Lecture 2. Math Review (2)

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Math Review 17

$$\begin{cases} \wedge : \min \\ \vee : \max \end{cases} \qquad \begin{cases} x^+ : \max(x,0), \text{ positive part } \geq 0 \\ x^- : \min(x,0), \text{ negative part } \leq 0 \end{cases}$$

- For $X \sim U(20, 40)$, Evaluate $\mathbb{E}[X \wedge 25]$ and $\mathbb{E}[(25 - X)^+]$

$$\text{pdf } f(x) = \begin{cases} 1/20 & 20 \leq x \leq 40 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X \wedge 25] = \int_{-\infty}^{\infty} \min(x,25) f(x)\,dx = \int_{20}^{40} \min(x,25) \cdot \frac{1}{20}\,dx = \frac{1}{20}\left\{\int_{20}^{25} x\,dx + \int_{25}^{40} 25\,dx\right\}$$

$$= \frac{1}{20}\left(\frac{1}{2}x^2 \Big|_{20}^{25} + 25x \Big|_{25}^{40} \right) = 24.375$$

$$\mathbb{E}((25-X)^+) = \int_{-\infty}^{\infty} \max(25-x,0) f(x)\,dx = \int_{20}^{40} \max(25-x) \cdot \frac{1}{20}\,dx$$

$$= \frac{1}{20}\left\{\int_{20}^{25} (25-x)\,dx + \int_{25}^{40} 0\,dx\right\} = \frac{1}{20}\left[25x - \frac{1}{2}x^2\right]_{20}^{25} = \frac{1}{20} \cdot 12.5 = \frac{5}{8}$$

Math Review 18 Poisson Distribution : $p(x) = \mathbb{P}(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$ for $x = 1, 2, 3, 4, \cdots$

- For $X \sim Poi(8)$, $\lambda = 8$.
  - $\mathbb{P}(X = 0) = \frac{e^{-8} \cdot 8^0}{0!} = e^{-8}$
  - $\mathbb{P}(2 \leq X \leq 4) = \frac{e^{-8} \cdot 8^2}{2!} + \frac{e^{-8} \cdot 8^3}{3!} + \frac{e^{-8} \cdot 8^4}{4!} = e^{-8} \times 288$
  - $\mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leqslant 2)$

    $= 1 - \frac{e^{-8} \cdot 8^2}{2!} - \frac{e^{-8} \cdot 8^1}{1!} - \frac{e^{-8} \cdot 8^0}{0!} = 1 - e^{-8} \times 41$

exp $f(x) = \lambda e^{-\lambda x}$ $\quad$ $\int uv' = uv - \int u'v$

- For $X \sim exp(7)$, Evaluate $\mathbb{E}[max(X, 7)]$

$f(x) = 7e^{-7x}$, $\mathbb{E}[max(X,7)] = \int_{-\infty}^{\infty} max(x,7) f(x) dx = \int_0^7 7f(x)dx + \int_7^\infty x f(x) dx$

$= 7\left[-e^{-7x}\right]_0^7 + \left[-xe^{-7x}\right]_7^\infty - \int_7^\infty e^{-7x} dx = 7\left[-e^{-7x}\right]_0^7 + \left[-xe^{-7x}\right]_7^\infty - \left[-\frac{1}{7}e^{-7x}\right]_7^\infty$

$= -7e^{-49} + 7e^0 + 0 + 7e^{-49} + 0 + \frac{1}{7}e^{-49} = 7 + \frac{1}{7}e^{-49}$.

Math Review 20

$f(x) = \lambda e^{-\lambda x}$, $F(x) = 1 - e^{-\lambda x}$ $(x \geq 0)$

- For $X \sim exp(8)$, find x^* such that $F(x^*) = 0.6$

$f(x) = 8e^{-8x}$, $F(x) = 1 - e^{-8x}$

$F(x^*) = 0.6 \Rightarrow 0.4 = e^{-8x^*} \Rightarrow \ln 0.4 = -8x^*$, $x^* = -\frac{1}{8}\ln 0.4$

Math Review 21

- For $X \sim U(10, 20)$, find x^* such that $F(x^*) = 0.7$

$$f(x) = \begin{cases} 1/10 & 10 \le x \le 20 \\ 0 & \text{o.w.} \end{cases} \qquad F(x) = \begin{cases} 0 & x < 10 \\ \frac{x-10}{10} & 10 \le x \le 20 \\ 1 & x > 20 \end{cases}$$

$$F(x^*) = \frac{x^* - 10}{10} = 0.7 \Rightarrow x^* = 17.$$

blank

- Matrix Multiplication

가로 · 세로 .

$$\begin{array}{cccc} 1\times 2 & 2\times 2 & & 1\times 2 \\ \begin{pmatrix} .6 & .4 \end{pmatrix} & \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} & = & \begin{pmatrix} (.6\times.7+.4\times.5) & (.6\times.3+.4\times.5) \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} .62 & .38 \end{pmatrix}$$

- Solve

$$\underset{m\times n}{A}\ \underset{n\times 1}{x} = \underset{m\times 1}{b}$$

\# eq : m
\# var : n

$$\begin{cases} m>n \\ m=n \\ m<n \end{cases}$$

same!

$$\left.\begin{array}{c} \begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}.7 & .3\\ .5 & .5\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2\end{pmatrix} \\ \pi_1+\pi_2=1 \end{array}\right\} \Rightarrow \begin{array}{c} 0.7\pi_1 + 0.5\pi_2 = \pi_1 \\ 0.3\pi_1 + 0.5\pi_2 = \pi_2 \\ \pi_1+\pi_2 = 1 \end{array}$$

$$0.7\pi_1 + 0.5(1-\pi_1) = \pi_1$$

$$0.5 = 0.8\pi_1 \Rightarrow \pi_1 = \frac{5}{8},\ \pi_2 = \frac{3}{8}$$

stationary distribution

$$\left(\pi P = \pi \quad \& \quad \sum \pi_i = 1\right)$$

- Solve the following system of equations

$$
\begin{aligned}
x &= y \\
y &= 0.5z \\
z &= 0.6 + 0.4x \\
x + y + z &= 1
\end{aligned}
$$

no sol...?

- Solve stationary distribution.

$$(\pi_0 \quad \pi_1 \quad \pi_2)\begin{pmatrix}-2 & 2 & \\ 3 & -5 & 2 \\ & 3 & -3\end{pmatrix} = (0 \quad 0 \quad 0)$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\begin{aligned}-2\pi_0 + 3\pi_1 &= 0 \\ 2\pi_0 - 5\pi_1 + 3\pi_2 &= 0 \\ 2\pi_1 - 3\pi_2 &= 0\end{aligned} \Rightarrow \begin{aligned}2\pi_0 &= 3\pi_1 \\ 2\pi_1 &= 3\pi_2\end{aligned} \Rightarrow \begin{aligned}\pi_0 &= \tfrac{3}{2}\pi_1 \\ \pi_2 &= \tfrac{2}{3}\pi_1\end{aligned}$$

$$\Rightarrow \tfrac{3}{2}\pi_1 + \pi_1 + \tfrac{2}{3}\pi_1 = 1, \quad \pi_0 = \frac{9}{19}, \quad \pi_1 = \frac{6}{19}, \quad \pi_2 = \frac{4}{19}$$

$$P = \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix}$$

- What is P^2 ?

$$P^2 = \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} \begin{pmatrix} .7 & .3 \\ .5 & .5 \end{pmatrix} = \begin{pmatrix} .64 & .36 \\ .60 & .40 \end{pmatrix}$$

- Solve

$$10 \times \begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix} \begin{pmatrix} .7 & .3 & & \\ .5 & .5 & & \\ & & .6 & .4 \\ & & .3 & .7 \end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix} \times 10$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$$\left.\begin{aligned} 7\pi_1 + 5\pi_2 &= 10\pi_1 \\ 3\pi_1 + 5\pi_2 &= 10\pi_2 \\ 6\pi_3 + 3\pi_4 &= 10\pi_3 \\ 4\pi_3 + 7\pi_4 &= 10\pi_4 \end{aligned}\right\} \quad \left.\begin{aligned} 3\pi_1 &= 5\pi_2 \\ 4\pi_3 &= 3\pi_4 \end{aligned}\right\} \quad \begin{aligned} &\pi_1 + \tfrac{3}{5}\pi_1 + \pi_3 + \tfrac{4}{3}\pi_3 = 1 \\ &\tfrac{8}{5}\pi_1 + \tfrac{7}{3}\pi_3 = 1 \quad \therefore \text{ not unique sol.} \end{aligned}$$

blank how to represent solution

$$\pi_1 + \frac{3}{5}\pi_1 + \pi_3 + \frac{4}{3}\pi_3 = \frac{8}{5}\pi_1 + \frac{7}{3}\pi_3 = 1$$

i) Let $p := \pi_1$ $\quad p \in [0,1]$

$\pi_1 = p \qquad \pi_3 = \frac{3}{7}\left(1 - \frac{8}{5}p\right)$

$\pi_2 = \frac{3}{5}p \qquad \pi_4 = \frac{4}{7}\left(1 - \frac{8}{5}p\right)$

$$\pi = \left[p,\ \frac{3}{5}p,\ \frac{3}{7}\left(1-\frac{8}{5}p\right),\ \frac{4}{7}\left(1-\frac{8}{5}p\right)\right]$$

for any $p \in [0,1]$

ii) Let $p := \pi_1 + \pi_2 = \frac{8}{5}\pi_1$

$\pi_1 = \frac{5}{8}p \qquad \pi_3 = \frac{3}{7}(1-p)$

$\pi_2 = \frac{3}{5}\pi_1 = \frac{3}{8}p \qquad \pi_4 = \frac{4}{7}(1-p)$

$$\pi = \left[\frac{5}{8}p,\ \frac{3}{8}p,\ \frac{3}{7}(1-p),\ \frac{4}{7}(1-p)\right]$$

for any $p \in [0,1]$

Math Review 23 - Express infinite vector math form

- Solve the following and express π_i for $i = 0, 1, 2, \ldots$

$$
\begin{aligned}
\pi_0 + \pi_1 + \pi_2 + \ldots &= 1 \\
\underline{0.02\pi_0 + 0.02\pi_1 + 0.02\pi_2 + \ldots} &= \pi_0 \\
0.98\pi_0 &= \pi_1 \\
0.98\pi_1 &= \pi_2 \\
0.98\pi_2 &= \pi_3 \\
\ldots &= \ldots
\end{aligned}
$$

$= 0.02(\pi_0 + \pi_1 + \cdots)$

$= 0.02 \times 1 = \pi_0$

$\pi_0 = 0.02$

$\pi_1 = 0.02 \times 0.98$

$\pi_2 = 0.02 \times 0.98^2$

$\vdots$

$\pi_i = 0.02 \times 0.98^i$

for $i = 0, 1, 2, \cdots$

blank

Math Review 24 - Infinite Geometric Series

- When $|r| < 1$, $S = a + ar + ar^2 + ar^3 + \ldots$ 등비수열의 합.

$$
\begin{array}{rl}
 & S = a + ar + ar^2 + ar^3 + \cdots \\
- & rS = \phantom{a + {}} ar + ar^2 + ar^3 + \cdots \\
\hline
 & (1-r)S = a. \Rightarrow S = \dfrac{a}{1-r}
\end{array}
$$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Math Review 25 - Geometric Series

- When $r \neq 1$, $S = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$

$$S = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$- \quad rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$(1-r)S = a - ar^n$$

$$\therefore S = \frac{a}{1-r}(1-r^n)$$

Math Review 26 - Power Series

- When $|r| < 1$, $S = r + 2r^2 + 3r^3 + 4r^4 + \ldots$

$$S = r + 2r^2 + 3r^3 + \cdots$$

$$- \quad rS = r^2 + 2r^3 + \cdots$$

$$(1-r)S = r + r^2 + r^3 + \cdots$$

$$= \frac{r}{1-r}$$

$$\sum_{0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\therefore S = \frac{r}{(1-r)^2}$$

blank

Math Review 27 - Conditional Probabilities

- Show that $\mathbb{P}(A|B\cap C)\mathbb{P}(B|C) = \mathbb{P}(A\cap B|C)$

LHS)

$$\mathbb{P}(A|B\cap C)\mathbb{P}(B|C) = \frac{\mathbb{P}(A\cap B\cap C)}{\mathbb{P}(B\cap C)} \times \frac{\mathbb{P}(B\cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(A\cap B\cap C)}{\mathbb{P}(C)}$$

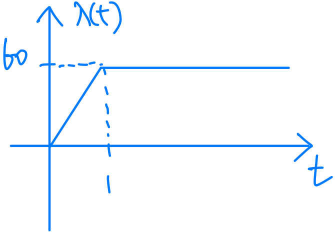
RHS)

$$\mathbb{P}(A\cap B|C) = \frac{\mathbb{P}(A\cap B\cap C)}{\mathbb{P}(C)}$$

blank

Math Review 28 - Formulation of time varying function

- During the first hour $(0 \leq t \leq 1)$, $\lambda(t)$ increases linearly from 0 to 60. After the first hour, $\lambda(t)$ is constant at 60. Draw plot for $\lambda(t)$ and express the function in math form.



$$\lambda(t) = \begin{cases} 60t & (0 \leq t \leq 1) \\ 60 & (t > 1) \end{cases}$$

blank

Exercise 1

- For $X \sim U(a, b)$, show pdf $\rightarrow$ cdf

pdf $f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{o.w.} \end{cases}$

cdf $F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$

Exercise 2

- For $X \sim exp(\lambda)$, what is pdf $f(x)$?

$f(x) = \lambda e^{-\lambda x}$

Exercise 3

- For $X \sim exp(\lambda)$, show pdf $\rightarrow$ cdf.

pdf $f(x) = \lambda e^{-\lambda x} \quad (x \geq 0)$

cdf $F(x) = \mathbb{P}(X < x) = 1 - \mathbb{P}(X \geq x) = 1 - \int_x^{\infty} \lambda e^{-\lambda y} dy = 1 - \left[-e^{-\lambda y}\right]_x^{\infty}$

$= 1 - (0 + e^{-\lambda x}) = 1 - e^{-\lambda x} \quad (x \geq 0)$ ∎

Exercise 4

- For $X \sim exp(\lambda)$, show that $\mathbb{E}X = 1/\lambda$

$x \geq 0$, $f(x) = \lambda e^{-\lambda x}$.

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} \lambda x e^{-\lambda x} dx = \left[-x e^{-\lambda x}\right]_0^{\infty} - \int_0^{\infty} -e^{-\lambda x} dx = \int_0^{\infty} e^{-\lambda x} dx$$

$$= \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^{\infty} = \frac{1}{\lambda}$$

Exercise 5

- For $X \sim exp(\lambda)$, show that $Var(X) = 1/\lambda^2$
- (hint) (You may prove this by continuing the following:)

pf) Since $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$ and we know $\mathbb{E}X = 1/\lambda$ from **hw1.#4**, we need to know what $\mathbb{E}X^2$ is.

$$\begin{aligned} \mathbb{E}X^2 &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx \\ &= \lambda \left(x^2 \cdot -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty} - \int_0^{\infty} 2x \cdot -\frac{1}{\lambda} e^{-\lambda x} dx \right) \\ &= \ldots \end{aligned}$$

blank

$$\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx = \left[-x^2 e^{-\lambda x}\right]_0^\infty + \int_0^\infty +2x e^{-\lambda x}\,dx$$

$$= 2\times\left\{ -\frac{1}{\lambda} x e^{-\lambda x}\Big|_0^\infty + \int_0^\infty +\frac{1}{\lambda} e^{-\lambda x}\,dx \right\} = 2\left[-\frac{1}{\lambda^2}e^{-\lambda x}\right]_0^\infty = 2\left(0+\frac{1}{\lambda^2}\right) = \frac{2}{\lambda^2}$$

$$\mathrm{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$\mathrm{sd}X = \frac{1}{\lambda}$$

- Solution

$$
\begin{aligned}
&= \lambda \left((0-0) + \frac{2}{\lambda} \int_0^\infty x e^{-\lambda x} dx \right) = 2 \int_0^\infty x e^{-\lambda x} dx \\
&= 2 \left[x \cdot -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty - \int_0^\infty -\frac{1}{\lambda} e^{-\lambda x} dx \right] \\
&= 2 \left[\infty \cdot -\frac{1}{\lambda} e^{-\lambda \infty} - \left(0 \cdot -\frac{1}{\lambda} e^{-\lambda 0} \right) + \frac{1}{\lambda} \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty \right] \\
&= 2 \left[0 - 0 + \frac{1}{\lambda} \left[-\frac{1}{\lambda} e^{-\lambda \infty} + \frac{1}{\lambda} e^{-\lambda 0} \right] \right] \\
&= 2 \cdot \frac{1}{\lambda} \left(0 + \frac{1}{\lambda} \right) = \frac{2}{\lambda^2}
\end{aligned}
$$

Since $Var(x) = \mathbb{E}X^2 - (\mathbb{E}X)^2, Var(x) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Exercise 6

- Show that Exponential distribution is memoryless.

r.v. X is memoryless when $\mathbb{P}(X > s+t \mid X > t) = \mathbb{P}(X > s)$ for $(s, t \geq 0)$

$F(x) = 1 - e^{-\lambda x} = \mathbb{P}(X \leq x)$

$$\mathbb{P}(X > s+t \mid X > t) = \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > t)} = \frac{1 - F(s+t)}{1 - F(t)}$$

$$= \frac{e^{-(s+t)\lambda}}{e^{-t\lambda}} = e^{-(s+t-t)\lambda} = e^{-s\lambda} = 1 - F(s)$$

$$= 1 - \mathbb{P}(X \leq s) = \mathbb{P}(X > s)$$

Exercise 7

- For $X \sim Poi(\lambda)$, show that $\mathbb{E}X = \lambda$

$$\mathbb{P}(X=k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad (k = 0, 1, 2, \cdots)$$

$$\mathbb{E}X = \sum_{0}^{\infty} x\,p(x) = \sum_{0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{1}^{\infty} \lambda \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} = \sum_{0}^{\infty} \lambda \cdot \frac{\lambda^y e^{-\lambda}}{y!} = \lambda \sum_{0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!}$$

$$= \lambda \sum_{0}^{\infty} p(x) = \lambda$$

Exercise 8

- For $X \sim Poi(\lambda)$, show that $Var(X) = \lambda$
- (hint) (you may prove by continuing following.)

pf) Since $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$ and we know $\mathbb{E}X = \lambda$ from **hw1.#7**, we need to know what $\mathbb{E}X^2$ is.

$$\begin{aligned}
\mathbb{E}X^2 &= \sum_{x=-\infty}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} x^2 \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \left(0^2 \frac{\lambda^0}{0!} + \sum_{x=1}^{\infty} x^2 \frac{\lambda^x}{x!} \right) \\
&= e^{-\lambda} \left(\sum_{x=1}^{\infty} x \frac{\lambda^x}{(x-1)!} \right) \\
&= e^{-\lambda} \left(\sum_{x=1}^{\infty} (x-1+1) \frac{\lambda^x}{(x-1)!} \right) \\
&= e^{-\lambda} \left(\sum_{x=1}^{\infty} (x-1) \frac{\lambda^x}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \right) \\
&= e^{-\lambda} \left(\sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \right) \\
&= \ldots
\end{aligned}$$

blank

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, \cdots \qquad\qquad x - 1 = y$$

$$\mathbb{E}X^2 = \sum_{x=0}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} \lambda x \cdot \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} = \sum_{y=0}^{\infty} \lambda (y+1) \frac{\lambda^y e^{-\lambda}}{y!}$$

$$= \lambda \left\{ \sum_{y=0}^{\infty} y \frac{\lambda^y e^{-\lambda}}{y!} + 1 \right\} = \lambda \left\{ \mathbb{E}X + 1 \right\} = \lambda(\lambda + 1) = \lambda^2 + \lambda$$

$$\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

- Solution

$$
\begin{aligned}
&= e^{-\lambda}\left[\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}\right] \\
&= e^{-\lambda}\left[\lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^{y}}{y!} + \lambda \sum_{z=0}^{\infty} \frac{\lambda^{z}}{z!}\right], \text{ where } y := x-2,\, z := x-1 \\
&= e^{-\lambda}\left[\lambda^2 \cdot e^{\lambda} + \lambda \cdot e^{\lambda}\right] \\
&= \lambda^2 + \lambda
\end{aligned}
$$

$$Var(x) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

(Note that the property of exponential function is used, i.e. $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$)

Exercise 9

- Let X be a discrete random variable with $\mathbb{P}(X = i) = ci$ for positive, odd integers $i < 8$; otherwise, the probability is zero.
  - *(a)* Compute the value of c. (by using the one of the properties of pmf)
  - *(b)* What is the mean of X?
  - *(c)* What is the second moment of X? (i.e. What is $\mathbb{E}X^2$?)
  - *(d)* What is the variance of X?
  - *(e)* Compute $\mathbb{E}[(X-3)^+]$

a) $i = 1, 3, 5, 7 \Rightarrow c(1+3+5+7) = 1 \quad \therefore c = \frac{1}{16}$

b) $\mathbb{E}X = \sum_{-\infty}^{\infty} x p(x) = \frac{1}{16}(1^2+3^2+5^2+7^2) = 21/4$

c) $\mathbb{E}X^2 = \sum_{-\infty}^{\infty} x^2 p(x) = \frac{1}{16}(1^3+3^3+5^3+7^3) = 496/16 = 31$

d) $\mathrm{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 31 - \left(\frac{21}{4}\right)^2 = 3.4375$

blank

e) $(x-3)^+ = \begin{cases} 0 & x \le 3 \\ x-3 & x > 3 \end{cases}$

$$\mathbb{E}\left((X-3)^+\right) = \sum_{-\infty}^{\infty} (x-3)^+ p(x) = (5-3)\frac{5}{16} + (7-3)\frac{7}{16}$$

$$= \frac{1}{16}(10+28) = \frac{19}{8}$$

- Solution

(a) Since $\sum_{i=-\infty}^{\infty} p(i) = \sum_{i=-\infty}^{\infty} \mathbb{P}(X=i) = 1,$
$\mathbb{P}(X=1)+\mathbb{P}(X=3)+\mathbb{P}(X=5)+\mathbb{P}(X=7) = c+3c+5c+7c = 1$
$\therefore\ c = 1/16$

(b)

$$\mathbb{E}X = \sum_{i=-\infty}^{\infty} i\cdot p(i) = \sum_{i=-\infty}^{\infty} i\cdot \mathbb{P}(X=i) = 1\cdot p(1)+3\cdot p(3)+5\cdot p(5)+7\cdot p(7)$$
$$= \frac{1}{16}(1^2+3^2+5^2+7^2) = \frac{84}{16} = \frac{21}{4}$$

(c)

$$\mathbb{E}X^2 = \sum_{i=-\infty}^{\infty} i^2\cdot p(i) = \sum_{i=-\infty}^{\infty} i^2\cdot \mathbb{P}(X=i) = 1^2\cdot p(1)+3^2\cdot p(3)+5^2\cdot p(5)+7^2\cdot p(7)$$
$$= \frac{1}{16}(1^3+3^3+5^3+7^3) = \frac{496}{16} = \frac{124}{4} = 31$$

(d) $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 31 - \left(\frac{21}{4}\right)^2 = 3.4375$

(e)

$$\mathbb{E}[(X-3)^+] = \mathbb{E}[max(X-3,0)] = \sum_{i=-\infty}^{\infty} max(i-3,0) \cdot \mathbb{P}(X=i)$$
$$= max(1-3,0) \cdot \mathbb{P}(X=1) + max(3-3,0) \cdot \mathbb{P}(X=3)$$
$$+ max(5-3,0) \cdot \mathbb{P}(X=5) + max(7-3,0) \cdot \mathbb{P}(X=7)$$
$$= 2 \cdot \frac{5}{16} + 4 \cdot \frac{7}{16} = \frac{38}{16} = \frac{19}{8}$$

Exercise 10

$X \sim Poi(5)$

- Let X be a Poisson random variable with parameter 5, and let $Y = min(X, 2)$
  - *(a)* What is the pmf of X?
  - *(b)* What is the mean of X?
  - *(c)* What is the variance of X?
  - *(d)* What is the pmf of Y? (i.e. Specify $\mathbb{P}(Y = i)$ for $i = 0, 1, 2, \ldots$)
  - *(e)* Compute $E[Y]$?

a) $p(x) = \frac{5^x e^{-5}}{x!}$

$Y = \begin{cases} X & i \le 2 \\ 2 & i > 2 \end{cases}$

b) $\lambda = 5$

c) $\lambda = 5$

d) $\mathbb{P}(Y=y)$

$= \mathbb{P}(\min(X,2) = y)$ $\begin{cases} p(0) = e^{-5} & y=0 \\ p(1) = 5e^{-5} & y=1 \\ p(2) = 1-6e^{-5} & y=2 \\ p(i) = 0, \ i \ge 3 & \end{cases}$

e) $\sum_{-\infty}^{\infty} x p(x)$

$= 0 \cdot e^{-5} + 1 \cdot 5e^{-5}$

$+ 2(1-6e^{-5})$

$= 2 - 7e^{-5}$

- Solution

(a) $p(x) = \mathbb{P}(X = x) = \frac{e^{-5}5^x}{x!}$

(b) $5 \quad (= \lambda)$

(c) $5 \quad (= \lambda)$

(d) We have $p(y) = \mathbb{P}(Y = y) = \mathbb{P}(min(X, 2) = y)$. It follows:

- $p(0) = \mathbb{P}(Y = 0) = \mathbb{P}(min(X, 2), 0) = \mathbb{P}(X = 0) = \frac{e^{-5}5^0}{0!} = e^{-5}$
- $p(1) = \mathbb{P}(Y = 1) = \mathbb{P}(min(X, 2), 1) = \mathbb{P}(X = 1) = \frac{e^{-5}5^1}{1!} = 5e^{-5}$
- $p(2) = \mathbb{P}(Y = 2) = \mathbb{P}(min(X, 2) = 2) = \mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) = 1 - e^{-5} - 5e^{-5} = 1 - 6e^{-5}$
- $p(3) = \mathbb{P}(Y = 3) = 0 \quad (\because min(X, 2) \leq 2$ always), also $p(i) = 0$ for all $i \geq 3$
- Therefore,

$$p(y) = \left(\begin{array}{c} e^{-5} \text{ for } y = 0 \\ 5e^{-5} \text{ for } y = 1 \\ 1 - 6e^{-5} \text{ for } y \geq 2 \end{array} \right)$$

(e)

$$\mathbb{E}Y = \sum_{i=-\infty}^{\infty} yp(y) = 0\cdot p(0)+1\cdot p(1)+2\cdot p(2) = 5e^{-5}+2(1-6e^{-5}) = 2-7e^{-5}$$

Exercise 11

- Let Y be a random variable with pdf ce^{-3y} for $y > 0$, and zero otherwise.
  - *(a)* Determine c.
  - *(b)* What is the mean, variance, and squared coefficient of variation of Y?
  - *(c)* Compute $\mathbb{P}(Y > 4)$
  - *(d)* Compute $\mathbb{P}(Y > 7|Y > 3)$
  - *(e)* Your answer for (c) and answer for (d) should be same. Discuss shortly why.
  - *(f)* What is the point x^* such that $\mathbb{P}(Y > x^*) = 2/3$?

a) $\int_{-\infty}^{\infty} ce^{-3y}\,dy = \int_0^{\infty} ce^{-3y}\,dy = \left[-\frac{1}{3}ce^{-3y}\right]_0^{\infty} = \frac{1}{3}c = 1 \quad \therefore c = 3$

b) $f(x) = \begin{cases} 3e^{-3y} & y>0 \\ 0 & \text{o.w} \end{cases}$ $\quad Y \sim \exp(3)$ $\quad \mathbb{E}Y = \frac{1}{3}$ $\quad \text{Var}X = \left(\frac{1}{3}\right)^2$

$CV^2 = \frac{(1/3)^2}{(1/3)^2} = 1$

c) $\mathbb{P}(Y>4) = 1 - \mathbb{P}(Y<4) = 1 - \left(1 - e^{-3\cdot 4}\right) = e^{-12}$

d) $\mathbb{P}(Y > 4+3 \mid Y>3) = \mathbb{P}(Y>4) = e^{-12}$

e) $\exp(\lambda)$ is memoryless

blank

f) $\mathbb{P}(Y > x^*) = 1 - \mathbb{P}(Y < x^*) = 1 - F(x^*) = e^{-3x^*} = 2/3$

$\Rightarrow -3x^* = \ln\left(\frac{2}{3}\right) \quad \therefore x^* = -\frac{1}{3}\ln(2/3)$

- Solution

(a)

$$\int_{-\infty}^{\infty} p(y)dy = 1$$
$$\rightarrow \int_{0}^{\infty} ce^{-3y}dy = c \cdot -\frac{1}{3}e^{-3y}\Big|_0^{\infty} = -\frac{c}{3}\left[0 - e^{-3\cdot 0}\right] = \frac{c}{3} = 1$$
$$\therefore c = 3$$

(b) pdf of $\begin{cases} 3e^{-3y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$ implies $Y \sim exp(3)$. Thus, $\mathbb{E}Y = \frac{1}{3}, VarY = \frac{1}{3^2}$,

and sqr-cv = $\frac{Var(Y)}{(\mathbb{E}Y)^2} = \frac{1/3^2}{(1/3)^2} = 1$

(c) cdf of $exp(3)$ is $\begin{cases} 1 - e^{-3y} & \text{for } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$ Thus,

$$\mathbb{P}(Y > 4) = 1 - \mathbb{P}(Y \leq 4) = 1 - (1 - e^{-3 \cdot 4}) = e^{-12}$$

(d) $\mathbb{P}(Y > 7 | Y > 3) = \frac{\mathbb{P}(Y>7, Y>3)}{\mathbb{P}(Y>3)} = \frac{1-(1-e^{-3\cdot 7})}{1-(1-e^{-3\cdot 3})} = \frac{e^{-21}}{e^{-9}} = e^{-12}$

(e) memoryless property

(f)

$$\mathbb{P}(Y > x^*) = 1 - \mathbb{P}(Y \leq x^*) = 1 - (1 - e^{-3x^*}) = \frac{2}{3}$$

$$\rightarrow e^{-3x^*} = \frac{2}{3}$$

$$\rightarrow -3x^* = ln(2/3)$$

$$\rightarrow x^* = -\frac{1}{3} ln(2/3)$$

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"Man can learn nothing unless he proceeds from the known to the unknown.
- Claude Bernard"
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