Lecture 3. Newsvendor Model (1)

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Newsvendor Model

- Highly related to supply chain, logistics, and warehousing.
- Optimal ordering on perishable goods/services. 부패재

Motivational Example

- Your brother will start part time job of selling newspapers at subway station in the morning. You are asked how many he should prepare for selling.

Input of Newsvendor Model

- What information do you need in order to give him a good advice?
  1. How many customers (demand)
  2. How much to sell at (retail price)
  3. How much to pay to wholesaler (material cost)
  4. What happens to unsold item (salvage value)

  retail price
  wholesale price
  value of unsold item
  avg # sales
- Suppose your brother gave the following information…
  1. [11, 15] w/ each 20% of chance
  2. $2
  3. $1
  4. $0.5

Objective of your brother

- Maximize expected profit
-
-

- Assume that a discrete r.v D follows the following distribution.

d	20	25	30	35
$\mathbb{P}[D=d]$	0.1	0.2	0.4	0.3
$30 \wedge d$	20	25	30	30
$(30-d)^+$	10	5	0	0
$24 \wedge d$	20	24	24	24
$(24-d)^+$	4	0	0	0

- $\mathbb{E}[30 \wedge D] =$ $2+5+12+9=28$
- $\mathbb{E}[(30-D)^+] =$ $1+1+0+0=2$
- $\mathbb{E}[24 \wedge D] =$ $10+4.8+9.6+7.2=21.6$
- $\mathbb{E}[(24-D)^+] =$ 0.4

Solving the newsvendor (your bro) problem (naive way)

profit = revenue − cost = (sales revenue + salvage revenue) − (material cost)

$$= \left(\min(X,D)\times 2 + (X-D)^{+}\times 0.5\right) - X\times 1$$

(X) preparation (Stock) \ (D) demand	11 (20%)	12 (20%)	13 (20%)	14 (20%)	15 (20%)	Expected Profit
11	$11\cdot2+0\cdot0.5-11\cdot1$ $=11$	11	11	11	11	11
12	$11\cdot2+1\cdot0.5-12\cdot1$ $=10.5$	$12\cdot2+0\cdot0.5-12\cdot1$ $=12$	12	12	12	11.7
13	10	11.5	13	13	13	12.1
14	9.5	11	12.5	14	14	12.2
15	9	10.5	12	13.5	15	12

stock > demand

over prepared

overstock

overstock unit cost = 0.5

under prepared

stock < demand

understock!

understock unit cost = 1

Economic cost affecting decision

Stock: X, Demand: D

- Cost of overstock
  - Economic cost incurred by having overstock
  - (extra item that exceeds the demand; when Stock $>$ Demand)
  - e.g. holding cost, perishable item.
  - Overstock by how many? $(X-D)^+$
- Cost of understock
  - Economic cost incurred by having understock
  - (insufficient stock than demand; when Stock $<$ Demand)
  - e.g. Lost sale as opportunity cost
  - e.g. reputation loss, lost revenue, cancellation reward.
  - Understock by how many? $(D-X)^+$
- **Newsvendor model is problem of balancing between overstock and understock cost!**

Mathematical representation

- If demand D is random and you prepare X unit, then
  - # of sales: $\min(X, D)$
  - # of unit for overstock: $(X-D)^+$
  - # of unit for understock: $(D-X)^+$
- If retail price is p, material cost is c, and salvage price is s, then
  - C_o unit cost for overstock: $(C-S)$
  - C_u unit cost for understock: $(p-c)$
- Newsvendor model is to find an optimal value x^* that minimize total expected economic cost by solving:

$$x^* = \text{argmin}_X \; c_o \mathbb{E}[(X-D)^+] + c_u \mathbb{E}[(D-X)^+]$$

- Calculate expected profit using

$$\mathbb{E}[\text{profit}] = \mathbb{E}(\text{sale rev.}) + \mathbb{E}(\text{salvage rev.}) - \mathbb{E}(\text{material cost})$$

Solution to Newsvendor model

Theorem

- For continuous distribution, find y s.t. $F(y) = \frac{c_u}{c_o+c_u}$
- For discrete distribution, smallest y such that $\boxed{F(y) \geq \frac{c_u}{c_o+c_u}}$

- Let's solve your brother's problem with this method!

discrete distribution $\Rightarrow$ $C_o = 0.5$ $C_u = 1.$ $F(y) \geq 2/3$

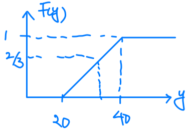
y	$F(y) = \mathbb{P}(Y \leq y)$
11	0.2
12	0.4
13	0.6
14	0.8 smallest y meets $F(y) \geq 2/3$
15	1

- Information
  - Demand $X \sim U(20, 40)$ gallons
  - retail | \$ 2/ gallons
  - material | \$ 1/ gallons
  - salvage | \$ 0.5/ gallons

$C_u = \$1$

$C_o = \$0.5$

$F(y) = \frac{1}{1.5} = 2/3$



$y = \frac{100}{3}$

$\mathbb{E}[\text{profit}] = \underset{\text{per unit}}{\text{sales price}} \times \mathbb{E}[\#\text{ Sales}] + \underset{\text{per unit}}{\text{salvage price}} \times \mathbb{E}\left[\begin{matrix}\#\text{ salvage}\\ \text{item}\end{matrix}\right]$

$- \underset{\text{per unit}}{\text{material price}} \times \mathbb{E}[\#\text{ stocks}]$

$= 2 \cdot \mathbb{E}\left[\min\left(\frac{100}{3}, D\right)\right] + 0.5\, \mathbb{E}\left[\left(\frac{100}{3} - D\right)^+\right] - 1 \cdot \frac{100}{3}$

blank

$$= 2\left(\int_{20}^{100/3} \frac{1}{20}x\,dx + \int_{100/3}^{40} 100/3 \cdot \frac{1}{20}\,dx\right) + \frac{1}{2}\left(\int_{20}^{100/3} \frac{1}{20}\left(\frac{100}{3} - x\right)dx\right) - 100/3$$

$$= 2\left(\frac{1}{40}x^2\Big|_{20}^{100/3} + \frac{5}{3}x\Big|_{100/3}^{40}\right) + \frac{1}{2}\left(\frac{5}{3}x - \frac{1}{40}x^2\Big|_{20}^{100/3}\right) - 100/3$$

$$= 2\left(\frac{1}{40}\left(\frac{10000}{9} - \frac{3600}{9}\right) + \frac{5}{3}\left(\frac{20}{3}\right)\right) + \frac{1}{2}\left(\frac{5}{3}\left(\frac{40}{3}\right) - \frac{1}{40}\left(\frac{10000}{9} - \frac{3600}{9}\right)\right) - 100/3$$

$$= 2\cdot\left(\frac{160}{9} + \frac{100}{9}\right) + \frac{1}{2}\left(\frac{40}{9}\right) - \frac{100}{3} = \frac{520}{9} + \frac{20}{9} - \frac{100}{3} = \frac{520+20-300}{9} = \frac{240}{9} = \frac{80}{3}$$

Exercise 1

- The lemonade sells for $18 per gallon (retail price (p)) but only costs $3 per gallon (material price (c)) to make. If we run out of lemonade during the game, it will be impossible to get more. On the other hand, leftover lemonade has a value of $1 (salvage value ($s$)). Let D be the demand of the lemondade.

  - c_u, c_o $\begin{cases} C_u = 18 - 3 = 15 \\ C_o = 3 - 1 = 2 \end{cases}$

  - $\mathbb{E}[\text{profit}(x)]$= $\mathbb{E}[\text{sales revenue}] + \mathbb{E}[\text{salvage revenue}] - \mathbb{E}[\text{material cost}]$

    # stock

    $= 18 \cdot \mathbb{E}[\min(D, x)^+] + 1 \cdot \mathbb{E}[(x - D)^+] - 3 \cdot x$

Exercise 2

retail price　　material price

- The selling price of lettuce salad is \$6, the buying price of one unit of lettuce is \$1. Of course, leftover lettuce of a day cannot be used for future salad and you have to pay 50 cents per unit of lettuce for disposal.

  \$ -0.5 salvage value

  - c_u, c_o $\begin{cases} C_u = 6 - 1 = 5 \\ C_o = 1 - (-0.5) = 1.5 \end{cases}$

  - $\mathbb{E}[\text{profit}(x)] = \underbrace{6 \cdot \mathbb{E}[\min(D, x)]}_{\text{sales revenue}} \underbrace{- 0.5 \cdot \mathbb{E}[(x - D)^+]}_{\text{salvage revenue}} \underbrace{- 1 \cdot x}_{\text{material cost}}$

Exercise 3

- Give an example of newsvendor model scenario in your daily life and identify understock cost and overstock cost. Which cost is high? The relative difference in the two costs affects your decision making?

(Suggested discussion)

In basic newsvendor scenario of retail business (like your brother being newsboy at subway station), understock incurs due to lack of supply to realized demand. Understock will result in losing sale opportunities. Unit understock cost is high when the difference between unit retail price and unit material cost is high.

Overstock incurs when supply exceed the realized demand. When overstock cost is high, then it implies either the salvage value of leftover item is very low or even incur additional cost like disposal cost. Unit overstock cost is high when the difference between unit material cost and unit salvage value is high.

Suppose you are in business of selling luxury items. This business has good profit margin for selling an item. For example, you get an item at 100 dollars but sells at 1000 dollars to customer and let the salvage value zero. Your unit understock cost is 900 dollars, i.e. you lose 900 dollars whenever you have a customer but no item. Your overstock cost is 100 dollars. Your unit understock cost is much higher than unit overstock cost. Thus, you do not want to have understock situations. Therefore, you must prepare stocks very aggressively.

Another example can be selling drinks on a beach. Drinks are quite expensive, however, material cost is low. So, understock cost is far higher than overstock cost. Thus, vendors seem to always have plenty to sell.

Think about outlet vs department store. Which business have relatively higher/lower understock/overstock cost structure? How often, as a customer, you can't find the right size of item for you and walk away from the store?

Also, think about example in the opposite side.

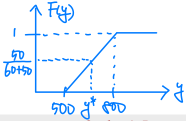
Exercise 4

- Next month's production at a manufacturing company will use a certain solvent for part of its production process. Assume that the solvent costs \$50 per liter. Due to short product lifecycle, unused solvent cannot be used in following months. There will be a \$10 disposal charge for each liter of solvent left over at the end of the month. If there is a shortage of solvent, the production process is seriously disrupted at a cost of \$100 per liter of shortage. Assume that the demand is governed by the continuous uniform distribution varying between 500 and 800 liters. Identify the i) overstock cost per unit, ii) understock cost per unit, iii) decide how many units to order. iv) Suppose your order quantity is 600 liters, then what would be the expected total cost?[1]

i) overstock cost per unit = $50+10=60$

ii) understock cost per unit = $100-50=50$

iii)



$y^* = 500 + (800-500)\cdot\frac{50}{60+50} \approx 636$

blank

iv) Total Cost $= \mathbb{E}[\text{Ordering Cost}] + \mathbb{E}[\text{Shortage Cost}] + \mathbb{E}[\text{Disposal Cost}]$

$$= 50 \times 600 + 100 \times \mathbb{E}[(D-600)^+] + 10 \times \mathbb{E}[(600-D)^+]$$

$D \sim U(500, 800)$

Eq 1. $\mathbb{E}[(D-600)^+] = \int_{600}^{800} (x-600) \cdot \frac{1}{300}\, dx = \frac{1}{300}\left[\frac{1}{2}x^2 - 600x\right]_{600}^{800} = \frac{200}{3}$

Eq 2. $\mathbb{E}[(600-D)^+] = \int_{500}^{600} (600-x) \frac{1}{300}\, dx = \frac{1}{300}\left[600x - \frac{1}{2}x^2\right]_{500}^{600} = \frac{50}{3}$

Based on Eq 1. & Eq 2.,

Total Cost $= 50 \times 600 + 100 \times \frac{200}{3} + 10 \times \frac{50}{3} \approx 36833$

Solution

- The stock is the number of unit of solvent you buy. The demand is known as $U(500, 800)$.
- i) $c_o = 50 + 10$. For every unit of overstock, you had to pay 50 dollars for buying solvent, and you still have to get rid of them by paying extra 10 dollars.
- ii) $c_u = 100 - 50$. For every unit of understock, it will incur 100 dollars of unit shortage cost, but at least you saved 50 dollars by not preparing the solvent. Thus, the understock cost is 50 dollars.
- iii) We shall find y such that $F(y) = \frac{c_u}{c_o + c_u} = \frac{50}{60+50}$ where $F(y) = \frac{y-500}{800-500}$ because the demand follows $U(500, 800)$. Thus, $y = 5/11 * 300 + 500 = 636$.

- iv)
  - $\mathbb{E}[\text{Total Cost}] = \mathbb{E}[Order\_cost] + \mathbb{E}[Shortage\_cost] + \mathbb{E}[Disposal\_cost]$
  - $\mathbb{E}[\text{Order Cost}]$ = (order cost per unit)*(number of order) = $50 * 600 = 30000$
  - $\mathbb{E}[\text{Shortage Cost}]$ = (Shortage cost per unit)*$\mathbb{E}(number\_of\_shortage)$ = $100 * \mathbb{E}[(D-600)^+] = 100 \times \frac{200}{3} = \frac{20000}{3}$
  - $\mathbb{E}[\text{Disposal Cost}]$ = (Disposal cost per unit)*$\mathbb{E}(number\_of\_disposal)$ = $10 * \mathbb{E}[(600-D)^+] = 10 \times \frac{50}{3} = \frac{500}{3}$
  - $\therefore \mathbb{E}[\text{Total Cost}] = 30000 + \frac{20000}{3} + \frac{500}{3} = \frac{1}{3}(90000 + 20000 + 500) = \frac{110500}{3} \doteqdot 36833$

$$
\begin{aligned}
\mathbb{E}[(600-D)^+] &= \int_{-\infty}^{\infty} (600-y)^+ f(y)dy \\
&= \int_{500}^{800} (600-y)^+ \frac{1}{300} dy \\
&= \int_{500}^{600} (600-y) \frac{1}{300} dy + \int_{600}^{800} 0 \cdot \frac{1}{300} dy \\
&= 2y\Big|_{500}^{600} - \frac{1}{300} \cdot \frac{1}{2} y^2 \Big|_{500}^{600} \\
&= 2(600-500) - \frac{1}{600}(360000 - 250000) \\
&= 200 - \frac{1}{600}(110000) = 200 - \frac{1100}{6} = \frac{100}{6} = \frac{50}{3}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[(D-600)^+] &= \int_{-\infty}^{\infty} (y-600)^+ f(y)dy \\
&= \int_{500}^{800} (y-600)^+ \frac{1}{300} dy \\
&= \int_{500}^{600} 0 \cdot \frac{1}{300} dy + \int_{600}^{800} (y-600) \frac{1}{300} dy \\
&= \frac{1}{300} \cdot \frac{1}{2} y^2 \Big|_{600}^{800} - 2y \Big|_{600}^{800} \\
&= \frac{1}{600} (800^2 - 600^2) - 2(800-600) \\
&= \frac{1}{600} (640000 - 360000) - 400 \\
&= \frac{2800}{6} - 400 = \frac{400}{6} = \frac{200}{3}
\end{aligned}
$$

Exercise 5

- In page 9 and 10 of lecture note 3, finish evaluating the expected profit. You should arrive to a number.

$D \sim U(20,40)$

$$\mathbb{E}[\text{profit}] = 2\cdot\mathbb{E}\left[\min\left(\tfrac{100}{3}, D\right)\right] + 0.5\,\mathbb{E}\left[\left(\tfrac{100}{3}-D\right)^+\right] - 1\cdot\tfrac{100}{3}$$

$$(1)\ \mathbb{E}\left[\min\left(\tfrac{100}{3}, D\right)\right] = \int_{20}^{100/3} \tfrac{1}{20}x\,dx + \int_{100/3}^{40} \tfrac{100}{3}\cdot\tfrac{1}{20}\,dx = \frac{260}{9}$$

$$(2)\ \mathbb{E}\left[\left(\tfrac{100}{3}-D\right)^+\right] = \int_{20}^{100/3} \tfrac{1}{20}\left(\tfrac{100}{3}-x\right)dx = \frac{40}{9}$$

$$\mathbb{E}[\text{profit}] = 2\times\frac{260}{9} + 0.5\times\frac{40}{9} - \frac{100}{3} = \frac{80}{3}.$$

Solution

$D \sim U(20, 40)$, and it follows

$$
\begin{aligned}
\mathbb{E}[\text{profit}] &= \mathbb{E}\left[2 \cdot min\left(D, \frac{100}{3}\right)\right] + \mathbb{E}\left[0.5 \cdot max\left(\frac{100}{3} - D, 0\right)\right] - \mathbb{E}\left[1 \cdot \frac{100}{3}\right] \\
&= 2\left[\int_{20}^{\frac{100}{3}} x \frac{1}{20} dx + \int_{\frac{100}{3}}^{40} \frac{100}{3} \cdot \frac{1}{20} dx\right] + 0.5 \int_{20}^{\frac{100}{3}} \left(\frac{100}{3} - x\right) \frac{1}{20} dx - \frac{100}{3} \\
&= 2\left[\frac{1}{40} x^2 \Big|_{20}^{\frac{100}{3}} + \frac{5}{3}\left(40 - \frac{100}{3}\right)\right] + 0.5\left[\frac{5}{3}\left(\frac{100}{3} - 20\right) - \frac{1}{40} x^2 \Big|_{20}^{\frac{100}{3}}\right] - \frac{100}{3} \\
&= 2\left[\frac{1}{40}\left(\frac{10000}{9} - \frac{3600}{9}\right) + \frac{5}{3} \cdot \frac{20}{3}\right] + 0.5\left[\frac{5}{3}\left(\frac{40}{3}\right) - \frac{1}{40}\left(\frac{10000}{9} - \frac{3600}{9}\right)\right] - \frac{100}{3} \\
&= 2\left[\frac{160}{9} + \frac{100}{9}\right] + 0.5\left[\frac{200}{9} - \frac{160}{9}\right] - \frac{300}{9} \\
&= \frac{520}{9} + \frac{20}{9} - \frac{300}{9} = \frac{240}{9} = \frac{80}{3}
\end{aligned}
$$

Exercise 6

$X \sim \exp(\frac{1}{5})$, $\lambda = 1/5$

- In page 11 of lecture note 3, assume that the demand of lemonade follows exponential distribution with mean 5 gallon.
  - *a)* Identify unit overstock cost $3-1=2$
  - *b)* Identify unit understock cost $18-3=15$
  - *c)* Find optimal order quantity
  - *d)* What is the expected profit when optimal quantity is ordered?

c) $F(y^*) = 1-e^{-\lambda y^*} = 1-e^{-y^*/5} = \frac{15}{2+15}$, $y^* = -5\ln(2/17) \approx 10.7$

d) $\mathbb{E}[\text{sales revenue}] + \mathbb{E}[\text{salvage revenue}] - \mathbb{E}[\text{material cost}]$

$= 18 \cdot \mathbb{E}[\min(D, 10.7)] + 1 \cdot \mathbb{E}[(10.7-D)^+] - 3 \times 10.7$

(i) $\mathbb{E}[\min(D, 10.7)] = \int_0^{10.7} y(\frac{1}{5}e^{-y/5})\,dy + \int_{10.7}^{\infty} 10.7 \times \frac{1}{5}e^{-y/5}\,dy = 4.41$

ii) $\mathbb{E}[(10.7-D)^+] = \int_0^{10.7} (10.7-y)\frac{1}{5}e^{-y/5}\,dy = 6.29$

$\therefore$ $\mathbb{E}[\text{sales revenue}] = 18 \times 4.41 + 6.29 - 3 \times 10.7 = 53.57$

Solution

a) 2

b) 15

c) $1 - e^{-\frac{1}{5}y} = \frac{15}{15+2} \Rightarrow -\frac{1}{5}y = ln(\frac{2}{17}) \Rightarrow y \doteqdot 10.7$

d)

$$\mathbb{E}[\text{profit}] = 18 \cdot \mathbb{E}[min(10.7, D)] + 1 \cdot \mathbb{E}[max(10.7 - D, 0)] - 3 \times 10.7$$

$$= \underbrace{18 \left[\int_0^{10.7} x \left(\frac{1}{5} e^{-\frac{1}{5}x} \right) dx + \int_{10.7}^{\infty} 10.7 \left(\frac{1}{5} e^{-\frac{1}{5}x} \right) dx \right]}_{①} + \underbrace{\int_0^{10.7} (10.7 - x) \left(\frac{1}{5} e^{-\frac{1}{5}x} \right) dx}_{②} - 32.1$$

$$① = 18 \left[\left. (-xe^{-x/5} - 5e^{-x/5}) \right|_0^{10.7} + \left. (-10.7e^{-x/5}) \right|_{10.7}^{\infty} \right]$$

$$= 18 \left[(-10.7e^{-10.7/5} - 5e^{-10.7/5}) - (0 - 5) + (0 - (-10.7e^{-10.7/5})) \right]$$

$$= 18 \times (5 - 5e^{-10.7/5}) = 18 \times 4.41 = 79.38$$

$$② = 10.7 \int_0^{10.7} \frac{1}{5} e^{-\frac{1}{5}x} dx - \frac{1}{5} \int_0^{10.7} xe^{-\frac{1}{5}x} dx$$

$$= 10.7 \left[-e^{-x/5} \right]_0^{10.7} - \frac{1}{5} \left[-5e^{-x/5}(x + 5) \right]_0^{10.7}$$

$$= 10.7 \left[-e^{-10.7/5} + 1 \right] - \frac{1}{5} \left[-5e^{-10.7/5} \times (15.7) - (-5 \times 5) \right]$$

$$= 9.44 - 3.15 = 6.29$$

$$\therefore \mathbb{E}[\text{profit}] = 79.38 + 6.29 - 32.1 = 53.57$$

Exercise 7

- In page 12 of lecture note 3, assume that the demand of lettuce follows following discrete distribution: $\mathbb{P}(D=5)=0.1; \mathbb{P}(D=6)=0.2; \mathbb{P}(D=7)=0.2; \mathbb{P}(D=8)=0.2; \mathbb{P}(D=9)=0.2; \mathbb{P}(D=10)=0.1\}$
  - *a)* Identify unit overstock cost $1-(-0.5)=1.5$
  - *b)* Identify unit understock cost $6-1=5$
  - *c)* Find optimal order quantity
  - *d)* What is the expected profit when the optimal quantity is ordered?

c) $F(y^*) \geq \frac{5}{5+1.5} \approx 0.77$, $y^* \geq 9$. $\therefore$ optimal quantity is 9.

d) $6 \cdot \mathbb{E}\left[\min(D,9)\right] - 0.5 \cdot \mathbb{E}\left[(9-D)^+\right] - 1 \cdot 9$

$= 6 \cdot \left[\sum_{y=5}^{8} y \cdot \mathbb{P}(x=y) + \sum_{y=9}^{10} 9 \cdot \mathbb{P}(x=y)\right] - 0.5 \left[\sum_{y=5}^{9} (9-y)\mathbb{P}(x=y)\right] - 9$

$= 6 \cdot \left[0.5+1.2+1.4+1.6+1.8+0.9\right] - 0.5\left[0.4+0.6+0.4+0.2+0\right] - 9$

$= 6 \times 7.4 - 0.5 \times 1.6 - 9$

$= 44.4 - 0.8 - 9 = 34.6$

Solution

a) 1.5

b) 5

c)

	5	6	7	8	9	10
pmf: $\mathbb{P}(D = d)$	.1	.2	.2	.2	.2	.1
cdf: $\mathbb{P}(D \leq d)$	.2	.3	.5	.7	.9	1.0

Find the smallest y s.t. $F(y) \geq \frac{5}{1.5+5} \doteq 0.77$ implies $y = 9$

d)

$$
\begin{aligned}
\mathbb{E}[\text{profit}] &= 6\mathbb{E}[min(D,9)] - 0.5\mathbb{E}\left[(9-D)^+\right] - 1 \times 9 \\
&= 6[0.1 \times 5 + 0.2 \times 6 + 0.2 \times 7 + 0.2 \times 8 + 0.2 \times 9 + 0.1 \times 9] \\
&\quad - 0.5[0.1 \times 4 + 0.2 \times 3 + 0.2 \times 2 + 0.2 \times 1] - 9 \\
&= 6 \times 7.4 - 0.5 \times \underset{1.6}{\cancel{1.5}} - 9 = 44.4 - 0.75 - 9 = \underset{34.6}{\cancel{34.65}}
\end{aligned}
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"An idea is always a generalization, and generalization is a property of thinking.
To generalize means to think. - George Wilhelm Friedrich Hegel"
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