Lecture 8. Discrete Time Markov Chain (3)

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Simple Random Walk

- Suppose you toss a coin at each time n and you go up if head and you go down if tail.
  - Let the state space $S = \{..., -1, 0, 1, 2...\}$, and X_n is the position after nth toss of the coin.
  - Suppose the probability of getting head of the coin is p. (and let $q = 1 - p$ for tail)
- Transition Diagram
- Transition Matrix

Simple Random Walk

- (Taxonomy) Symmetric/Asymmetric.

- (Taxonomy) 1D/2D/3D

- Drunken man, Jumping frog, Casino, Stock Market.

Simple Random Walk - Questions to be asked

- Will I ever get back to where I am? (Prob. of ever getting back to original place.)
- Do I stand a chance to get to where I want to go?
- How long does it take for a drunken man gets home?
- Can I beat the Casino?
- I have \$50 and I bet \$1 every 30 seconds with $p = 18/38$ on Casino. Can I survive for 30 minutes?
- What is the chance of doubling stock price within 1 year?

A few definitions (4) - Classifications of state

- A state i is said to be **recurrent** if, starting from i, the probability of getting back to i is 1.
  (*There is always a way to get back to state* i).
- A state i is said to be **absorbing state**, as a special case of recurrent state, if $\mathbf{P}_{ii} = 1$.
  (*You can never leave the state* i *once you get there*).
- A state i is said to be **transient** if, starting from i, the probability of getting back to i is less than 1.
  (*It is possible that the process cannot come back to state* i)

- Remark: *Recurrence* and *Transience* are class property
  - If $i \leftrightarrow j$, then i is recurrent if and only if j is recurrent.
  - If $i \leftrightarrow j$, then i is transient if and only if j is transient.

Example

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{pmatrix}$$

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1/2 & 1/6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A few remarks

- In a MC with finite states space, not all states can be transient.
  (i.e., $\exists$ at least one recurrent state)
- A recurrent state is accessible from all states in its class, but is not accessible from recurrent states in other classes.
- A transient state is not accessible from any recurrent state.
- At least one, possibly more, recurrent states are accessible from a given transient state.

Random Walk - Classifications of States?

1. $S = \{..., -1, 0, 1, 2, ...\}$ and $p \neq 0.5$

2. $S = \{..., -1, 0, 1, 2, ...\}$ and $p = 0.5$

3. $S = \{0, 1, 2, ...\}$ and $p > 0.5$

4. $S = \{0, 1, 2, ...\}$ and $p = 0.5$

5. $S = \{0, 1, 2, ...\}$ and $p < 0.5$

Random Walk - Stationary Distribution?!

#5. $S = \{0, 1, 2, ...\}$ and $p = 1/3$, using flow balance equation.

Random Walk - Stationary Distribution?!

Why not #2,#4?

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Exercise 1-3

- For each of the following transition matrices, do the following:
  - (*i*) Determine whether the Markov chain is irreducible
  - (*ii*) Find a stationary distribution; is the stationary distribution unique?
  - (*iii*) Determine whether the Markov chain is periodic
  - (*iv*) Give the period of each state.
  - (*v*) Without using any software package, find $\mathbf{P}^{100}$ approximately.
  - (*vi*) Using software (e.g. Python / R) or web-program (e.g. Wolfram-Alpha), make sure your answer in (v) is correct.

Exercise 1

$$\mathbf{P} = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

(Solution)

$$\mathbf{P} = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

(i) **Irreducible** because all states communicate (there is only one class in MC)

(ii) **It is unique** because MC is irreducible. Setting $\pi P = \pi$ and $\sum(\pi) = 1$, $\pi = (1/3, 1/3, 1/3)$

(iii) **Aperiodic** because there is no state with period more than 1 (see (iv)).

(iv) **1,1,1**

(v) Because this MC is finite, aperiodic, and irreducible,

$$\mathbf{P}^{100} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Exercise 2

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

(Solution)

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

(i) two classes: $\{1, 2\}, \{3, 4\}$, so it is not irreducible.

(ii) **It is not unique** because not irreducible. Out of infinite number of stationary distribution, if I may list a few, $(5/13, 8/13, 0, 0)$ and $(0, 0, 1/3, 2/3)$ are stationary distributions. One may generalize stationary distributions as linear combination of the two such as $p(5/13, 8/13, 0, 0) + (1 - p)(0, 0, 1/3, 2/3)$, where $0 \leq p \leq 1$.

(iii) Aperiodic because all states have period 1 (see (iv)).

(iv) **1,1,1,1**

(v) Markov chain can be decomposed into two sub-Markov chains with state space $\{1,2\}$ and $\{3,4\}$ respectively. Each of them (the sub-Markov Chains) is **irreducible** and **aperiodic**, having **finite** state space,

$$\lim_{n\to\infty} P_{ij}^n = \pi_j \quad \text{for } i,j \in \{1,2\} \text{ or } i,j \in \{3,4\}$$

where $\{\pi_1, \pi_2\}$ is the unique stationary distribution for the sub-Markov chain with state space $\{1,2\}$ and $\{\pi_3, \pi_4\}$ is the unique stationary distribution for the sub-Markov chain with state space $\{3,4\}$. For all other (i,j) pairs such that $i \in \{1,2\}, j \in \{3,4\}$ or $i \in \{3,4\}, j \in \{1,2\}$, (off-block diagonal parts) we have $P_{ij}^n = 0$. Thus,

$$P^{100} \approx \begin{bmatrix} 5/13 & 8/13 & 0 & 0 \\ 5/13 & 8/13 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix}$$

Exercise 3

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0.75 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Solution)

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0.75 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) Three classes: $\{1, 2\}, \{3, 4\}, \{5\}$, so **not irreducible**

(ii) **It is not unique** because MC is not irreducible. Out of infinite number of stationary distributions, if I may list a few, $(5/13, 8/13, 0, 0, 0)$ and $(0, 0, 0, 0, 1)$ are stationary distributions. One may generalize all the stationary distribution as linear combination of the two such as $p(5/13, 8/13, 0, 0, 0) + (1 - p)(0, 0, 0, 0, 1)$, where $0 \leq p \leq 1$.

(iii) MC is **periodic** because there are states with period > 1. (see (iv))

(iv)**1,1,2,2,1**

(v) From the transition matrix, the DTMC will be absorbed either by state 1 and 2 or state 5.

$$(\pi_1, \pi_2)\begin{pmatrix} .2 & .8 \\ .5 & .5 \end{pmatrix} = (\pi_1, \pi_2)$$

Solving above equation, we have $(\pi_1, \pi_2) = (\frac{5}{13}, \frac{8}{13})$. Let X be the DTMC with transition matrix P. Starting $X_0 = 3$, let $f_{3,\{1,2\}}$ be the probabilty that the DTMC is absorbed into $\{1, 2\}$. Similarly starting at $X_0 = 4$, let $f_{4,\{1,2\}}$ be the probability that DTMC is absorbed into $\{1, 2\}$. Then, we have

$$f_{3,\{1,2\}} = \frac{1}{4} + \frac{3}{4} f_{4,\{1,2\}}$$
$$f_{4,\{1,2\}} = \frac{1}{2} f_{3,\{1,2\}}$$

which will give the solution $f_{3,\{1,2\}} = \frac{2}{5}$ and $f_{4,\{1,2\}} = \frac{1}{5}$. Therefore,

$$\mathbf{P}^{\mathbf{100}} \approx \begin{pmatrix} \frac{5}{13} & \frac{8}{13} & 0 & 0 & 0 \\ \frac{5}{13} & \frac{8}{13} & 0 & 0 & 0 \\ \frac{5}{13}(\frac{2}{5}) & \frac{8}{13}(\frac{2}{5}) & 0 & 0 & (\frac{3}{5}) \\ \frac{5}{13}(\frac{1}{5}) & \frac{8}{13}(\frac{1}{5}) & 0 & 0 & (\frac{4}{5}) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 4

Suppose you are modelling your economic status using DTMC. There are five possible economic status: Trillionaire (T), Billionaire (B), Millionaire (M), Thousandaire (Th), Bankrupt (Bk). Your economic status changes every month. Once you are bankrupt, there is no chance of coming back to other states. Also, once you become a billionaire or a trillionaire, you will never be millionaire, thousandaire, or bankrupt. Other chances of transitions are as follows. (Chances for transitions itself are omitted.)

Transition	Probability
T $\rightarrow$ B	0.9
B $\rightarrow$ T	0.3
M $\rightarrow$ B	0.1
M $\rightarrow$ Th	0.5
Th $\rightarrow$ M	0.2
Th $\rightarrow$ Bk	0.3

- *(a)* Is this chain irreducible? Explain your answer.
- *(b)* What is the transition probability matrix of this DTMC model?
- *(c)* Compute $\mathbf{P}^{359}$.
- *(d)* What is the probability that you will become a trillionaire (T) 30 years later given that you are a thousandaire (Th) now?

(Solution)

(a) **No.** Not all states is accessible from any state. Ex) State T cannot reach State M.

(b) P is the following $\left(\begin{array}{c|ccccc} T & .1 & .9 & 0 & 0 & 0 \\ B & .3 & .7 & 0 & 0 & 0 \\ M & 0 & .1 & .4 & .5 & 0 \\ Th & 0 & 0 & .2 & .5 & .3 \\ Bk & 0 & 0 & 0 & 0 & 1 \end{array}\right)$

(c) Define $f_{i,j}$ to be probability of starting in state i and being absorbed by state j. We can write the following equations to find absorption probabilities.

$$f_{M,Bk} = .4f_{M,Bk} + .5f_{Th,Bk}$$
$$f_{Th,Bk} = .3 + .5f_{Th,Bk} + .2f_{M,Bk}$$

Hence, $f_{M,Bk} = .75$ and $f_{Th,Bk} = .9$. We can now approximate

$$\mathbf{P}^{359} = \begin{pmatrix} .25 & .75 & 0 & 0 & 0 \\ .25 & .75 & 0 & 0 & 0 \\ (.25)(.25) & (.25)(.75) & 0 & 0 & .75 \\ (.1)(.25) & (.1)(.75) & 0 & 0 & .9 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) Approximately, $P^{359} \approx P^{360}$. $\mathbf{Pr}\{X_{360} = T | X_0 = Th\} = .025$

Exercise 5

- In L08,p12. Let $p = 1/4$ and calculate the stationary distribution.

(Solution)

We can write the following balance equation

$$\text{State 0: } \frac{3}{4}\pi_0 + \frac{3}{4}\pi_1 = \pi_0$$
$$\text{State 1: } \frac{1}{4}\pi_0 + \frac{3}{4}\pi_2 = \pi_1$$
$$\text{State 2: } \frac{1}{4}\pi_1 + \frac{3}{4}\pi_3 = \pi_2$$
$$\dots$$

Which can be simplified as

$$\pi_1 = \frac{1}{3}\pi_0$$
$$\pi_2 = \frac{1}{3}\pi_1$$
$$\dots$$

Since the sum of stationary distribution equals to one, we can write $\pi_0 + \pi_1 + \pi_2 + \dots = \pi_0(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots) = \pi_0(\frac{1}{1-1/3}) = 1$. So $\pi_0 = \frac{2}{3}$. Therefore, the unique stationary distribution is $\pi_i = \frac{2}{3}(\frac{1}{3})^i$ for $i = 0, 1, 2, 3\dots$

Exercise 6

[Read the following discussions regarding MC and widen your understanding - this exercise looks long, but not demanding at all.]

From the classes and homeworks, we have learned that MC is a powerful tool to analyse the situation when the stochastic evolutions depends only on the most recent information. But this fact also sounds like the limited possibility of applying MC to real world, because not all stochastic evolutions should depend only on the most recent information. Following exercise tackles the limitation.

Suppose you love coffee. The longer you don't drink coffee, the more you want to drink coffee. Consider the discrete time stochastic process $\{X_n, n \geq 0\}$ with state space $S_X = \{C, NC\}$. $X_n = C$ implies you drink coffee on n-th day, and $X_n = NC$ implies no coffee on n-th day. The stochastic evolution of your coffee drinking habit is described following:

> *If you drank coffee yesterday and today, the chance of you drinking coffee tomorrow is 0.2. If you did not drink coffee yesterday but drank coffee today, then the chance of drinking coffee tomorrow is 0.4. If you drank coffee yesterday but not today, then chance of drinking coffee tomorrow is 0.6. If you did not drink coffee yesterday and today, then you will drink coffee tomorrow with probability 0.8.*

The above statements can be expressed mathematically as following. Fill in the blank with decimal number.

$$\mathbb{P}(X_{n+1} = C | X_{n-1} = C, X_n = C) = (.2) \tag{1}$$
$$\mathbb{P}(X_{n+1} = C | X_{n-1} = NC, X_n = C) = (.4) \tag{2}$$
$$\mathbb{P}(X_{n+1} = C | X_{n-1} = C, X_n = NC) = (.6) \tag{3}$$
$$\mathbb{P}(X_{n+1} = C | X_{n-1} = NC, X_n = NC) = (.8) \tag{4}$$

Obviously, the stochastic process $\{X_n, n \geq 0\}$ is not MC. The way to prove it is simply to show one counter example as following:

$$\mathbb{P}(X_{n+1} = C | X_{n-1} = C, X_n = C) \neq \mathbb{P}(X_{n+1} = C | X_{n-1} = NC, X_n = C)$$

Now, letting $Y_n = (X_{n-1}, X_n)$ and consider the discrete time stochastic process $\{Y_n, n \geq 1\}$ with space space $S_Y = \{(C, C), (NC, C), (C, NC), (NC, NC)\}$. I want to show that Y_n is MC! As a first step, fill in the following blanks with decimal numbers.

$$\mathbb{P}(X_n = C, X_{n+1} = C | X_{n-1} = C, X_n = C) = (.2) \tag{5}$$
$$\mathbb{P}(X_n = C, X_{n+1} = NC | X_{n-1} = C, X_n = C) = (.8) \tag{6}$$
$$\mathbb{P}(X_n = C, X_{n+1} = C | X_{n-1} = NC, X_n = C) = (.4) \tag{7}$$
$$\mathbb{P}(X_n = C, X_{n+1} = NC | X_{n-1} = NC, X_n = C) = (.6) \tag{8}$$
$$\mathbb{P}(X_n = NC, X_{n+1} = C | X_{n-1} = C, X_n = NC) = (.6) \tag{9}$$
$$\mathbb{P}(X_n = NC, X_{n+1} = NC | X_{n-1} = C, X_n = NC) = (.4) \tag{10}$$
$$\mathbb{P}(X_n = NC, X_{n+1} = C | X_{n-1} = NC, X_n = NC) = (.8) \tag{11}$$
$$\mathbb{P}(X_n = NC, X_{n+1} = NC | X_{n-1} = NC, X_n = NC) = (.2) \tag{12}$$

Again, fill in the following more blanks with decimal numbers. As a hint, what is the probability of not drinking coffee today given that you drink coffee today? (This sentence is NOT typo)

$$\mathbb{P}(X_n = NC, X_{n+1} = C | X_{n-1} = C, X_n = C) = (0) \tag{13}$$
$$\mathbb{P}(X_n = NC, X_{n+1} = NC | X_{n-1} = C, X_n = C) = (0) \tag{14}$$
$$\mathbb{P}(X_n = NC, X_{n+1} = C | X_{n-1} = NC, X_n = C) = (0) \tag{15}$$
$$\mathbb{P}(X_n = NC, X_{n+1} = NC | X_{n-1} = NC, X_n = C) = (0) \tag{16}$$
$$\mathbb{P}(X_n = C, X_{n+1} = C | X_{n-1} = C, X_n = NC) = (0) \tag{17}$$
$$\mathbb{P}(X_n = C, X_{n+1} = NC | X_{n-1} = C, X_n = NC) = (0) \tag{18}$$
$$\mathbb{P}(X_n = C, X_{n+1} = C | X_{n-1} = NC, X_n = NC) = (0) \tag{19}$$
$$\mathbb{P}(X_n = C, X_{n+1} = NC | X_{n-1} = NC, X_n = NC) = (0) \tag{20}$$

Did you fill all the blanks in (13)-(20) with zero? Note that the equation (5) can be expressed by Y_n as following:

$$\mathbb{P}[X_n = C, X_{n+1} = C | X_{n-1} = C, X_n = C] \tag{21}$$
$$= \mathbb{P}[(X_n, X_{n+1}) = (C, C) | (X_{n-1}, X_n) = (C, C)] \tag{22}$$
$$= \mathbb{P}[Y_{n+1} = (C, C) | Y_n = (C, C)] \tag{23}$$

You can easily express (5)-(20) by using Y_n and Y_{n+1}. Then you know that Y_n is MC. Complete the following 4 by 4 transition matrix of Y_n.

$$\mathbf{P} = \begin{array}{c} (C,C) \\ (NC,C) \\ (C,NC) \\ (NC,NC) \end{array} \begin{pmatrix} .2 & & .8 & \\ .4 & & .6 & \\ & .6 & & .4 \\ & .8 & & .2 \end{pmatrix}$$

Do you see that the first element $.2$ corresponds to quantity in (23)? As we completed transition matrix of Y_n without conflicting information, we know that Y_n has the Markov property and is thus Markov chain.

Q1) Suppose today is nth day, then Y_n contains information of yesterday and today. Y_{n+1} contains information of today and tomorrow and Y_{n+2} contains information of tomorrow and the day next tomorrow. Suppose *you drank coffee yesterday and today*, then what is the probability that *you will not drink coffee tomorrow but will drink coffee the day next tomorrow*?

Ans for Q1) This can be written as $\mathbb{P}(Y_{n+2} = (NC, C)|Y_n = (C, C))$. We can consider path of the transition from $Y_n = (C, C)$ to $Y_{n+2} = (NC, C)$. Only possible path is $Y_n = (C, C) \to Y_{n+1} = (C, NC) \to Y_{n+2} = (NC, C)$.

Therefore the answer is
$\mathbb{P}[Y_{n+2} = (NC, C)|Y_n = (C, C)] = \mathbb{P}[Y_{n+2} = (NC, C)|Y_{n+1} = (C, NC)] \cdot \mathbb{P}[Y_{n+1} = (C, NC)|Y_n = (C, C)] = 0.6 \cdot 0.8 = 0.48.$

Q2) Suppose today is nth day and you drank coffee yesterday and today, then what is the probability that *you will drink coffee in three days* from now?

Ans for Q2) Calculate probability of $\mathbb{P}(Y_{n+3} = (C, C)$ or $Y_{n+3} = (NC, C)|Y_n = (C, C))$. This is equal to $\mathbb{P}(Y_{n+3} = (C, C)|Y_n = (C, C)) + \mathbb{P}(Y_{n+3} = (NC, C)|Y_n = (C, C))$. We can do calculation of $\mathbf{P}^3$. Also, we can consider path from $Y_n = (C, C)$ to $Y_{n+3} = (C, C)$, and $Y_n = (C, C)$ to $Y_{n+3} = (NC, C)$. Which will save more time and which is less prone to error is your decision to make. If we calculate $\mathbf{P}^3$,

$$\mathbf{P}^3 = \begin{pmatrix} .2 & .352 & .32 & .128 \\ .16 & .384 & .28 & .176 \\ .176 & .28 & .384 & .16 \\ .128 & .32 & .352 & .2 \end{pmatrix}$$

$\therefore 0.2 + 0.352 = 0.552$

- What did you learn from this example? Although the MC imposes the very strong assumption of “Evolution depends only on the very recent information”, you can get around this assumption as long as the future evolution depends only on some recent history. This widens the applicability of MC. Think about example that you first thought that it cannot be modelled as DTMC, but now can be modelled by this technique.
- For example, there are very large group of people who believe that tomorrow’s stock price only depends on today’s stock price. On the other hand, there are also very large group of people who believe that tomorrow’s stock price only depends on the prices of recent five days. Both of the groups can utilize MC modelling to translate their beliefs. (Although people in the second group have to spend more time to design a MC)

Exercise 7

Find $\mathbf{P}^{100}$ for

$$\mathbf{P} = \begin{pmatrix} 1 & & & & \\ .6 & & .4 & & \\ & .6 & & .4 & \\ & & .6 & & .4 \\ & & & & 1 \end{pmatrix}$$

\

(Solution)

$$\mathbf{P} = \begin{pmatrix} 1 & & & & \\ .877 & 0 & 0 & 0 & .123 \\ .692 & 0 & 0 & 0 & .301 \\ .415 & 0 & 0 & 0 & .585 \\ & & & & 1 \end{pmatrix}$$

Exercise 8

- Suppose you have \$3 and keep betting \$1 with winning probability 0.4 until your wealth becomes either \$0 or \$4. What is the chance of you will make \$4 before being broken?

(Solution)

Let X_n be your wealth after n times of betting. $X_0 = 3$ is your beginning wealth. Set up this as DTMC and you will have transition matrix same as the previous problem. Thus, the probability that you will make \$4 before being broken is the probability that starting from 4th state (4th state corresponds to wealth being \$3) and absorbed into 5th state. (5th state corresponds to wealth being \$4).

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"If I only had an hour to chop down a tree, I would spend the first 45 minutes
sharpening my axe. - A. Lincoln"
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