Lecture 7. Discrete Time Markov Chain (2)

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Motivation - Stationary Distribution

Remind our "Soda example" with transition probability matrix.

$$\mathbf{P} = \begin{array}{c} Coke \\ Pepsi \end{array} \begin{pmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{pmatrix}$$

- Suppose we happen to have a distribution of X_k, as expressed as $a_k = (5/8\ 3/8)$, then what is a_{k+1}?
- Then, what is a_{k+2}?
- Then, what is a_∞?
- Once you get into "this" distribution, it does not change over time any more!

Definition

Definition - Stationary Distribution

- For a DTMC with state space S and transition probability matrix **P**, $\pi = (\pi_i, i \in S)$ is said to be a **stationary distribution** if
  - $\pi_i \geq 0$ for all $i \in S$ and $\sum_{\pi \in S} \pi_i = 1$
  - $\pi = \pi P$
- Observations
  - Dynamic equilibrium, Steady states.
  - Under stationary condition, $(\text{inflow})_i$=$(\text{outflow})_i$ for all $i \in S$.
- Either unique or ∞ number of stationary distributions.

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How to compute stationary distibution? (1)

- By definition

How to compute stationary distibution? (2)

- By flow balance equation

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Motivation - Limiting probabilities

- n-step Transition probability: $\mathbb{P}(X_{k+n} = j | X_k = i) = \mathbf{P}_{ij}^n$

- Letting $n \to \infty$ to see what happens!

```r
P <- matrix(c(0.7, 0.5, 0.3, 0.5), ncol=2)
library(expm) # powering matrix
```

```r
P
```

```
##      [,1] [,2]
## [1,]  0.7  0.3
## [2,]  0.5  0.5
```

```r
P%^%2
```

```
##      [,1] [,2]
## [1,] 0.64 0.36
## [2,] 0.60 0.40
```

```r
P%^%3
```

```
##       [,1]  [,2]
## [1,] 0.628 0.372
## [2,] 0.620 0.380
```

```r
P%^%10
```

```
##           [,1]      [,2]
## [1,] 0.6250000 0.3750000
## [2,] 0.6249999 0.3750001
```

- The limit may or may not exist.

```r
P <- matrix(c(1, 0, 0, 1), ncol=2)
P
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

```r
P%^%10
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

```r
P <- matrix(c(0, 1, 1, 0), ncol=2)
P
```

```
##      [,1] [,2]
## [1,]    0    1
## [2,]    1    0
```

```r
P%^%10
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

```r
P%^%11
```

```
##      [,1] [,2]
## [1,]    0    1
## [2,]    1    0
```

When chain goes well

- Demonstration of soda problem

- Observations
    1. Limiting probability exists
    2. Limiting probability is independent on the initial state
    3. Stationary distribution is unique
- So what? (in this example)
    - Initial distribution does not matter in long run.
    - LONG RUN AVERAGE! (fraction of time, revenue, cost, $\cdots$)
    - Calculate limiting prob. by stationary dist.

How to make use of [stationary dist. = limiting prob.]

- If I do this for 10 years (3650 days) from now, then how many days I will drink Pepsi?
- Suppose Pepsi is \$1 and Coke is \$1.5. How much on average I spend on soda in a month?

Usage of stationary distribution

- Suppose there are a billion customers (who has same type of consuming pattern) like me in the world. You are working for Pepsi and like to boost Pepsi $\rightarrow$ Pepsi probability from 0.5 to 0.6 by marketing. On average, how much additional revenue will be generated by this change for a day?

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When things are not going well 1 - Periodic MC

- Transition diagram
- Demonstration

$$\mathbf{P} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix}$$

- Observations
  1. Limiting probability NOT exists
  2. Stationary distribution is unique
- Remedy
  - $\lim_{n\to\infty} \frac{\mathbf{P}^{n+1}+\mathbf{P}^{n+2}+\ldots+\mathbf{P}^{n+d}}{d}$ exists and each row is same as the stationary distribution.

When things are not going well 2 - Reducible MC

- Transition diagram
- Demonstration

$$\mathbf{P} = \begin{array}{c} Coke \\ Pepsi \\ Bud \\ Miller \end{array} \begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$

- Observations
  1. Limiting probability exists.
  2. But limiting probability depends on the initial state.
  3. Stationary distribution is not unique (∞)

Stationary distribution is not unique

Summary of observations

For a *finite* state space MC,

MC	Limiting	Stationary	Remark
Irreducible Aperiodic	Exists, indep. of initial state	Unique	NICE!
Irreducible Periodic	Not Exists	Unique	Remedy by average of d
Reducible Aperiodic	Exists, dependent on initial state	∞	Deeper look

A few definitions (1)

- Accessibility
  - **Def.** A state i can **reach** state j and write $i \to j$ if $\exists n$ s.t. $P_{ij}^n > 0$.
    -
    -
    -
  - State i and j are said to **communicate** and write $i \leftrightarrow j$ if $i \to j$ and $j \to i$.
  - A group of states that communicate is said to be a **class**.

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A few definitions (2)

- Reducibility
  - MC X_n is said to be **irreducible** if all states communicate.
  - MC X_n is said to be **irreducible** if $\exists$ only one class.

A few definitions (3)

- Periodicity
  - For a state $i \in S$, **period** $d(i) := gcd\{n, P_{ii}^n > 0\}$.
  - MC X_n is said to be **periodic** if $\exists\, i$ with $d(i) > 1$.
  - MC X_n is said to be **aperiodic** if not *periodic*.
  - Remark: Periodicity is class property.
    (Class shares period; $i \leftrightarrow j \Rightarrow d(i) = d(j)$)

So, when does it go well?

Theorem

- If a *finite state* DTMC X_n is *aperiodic* and *irreducible*, then all of the followings hold:
  - Limiting probabilities exists
  - Stationary distribution is unique.
  - Stationary distribution = Limiting probabilities.
- Above theorem can be also written as:
  - Finite, Aperiodic, Irreducible $\Rightarrow \lim_{n\to\infty} \mathbf{P}_{ij}^{n} = \pi_j, \forall i, j \in S$

Remark

- In these "nice" cases, we can talk about things like "The long-run fraction of time that the MC spends in each state".
- In these "nice" cases, we can calculate limiting probability by solving stationary distribution.

Your work

- Write your python codes to replicate what have been discussed above.

Exercise 1

- A store stocks a particular item. The demand for the product each day is 1 item with probability 1/6, 2 items with probability 3/6, and 3 items with probability 2/6. Assume that the daily demands are independent and identically distributed. Each evening if the remaining stock is less than 3 items, the store orders enough to bring the total stock up to 6 items. These items reach the store before the beginning of the following day. Assume that any demand is lost when the item is out of stock.
- *(a)* Let X_n be the amount in stock at the beginning of day n; assume that $X_0 = 5$.
  - (a1) Is this process Markov Chain?
  - (a2) Give the state space and initial distribution a_0 in a row vector.
  - (a3) Transition matrix.
- *(b)* Let Y_n be the amount in stock at the end of day n; assume that $Y_0 = 2$.
  - (b1) Is this process Markov Chain?
  - (b2) Give the state space and initial distribution a_0 in a row vector
  - (b3) Transition matrix.
- *(c)* Let P_X be the transition matrix obtained at (a3). Calculate P_X^2.
- *(d)* Discuss the difference of (a) and (b). What is the pros/cons on (a) and (b)?
- *(e)* What is $\mathbb{P}(X_2 = 6 | X_0 = 5)$?
- *(f)* Assume the initial distribution is $(0, 0, 0.5, 0.5)$. Find $\mathbb{P}(X_2 = 6)$.

(Solution)

(a1) **Yes, X_n is a markov chain** because how much item will be in stock tomorrow only depends on current stock level and today's demand and it does NOT depend on yesterday's stock level and yesterday's demand.

(a2) state space is $S = \{3, 4, 5, 6\}$ and initial distribution is $a_0 = (0, 0, 1, 0)$

(a3) Transition matrix is

$$\mathbf{P} = \begin{pmatrix} & & & 1 \\ 1/6 & & & 5/6 \\ 3/6 & 1/6 & & 2/6 \\ 2/6 & 3/6 & 1/6 & \end{pmatrix}$$

(b1) **Yes, Y_n is a markov chain** because given the reordering policy, how much inventory will be left tomorrow, depends on how much inventory is left over today and tomorrow's random demand.

(b2) state space is S = $S = \{0, 1, 2, 3, 4, 5\}$ and initial distribution is $a_0 = (0, 0, 1, 0, 0, 0)$

(b3) Transition matrix is

$$\mathbf{P} = \begin{pmatrix} & & & 2/6 & 3/6 & 1/6 \\ & & & 2/6 & 3/6 & 1/6 \\ & & & 2/6 & 3/6 & 1/6 \\ 2/6 & 3/6 & 1/6 & & & \\ & 2/6 & 3/6 & 1/6 & & \\ & & 2/6 & 3/6 & 1/6 & \end{pmatrix}$$

(c)

$$\mathbf{P}_X^2 = \begin{pmatrix} 1/3 & 1/2 & 1/6 & 0 \\ 10/36 & 15/36 & 5/36 & 6/36 \\ 5/36 & 6/36 & 2/36 & 23/36 \\ 6/36 & 1/36 & 0 & 29/36 \end{pmatrix}$$

(d) Both (a) and (b) captures how this business works, and they are both Discrete Time Markov Chain. Since (a) has less states than (b). Defining state space concisely makes further analysis easier (For example, doing P_X^2 is less time consuming than doing P_Y^2

(e) $P(X_2 = 6|X_0 = 5) = 23/36 =$ **0.6389**

(f) 0.7222

Method 1. (matrix algebra)
$(0\ 0\ 0.5\ 0.5)P_X^2 = (11/72\ 7/72\ 1/36\ 13/18)$
The answer is the third element in the vector.
$\therefore$ 0.7222

Method 2. (path perspective)

$$\begin{aligned}\mathbb{P}(X_2 = 6) &= \sum_{i=3}^{6} \mathbb{P}(X_2 = 6 | X_0 = i)\mathbb{P}(X_0 = i) \\ &= 0 \times 0 + 0.1667 \times 0 + 0.6389 \times 0.5 + 0.8056 \times 0.5 = 0.7222\end{aligned}$$

Exercise 2

- Consider two stocks. Stock 1 always sells for \$10 or \$20. If stock 1 is selling for \$10 today, there is a .80 chance that it will sell for \$10 for tomorrow. If it is selling for \$20 today, there is a .90 chance that it will sell for \$20 tomorrow. Stock 2 always sells for \$10 or \$25. If stock 2 sells today for \$10 there is a .90 chance that it will sell tomorrow for \$10. If it sells today for \$25, there is a .85 chance that it will sell tomorrow for 25. Let X_n denote the price of the 1st stock and Y_n denote the price of the 2nd stock during the nth day. Assume that $\{X_n : n \geq 0\}$ and $\{Y_n : n \geq 0\}$ are discrete time Markov chains.
  - *(a)* What is the transition matrix for $\{X_n : n \geq 0\}$? Is $\{X_n : n \geq 0\}$ irreducible?
  - *(b)* What is the transition matrix for $\{Y_n : n \geq 0\}$? Is $\{Y_n : n \geq 0\}$ irreducible?
  - *(c)* What is the stationary distribution of $\{X_n : n \geq 0\}$?
  - *(d)* What is the stationary distribution of $\{Y_n : n \geq 0\}$?
  - *(e)* On January 1st, your grand parents decide to give you a gift of 300 shares of either Stock 1 or Stock 2. You are to pick one stock. Once you pick the stock you cannot change your mind. To take advantage of a certain tax law, your grand parents dictate that one share of the chosen stock is sold on each trading day. Which stock should you pick to maximize your gift account by the end of the year? (Explain your answer.)

(Solution)

(a) Let's call the state where the stock sells for \$10, state 1, and where it sells for \$20, state 2. Using the probabilities in the problem statement, and making sure the rows add up to one, we can conclude that the transition matrix is

$$\mathbf{P_X} = \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}.$$

As every state communicates with every other state, **the chain is irreducible.**

(b) Let's call the state where the stock sells for \$10, state 1 and where it sells for \$25, state 2. Using the probabilities in the problem statement, and making sure the rows add up to one, we can conclude that the transition matrix is

$$\mathbf{P_Y} = \begin{pmatrix} 0.9 & 0.1 \\ 0.15 & 0.85 \end{pmatrix}.$$

As every state communicates with every other state, **the chain is irreducible.**

(c) To get the stationary distribution we must make sure the chain satisfies the flow balance equations, hence,

$$\pi_X = \pi_X \mathbf{P_X}$$
$$\pi_{X1} = 0.8\pi_{X1} + 0.1\pi_{X2}$$
$$\pi_{X2} = 0.2\pi_{X1} + 0.9\pi_{X2}$$

We also know that

$$\pi_{X1} + \pi_{X2} = 1$$

We can solve this system of equations by substitution of $\pi_{X1} = 1 - \pi_{X2}$ in any of the flow equations and get:

$$\pi_X = (1/3,\ 2/3)$$

Sanity Check: Both $\pi_X \mathbf{P_X} = \pi_X$ and $\sum_i \pi_{X_i} = 1$ hold.

(d) Again, to get the stationary distribution we must make sure the chain satisfies the flow balance equations, hence,

$$\pi_Y = \pi_Y \mathbf{P_Y}$$
$$\pi_{Y1} = 0.9\pi_{Y1} + 0.15\pi_{Y2}$$
$$\pi_{Y2} = 0.1\pi_{Y1} + 0.85\pi_{Y2}$$

Plus we also know that

$$\pi_{Y1} + \pi_{Y2} = 1$$

We can solve this system of equations by substitution of $\pi_{Y1} = 1 - \pi_{Y2}$ in any of the flow equations and get:

$$\pi_Y = (3/5,\, 2/5)$$

Sanity Check: Both $\pi_Y \mathbf{P_Y} = \pi_Y$ and $\sum_i \pi_{Y_i} = 1$ hold.

(e) So assuming the year has at least 300 trading days, what we would like to choose is the stock that gives the highest expected return. In order to calculate this we must assume that the market is operating in the steady state, thus for Stock 1:

$$\mathbb{E}\left(\sum_{i=1}^{300} X_i\right) = \sum_{i=1}^{300} \mathbb{E}\left(X_i\right) = 300\left(10 \cdot \frac{1}{3} + 20 \cdot \frac{2}{3}\right) = 5000$$

Similarly for Stock 2:

$$\mathbb{E}\left(\sum_{i=1}^{300} Y_i\right) = \sum_{i=1}^{300} \mathbb{E}\left(Y_i\right) = 300\left(10 \cdot \frac{3}{5} + 25 \cdot \frac{2}{5}\right) = 4800$$

So we would **prefer to get Stock 1 over Stock 2.**

Exercise 3

- Suppose each morning a factory posts the number of days worked in a row without any injuries (i.e., consecutive days without injury). Assume that each day is injury free with probability 98/100. Furthermore, assume that whether tomorrow is injury free or not is independent of which of the preceding days were injury free. Let $X_0 = 0$ be the morning the factory first opened. Let X_n be the number posted on the morning after n full days of work.
  - *(a)* Is $\{X_n : n \geq 0\}$ a Markov chain? If so, give its state space, initial distribution, and transition matrix P. If not, show that it is not a Markov chain.
  - *(b)* Is the Markov chain irreducible? Explain.
  - *(c)* Is the Markov chain periodic or aperiodic? Explain and if it is periodic, also give the period.
  - *(d)* Find the stationary distribution.
  - *(e)* Is the Markov chain recurrent? If so, why? If not, why not?

(Solution)

(a) **Yes, it is a Markov Chain** because the probabilities of going to a new state will depend only on the current state and the pure randomness of a given day whether the n-th day was accident free or not. The state space is $S = \{0, 1, 2, ...\}$ Since $X_0 = 0$, initial distribution is $a_0 = (1, 0, 0...)$ And finally we can construct the probabilities as

$$\mathbf{P}_{ij} = \begin{cases} 0.98 & j = i + 1 \\ 0.02 & j = 0 \\ 0 & otherwise \end{cases} \tag{1}$$

Note that above equation (1) contains information that is enough to write transition probability matrix. Especially when the size of transition matrix is infinite, above description is concise way to describe transition probability matrix. Of course, equation (1) contains equivalent information that transition diagram contains.

(b) **Yes**, it is. Every state n, is accessible from 0, with positive probability 0.98^n that state will be visited in n steps starting from 0. Similarly, state 0 is accesible from every other state since the probability to go to it in one step is 0.02 from any state, so 0 communicates with every other state. Therefore, every state communicates with every other state, since from any state a you could go to 0, and from zero to any state b.

(c) The Markov chain is **aperiodic**. Starting in state 0, you can go to state zero in 1 step with probability 0.02, hence state zero has a period of 1. Since the chain is irreducible, all the states form a single class, and since all states in the same class have the same period, then every state has period 1. Thus it is aperiodic.

(d)
Method 1. By using the transition matrix, we can do the following.

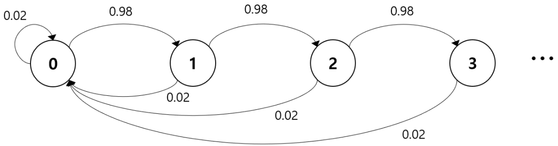
$$(\pi_0, \pi_1, \pi_2, \pi_3 \ldots) \cdot \begin{pmatrix} 0.02 & 0.98 & & & & \\ 0.02 & & 0.98 & & & \\ 0.02 & & & 0.98 & & \\ 0.02 & & & & \ldots & \\ 0.02 & & & & & \ldots \\ \ldots & & & & & \end{pmatrix} = (\pi_0, \pi_1, \pi_2, \pi_3 \ldots)$$

Then, we have following:

$$0.02(\pi_0 + \pi_1 + \ldots) = \pi_0 \tag{2}$$
$$0.98\pi_0 = \pi_1 \tag{3}$$
$$0.98\pi_1 = \pi_2 \tag{4}$$
$$0.98\pi_3 = \pi_3 \tag{5}$$
$$\ldots$$

The equation (2) gives $\pi_0 = 0.02$ because summation of all π_i is equal to 1. Plugging this into equation (3) gives us $\pi_1 = 0.02 * 0.98$. Plugging this to the next equation and so on, we have $\pi_i = 0.02 * 0.98^i$ of stationary distribution.

Method 2. In order to use flow balance equation, generally we need to draw transition diagram first.



Write $(Inflow)_i = (Outflow)_i$ for some states as following:

$$\text{State 0:} \qquad 0.02(\pi_1 + \pi_2 + ...) = 0.98\pi_0 \tag{6}$$

$$\text{State 1:} \qquad 0.98\pi_0 = 0.98\pi_1 + 0.02\pi_1 \tag{7}$$

$$\text{State 2:} \qquad 0.98\pi_1 = 0.98\pi_2 + 0.02\pi_2 \tag{8}$$

$$\text{State 3:} \qquad 0.98\pi_2 = 0.98\pi_3 + 0.02\pi_3 \tag{9}$$

$$\ldots$$

Again, the general tip is "try first a few until you can generalize the pattern' '. From equation (6), we get $\pi_0 = 0.02$. From (7), we get $\pi_1 = 0.02 \cdot 0.98$. In the same way, we have $\pi_i = 0.02 \cdot 0.98^i$ of stationary distribution.

(e) **Yes** it is, because it is irreducible (see (b)) and a stationary distribution exists (see (d)).

Exercise 4

- Consider the following transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 & 0 \end{pmatrix}.$$

- *(a)* Is the Markov chain periodic? Give the period of each state.
- *(b)* Is $(\pi_1, \pi_2, \pi_3, \pi_4) = (33/96, 27/96, 15/96, 21/96)$ the stationary distribution of the Markov Chain? (First state the mathematical definition of stationary distribution and show that given distribution satisfies the property)
- *(c)* Is $\mathbf{P}_{11}^{100} = \pi_1$? Is $\mathbf{P}_{11}^{101} = \pi_1$? Give an expression for π_1 in terms of $\mathbf{P}_{11}^{100}$ and $\mathbf{P}_{11}^{101}$.

(Solution)

(a) **Yes** it is, each state has period 2. Note that for any number n and any state i,

$$\mathbf{P}_{ii}^{2n} > 0$$

$$\mathbf{P}_{ii}^{2n-1} = 0$$

(b) This question basically asks you to do the "sanity check". We need to verify the following:

$$\sum_{i=1}^{4} \pi_i = 33/96 + 27/96 + 15/96 + 21/96 = 1$$

So π can be a probability distribution, we must also verify if $\pi = \pi\mathbf{P}$ for this to be a stationary distribution so,

$$(33/96 \;\; 27/96 \;\; 15/96 \;\; 21/96) \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.7 & 0 & 0.3 \\ 0.8 & 0 & 0.2 & 0 \end{pmatrix} = (33/96 \;\; 27/96 \;\; 15/96$$

Thus, π **is a stationary distribution for** X_n**.**

(c) You can verify using a calculator (or Python / R) that $\mathbf{P}_{11}^{100} = 0.6875 = 11/16$ and $\mathbf{P}_{11}^{101} = 0$. Now notice $\pi_1 = 33/96 = 0.34375$, hence we can conclude,

$$\pi_1 = \frac{\mathbf{P}_{11}^{100} + \mathbf{P}_{11}^{101}}{2}$$

This is the "remedy" method discussed during the lecture.

Exercise 5

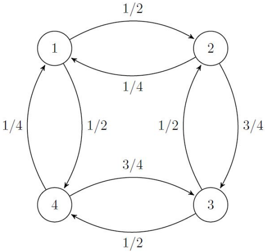
- Let $X = \{X_n : n = 0, 1, 2, ...\}$ be a discrete time Markov chain on state space $S = \{1, 2, 3, 4\}$ with transition matrix.

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix}$$

- *(a)* Draw a transition diagram
- *(b)* Find $\mathbb{P}(X_2 = 4 | X_0 = 2)$
- *(c)* Find $\mathbb{P}(X_2 = 2, X_4 = 4, X_5 = 1 | X_0 = 2)$
- *(d)* What is the period of each state?
- *(e)* Let $\pi = (1/8, 1/4, 3/8, 1/4)$. Is π the unique stationary distribution of X? Explain your answer.
- *(f)* Let $\mathbf{P}^n$ be the n-th power of $\mathbf{P}$. Does $\lim_{n\to\infty} \mathbf{P}^n_{14} = 1/4$ hold? Explain your answer.

(Solution)

(a) Using the information in the transition matrix,



(b) So the probability we want is $\mathbb{P}(X_2 = 4 | X_0 = 2) = \mathbf{P}^2_{24}$. By considering the only two two-step paths from state 2 to state 4, we get

$$\begin{aligned}\mathbf{P}^2_{24} &= \mathbf{P}_{21} \cdot \mathbf{P}_{14} + \mathbf{P}_{23} \cdot \mathbf{P}_{34} \\ &= \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{2}\end{aligned}$$

(c) Now we can rewrite this probability as following. The step from (10) to (11) is called "the law of iterated conditional expectation", and the step from (11) to (12) is by Markov property.

$$
\begin{aligned}
&\mathbb{P}(X_2 = 2, X_4 = 4, X_5 = 1 | X_0 = 2) \\
&= \mathbb{P}(X_5 = 1 | X_0 = 2, X_2 = 2, X_4 = 4)\mathbb{P}(X_4 = 4 | X_0 = 2, X_2 = 2)\mathbb{P}(X_2 = \\
&= \mathbb{P}(X_5 = 1 | X_4 = 4)\mathbb{P}(X_4 = 4 | X_2 = 2)\mathbb{P}(X_2 = 2 | X_0 = 2) \\
&= \mathbf{P}_{41} \cdot \mathbf{P}^2_{24} \cdot \mathbf{P}^2_{22}
\end{aligned}
$$

One can do the matrix algebra to get $\mathbf{P}^2$ in order to evaluate $\mathbf{P}^2_{22}$, but we use alternative way here. Considering only possible paths is more efficient way to do it. Note that there are only two possible two-step paths from state 2 back to state 2, thus

$$
\begin{aligned}
\mathbb{P}(X_2 = 2, X_4 = 4, X_5 = 1 | X_0 = 2) &= \mathbf{P}_{41} \cdot \mathbf{P}^2_{24} \cdot (\mathbf{P}_{21} \cdot \mathbf{P}_{12} + \mathbf{P}_{23} \cdot \mathbf{P}_{32}) \\
&= \frac{1}{4} \cdot \frac{1}{2} \cdot \left(\frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2}\right) = \frac{1}{16}
\end{aligned}
$$

(d) Each state has period 2. Note that for any number n and any state i,

$$\mathbf{P}_{ii}^{2n} > 0$$

$$\mathbf{P}_{ii}^{2n-1} = 0$$

(e) **Yes,** it is. We need to verify the following:
$\sum_{i=1}^{4} \pi_i = 1/8 + 1/4 + 3/8 + 1/4 = 1$. So π can be a probability distribution, we must also verify if $\pi = \pi\mathbf{P}$ for this to be a stationary distribution so,

$$(1/8,\ 1/4,\ 3/8,\ 1/4)\begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix} = (1/8,\ 1/4,\ 3/8,\ 1/4).$$

Thus, π is indeed a stationary distribution for X_n. Furthermore, as X_n is irreducible, (because every state communicates with every other state), the stationary distribution is unique.

(f) **No**, it doesn't. You can verify using a calculator (or Python/R) that for any natural number r,

$$\mathbf{P^{2r-1}} = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix}$$

and

$$\mathbf{P^{2r}} = \begin{pmatrix} 1/4 & 0 & 3/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

Hence, for all even n, $P_{14}^n = 1/4$ but for any odd n, $P_{14}^n = 0$ hence $\lim_{n\to\infty} P_{14}^n$ does not converge. However, as a remedy, we know following holds, because the period of this Markov chain is 2. $\lim_{n\to\infty} \frac{P_{14}^n + P_{14}^{n+1}}{2} = \pi_4 = 1/4$.

Exercise 6

- Suppose that the weekly demand D of a non-perishable product is independent and indentically distributed with the following distribution.

d	10	20	30
$\mathbb{P}(D=d)$	.2	.5	.3

- Unused items from one week can be used in the following week. The management decides to use the following inventory policy: whenever the inventory level in Friday evening is less than or equal to 10, an order is made and it will arrive by Monday morning. The order-up-to quantity is set to be 30. Model this as Markov chain with state space $\{20, 30\}$ and solve following.
  - *(a)* If this week (week 0) starts with inventory 20, what is the probability that week 2 has starting inventory level 10?
  - *(b)* If this week (week 0) starts with inventory 20, what is the probability that week 2 has starting inventory level 20?
  - *(c)* If this week (week 0) starts with inventory 20, what is the probability that week 100 has starting inventory level 10?
  - *(d)* If this week (week 0) starts with inventory 20, what is the probability that week 100 has starting inventory level 20?
  - *(e)* Suppose that each item sells at \$200. Each item costs \$100 to order, and each leftover item by Friday evening has a holding cost of \$20. Suppose that each order has a fixed cost \$500. Find

(Solution)

(a) 0 (not possible to go to inventory level 10 at the beginning of any week, this is why the state space is given as $\{20, 30\}$)

(b) Given inventory policy and demand distribution, it's only possible to start the week with either 20 or 30 units. $S = \{20, 30\}$. And the requested probability is $\mathbf{P}^2_{20,20}$ (i.e. to go from 20 to 20 in two steps). First we construct $\mathbf{P}$ as follows:

$$\mathbf{P}_{20,20} = 0$$
$$\mathbf{P}_{20,30} = 0.2 + 0.5 + 0.3 = 1$$
$$\mathbf{P}_{30,20} = 0.2 \ \text{ (when demand is only 10)}$$
$$\mathbf{P}_{30,30} = 0.3 + 0.5 = 0.8 \ \text{ (when demand is either 20 or 30)}$$

Hence,

$$\mathbf{P}^2 = \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix}^2 = \begin{pmatrix} 0.2 & 0.8 \\ 0.16 & 0.84 \end{pmatrix}$$

So the requested probability is $\mathbf{P}^2_{20,20} = 0.2$

(c) 0 (see (a) for explanation)

(d) This can be done by calculating stationary distribution. (The procedure to get stationary distribution is omitted.)

$$\mathbf{P}^{100} = \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix}^{100} \approx \begin{pmatrix} 1/6 & 5/6 \\ 1/6 & 5/6 \end{pmatrix}$$

So the requested probability is $\mathbf{P}^{100}_{20,20} = 1/6$.

(e) To solve this let's verify whether $\pi = (1/6, 5/6)$ (from (d)) is the stationary distribution. Clearly $\sum_i \pi_i = 1$, also,

$$(1/6, 5/6) \cdot \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix} = (1/6, 5/6)$$

Since the chain is irreducible (check why!) this is the unique stationary distribution. Next, let's call $f(y)$ the expected profit if at the beginning of the week my inventory is i. Fixing y to each possible value (20 or 30),we can build an equation of the form.

$$\text{Profit}_D = \text{Revenue} - \text{Cost}$$
$$\text{Profit}_D = \text{sale rev} - \text{fixed order cost} - \text{holding cost} - \text{variable order cost}$$

For $y = 20$,

$$D = 10: \ Profit_{10} = 200(10) - 500 - 20(10) - 100(20) = -700$$
$$D = 20: \ Profit_{20} = 200(20) - 500 - 100(30) = 500$$
$$D = 30: \ Profit_{30} = 200(20) - 500 - 100(30) = 500$$

$$\therefore f(20) = -700 \cdot 0.2 + 500 \cdot 0.5 + 500 \cdot 0.3 = 260$$

For $y = 30$,

$$D = 10: \ Profit_{10} = 200(10) - 20(20) = 1600$$
$$D = 20: \ Profit_{20} = 200(20) - 500 - 20(10) - 100(20) = 1300$$
$$D = 30: \ Profit_{30} = 200(20) - 500 - 100(30) = 2500$$

$\therefore f(30) = 1600 \cdot 0.2 + 1300 \cdot 0.5 + 2500 \cdot 0.3 = 1720$ Now we can calculate the expected profit using all this information

$$\mathbb{E}\left[Profit\right] = \pi_{20} \cdot f(20) + \pi_{30} \cdot f(30) = 260 \cdot \frac{1}{6} + 1720 \cdot \frac{5}{6} = \$1476.66$$

Exercise 7

- Suppose we are looking at discrete time stochastic process $\{X_n : n \geq 0, n \in \mathbb{N}\}$. The proper way to show that X_n is Markov Chain is by showing that "Only most recent information matters' ', or mathematically, proving the Markov property of $\mathbb{P}(X_{n+1} = j | X_0 = i_0, X_1 = i_1, ..., X_n = i) = \mathbb{P}(X_{n+1} = j | X_n = i)$ for all i and j.
- Then how can we show that X_n is NOT Markov Chain? You can do it by showing that "Tomorrow depends not only on today, but also on what happened on yesterday' '. Or, mathematically, one way is by showing that $\mathbb{P}(X_{n+1} = k | X_n = j, X_{n-1} = i_0) \neq \mathbb{P}(X_{n+1} = k | X_n = j, X_{n-1} = i_1)$ for some i_0, i_1, j, k.
- Suppose that there are only rainy day and sunny day in Seoul, (rainy day is a day when there is any size of rain drop in anywhere inside Seoul, and not rainy day is sunny day) define this process as discrete time stochastic process. Do you think this discrete time stochastic process is Markov Chain? Define state space. Give an argument of "Tomorrow depends not only on today, but also on what happened on yesterday' ' so that this is not Markov Chain. Can you express your argument in mathematical form?

(Solution)

This question is asking you precise mathematical way to show that **this discrete time stochastic process is not a Markov Chain**. Let $\{X_n : n \geq 0, n \in \mathbb{N}\}$ be a discrete time stochastic process with state space $\{R, S\}$, i.e. $X_n = R$ implies it's Rainy day on day n; $X_n = S$ implies it's Sunny day on day n. $\{X_n : n \geq 0, n \in \mathbb{N}\}$ is NOT Markov Chain because
$\mathbb{P}(X_{n+1} = R | X_n = S, X_{n-1} = S) \neq \mathbb{P}(X_{n+1} = R | X_n = S, X_{n-1} = R)$. See the last mathematical expression in the second paragraph of the question. You just need one line of mathematical statement.

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"If I only had an hour to chop down a tree, I would spend the first 45 minutes
sharpening my axe. - A. Lincoln"
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