Lecture 2. Math Review (2)

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Math Review 17

- For $X \sim U(20, 40)$, Evaluate $\mathbb{E}[X \wedge 25]$ and $\mathbb{E}[(25 - X)^+]$

Math Review 18

- For $X \sim Poi(8)$,
  - $\mathbb{P}(X = 0) =$
  - $\mathbb{P}(2 \leq X \leq 4) =$
  - $\mathbb{P}(X > 2) =$

Math Review 19

- For $X \sim exp(7)$, Evaluate $\mathbb{E}[max(X, 7)]$

Math Review 20

- For $X \sim exp(8)$, find x^* such that $F(x^*) = 0.6$

Math Review 21

- For $X \sim U(10, 20)$, find x^* such that $F(x^*) = 0.7$

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- Matrix Multiplication

$$(.6 \quad .4) \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} =$$

- Solve

$$\begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}.7 & .3\\ .5 & .5\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}$$

$$\pi_1 + \pi_2 = 1$$

- Solve the following system of equations

$$
\begin{aligned}
x &= y \\
y &= 0.5z \\
z &= 0.6 + 0.4x \\
x + y + z &= 1
\end{aligned}
$$

- Solve

$$\begin{pmatrix}\pi_0 & \pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}-2 & 2 & \\ 3 & -5 & 2 \\ & 3 & -3\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\end{pmatrix}$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$P = \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix}$$

- What is P^2 ?

- Solve

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}\begin{pmatrix} .7 & .3 & & \\ .5 & .5 & & \\ & & .6 & .4 \\ & & .3 & .7 \end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

blank

Math Review 23 - Express infinite vector math form

- Solve the following and express π_i for $i = 0, 1, 2, ...$

$$
\begin{aligned}
\pi_0 + \pi_1 + \pi_2 + ... &= 1 \\
0.02\pi_0 + 0.02\pi_1 + 0.02\pi_2 + ... &= \pi_0 \\
0.98\pi_0 &= \pi_1 \\
0.98\pi_1 &= \pi_2 \\
0.98\pi_2 &= \pi_3 \\
... &= ...
\end{aligned}
$$

blank

Math Review 24 - Infinite Geometric Series

- When $|r| < 1$, $S = a + ar + ar^2 + ar^3 + \ldots$

Math Review 25 - Geometric Series

- When $r \neq 1$, $S = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$

Math Review 26 - Power Series

- When $|r| < 1$, $S = r + 2r^2 + 3r^3 + 4r^4 + \ldots$

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Math Review 27 - Conditional Probabilities

- Show that $\mathbb{P}(A|B \cap C)\mathbb{P}(B|C) = \mathbb{P}(A \cap B|C)$

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Math Review 28 - Formulation of time varying function

- During the first hour $(0 \le t \le 1)$, $\lambda(t)$ increases linearly from 0 to 60. After the first hour, $\lambda(t)$ is constant at 60. Draw plot for $\lambda(t)$ and express the function in math form.

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Exercise 1

- For $X \sim U(a, b)$, show pdf $\rightarrow$ cdf

Exercise 2

- For $X \sim exp(\lambda)$, what is pdf $f(x)$?

Exercise 3

- For $X \sim exp(\lambda)$, show pdf $\rightarrow$ cdf.

Exercise 4

- For $X \sim exp(\lambda)$, show that $\mathbb{E}X = 1/\lambda$

Exercise 5

- For $X \sim exp(\lambda)$, show that $Var(X) = 1/\lambda^2$
- (hint) (You may prove this by continuing the following:)

pf) Since $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$ and we know $\mathbb{E}X = 1/\lambda$ from **hw1.#4**, we need to know what $\mathbb{E}X^2$ is.

$$
\begin{aligned}
\mathbb{E}X^2 &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx \\
&= \lambda \left(x^2 \cdot -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty} - \int_0^{\infty} 2x \cdot -\frac{1}{\lambda} e^{-\lambda x} dx \right) \\
&= \ldots
\end{aligned}
$$

- Solution

$$
\begin{aligned}
&= \lambda \left((0-0) + \frac{2}{\lambda} \int_0^\infty x e^{-\lambda x} dx \right) = 2 \int_0^\infty x e^{-\lambda x} dx \\
&= 2 \left[x \cdot -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty - \int_0^\infty -\frac{1}{\lambda} e^{-\lambda x} dx \right] \\
&= 2 \left[\infty \cdot -\frac{1}{\lambda} e^{-\lambda \infty} - \left(0 \cdot -\frac{1}{\lambda} e^{-\lambda 0} \right) + \frac{1}{\lambda} \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty \right] \\
&= 2 \left[0 - 0 + \frac{1}{\lambda} \left[-\frac{1}{\lambda} e^{-\lambda \infty} + \frac{1}{\lambda} e^{-\lambda 0} \right] \right] \\
&= 2 \cdot \frac{1}{\lambda} \left(0 + \frac{1}{\lambda} \right) = \frac{2}{\lambda^2}
\end{aligned}
$$

Since $Var(x) = \mathbb{E}X^2 - (\mathbb{E}X)^2, Var(x) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Exercise 6

- Show that Exponential distribution is memoryless.

Exercise 7

- For $X \sim Poi(\lambda)$, show that $\mathbb{E}X = \lambda$

Exercise 8

- For $X \sim Poi(\lambda)$, show that $Var(X) = \lambda$
- (hint) (you may prove by continuing following.)

pf) Since $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$ and we know $\mathbb{E}X = \lambda$ from **hw1.#7**, we need to know what $\mathbb{E}X^2$ is.

$$
\begin{aligned}
\mathbb{E}X^2 &= \sum_{x=-\infty}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} x^2 \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \left(0^2 \frac{\lambda^0}{0!} + \sum_{x=1}^{\infty} x^2 \frac{\lambda^x}{x!} \right) \\
&= e^{-\lambda} \left(\sum_{x=1}^{\infty} x \frac{\lambda^x}{(x-1)!} \right) \\
&= e^{-\lambda} \left(\sum_{x=1}^{\infty} (x-1+1) \frac{\lambda^x}{(x-1)!} \right) \\
&= e^{-\lambda} \left(\sum_{x=1}^{\infty} (x-1) \frac{\lambda^x}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \right) \\
&= e^{-\lambda} \left(\sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \right) \\
&= \dots
\end{aligned}
$$

- Solution

$$
\begin{aligned}
&= e^{-\lambda}\left[\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}\right] \\
&= e^{-\lambda}\left[\lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^{y}}{y!} + \lambda \sum_{z=0}^{\infty} \frac{\lambda^{z}}{z!}\right], \text{ where } y := x-2, z := x-1 \\
&= e^{-\lambda}\left[\lambda^2 \cdot e^{\lambda} + \lambda \cdot e^{\lambda}\right] \\
&= \lambda^2 + \lambda
\end{aligned}
$$

$$
Var(x) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda
$$

(Note that the property of exponential function is used, i.e. $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$)

Exercise 9

- Let X be a discrete random variable with $\mathbb{P}(X = i) = ci$ for positive, odd integers $i < 8$; otherwise, the probability is zero.
  - *(a)* Compute the value of c. (by using the one of the properties of pmf)
  - *(b)* What is the mean of X?
  - *(c)* What is the second moment of X? (i.e. What is $\mathbb{E}X^2$?)
  - *(d)* What is the variance of X?
  - *(e)* Compute $\mathbb{E}[(X-3)^+]$

- Solution

(a) Since $\sum_{i=-\infty}^{\infty} p(i) = \sum_{i=-\infty}^{\infty} \mathbb{P}(X=i) = 1,$
$\mathbb{P}(X=1) + \mathbb{P}(X=3) + \mathbb{P}(X=5) + \mathbb{P}(X=7) = c + 3c + 5c + 7c = 1$
$\therefore\ c = 1/16$

(b)

$$\mathbb{E}X = \sum_{i=-\infty}^{\infty} i \cdot p(i) = \sum_{i=-\infty}^{\infty} i \cdot \mathbb{P}(X=i) = 1 \cdot p(1) + 3 \cdot p(3) + 5 \cdot p(5) + 7 \cdot p(7)$$

$$= \frac{1}{16}(1^2 + 3^2 + 5^2 + 7^2) = \frac{84}{16} = \frac{21}{4}$$

(c)

$$\mathbb{E}X^2 = \sum_{i=-\infty}^{\infty} i^2 \cdot p(i) = \sum_{i=-\infty}^{\infty} i^2 \cdot \mathbb{P}(X=i) = 1^2 \cdot p(1) + 3^2 \cdot p(3) + 5^2 \cdot p(5) + 7^2 \cdot p(7)$$

$$= \frac{1}{16}(1^3 + 3^3 + 5^3 + 7^3) = \frac{496}{16} = \frac{124}{4} = 31$$

(d) $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 31 - \left(\frac{21}{4}\right)^2 = 3.4375$

(e)

$$\begin{aligned}
\mathbb{E}[(X-3)^+] = \mathbb{E}[max(X-3,0)] &= \sum_{i=-\infty}^{\infty} max(i-3,0) \cdot \mathbb{P}(X=i) \\
&= max(1-3,0) \cdot \mathbb{P}(X=1) + max(3-3,0) \cdot \mathbb{P}(X=3) \\
&\quad + max(5-3,0) \cdot \mathbb{P}(X=5) + max(7-3,0) \cdot \mathbb{P}(X=7) \\
&= 2 \cdot \frac{5}{16} + 4 \cdot \frac{7}{16} = \frac{38}{16} = \frac{19}{8}
\end{aligned}$$

Exercise 10

- Let X be a Poisson random variable with parameter 5, and let $Y = min(X, 2)$
  - *(a)* What is the pmf of X?
  - *(b)* What is the mean of X?
  - *(c)* What is the variance of X?
  - *(d)* What is the pmf of Y? (i.e. Specify $\mathbb{P}(Y = i)$ for $i = 0, 1, 2, \ldots$)
  - *(e)* Compute $E[Y]$?

- Solution

(a) $p(x) = \mathbb{P}(X = x) = \frac{e^{-5}5^x}{x!}$

(b) $5 \quad (= \lambda)$

(c) $5 \quad (= \lambda)$

(d) We have $p(y) = \mathbb{P}(Y = y) = \mathbb{P}(min(X, 2) = y)$. It follows:

- $p(0) = \mathbb{P}(Y = 0) = \mathbb{P}(min(X, 2), 0) = \mathbb{P}(X = 0) = \frac{e^{-5}5^0}{0!} = e^{-5}$
- $p(1) = \mathbb{P}(Y = 1) = \mathbb{P}(min(X, 2), 1) = \mathbb{P}(X = 1) = \frac{e^{-5}5^1}{1!} = 5e^{-5}$
- $p(2) = \mathbb{P}(Y = 2) = \mathbb{P}(min(X, 2) = 2) = \mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) = 1 - e^{-5} - 5e^{-5} = 1 - 6e^{-5}$
- $p(3) = \mathbb{P}(Y = 3) = 0 \quad (\because min(X, 2) \leq 2$ always), also $p(i) = 0$ for all $i \geq 3$
- Therefore,

$$p(y) = \left(\begin{array}{c} e^{-5} \text{ for } y = 0 \\ 5e^{-5} \text{ for } y = 1 \\ 1 - 6e^{-5} \text{ for } y \geq 2 \end{array} \right)$$

(e)

$$\mathbb{E}Y = \sum_{i=-\infty}^{\infty} yp(y) = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) = 5e^{-5} + 2(1 - 6e^{-5}) = 2 - 7e^{-5}$$

Exercise 11

- Let Y be a random variable with pdf ce^{-3y} for $y > 0$, and zero otherwise.
  - *(a)* Determine c.
  - *(b)* What is the mean, variance, and squared coefficient of variation of Y?
  - *(c)* Compute $\mathbb{P}(Y > 4)$
  - *(d)* Compute $\mathbb{P}(Y > 7 | Y > 3)$
  - *(e)* Your answer for (c) and answer for (d) should be same. Discuss shortly why.
  - *(f)* What is the point x^* such that $\mathbb{P}(Y > x^*) = 2/3$?

- Solution

(a)

$$\int_{-\infty}^{\infty} p(y)dy = 1$$

$$\rightarrow \int_{0}^{\infty} ce^{-3y}dy = c \cdot -\frac{1}{3}e^{-3y}\Big|_{0}^{\infty} = -\frac{c}{3}\left[0 - e^{-3\cdot 0}\right] = \frac{c}{3} = 1$$

$$\therefore c = 3$$

(b) pdf of $\begin{cases} 3e^{-3y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$ implies $Y \sim exp(3)$. Thus, $\mathbb{E}Y = \frac{1}{3}, VarY = \frac{1}{3^2}$, and sqr-cv = $\frac{Var(Y)}{(\mathbb{E}Y)^2} = \frac{1/3^2}{(1/3)^2} = 1$

(c) cdf of $exp(3)$ is $\begin{cases} 1 - e^{-3y} & \text{for } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$ Thus,

$$\mathbb{P}(Y > 4) = 1 - \mathbb{P}(Y \leq 4) = 1 - (1 - e^{-3\cdot 4}) = e^{-12}$$

(d) $\mathbb{P}(Y > 7|Y > 3) = \frac{\mathbb{P}(Y>7, Y>3)}{\mathbb{P}(Y>3)} = \frac{1-(1-e^{-3\cdot 7})}{1-(1-e^{-3\cdot 3})} = \frac{e^{-21}}{e^{-9}} = e^{-12}$

(e) memoryless property

(f)

$$\mathbb{P}(Y > x^*) = 1 - \mathbb{P}(Y \leq x^*) = 1 - (1 - e^{-3x^*}) = \frac{2}{3}$$

$$\rightarrow e^{-3x^*} = \frac{2}{3}$$

$$\rightarrow -3x^* = ln(2/3)$$

$$\rightarrow x^* = -\frac{1}{3}ln(2/3)$$

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"Man can learn nothing unless he proceeds from the known to the unknown.
- Claude Bernard"
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