by 9:10am.

blank

blank

Exercise 1

- Calculate $\mathbf{P}^{100}$ for the following DTMC. Make sure you can retrieve every element of the $\mathbf{P}^{100}$ matrix.



(Solution)

$$\mathbf{P} = \begin{pmatrix} .2 & .8 & 0 & 0 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 & 0 & 0 \\ .5 & 0 & 0 & 0 & 0 & .5 & 0 \\ 0 & 0 & .8 & 0 & .2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .2 & .8 \\ 0 & 0 & 0 & 0 & 0 & .5 & .5 \end{pmatrix}.$$

Step 1) Identifying classes.
Recurrent classes: $\{1,2\}$, $\{6,7\}$. Transient classes: $\{3\}$, $\{4\}$, $\{5\}$
(So far, how many classes do you see in this example? There are two recurrent classes and three transient classes. Therefore, there are total five classes.)

Step 2) Set up p^{100} and identify as many zeros as possible.

$$\mathbf{P}^{100} \approx \lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & & \\ & & 0 & 0 & 0 & & \\ & & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x \end{pmatrix}.$$

Step 3) Identify the "x" in above matrix. The "x" are limiting probability from a recurrent class to the recurrent class itself. Thus, we can find limiting probability by getting stationary distribution of the small Markov chain of recurrent class. \
For the first recurrent class of $\{1, 2\}$,

$$
\begin{aligned}
\pi_1 &= 0.2\pi_1 + 0.5\pi_2 \\
\pi_2 &= 0.8\pi_1 + 0.5\pi_2 \\
\pi_1 + \pi_2 &= 1
\end{aligned}
$$

Thus, $(\pi_1, \pi_2) = (5/13, 8/13)$. Similarly, we have $(\pi_6, \pi_7) = (5/13, 8/13)$. Now you have identified the "x"s in the previous matrix arriving to

$$
\mathbf{P}^{100} \approx \lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix}
5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\
5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & & \\
 & & 0 & 0 & 0 & & \\
 & & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \\
0 & 0 & 0 & 0 & 0 & 5/13 & 8/13
\end{pmatrix}.
$$

Step 4) We shall find the limiting probability going from transient state to recurrent class, i.e. $f_{3,\{1,2\}}, f_{4,\{1,2\}}, f_{5,\{1,2\}}$. Once you find these three numbers out, you will automatically get $f_{3,\{6,7\}}, f_{4,\{6,7\}}, f_{5,\{6,7\}}$.

$$\mathbf{P}^{100} \approx \lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 5/13 f_{3,\{1,2\}} & 8/13 f_{3,\{1,2\}} & 0 & 0 & 0 & 5/13 f_{3,\{6,7\}} & 8/13 f_{3,\{6,7\}} \\ 5/13 f_{4,\{1,2\}} & 8/13 f_{4,\{1,2\}} & 0 & 0 & 0 & 5/13 f_{4,\{6,7\}} & 8/13 f_{4,\{6,7\}} \\ 5/13 f_{5,\{1,2\}} & 8/13 f_{5,\{1,2\}} & 0 & 0 & 0 & 5/13 f_{5,\{6,7\}} & 8/13 f_{5,\{6,7\}} \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \end{pmatrix}$$

Step 5) It remains us to find $f_{3,\{1,2\}}, f_{4,\{1,2\}}, f_{5,\{1,2\}}$ as explained above. So let's begin filling this out by calculating the absorption probabilities into class $\{1,2\}$

$$f_{3,\{1,2\}} = 0.5$$
$$f_{4,\{1,2\}} = 0.8f_{3,\{1,2\}} + 0.2f_{5,\{1,2\}}$$
$$f_{5,\{1,2\}} = 1$$

Solving by substituting the first and third equations in the second, we get

$$f_{3,\{1,2\}} = 1/2$$
$$f_{4,\{1,2\}} = 3/5$$
$$f_{5,\{1,2\}} = 1$$

Obviously, the absorption probabilities into class $\{6,7\}$, can be calculated directly since for the transient states the sum of all absorption probabilities is 1.

$$f_{3,\{6,7\}} = 1 - f_{3,\{1,2\}} = 1/2$$
$$f_{4,\{6,7\}} = 1 - f_{4,\{1,2\}} = 2/5$$
$$f_{5,\{6,7\}} = 1 - f_{5,\{1,2\}} = 0$$

Step 6) Deliver solution.

$$\mathbf{P}^{100} \approx \lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 5/26 & 4/13 & 0 & 0 & 0 & 5/26 & 4/13 \\ 3/13 & 24/65 & 0 & 0 & 0 & 2/13 & 16/65 \\ 5/13 & 8/13 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \\ 0 & 0 & 0 & 0 & 0 & 5/13 & 8/13 \end{pmatrix}$$

Exercise 2

- Carefully state the definitions for the terms below.
  - Reach
  - Communicate
  - Class
  - Irreducible
  - Reducible
  - Period
  - Periodic
  - Aperiodic
  - Recurrent
  - Absorbing state
  - Transient

(Solution)

- A state i can ***reach*** a state j and write $i \to j$ if $\exists n$ s.t. $P_{ij}^n > 0$.
- State i and j are said to ***communicate*** and write $i \leftrightarrow j$ if $i \to j$ and $j \to i$.
- A group of states that communicate is said to be a ***class***.
- MC X_n is said to be ***irreducible*** if all states communicate. (Or, equivalently, MC X_n is said to be ***irreducible*** if $\exists$ only one class.)
- MC X_n is said to be ***reducible*** if there are more than one class in MC.
- For a state $i \in S$, ***period*** $d(i) := gcd\{n, P_{ii}^n > 0\}$
- MC X_n is said to be ***periodic*** if $\exists\, i$ with $d(i) > 1$
- MC X_n is said to be ***aperiodic*** if not ***periodic***. (Or, equivalently, MC X_n is said to be ***aperiodic*** if all states have period of 1.)
- A state i is said to be ***recurrent*** if, starting from i, the probability of getting back to i in some future is 1.
- A state i is said to be ***absorbing state***, as a special case of recurrent state, if $p_{ii} = 1$.
- A state i is said to be ***transient*** if, starting from i, the probability of getting back to i is less than 1. (Remark. The chance of staying in transient class after infinite amount of time is zero.)

Exercise 3

(Solutions to questions below can be all found in the lecture note.)

- What is pmf and cdf for the Poisson distribution?
- For $X \sim Poi(\lambda)$, show that $\mathbb{E}X = \lambda$.
- What is the definition of the stochastic matrix? What is the definition of the doubly stochastic matrix and what is an important theorem regarding its stationary distribution?
- Suppose $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$ and they are independent, then what is the distribution of the random variable $min(X_1, X_2)$? Prove your answer.
- Prove that the time to first arrival in $PP(\lambda)$ follows $exp(\lambda)$.

```
"Quality is never an accident. It is always the result of intelligent effort.
- John Ruskin."
```