Lecture 09. Poisson Process

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Math Review - Exponential distribution AGAIN?!

- pdf, cdf
- $\mathbb{E}X, Var(X), c_X, c_X^2$
- Memoryless property
- $X_1 \sim exp(\lambda_1), X_2 \sim exp(\lambda_2)$, and they are indep.
  $\Rightarrow \mathbb{P}(X_1 < X_2) = ?$
- [NEW] $X_1 \sim exp(\lambda_1), X_2 \sim exp(\lambda_2)$, and they are indep.
  $\Rightarrow min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$

Examples

- Two servers, A and B, with service times following $X_A \sim \exp(\lambda_A)$ and $X_B \sim \exp(\lambda_B)$. Their service times are independent.
  - The time for one customer finishes?
  - If $X_A \sim \exp(1/2)$, $X_B \sim \exp(1/2)$ then $\mathbb{E}[min(X_A, X_B)]$?
  - If $X_A \sim \exp(1/2)$, $X_B \sim \exp(1/4)$ then $\mathbb{E}[min(X_A, X_B)]$?

- Two servers, A, B with $X_A \sim \exp(\lambda_A)$, $X_B \sim \exp(\lambda_B)$ serving John and Paul, respectively. Their service times are independent.
  - On average, how long does it take to clear the system?

Poisson Process - Motivation

- In a call center for Amazon, there are 100 incoming calls per minute on average. Answer the followings in most naive and plain but intuitive way.
  - $\mathbb{E}$[# of calls bet'n 9:00 and 9:01]?
  - $\mathbb{E}$[# of calls bet'n 9:05 and 9:08]?
  - Suppose 150 calls bet'n 9:00 and 9:02, $\mathbb{E}$[# of calls bet'n 9:03 and 9:05]?
  - Suppose 800 calls bet'n 9:00 and 9:10, $\mathbb{E}$[# of calls bet'n 9:05 and 9:10]?
  - Suppose 900 calls bet'n 9:10 and 9:20, $\mathbb{E}$[# of calls bet'n 9:18 and 9:20]?
  - Suppose 850 calls bet'n 9:00 and 9:10, $\mathbb{E}$[# of calls bet'n 9:08 and 9:12]?

- Let $t = 0$ at 9:00, $t = 1$ at 9:01,… and consider a function $N(t)$ that counts # of total calls since $t = 0$.
- Go back to previous slide and express in terms of $N(\cdot)$.
- Observations on $N(t)$ in common sense. (not PP yet!)
  - $N(0) = 0$
  - $N(t)$ is non-decreasing function
  - $N(t)$ has a nonnegative value.
  - For $t_1 \leq t_2 \leq t_3 \leq t_4$, $N(t_2) - N(t_1)$ is indep. of $N(t_4) - N(t_3)$
  - For $t_1 \leq t_2$, $\mathbb{E}[N(t_2) - N(t_1)] = \mathbb{E}[N(t_2 - t_1)]$

Math Review - Poisson distribution

Definition - Poisson distribution

- A discrete random variable X is said to follow **Poisson distribution** with parameter λ, and write $X \sim Poi(\lambda)$, if pmf

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for } k = 0, 1, 2, ...$$

- What is cdf of $Poi(\lambda)$?
- $\mathbb{E}X = Var(X) = \lambda$.

Definition

Definition - Poisson process

- $N = \{N(t), t \geq 0\}$ is said to be **counting process** if $N(0) = 0$, and the integer-valued function $N(t)$ is non-decreasing.
- A counting process $N = \{N(t), t \geq 0\}$ is said to be a **Poisson process** with rate λ, and write $PP(\lambda)$ if
  1. $N(t_2) - N(t_1) \sim poi(\lambda(t_2 - t_1))$ for any $0 \leq t_1 \leq t_2$
  2. For $t_1 \leq t_2 \leq t_3 \leq t_4$, $N(t_2) - N(t_1)$ is indep. of $N(t_4) - N(t_3)$

- Above statements can be understood as:
  1. (Amount of increment follows Poisson dist. with parameter proportional to time length)
  2. (Increments in non-overlapping intervals are independent)

Example

- Suppose N is $PP(\lambda = 2/minutes)$
  1. The probability that there are 4 arrivals in the first 3 minutes.

  2. The probability that two arrivals in [0,2] and at least 3 arrivals in [1,3].

blank

- Suppose N is $PP(\lambda = 2/minutes)$
  1. The probability that there is no arrival in [0,4]?

  2. The probability that the first arrival takes at least 4 minutes?

  3. The probability that the first arrival takes at least t minutes?

  4. What is the distribution of T_1?

Theorem

- Time to first arrival in $PP(\lambda)$ follows $exp(\lambda)$

Merging PP

Example

- Consider the north confluence area of I75 and I85.
  - Suppose, for South bound, N_{I75S} is $PP(100/min)$ and N_{I85S} is $PP(200/min)$.
  - Then, at the north confluence point, we have $N_{I75S} + N_{I85S}$, which is $PP(300/min)$ traffic heading South.

Theorem

- Suppose $N_A = \{N_A(t), t \geq 0\}$ is $PP(\lambda_A)$ and $N_B = \{N_B(t), t \geq 0\}$ is $PP(\lambda_B)$, and they are independent. Then $N = \{N(t) = N_A(t) + N_B(t)\}$ is $PP(\lambda_A + \lambda_B)$

Thinning PP

Example

- Consider the north confluence area of I75 and I85.
  - Suppose, for North bound, we have $PP(400/min)$ heading North.
  - Each car heading north chooses I75N with probability 0.4 and I85N with probability 0.6.
  - Then, we have N_{I75N}, which is $PP(160/min)$ and N_{I85N}, which is $PP(240/min)$.

Theorem

- Suppose $N = \{N(t), t \geq 0\}$ is $PP(\lambda)$ and each arrival will choose subset A or B, with probability p and $q = 1 - p$, respectively. Then, $N_A = \{N_A(t), t \geq 0\}$ is $PP(p\lambda)$ and $N_B = \{N_B(t), t \geq 0\}$ is $PP(q\lambda)$

blank

Criticism and Remedy of Poisson Process

- Criticism: Arrival process may not be *time homogeneous* for systems such as…
  - Call center traffic in the morning vs afternoon.
  - Hospital emergency patients in Friday night vs Tuesday morning
  - Aircraft arrival rate in an airport during holiday vs today

- Remedy: Why don't we make λ to be *time-dependent*?
  - We call this process ''Non-homogeneous Poisson process".
  - KEY function: $\lambda(t)$ as a function for arrival rate that changes over time.
  - We can still use very similar mathematical technique of (Stationary or Homogeneous) Poisson Process!

Review: Definitions of Poisson Process

Definition (ver 1)

- A counting process $N = \{N(t), t \geq 0\}$ is said to be a **Poisson process** with rate λ, and write $PP(\lambda)$ if
  1. $N(0) = 0$
  2. $N(t)$ has independent increment.
  3. $\mathbb{P}(N(t+s) - N(s) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ for $n = 0, 1, 2, ...$

Definition (ver 2)

- A counting process $N = \{N(t), t \geq 0\}$ is said to be a **Poisson process** with rate λ, and write $PP(\lambda)$ if
  1. $N(0) = 0$
  2. $N(t)$ has independent increment.
  3. $N(t+s) - N(s) \sim Poisson(\lambda t)$

Definition of Non-homogeneous Poisson Process

Definition (ver 2 of PP)

- A counting process $N = \{N(t), t \geq 0\}$ is said to be a *Poisson process* with rate λ, and write $PP(\lambda)$ if
  1. $N(0) = 0$
  2. $N(t)$ has independent increment.
  3. $N(t+s) - N(s) \sim Poisson(\lambda t)$

Definition - Non-homogeneous Poisson process

- A counting process $N = \{N(t), t \geq 0\}$ is said to be a Non-homogeneous Poisson process with rate function $\lambda(t), t \geq 0$, and write $NHPP(\lambda(t))$ if
  1. $N(0) = 0$
  2. $N(t)$ has independent increment.
  3. $N(t+s) - N(s) \sim Poisson(\int_s^{t+s} \lambda(u)du)$

- Remark: PP is special case of NHPP where rate function of NHPP is a constant.

Example

- Assume that call arrival to a call center follows a NHPP.
  - The call center opens from 9 am to 5 pm.
  - During the first hour, the arrival rate increases linearly from 0 at 9 am to 60 calls per hour at 10 am.
  - After 10 am, the arrival rate is constant at 60 calls per hour.

(a) Plot the arrival rate function $\lambda(t)$ as a function of time t; indicate clearly the time unit used.

(b) Find the probability that exactly 5 calls have arrived by 9:10am.

Example

(c) Find the probability that exactly k calls will arrive between 9:50 am to 10:10 am.

Exercise 1

- Nortel in Canada operates a call center for customer service. Assume that each caller speaks either English or French, but not both. Suppose that the arrival for each type of calls follows a Poisson process. The arrival rates for English and French calls are 2 and 1 calls per minute, respectively. Assume that call arrivals for different types are independent.
  - *(a)* What is the probability that the 2nd English call will arrive after minute 5?
  - *(b)* Find the probability that, in first 2 minutes, there are at least 2 English calls and exactly 1 French call?
  - *(c)* What is the expected time for the 1st call that can be answered by a bilingual operator (that speaks both English and French) to arrive?
  - *(d)* Suppose that the call center is staffed by bilingual operators that speak both English and French. Let $N(t)$ be the total number of calls of both types that arrive during $(0, t]$. What kind of a process is $N(t)$? What is the mean and variance of $N(t)$?
  - *(e)* What is the probability that no calls arrive in a 10 minute interval?
  - *(f)* What is the probability that at least two calls arrive in a 10 minute interval?
  - *(g)* What is the probability that two calls arrive in the interval 0 to 10 minutes and three calls arrive in the interval from 5 to 15 minutes?
  - *(h)* Given that 4 calls arrived in 10 minutes, what is the probability that all of these calls arrived in the first 5 minutes?
  - *(i)* Find the probability that, in the first 2 minutes, there are at most 1 call, and, in the first 4 minutes, there are exactly 3 calls?

(Solution)

Let N_E denote the Poisson process corresponding to English calls, and let $N_F(\cdot)$ denote the Poisson process corresponding to French calls. Then, N_E is $PP(2/min)$ and N_F is $PP(1/min)$.

(a)

$$
\begin{aligned}
& \mathbb{P}[N_E(5) - N_E(0) < 2] = \mathbb{P}[Poi(2(5-0)) < 2] \\
= \ & \mathbb{P}[Poi(10) = 0] + \mathbb{P}[Poi(10) = 1] = e^{-10} + 10e^{-10} = 11e^{-10}
\end{aligned}
$$

(b)

$$
\begin{aligned}
& \mathbb{P}[N_E(2) - N_E(0) \geq 2, N_F(2) - N_F(0) = 1] \\
= \ & \mathbb{P}[N_E(2) - N_E(0) \geq 2]\mathbb{P}[N_F(2) - N_F(0) = 1] \\
= \ & \mathbb{P}[Poi(2(2-0)) \geq 2]\mathbb{P}[Poi(1(2-0)) = 1] \\
= \ & [1 - \mathbb{P}(Poi(4) = 0) - \mathbb{P}(Poi(4) = 1)][\mathbb{P}(Poi(2) = 0)] \\
= \ & (1 - e^{-4} - 4e^{-4})(2e^{-2}) \\
= \ & (1 - 5e^{-4})(2e^{-2})
\end{aligned}
$$

(c) Note that [the arrival time of the first English call] is exponential distribution with parameter 2/min, and [the arrival time of the first French call] is exponential distribution with parameter 1/min. Hence [the arrival time of the first call] is minimum of the two times, which we now know is exponential with parameter 3/min (since the two processes are independent). Thus, the expected arrival time of the first call is just $1/3$ minutes.

(d) $N = N_E + N_F$, and N_E and N_F are independent, so N is $PP((1+2)/min)$ by merging principle. Thus, $N(t) - N(0) \sim Poi(\lambda(t-0)) = Poi(3t)$, and $\mathbb{E}[N(t)] = 3t$ and $Var(N(t)) = 3t$ because $N(0) = 0$ always by the definition of counting process.

(e) The probability that no calls arrive in a 10 minute interval is just $\mathbb{P}(N(10) - N(0) = 0) = \mathbb{P}[Poi(3(10-0)) = 0] = \frac{30^0 e^{-30}}{0!} e^{-30}$

(f)

$$\begin{aligned} & \mathbb{P}(N(10) - N(0) \geq 2) = \mathbb{P}[Poi(3(10-0)) \geq 2] \\ = & \ 1 - \mathbb{P}[Poi(30) = 0] - \mathbb{P}[Poi(30) = 1] \\ = & \ 1 - 1e^{-30} - 30e^{-30} = 1 - 31e^{-30} \end{aligned}$$

(g)

$$
\begin{aligned}
& \mathbb{P}(N(10)=2, N(15)-N(5)=3) \\
= \; & \mathbb{P}(N(5)-N(0)=2, N(10)-N(5)=0, N(15)-N(10)=3) \\
& +\mathbb{P}(N(5)-N(0)=1, N(10)-N(5)=1, N(15)-N(10)=2) \\
& +\mathbb{P}(N(5)-N(0)=0, N(10)-N(5)=2, N(15)-N(10)=1) \\
= \; & \mathbb{P}(N(5)-N(0)=2)\mathbb{P}(N(10)-N(5)=0)\mathbb{P}(N(15)-N(10)=3) \\
& +\mathbb{P}(N(5)-N(0)=1)\mathbb{P}(N(10)-N(5)=1)\mathbb{P}(N(15)-N(10)=2) \\
& +\mathbb{P}(N(5)-N(0)=0)\mathbb{P}(N(10)-N(5)=2)\mathbb{P}(N(15)-N(10)=1) \\
= \; & \mathbb{P}(Poi(15)=2)\mathbb{P}(Poi(15)=0)\mathbb{P}(Poi(15)=3) \\
& +\mathbb{P}(Poi(15)=1)\mathbb{P}(Poi(15)=1)\mathbb{P}(Poi(15)=2) \\
& +\mathbb{P}(Poi(15)=0)\mathbb{P}(Poi(15)=2)\mathbb{P}(Poi(15)=1) \\
= \; & (\frac{15^2}{2}e^{-15})(e^{-15})(\frac{15^3}{6}e^{-15}) + (15e^{-15})(15e^{-15})(\frac{15^2}{2}e^{-15}) \\
& +(e^{-15})(\frac{15^2}{2}e^{-15})(15e^{-15})
\end{aligned}
$$

(h) $(5/10)^4$

(i)

$$
\begin{aligned}
& \mathbb{P}(N(2) \leq 1, N(4) = 3) \\
= \;& \mathbb{P}(N(2) = 0, N(4) = 3) + \mathbb{P}(N(2) = 1, N(4) = 3) \\
= \;& \mathbb{P}(N(2) - N(0) = 0, N(4) - N(2) = 3) + \mathbb{P}(N(2) - N(0) = 1, N(4) - \\
= \;& \mathbb{P}(Poi(6) = 0)\mathbb{P}(Poi(6) = 3) + \mathbb{P}(Poi(6) = 1)\mathbb{P}(Poi(6) = 2) \\
= \;& e^{-6}(\frac{6^3}{3!}e^{-6}) + (6e^{-6})(\frac{6^2}{2}e^{-6}) \\
= \;& 6^2 e^{-12} + 6^3 e^{-12}/2
\end{aligned}
$$

Exercise 2

- Calls to a center follow a Poisson process with rate 120 calls per hour. Each call has probability 1/4 from a male customer. The call center opens at 9am each morning.
  - *(a)* What is the probability that there are no incoming calls in the first 2 minutes?
  - *(b)* What is the probability that there are exactly one calls in the first 2 minutes and exactly three calls from minute 1 to minute 4?
  - *(c)* What is the probability that during the first 2 minutes there are 1 call from male customer and 2 calls from female customers?
  - *(d)* What is the expected elapse of time from 9am until the second female customer arrives?
  - *(e)* What is the probability that the second call arrives after 9:06am?

(Solution)

(a) Let $N(t)$ be the total number of calls by minute t, then N is $PP(2/min)$.
$\mathbb{P}(N(2)=0)=\mathbb{P}(Poi(4)=0)=e^{-2\lambda}=e^{-4}$

(b)

$$\begin{aligned}
&\mathbb{P}(N(2)=1, N(4)-N(1)=3)\\
=&\mathbb{P}(N(1)-N(0)=1, N(2)-N(1)=0, N(4)-N(2)=3)\\
&+\mathbb{P}(N(1)-N(0)=0, N(2)-N(1)=1, N(4)-N(2)=2)\\
=&\lambda e^{-\lambda}\times e^{-\lambda}\times e^{-2\lambda}\frac{(2\lambda)^3}{6}+e^{-\lambda}\times\lambda e^{-\lambda}\times e^{-2\lambda}\frac{(2\lambda)^2}{2}\\
=&\frac{112}{3}e^{-8}
\end{aligned}$$

(c) By thinning principle, we have that N_M is $PP(0.5/min)$ and N_F is $PP(1.5/min)$. $\mathbb{P}(N_M(2) = 1, N_F(2) = 2) = \mathbb{P}(Poi(1) = 1)\mathbb{P}(Poi(3) = 2) = e^{-1} \times \frac{9}{2}e^{-3} = \frac{9}{2}e^{-4}$

Alternative approach - Not using thinning principle. Let N be the number of calls received in the call center, regardless of the gender of the customer.
$\mathbb{P}(N(2) - N(0) = 3) = \mathbb{P}(Poi(2 \cdot (2 - 0)) = 3) = \mathbb{P}(Poi(4) = 3) = \frac{e^{-4} \cdot 4^3}{3!}$
Among the three calls, there is only one call from a male customer with a probability of 1/4. $\mathbb{P}(Poi(4) = 3) \times {}_3\mathrm{C}_1 \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 = \frac{\mathrm{e}^{-4} \cdot 4^3}{3!} \times 3 \times \frac{3^2}{4^3} = \frac{9}{2}\mathrm{e}^{-4}$

(d) Let T_i be the interarrival time between the $(i-1)$th female customer and the ith female customer, which follows exponential distribution with rate $\frac{3}{2}$:
$\mathbb{P}[T_1 + T_2] = \mathbb{E}[T_1] + \mathbb{E}[T_2] = 2 \times \frac{1}{1.5} = \frac{4}{3}$ minutes.

(e)

$$
\begin{aligned}
& \mathbb{P}(N(6) < 2) = \mathbb{P}(N(6) = 0) + \mathbb{P}(N(6) = 1) \\
= \;& \mathbb{P}(Poi(12) = 0) + \mathbb{P}(Poi(12) = 1) \\
= \;& e^{-12} + 12e^{-12} = 13e^{-12}
\end{aligned}
$$

Exercise 3

- Assume that call arrival to a call center follows a non-homogeneous Poisson process. The call center opens from 9am to 5pm. During the first hour, the arrival rate increases linearly from 0 at 9am to 60 calls per hour at 10am. After 10am, the arrival rate is constant at 60 calls per hour.
  - *(a)* Plot the arrival rate function $\lambda(t)$ as a function of time t; indicate clearly the time unit used.
  - *(b)* Find the probability that exactly 5 calls have arrived by 9:10am.
  - *(c)* What is the probability that the first call arrives after 9:20am?
  - *(d)* What is the probability that there are exactly one call between 11:00am and 11:05am and at least two calls between 11:03am and 11:06am?

(Solution)

(a) If we use time unites of minutes, then the rate function $\lambda(t) = \frac{1}{60}t$ for $0 \leq t \leq 60$, and $\lambda(t) = 1$ for $t > 60$

From now on, N is defined as $NHPP(\lambda(t))$ where $\lambda(t)$ is given at (a).

(b)

$$\mathbb{P}[N(10) - N(0) = 5] = \mathbb{P}[Poi(\int_0^{10} \lambda(t)dt) = 5]$$

$$= \mathbb{P}[Poi(\int_0^{10} \frac{1}{60}tdt) = 5] = \mathbb{P}[Poi(\frac{5}{6}) = 5] = \frac{(\frac{5}{6})^5}{5!}e^{-\frac{5}{6}}$$

(c)

$$\mathbb{P}[N(20) - N(0) = 0] = \mathbb{P}[Poi(\int_0^{20} \lambda(t)dt) = 0]$$

$$= \mathbb{P}[Poi(\int_0^{20} \frac{1}{60}tdt) = 0] = \mathbb{P}[Poi(\frac{10}{3}) = 0] = e^{-\frac{10}{3}}$$

(d)

$$
\begin{aligned}
& \mathbb{P}(N(125)-N(120)=1, N(126)-N(123)\geq 2) \\
= \;& \mathbb{P}(N(123)-N(120)=0, N(125)-N(123)=1, N(126)-N(125)\geq 1) \\
& +\mathbb{P}(N(123)-N(120)=1, N(125)-N(123)=0, N(126)-N(125)\geq 2) \\
= \;& \mathbb{P}(N(123)-N(120)=0, N(125)-N(123)=1, N(126)-N(125)\geq 1) \\
& +\mathbb{P}(N(123)-N(120)=1, N(125)-N(123)=0, N(126)-N(125)\geq 2)
\end{aligned}
$$

From above, $N(123)-N(120)\sim Poi(\int_{120}^{123} 1dt) = Poi(3)$, and similarly, we have $N(125)-N(123))\sim Poi(2)$ and $N(126)-N(125))\sim Poi(1)$. Hence, we can write

$$
\begin{aligned}
= \;& \mathbb{P}(Poi(3)=0)\mathbb{P}(Poi(2)=1)\mathbb{P}(Poi(1)\geq 1) \\
& +\mathbb{P}(Poi(3)=1)\mathbb{P}(Poi(2)=0)\mathbb{P}(Poi(1)\geq 2) \\
= \;& e^{-3}\cdot 2e^{-2}\cdot(1-e^{-1}) + 3e^{-3}\cdot e^{-2}\cdot(1-2e^{-1}) \\
= \;& 5e^{-5}-8e^{-6}
\end{aligned}
$$

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"Success isn't permanent, and failure isn't fatal. - Mike Ditka"
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