Study Record – ITM529 Stochastic Processes – 2025 Fall	
Name: Lee Jeong-yun 이정윤 21102052	Due Date: 2025-09-22

random variable
- discrete r.v. $\Rightarrow$ pmf, $\mathbb{P}(X=x)=p(x)$; $p(x)\geq 0$, $\sum p(x)=1$
- continuous r.v. $\Rightarrow$ pdf, $\mathbb{P}(a\leq X\leq b)=\int_a^b f(x)dx$; $f(x)\geq 0$, $\int_{-\infty}^{\infty} f(x)dx=1$

cdf, $F(x)=\mathbb{P}(X\leq x)=\mathbb{P}(-\infty< x\leq x)$

pmf, $\mathbb{E}X=\sum x p(x)$, $\mathbb{E}[g(X)]=\sum g(x)f(x)$. $F(x)=\sum_{-\infty}^{x} p(y)$

pdf, $\mathbb{E}X=\int_{-\infty}^{\infty} x f(x)dx$, $\mathbb{E}[g(X)]=\int_{-\infty}^{\infty} g(x)f(x)dx$ $F(x)=\int_{-\infty}^{x} f(y)dy$

$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)dx$ or $\sum x p(x)$

$Var(X) = \mathbb{E}X^2-(\mathbb{E}X)^2$

$sd(X) = \sqrt{Var(X)}$

coefficient of variation: $C_X = \frac{sd(X)}{\mathbb{E}X}$

$X\sim U(a,b)$, pdf $f(x)=\begin{cases}\frac{1}{b-a} & a\leq x\leq b\\ 0 & \text{o.w}\end{cases}$ cdf $F(x)=\begin{cases}0 & x\leq a\\ \frac{x-a}{b-a} & a< x\leq b\\ 1 & x>b\end{cases}$

$X\sim \exp(\lambda)$, pdf $f(x)=\begin{cases}\lambda e^{-\lambda x} & x\geq 0\\ 0 & \text{o.w}\end{cases}$ cdf $F(x)=\begin{cases}1-e^{-\lambda x} & x\geq 0\\ 0 & \text{o.w}\end{cases}$

$\mathbb{E}X=1/\lambda$ $Var(X)=1/\lambda^2$ ($\mathbb{E}X$: 1^{st} moment, $\mathbb{E}X^2$: 2^{nd} moment, $\cdots$, $\mathbb{E}X^n$: n^{th} moment)

$\mathbb{E}X=\int_{-\infty}^{\infty} x f(x)dx=\int_0^{\infty} x\lambda e^{-\lambda x}dx=\left[-xe^{-\lambda x}\right]_0^{\infty}+\int_0^{\infty} e^{-\lambda x}dx=\left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\infty}=0+\frac{1}{\lambda}=\frac{1}{\lambda}$

$\mathbb{E}X^2=\int_{-\infty}^{\infty} x^2 f(x)dx=\int_0^{\infty} x^2\cdot\lambda e^{-\lambda x}dx=\left[x^2\cdot(-e^{-\lambda x})\right]_0^{\infty}+\int_0^{\infty} 2xe^{-\lambda x}dx=\frac{2}{\lambda}\int_0^{\infty} x\cdot\lambda e^{-\lambda x}dx=\frac{2}{\lambda^2}$

$\therefore Var(X)=\frac{2}{\lambda^2}-\left(\frac{1}{\lambda}\right)^2=1/\lambda^2$ cf) n^{th} moment : $n!/\lambda^n$

r.v. $X\sim\exp(\lambda)$ is memoryless $\Rightarrow \mathbb{P}(X>s+t\mid X>t)=\mathbb{P}(X>s)$ for $(s,t\geq 0)$

pf) $\mathbb{P}(X>s+t\mid X>t)=\frac{\mathbb{P}(X>s+t\ \cap\ X>t)}{\mathbb{P}(X>t)}=\frac{\mathbb{P}(X>s+t)}{\mathbb{P}(X>t)}=\frac{1-F(s+t)}{1-F(t)}=\frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}=e^{-\lambda s}$

$\mathbb{P}(X>s)=1-F(s)=e^{-\lambda t}$

$\therefore \mathbb{P}(X>s+t\mid X>t)=\mathbb{P}(X>s)$ if $X\sim\exp(\lambda)$ ■

$X_1\sim\exp(\lambda_1)$ & $X_2\sim\exp(\lambda_2)$ $\rightarrow$ X_1 & X_2 independent $\rightarrow$ $\mathbb{P}(X_1<X_2)=\frac{\lambda_1}{\lambda_1+\lambda_2}$

$X_1\sim\exp(\lambda_1)$ & $X_2\sim\exp(\lambda_2)$ $\rightarrow$ X_1 & X_2 independent $\rightarrow$ $\min(X_1,X_2)\sim\exp(\lambda_1+\lambda_2)$

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$X\sim Poi(\lambda)$ pmf, $\mathbb{P}(X=k)=p(k)=\frac{\lambda^k e^{-\lambda}}{k!}$ for $k=0,1,2,\cdots$ $F(x)=\mathbb{P}(X\leq x)=\sum_{k=0}^{x}\frac{\lambda^k e^{-\lambda}}{k!}$

$\sum_0^\infty \mathbb{P}(X=k)=\sum_0^\infty \frac{\lambda^k e^{-\lambda}}{k!}=e^{-\lambda}\sum_0^\infty\frac{\lambda^k}{k!}=e^{-\lambda}\cdot e^{\lambda}=1$ ($\because \sum_0^\infty \frac{1}{k!}=e$, $\sum_0^\infty\frac{x^k}{k!}=e^x$ by Taylor series)

$\mathbb{E}X=\sum_0^\infty x p(x)=\sum_0^\infty x\cdot\frac{\lambda^x e^{-\lambda}}{x!}=\sum_{x=1}^\infty x\cdot\frac{\lambda^x e^{-\lambda}}{x!}=\sum_{x=1}^\infty \lambda\cdot\frac{\lambda^{x-1}e^{-\lambda}}{(x-1)!}$

$\Rightarrow y=x-1$, $\mathbb{E}X=\lambda\cdot\sum_{y=0}^\infty\frac{\lambda^y e^{-\lambda}}{y!}=\lambda$

$\mathbb{E}X^2=\sum_{x=0}^\infty x^2p(x)=\sum_{x=0}^\infty x^2\cdot\frac{\lambda^x e^{-\lambda}}{x!}=\sum_{x=1}^\infty \lambda x\cdot\frac{\lambda^{x-1}e^{-\lambda}}{(x-1)!}$

$\Rightarrow y=x-1$, $\mathbb{E}X^2=\sum_{y=0}^\infty\lambda(y+1)\frac{\lambda^y e^{-\lambda}}{y!}=\lambda\left\{\sum_{y=0}^\infty y\cdot\frac{\lambda^y e^{-\lambda}}{y!}+\sum_{y=0}^\infty\frac{\lambda^y e^{-\lambda}}{y!}\right\}=\lambda(\lambda+1)$

$Var(X)=\lambda(\lambda+1)-\lambda^2=\lambda$

Bayes' Rule $F_i\cap F_j=\emptyset$ for any $i\neq j$, $\mathbb{P}(F_1\cup\cdots\cup F_n)=1$

$\mathbb{P}(E)=\sum_{i=1}^n\mathbb{P}(E\cap F_i)=\sum_{i=1}^n\mathbb{P}(E|F_i)\mathbb{P}(F_i)$

$X\sim U(20,40)$ pdf $f(x)=\begin{cases}1/20 & 20\leq x\leq 40\\ 0 & o.w\end{cases}$ cf) $A\wedge B=\min(A,B)$ $A\vee B=\max(A,B)$

$X^+=\max(x,0)$ $X^-=\min(x,0)$

$\mathbb{E}(X\wedge 25)=\int_{-\infty}^\infty \min(x,25)f(x)dx=\int_{20}^{40}\min(x,25)\cdot\frac{1}{20}dx=\int_{20}^{25}\frac{1}{20}x\,dx+\int_{25}^{40}\frac{25}{20}dx$

$=\frac{1}{20}\left(\left[\frac12 x^2\right]_{20}^{25}+\left[25x\right]_{25}^{40}\right)=24.375$

$\mathbb{E}((25-X)^+)=\int_{-\infty}^\infty\max(25-x,0)f(x)dx=\int_{20}^{40}\max(25-x,0)\cdot\frac1{20}dx=\frac1{20}\left(\int_{20}^{25}25-x\,dx+\int_{25}^{40}0\,dx\right)$

$=\frac1{20}\left[25x-\frac12x^2\right]_{20}^{25}=\frac58$

Stationary distribution. $\pi P=\pi$, $\sum\pi_i=1$

representing not unique solution. let $\pi_1+\pi_2+\pi_3+\pi_4=\pi_1+\frac35\pi_1+\pi_3+\frac43\pi_3=1$

i) Let $p:=\pi_1$ $p\in[0,1]$

$\pi_1=p$

$\pi_2=\frac35\pi$

$\pi_3=\frac37(1-\frac85p)$

$\pi_4=\frac47(1-\frac85p)$

$\pi=[p,\frac35p,\frac37(1-\frac85p),\frac47(1-\frac85p)]$

for any $p\in[0,1]$

ii) Let $p:=\pi_1+\pi_2=\frac85\pi_1$

$\pi_1=\frac58p$

$\pi_2=\frac38p$

$\pi_3=\frac37(1-p)$

$\pi_4=\frac47(1-p)$

$\pi=[\frac58p,\frac38p,\frac37(1-p),\frac47(1-p)]$

for any $p\in[0,1]$

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$|r|<1$,

$$S = a + ar + ar^2 + \cdots$$
$$-\ rS = \quad ar + ar^2 + \cdots$$
$$(1-r)S = a$$

$$\therefore S = \sum_0^\infty ar^n = \frac{a}{1-r} \quad (|r|<1)$$

$r \neq 1$,

$$S = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$-\ rS = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$
$$(1-r)S = a - ar^n$$

$$\therefore S = \sum_1^n ar^{n-1} = \frac{a}{1-r}(1-r^n)$$

$|r|<1$

$$S = r + 2r^2 + 3r^3 + \cdots$$
$$-\ rS = \quad r^2 + 2r^3 + \cdots$$
$$(1-r)S = r + r^2 + r^3 + \cdots = \sum_0^\infty r \cdot r^n = \frac{r}{1-r}$$

$$\therefore S = \frac{r}{(1-r)^2}$$

Newsvendor Model ⋯ optimal ordering on perishable goods/services

information needed =

demand	[11, 15] w/ each 20% of chance
retail price	\$ 2
wholesale price / material cost	\$ 1
salvage value.	\$ 0.5

profit = revenue − cost = (sales revenue + salvage revenue) − (material cost)

$= (2 \times \min(X, D) + (X-D)^+ \times 0.5) - 1 \times X$

Stock < demand ⋯ understock unit cost \$ 1 = C_u

demand (D) / stock (X)	20% 11	20% 12	20% 13	20% 14	20% 15	Expected profit
11	2×11+0×0.5−1×11 =11	11	11	11	11	11
12	2×11+1×0.5−1×12 =10.5	2×12+0×0.5−1×12 =12	12	12	12	11.7
13	2×11+2×0.5−1×13 =10	2×12+1×0.5−1×13 =11.5	2×13+0×0.5−1×13 =13	13	13	12.1
14	2×11+3×0.5−1×14 =9.5	2×12+2×0.5−1×14 =11	2×13+1×0.5−1×14 =12.5	2×14+0×0.5−1×14 =14	14	12.2
15	2×11+4×0.5−1×15 =9	2×12+3×0.5−1×14 =10.5	2×13+2×0.5−1×15 =12	2×14+1×0.5−1×15 =13.5	2×15+0×0.5−1×15 =15	12

Stock > demand ⋯ overstock unit cost \$0.5 = C_o

continuous distribution, $F(y) = \frac{C_u}{C_o + C_u}$ discrete distribution, $F(y) \geq \frac{C_u}{C_o + C_u}$

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Queueing Theory Variables

Demand : inter-arrival time

Capacity : inter-departure times (service time), # servers , queue size.

⇒ notation : inter-arrival time / service time / # servers / queue size

- G | general distribution
- M | exponential distribution
- D | deterministic (constant)

"unstable" : # waiting customers for the system diverges to infinity or not converging

"stable" : not unstable.

"traffic intensity", ρ : demand / capacity

- if $\rho < 1$: stable
- $\rho = 1$: unstable, unless inter-arrival time & service times are deterministic.
- $\rho > 1$: unstable

M/M/1/∞ system. ⇒ $\rho = \lambda/\mu = \frac{1/\mathbb{E}(U)}{1/\mathbb{E}(V)}$, λ = arrival rate, μ = service rate

U = inter-arrival time. V = inter-departure time.

$\lambda = 1/\mathbb{E}U$, $\mu = 1/\mathbb{E}V$

"utilization of server, u" $= \min(1, \rho)$

Kingman's high-traffic approximation formula. : how long customer needs to wait prior to begin of the service ($\rho < 1$, but ρ close to 1.)

$$\mathbb{E}W_q = \mathbb{E}V \times \left(\frac{\rho}{1-\rho}\right) \times \left(\frac{C_a^2 + C_s^2}{2}\right),$$

W_q = long-run average waiting time

C_a = coefficient of variation of inter-arrival time

C_s = coefficient of variation of service time

Little's Law, $L = \lambda W$.

L = # people

λ = arrival rate

W = time spent

① $L_{queue} = \lambda \cdot W_{queue}$

② $L_{service} = \lambda \cdot W_{service}$

③ $L_{system} = L_{queue} + L_{service}$ & $W_{system} = W_{queue} + W_{service}$,

④ $L_{sys} = \lambda \cdot W_{sys}$

$W_{queue} = \mathbb{E}U$

$W_{service} = \mathbb{E}V$

system = queue + services

"Throughput, TH" : if system is stable & infinite waiting space, then throughput $= \lambda$

otherwise, throughput $< \lambda$

one server, $\rho = \frac{\lambda}{\mu}$ ⇒ $TH = \begin{cases} \lambda & (\text{if } \rho < 1) \\ \mu & (\text{if } \rho \geq 1) \end{cases} = \min(\lambda, \mu)$

* $\rho \leq 1$ & $\rho > 1$ is incorrect, in my opinion.
$\rho = 1$ is unstable status.
the result will not change, but I think the condition should be revised

two server, (parallel) $\rho = \frac{\lambda}{\mu_1 + \mu_2}$ ⇒ $TH = \begin{cases} \lambda & (\text{if } \rho < 1) \\ \mu_1 + \mu_2 & (\text{if } \rho \geq 1) \end{cases} = \min(\lambda, \mu_1 + \mu_2)$

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two server (in sequence)

$\xrightarrow{\lambda} \boxed{\mu_1} \rightarrow \boxed{\mu_2} \xrightarrow{TH} \;\approx\; \xrightarrow{\lambda} \boxed{\min(\mu_1,\mu_2)} \xrightarrow{TH}$

demand $= \lambda$, capacity $= \min(\mu_1, \mu_2)$

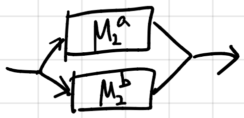
$$\rho = \frac{\lambda}{\mu_1 + \mu_2}, \quad TH = \begin{cases} \lambda & \text{if } \rho < 1 \\ \min(\mu_1, \mu_2) & \text{if } \rho \geq 1 \end{cases} = \min(\lambda, \mu_1, \mu_2)$$

n server (in sequence)

$\xrightarrow[8]{\lambda} \underset{15}{\boxed{\mu_1}} \rightarrow \underset{10}{\boxed{\mu_2}} \rightarrow \cdots \rightarrow \underset{12}{\boxed{\mu_n}} \xrightarrow{TH}$

$\mu_2 \Rightarrow$ bottleneck station. busiest server.

sol) line balancing. $\Rightarrow$



demand $= \lambda$, capacity $= \min(\mu_1, \mu_2, \cdots, \mu_n)$

$$\rho = \frac{\lambda}{\min(\mu_1, \mu_2, \cdots, \mu_n)} \qquad TH = \min(\lambda, \mu_1, \mu_2, \cdots, \mu_n)$$

Kingman's equation : $\mathbb{E}W_q = \mathbb{E}V \left(\frac{\rho}{1-\rho}\right)\left(\frac{C_a^2 + C_s^2}{2}\right)$ $\Rightarrow$ if ρ is closed to 1, $\mathbb{E}W_q$ is exponentially increasing

Highway traffic example : 100 cars/min in I-75 200 cars/min in I-85
3000 cars in merged area

using little's law ($= L = \lambda W$), 3000 cars $= (100+200)$ cars/min $\times$ 10 min.

Stochastic Process : Time + Random.
- discrete time : $X_n : n \geq 0,\ n \in \mathbb{N}$
- Continuous time : $X_t : t \geq 0,\ t \in \mathbb{R}^+$

State : value of X_n (discrete) State space : set of all possible values of X_n

Markov property : for discrete time stochastic process.
X_{n+1} depends only on state of X_n.

$X_{n+1} = f(X_n, \text{randomness})$.

$$\mathbb{P}(X_{n+1} = j \mid \underbrace{X_0 = i_0, X_1 = i_1, \cdots,}_{\text{irrelevant history, outdated info.}} X_n = i) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$$
$$= \mathbb{P}(X_3 = j \mid X_2 = i) = \mathbb{P}(X_2 = j \mid X_1 = i)$$

DTMC, Discrete Time Markov Chain : discrete time stochastic process with Markov chain

Transition Probability, $p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) = \mathbb{P}(X_n = j \mid X_{n-1} = i) = \mathbb{P}(X_1 = j \mid X_0 = i)$

Transition Probability $P = [p_{ij}] \in \mathbb{R}^{|S| \times |S|}$

Initial distribution : info of chain at time 0.
$a_0 :=$ dist of X_0 as a row vector.

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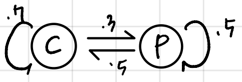
Soda example

$i \backslash j$	c	p
c	.7	.3
p	.5	.5

$\Rightarrow$ 

DTMC transition diagram

$\Rightarrow$ matrix A

$$\begin{array}{cc} & \begin{array}{cc} c & p \end{array} \ (\text{to}) \\ (\text{from}) \begin{array}{c} c\to \\ p\to \end{array} & \begin{bmatrix} .7 & .3 \\ .5 & .5 \end{bmatrix} \end{array}$$

DTMC transition matrix

$\mathbb{P}(X_{n+1}=p \mid X_n = c) = 0.3 = A_{cp}$

$S = \{c, p\}$
$|S| \times |S|$ matrix
row sum $=1$ … stochastic!
elements $\in [0,1]$

Stochastic Process : Time + Random.
- $\to$ Discrete Time : $\{X_n : n \geq 0,\ n \in \mathbb{N}\} = \{X_0, X_1, X_2, \cdots\}$
- $\to$ Continuous Time : $\{X_t : t \geq 0,\ t \in \mathbb{R}^+\} = \{X_0, \cdots, X_{\sqrt{2}}, \cdots, X_{3.14}, \cdots\}$

State: value of X_n. State Space S : a set of all possible states.

Markov property : the nearest future depends on the present.
for discret time $\{X_n : n \geq 0,\ n \in \mathbb{N}\} \Rightarrow X_{n+1}$ depends only on the state of X_n.
X_{n+1} is function of X_n & some randomness. $X_{n+1} = f(X_n, \text{random})$

$\Rightarrow \mathbb{P}(X_{n+1} = j \mid \underbrace{X_0 = i_0, X_1 = i_1, \cdots,}_{\text{outdated info.}} \underbrace{X_n = i}_{\text{most recent history}}) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$

Discrete Time Markov chain, DTMC
$\Rightarrow$ discrete time stochastic process with Markov property

Transition probability $P_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i) = \mathbb{P}(X_n = j \mid X_{n-1} = i) = \mathbb{P}(X_1 = j \mid X_0 = i)$
Transition probability matrix $P = [P_{ij}] \in \mathbb{R}^{|S| \times |S|}$
Initial Distribution : info of where the chain starts at time 0.
$P(X_0 = C) = 0.6,\ \mathbb{P}(X_1 = P) = 0.4 \Rightarrow a_0 = (1, 0)$ … $\mathbb{P}(X_1 = C) = a_0 \begin{pmatrix} 0.7 \\ 0.5 \end{pmatrix} = 0.62.$
$a_1 = a_0 P$. $a_2 = a_1 P = a_0 P^2$, $a_3 = a_2 P = a_0 P^3 = a_1 P^2$, …, $a_n = a_i P^{n-i}$

Stationary Distribution : State space S, transition probability matrix P,
$\pi = (\pi_i,\ i \in S)$ if $\pi_i \geq 0$, all $i \in S$. $\sum_{\pi \in S} \pi_i = 1$, $\pi = \pi P$
$\Rightarrow$ dynamic equilibrium, Steady states. $(\text{inflow})_i = (\text{outflow})_i$ for all $i \in S$ $\Rightarrow$ # of stationary dist. $= 0, 1, \infty$

Soda with stationary dist. $\pi = (\pi_c, \pi_p) \Rightarrow \pi_c, \pi_p \geq 0$, $\pi_c + \pi_p = 1$, $(\pi_c, \pi_p)\begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} = (\pi_c, \pi_p) \Rightarrow \pi_c = 5/8,\ \pi_p = 3/8$

$P = \begin{bmatrix} P_{cc} & P_{cp} \\ P_{pc} & P_{pp} \end{bmatrix} = \begin{bmatrix} .7 & .3 \\ .5 & .5 \end{bmatrix}$ $\quad$ $(\text{outflow})_{\text{coke}} = (\text{inflow})_{\text{coke}}$ & $\pi_c + \pi_p = 1 \Rightarrow \pi_c = 5/8,\ \pi_p = 3/8$
$\pi_c \cdot P_{cp} = \pi_p \cdot P_{pc}$

n-step transition probability : $\mathbb{P}(X_{k+n} = j \mid X_k = i) = P_{ij}^n$, $\lim_{n\to\infty} (\lambda\ 1-\lambda)\, P^n \simeq$ stationary dist.
Soda $\Rightarrow P = \begin{bmatrix} .7 & .3 \\ .5 & .5 \end{bmatrix}$. $P^\infty = \begin{bmatrix} 5/8 & 3/8 \\ 5/8 & 3/8 \end{bmatrix} = \begin{bmatrix} - \pi - \\ - \pi - \end{bmatrix}$ > identical rows! $\quad$ unique (only 1) stationary dist. exist!
not always P^∞ converges.

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DTMC with state space S & transition probability matrix P
$\pi = \pi(\pi_i, i \in S)$, $\pi_i \geq 0$ for all $i \in S$, $\sum \pi_i = 1$, $\pi P = \pi$ ⇒ stationary distribution
··· dynamic equilibrium. steady state. $(\text{inflow})_i = (\text{outflow})_i$, unique or inf solution.

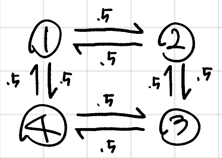
compute stationary dist. ① use $[\pi_i \geq 0, \sum \pi_i = 1, \pi P = \pi]$
② use $[(\text{inflow})_i = (\text{outflow})_i, \sum \pi_i = 1]$

n-step transition probability : $\mathbb{P}(X_{k+n} = j \mid X_k = i) = P_{ij}^n$ ··· from i state to j state in n-step.

soda case) $P = \begin{bmatrix} .7 & .3 \\ .5 & .5 \end{bmatrix}$ $P^{10} = \begin{bmatrix} 5/8 & 3/8 \\ 5/8 & 3/8 \end{bmatrix}$ identical rows.
initial dist. doesn't matter in long run. calculate limiting prob. by stationary dist.

$$(\pi_1\ \pi_2\ \pi_3\ \pi_4)\begin{pmatrix} 0 & .5 & 0 & .5 \\ .5 & 0 & .5 & 0 \\ 0 & .5 & 0 & .5 \\ .5 & 0 & .5 & 0 \end{pmatrix} = (\pi_1\ \pi_2\ \pi_3\ \pi_4)$$
⇒ $\pi = (1/4\ \ 1/4\ \ 1/4\ \ 1/4)$ ··· unique

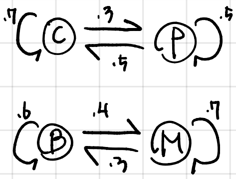
Periodic MC. limiting prob. NOT Exist.



$$\lim_{n\to\infty} \frac{P^{n+1} + P^{n+2} + \cdots + P^{n+d}}{d} = \begin{pmatrix} -\pi- \\ -\pi- \\ \vdots \end{pmatrix}$$
(d = period)
stationary dist unique

$$(\pi_C\ \pi_P\ \pi_B\ \pi_M)\begin{pmatrix} .7 & .3 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & .6 & .4 \\ 0 & 0 & .3 & .7 \end{pmatrix} = (\pi_C\ \pi_P\ \pi_B\ \pi_M)$$
⇒ $\alpha = \pi_C + \pi_P$ $1-\alpha = \pi_B + \pi_M$
$\pi = (5/8\,\alpha\ \ 3/8\,\alpha\ \ 3/7(1-\alpha)\ \ 4/7(1-\alpha))$ ··· ∞

2-reducible MC ⇒ 2 classes exist



limiting prob. exist
↳ depending on initial state.
stationary dist. not unique (∞)

Accessibility : A state i can reach state j in some future. & write $i \to j$ if $\exists\, n$ s.t. $P_{ij}^n > 0$.
state i & j are said to communicate & write $i \leftrightarrow j$ if $i \to j$, $j \to i$
Class: group of states communicate

Reducibility : MC X_n is irreducible if all states communicate. ($\exists$ only one class)

Periodicity : state $i \in S$, period $d(i) := \gcd\{n, P_{ii}^n > 0\}$
MC X_n is periodic if $\exists\, i$ with $d(i) > 1$
⇒ class property. class shares period. $i \leftrightarrow j \Rightarrow d(i) = d(j)$

If a finite state DTMC X_n is aperiodic & irreducible, { limiting prob. exist.
stationary dist. is unique.
stationary dist. = limiting prob.
⇒ The long-run fraction of time that the MC spends in each state.
calculate limiting prob. by solving stationary dist.

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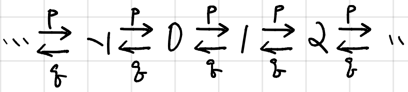
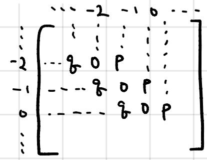
Simple Random walk : $S = \{\cdots, -1, 0, 1, 2, \cdots\}$,



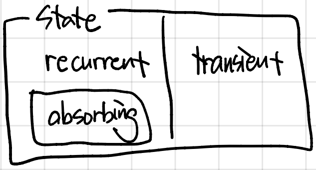


$q = 1-p$, if $p = q \Rightarrow$ symmetric

if $p \neq q \Rightarrow$ asymmetric

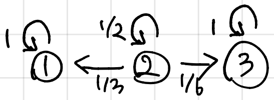
State i
- recurrent : prob. getting back from i to i = 1 (always be a way to get back to i)
- absorbing state : recurrent & $P_{ii} = 1$ (cannot leave state i)
- transient : prob. getting back from i to i < 1 (possible to be no way to get back to i)

recurrent & transient $\Rightarrow$ class property. ($i \leftrightarrow j$)



compute P^∞
1. transition diagram.
2. period, R/A/T
3. identify class
4. compute P^∞

ex) $P = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1/2 & 1/6 \\ 0 & 0 & 1 \end{pmatrix}$



period = 1 class = {1} {2} {3}

R/A/T A T A

$P^\infty = \begin{bmatrix} 1 & 0 & 0 \\ f_{2,1} & 0 & f_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$ $f_{2,1} = P^\infty_{2,1}$, $f_{2,3} = P^\infty_{2,3}$, $f_{2,1} + f_{2,3} = 1$

$f_{2,1} = \mathbb{P}(X_\infty = 1 | X_0 = 2) = \mathbb{P}(X_\infty = 1, X_1 = 1 | X_0 = 2) + \mathbb{P}(X_\infty = 1, X_1 = 2 | X_0 = 2) + \mathbb{P}(X_\infty = 1, X_1 = 3 | X_0 = 2)$

$= \mathbb{P}(X_\infty = 1 | X_1 = 1, X_0 = 2)\mathbb{P}(X_1 = 1 | X_0 = 2) +$
$\mathbb{P}(X_\infty = 1 | X_1 = 2, X_0 = 2)\mathbb{P}(X_1 = 2 | X_0 = 2) +$
$\mathbb{P}(X_\infty = 1 | X_1 = 3, X_0 = 2)\mathbb{P}(X_1 = 3 | X_0 = 2)$ ($\because \mathbb{P}(A \cap B | C) = \mathbb{P}(A | B \cap C)\mathbb{P}(B | C)$)

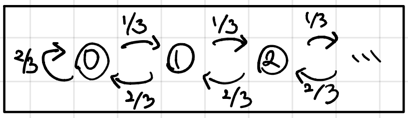
$= \mathbb{P}(X_\infty = 1 | X_1 = 1)\mathbb{P}(X_1 = 1 | X_0 = 2) +$
$\mathbb{P}(X_\infty = 1 | X_1 = 2)\mathbb{P}(X_1 = 2 | X_0 = 2) +$
$\mathbb{P}(X_\infty = 1 | X_1 = 3)\mathbb{P}(X_1 = 3 | X_0 = 2)$ ($\because$ Markov property)

$= 1 \times 1/3 + f_{2,1} \times 1/2 + 0 \times 1/6 \Rightarrow f_{2,1} = 2/3$ $f_{2,3} = 1/3$ ($\because f_{2,1} + f_{2,3} = 1$)

Remarks. In a MC finite state space, not all states can be transient.
A recurrent state is accessible from all states in its class, but is not accessible from recurrent states in other classes.
A transient state is not accessible from any recurrent state.
At least one, recurrent states are accessible from a given transient state.

1. $S = (-\infty, \infty)$	$p \neq 0.5$	all states are transient
2. $S = (-\infty, \infty)$	$p = 0.5$	all states are null recurrent ($\nexists$ stationary dist.)
3. $S = [0, \infty)$	$p > 0.5$	all states are transient
4. $S = [0, \infty)$	$p = 0.5$	all states are null recurrent ($\nexists$ stationary dist.)
5. $S = [0, \infty)$	$p < 0.5$	all states are positive recurrent. ($\exists$ stationary dist.)

5. $\Rightarrow S = \{0, 1, 2, \cdots\}$ $p = 1/3 < 0.5$



using flow equation,

	inflow	=	outflow	
State 0	$\frac{2}{3}\pi_0 + \frac{2}{3}\pi_1$	=	$\frac{2}{3}\pi_0 + \frac{1}{3}\pi_0$	$\Rightarrow \pi_1 = \frac{1}{2}\pi_0$
State 1	$\frac{1}{3}\pi_0 + \frac{2}{3}\pi_2$	=	$\frac{2}{3}\pi_1 + \frac{1}{3}\pi_1$	$\Rightarrow \pi_2 = 1/4\,\pi_0$
State 2	$\frac{1}{3}\pi_1 + \frac{2}{3}\pi_3$	=	$\frac{2}{3}\pi_2 + \frac{1}{3}\pi_2$	$\Rightarrow \pi_3 = 1/8\,\pi_0$

$\sum \pi = \pi_0 + \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0 + \frac{1}{8}\pi_0 + \cdots = \frac{1}{1-1/2}\pi_0 = 1 \Rightarrow \pi_0 = 1/2$

$\pi_i = (\frac{1}{2})^{i+1}$ for all $i \in \mathbb{R}^+$

$\pi = \{1/2, 1/4, 1/8, 1/16, \cdots\}$

reminder: $X \sim \exp(\lambda)$, $P(X>s+t \mid X>t) = P(X>s)$ for all $s,t \geq 0$
$X_1 \sim \exp(\lambda_1)$, $X_2 \sim \exp(\lambda_2)$, $P(X_1 < X_2) = \lambda_1/(\lambda_1+\lambda_2)$, $\min(X_1,X_2) \sim \exp(\lambda_1+\lambda_2)$
$X_1+X_2 = \min(X_1,X_2) + \max(X_1,X_2) \Rightarrow \mathbb{E}X_1 + \mathbb{E}X_2 = 1/(\lambda_1+\lambda_2) + \mathbb{E}(\max(X_1,X_2)) \Rightarrow 1/\lambda_1 + 1/\lambda_2 = 1/(\lambda_1+\lambda_2) + \mathbb{E}[\max(X_1,X_2)]$
$\Rightarrow \mathbb{E}[\max(X_1,X_2)] = 1/\lambda_1 + 1/\lambda_2 - 1/(\lambda_1+\lambda_2)$

Discrete random var $X \sim Poi(\lambda)$ if pmf $P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k=0,1,2,\cdots \Rightarrow \mathbb{E}X = Var(X) = \lambda$
$N = \{N(t), t \geq 0\}$ counting process, if $N(0)=0$ & $N(t)$ is non-decreasing with integer value.
$\Rightarrow$ Poisson Process with rate λ, $PP(\lambda)$. if $\begin{cases} N(t_2)-N(t_1) \sim Poi(\lambda(t_2-t_1)) \text{ for any } 0 \leq t_1 \leq t_2 \Rightarrow \text{stationary increment.} \\ \text{for } t_1 \leq t_2 \leq t_3 \leq t_4,\ N(t_2)-N(t_1) \ \&\ N(t_4)-N(t_3) \Rightarrow \text{independent. increment} \end{cases}$

N is $PP(\lambda = 2/min)$
- Prob. 4 arrival in first 3 min : $P[N(3)-N(0)=4] = P[X=4 \mid X \sim Poi(2(3-0)=6)] = \frac{6^4 e^{-6}}{4!}$
- Prob. 2 arrival in $[0,2]$ & at least 3 arrival in $[1,3]$

0–1	1–2	2–3
2	0	≥3
1	1	≥2
0	2	≥1

$P[N(2)-N(0)=2,\ N(3)-N(1) \geq 3] = P[N(1)-N(0)=2,\ N(2)-N(1)=0,\ N(3)-N(2) \geq 3]$ $P[N(1)-N(0)=1,\ N(2)-N(1)=1,\ N(3)-N(2) \geq 2]$
$P[N(1)-N(0)=0,\ N(2)-N(1)=2,\ N(3)-N(2) \geq 1]$
$= \frac{2^2e^{-2}}{2!} \cdot \frac{2^0 e^{-2}}{0!} \cdot \left(1 - \frac{2^0e^{-2}}{0!} - \frac{2^1e^{-2}}{1!} - \frac{2^2e^{-2}}{2!}\right) + \frac{2^1e^{-2}}{1!} \cdot \frac{2^1e^{-2}}{1!} \cdot \left(1 - \frac{2^0e^{-2}}{0!} - \frac{2^1e^{-2}}{1!}\right) + \frac{2^0e^{-2}}{0!} \cdot \frac{2^2e^{-2}}{2!} \cdot \left(1 - \frac{2^0e^{-2}}{0!}\right)$

- Prob no arrival in $[0,4]$: made in counting domain $\equiv$ Prob. 1st arrival at least 4 min : made in time domain
  $\Rightarrow P[N(4)-N(0)=0] = P[X=0 \mid X \sim Poi(2(4-0)=8)] = \frac{8^0 e^{-8}}{0!}$
- Prob. 1st arrival takes at least t min $\Rightarrow P[N(t)-N(0)=0] = P[X=0 \mid X \sim Poi(\lambda(t-0)=2t)] = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$
- Dist of T_1 := time to 1st arrival $\Rightarrow P[T_1 > t] = e^{-\lambda t}$, $P[T_1 \leq t] = 1 - P[T_1 > t] = 1 - e^{-\lambda t}$
  $\Rightarrow$ Time to 1st arrival in $PP(\lambda) \sim \exp(\lambda)$ pf) let T_1 be the time to first arrival.
  $P(T_1 \leq t) = P(N(t)-N(0) > 0) = 1 - P(N(t)-N(0)=0) = 1 - P(X=0 \mid X \sim Poi(\lambda t)) = 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = 1 - e^{-\lambda t}$
  $\therefore T_1 \sim \exp(\lambda)$

For $n>1$, let T_n denoted the elapsed time between $(n-1)^{th}$ & n^{th} event. $\{T_n, n=1,2,3,\cdots\}$ = sequence of interarrival times.
$\therefore T_n \sim \exp(\lambda)$ for all $n=1,2,3,\cdots$.
independent increments : independent of all that has previously occured.
stationary increments : same distribution as the OG process.
$\Rightarrow$ no memory.

Merging PP.
$N_A = \{N_A(t), t \geq 0\}$ is $PP(\lambda_A)$, $N_B = \{N_B(t), t \geq 0\}$ is $PP(\lambda_B)$, they are independent. $\Rightarrow N = \{N(t) = N_A(t) + N_B(t)\}$ is $PP(\lambda_A + \lambda_B)$

Thinning PP.
$N = \{N(t), t \geq 0\}$ is $PP(\lambda)$, each arrival choose subset A (prob $=p$) or B (prob $= q = 1-p$)
$\Rightarrow N_A = \{N_A(t), t \geq 0\}$ is $PP(p\lambda)$, $N_B = \{N_B(t), t \geq 0\}$ is $PP(q\lambda)$

counting process $N = \{N(t), t \geq 0\}$ with rate λ, $PP(\lambda)$ if $N(0)=0$, $N(t)$ has independent increment, $N(t+s)-N(s) \sim Poi(\lambda t)$
Non-homogeneous Poisson Process : $\lambda \to \lambda(t)$. function for arrival rate that changes over time.
$\Rightarrow$ counting process $N = \{N(t), t \geq 0\}$ is $NHPP(\lambda(t))$ if $\begin{cases} N(0)=0 \\ N(t) \text{ has independent increment.} \\ N(t+s)-N(s) \sim Poi\left(\int_s^{s+t} \lambda(u)du\right) \end{cases}$

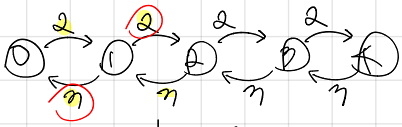
example. $\lambda(t) = \begin{cases} 60t & (0 \leq t \leq 1) \\ 60 & (t > 1) \end{cases}$ $P[N(\frac{7}{6}) - N(\frac{5}{6}) = k] = P\left[Poi\left(\int_{5/6}^{7/6} \lambda(u)du\right)\right]$
$= P\left[Poi\left(\int_{5/6}^{1} 60u\,du + \int_1^{7/6} 60\,du\right) = k\right] = P[Poi(115/6) = k] = \frac{(\frac{115}{6})^k e^{-115/6}}{k!}$

cf) $X_1 \sim exp(\lambda_1)$, $X_2 \sim exp(\lambda_2)$ X_1 & X_2 independent. $\Rightarrow P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$

total n customers $(n \geq 1)$ in 1-server post office. $T_A \sim exp(2)$, $T_S \sim exp(3)$ $\quad n \begin{cases} n-1 & \text{(service complete)} \\ n+1 & \text{(customer arrival)} \end{cases}$

$P(n \to n+1) = \frac{2}{5}$, $P(n \to n-1) = \frac{3}{5}$ distribution of time to transition = $\min(T_A, T_S) \sim exp(2+3)$

M/M/1/3. $S = \{0,1,2,3,4\}$ CTMC's rate diagram:



time to transition

from	
0	$exp(2)$
1	$exp(2+3)$
2	$exp(2+3)$
3	$exp(2+3)$
4	$exp(3)$

rate matrix G: $(\Sigma\, row = 0)$
$$\begin{bmatrix} -2 & 2 & & & \\ 3 & -5 & 2 & & \\ & 3 & -5 & 2 & \\ & & 3 & -5 & 2 \\ & & & 3 & -3 \end{bmatrix}$$

$\Rightarrow \pi G = 0$.

flow balance	inflow	=	outflow.
state 0	$3\pi_1$	=	$2\pi_0$
state 1	$2\pi_0 + 3\pi_2$	=	$3\pi_1 + 2\pi_1$
state 2	$2\pi_1 + 3\pi_3$	=	$3\pi_2 + 2\pi_2$
state 3	$2\pi_2 + 3\pi_4$	=	$3\pi_3 + 2\pi_3$
state 4	$2\pi_3$	=	$3\pi_4$

$\Sigma \pi_i = 1$, $\pi = [\, 81/211,\ 54/211\ \ 36/211\ \ 24/211\ \ 16/211 \,]$

CTMC:

continuous time stochastic process $\{X(t), t \geq 0\}$ in state space S

all $s, t \geq 0$, $i, j \in S$, $X(u) \in S$ for $0 \leq u < s$

$P[X(t+s) > t \mid X(s) = i, X(u) = x(u), 0 \leq u < s]$
$= P[X(t+s) > t \mid X(s) = i]$

$\Rightarrow$ depending on most recent info.

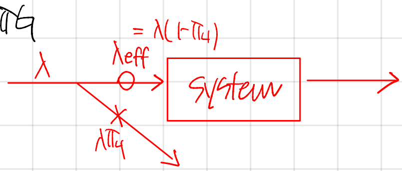
Time to transition $\sim$ exp. Markov property = exp. dist. time to transition.

cutting method	$L \to R$	=	$R \to L$
$\{0\} \to \{1,2,3,4\}$	$2\pi_0$	=	$3\pi_1$
$\{0,1\} \to \{2,3,4\}$	$2\pi_1$	=	$3\pi_2$
$\{0,1,2\} \to \{3,4\}$	$2\pi_2$	=	$3\pi_3$
$\{0,1,2,3\} \to \{4\}$	$2\pi_3$	=	$3\pi_4$

Stationary distribution, π $\Rightarrow \pi = \pi P(t)$ & $\pi G = 0$. $P(t)$ = CTMC transition matrix. $= e^{tG}$

proof). claim, $\pi = \pi P(t)$ for all $t \geq 0 \Rightarrow \pi G = 0$.

$\pi = \pi P(t) = \pi e^{tG} \Rightarrow \frac{d}{dt}\pi = \frac{d}{dt}\pi P(t) = \frac{d}{dt}\pi e^{tG} \Rightarrow 0 = \pi e^{tG} \cdot G \Rightarrow t = 0,\ 0 = \pi G$



system empty? π_0, system busy? $1-\pi_0$. customer not accepted? π_4.

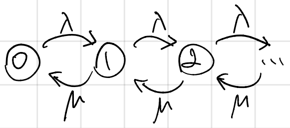
$L_{sys} = 0 \cdot \pi_0 + 1 \cdot \pi_1 + 2\pi_2 + 3\pi_3 + 4\pi_4$. $\quad L_{queue}: 0\pi_0 + 0\pi_1 + 1\pi_2 + 2\pi_3 + 3\pi_4$.

$W_{sys} = \frac{L_{sys}}{\lambda_{eff}}$ $\quad W_{queue} = \frac{L_{queue}}{\lambda_{eff}}$ $\quad W_{svc} = W_{sys} - W_{queue}$, $TH = \lambda_{eff}$, $\lambda_{eff} = \lambda(1-\pi_4)$

M/M/1/∞ Arrival is $PP(\lambda/min)$, Service is $PP(\mu/min)$ $\Rightarrow T_A \sim exp(\lambda)$, $T_S \sim exp(\mu)$ $\quad \rho = \frac{\lambda}{\mu} < 1$ for stability.

$X(t)$: # customers in the sys at time t. $\{X(t), t \geq 0\}$ is CTMC with state space $S = \{0,1,2,\cdots\}$

rate diagram:



stationary dist.	$L \to R$	=	$R \to L$
$\{0\} \to \{1,2,\cdots\}$	$\lambda\pi_0$	=	$\mu\pi_1$
$\{0,1\} \to \{2,3,\cdots\}$	$\lambda\pi_1$	=	$\mu\pi_2$
$\{0,1,2\} \to \{3,4,\cdots\}$	$\lambda\pi_2$	=	$\mu\pi_3$

$\pi_i = \left(\frac{\lambda}{\mu}\right)^i \pi_0$

$\sum_{i=0}^{\infty} \pi_i = \frac{\pi_0}{1-\frac{\lambda}{\mu}} = 1 \Rightarrow \pi_0 = 1 - \frac{\lambda}{\mu} = 1-\rho$, $\quad \pi_i = \left(\frac{\lambda}{\mu}\right)^i \left(1 - \frac{\lambda}{\mu}\right) = \rho^i(1-\rho)$

$L_{sys} = 0\pi_0 + 1\pi_1 + 2\pi_2 + \cdots = 0(1-\rho) + 1 \cdot \rho(1-\rho) + 2\rho^2(1-\rho) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$ $\quad (\because S - \rho S)$

$W_{sys} = L_{sys}/\lambda = \frac{\rho}{1-\rho} \times \frac{1}{\lambda} = \frac{\lambda}{\mu-\lambda} \times \frac{1}{\lambda} = \frac{1}{\mu - \lambda}$

$L_{queue} = 0\pi_0 + 0\pi_1 + 1\pi_2 + 2\pi_3 + \cdots = \rho^2/(1-\rho) = \frac{\lambda^2}{\mu(\mu-\lambda)}$ $\quad (\because S - \rho S)$

$W_{queue} = L_{queue}/\lambda = \frac{\rho^2}{1-\rho} \times \frac{1}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$ $\quad \left(W_q = E[V] \cdot \frac{\rho}{1-\rho}\left(\frac{C_a^2 + C_s^2}{2}\right)\right)$

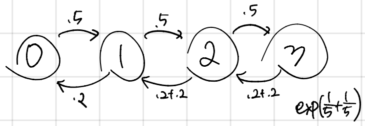
Stable dist $\Rightarrow \exists$ Stationary dist.
unstable dist $\Rightarrow \nexists$ Stationary dist

finite CTMC $\Rightarrow$ unique Stationary dist.
infinite Stable CTMC $\Rightarrow \exists$ Stat. dist.

M/M/2/1

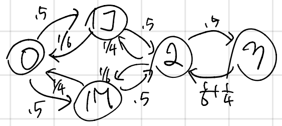
Arrival is $PP(\frac{1}{2})$, each service $\sim exp(\frac{1}{5})$

$S=\{0,1,2,3\}$



Arrival is $PP(1/2)$, John $\sim exp(\frac{1}{8})$ Mary $\sim exp(\frac{1}{4})$

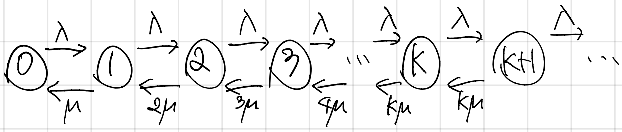
$S=\{0,1,2,3\}$



M/M/K/∞

Arrival is $PP(\lambda)$, service $\sim exp(\mu)$ $S=\{0,1,2,\cdots\}$

$\Rightarrow \lambda_n=\lambda$ $\Rightarrow \mu_n=\begin{cases} n\mu & (1\le n<k) \\ k\mu & (n\ge k)\end{cases}$



cutting method	L→R	=	R→L
$\{0,\cdots,n-1\}\to\{n,\cdots\}$	$\lambda_{n-1}\pi_{n-1}$	=	$\mu_n\pi_n$

$\Rightarrow \pi_n=\frac{\lambda_{n-1}}{\mu_n}\pi_{n-1}=\frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{n-1}}{\mu_1\mu_2\mu_3\cdots\mu_n}\pi_0=\begin{cases} n<k, & \pi_n=\frac{\lambda^n}{n!\mu^n}\pi_0=\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\pi_0 \\ n\ge k, & \pi_n=\frac{\lambda^n}{k!k^{n-k}\mu^n}\pi_0=\frac{1}{k!k^{n-k}}\left(\frac{\lambda}{\mu}\right)^n\pi_0 .\end{cases}$

$\sum_0^\infty \pi_n=1=\pi_0\left(\sum_{n=0}^{k-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\sum_{n=k}^{\infty}\frac{1}{k!k^{n-k}}\left(\frac{\lambda}{\mu}\right)^n\right)$

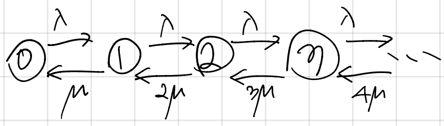
$L_q=\sum_k^\infty (n-k)\pi_n$, $W_q=\frac{L_q}{\lambda}$, $W_{sys}=W_q+\frac{1}{\mu}$, $L_{sys}=W_{sys}\times\lambda=L_q+\frac{\lambda}{\mu}$

M/M/∞/∞ … self service

$S=\{0,1,2,\cdots\}$ n= customer in system & service

λ_n, arrival rate $=\lambda$. μ_n, service rate $=n\mu$.

$\Rightarrow$ always stable ($\because$ #server $\to\infty$)



cutting method	L→R	R→L
$\{0\}\to\{1,2,3\cdots\}$	$\lambda\pi_0$	$\mu\pi_1$
$\{0,1\}\to\{2,3,4\cdots\}$	$\lambda\pi_1$	$2\mu\pi_2$
$\{0,1,2\}\to\{3,4,5,\cdots\}$	$\lambda\pi_2$	$3\mu\pi_3$

$\begin{cases}\pi_1=\frac{\lambda}{\mu}\pi_0 \\ \pi_2=\frac{1}{2}\left(\frac{\lambda}{\mu}\right)\pi_1=\frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2\pi_0 \\ \pi_3=\frac{1}{3}\left(\frac{\lambda}{\mu}\right)\pi_2=\frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3\pi_0\end{cases}$ $\Rightarrow \pi_i=\frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\pi_0$

$\sum_0^\infty \pi_i=\pi_0\sum_0^\infty\frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i=1$, $\pi_0=e^{-\frac{\lambda}{\mu}}$, ($\because \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}=e^\lambda$)

$\therefore \pi_i=\frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i e^{-\frac{\lambda}{\mu}} \Rightarrow Poi\left(\frac{\lambda}{\mu}\right)$

$L_{sys}=\frac{\lambda}{\mu}$. $L_q=0$. $W_{sys}=\frac{1}{\mu}$ $W_q=0$

$L_{svc}=\frac{\lambda}{\mu}$ $L_{svc}=\frac{1}{\mu}$