

Introduction to Linear Programming

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Linear Programming

- Linear Programming (LP)
 - Linear: Both the objective function and the constraints are expressed as linear combinations.
 - Programming: Here, “programming” means planning – solving problems of how to make plans to achieve a goal
- What is LP?
 - An optimization tool have been widely used in real industries to reduce costs and improve efficiency since the 1950s
 - Applications include production planning, logistics and transportation, investment portfolios, policy resource allocation, among many others.
- What problems does linear programming solve?
 - Allocating “limited resources” to “competing activities” in the most optimal way
 - ✓ Resources: budget, machines/equipment, time, manpower, raw materials, storage space, etc.
 - ✓ Activities: product manufacturing, marketing campaigns, route assignments, investment decisions, etc.
 - Once the activity levels are decided, the corresponding resource usage and outcomes (profit/cost) are determined.
 - The goal is to find the best possible option among all feasible choices.

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- Steps in LP Modeling
 - The general process of converting a problem description into a LP model is
 - ✓ 1. Understand the problem
 - Identify the background, objectives, available resources, and constraints.
 - ✓ 2. Set the objective
 - Decide what to maximize or minimize (e.g., profit, cost, time).
 - ✓ 3. Identify the constraints
 - Consider resource limitations, technical restrictions, and policy requirements.
 - ✓ 4. Define the decision variables
 - Represent the “decisions to be made” as variables.
 - ✓ 5. Formulate the mathematical model
 - Express the objective function and constraints in terms of these variables.

Linear Programming

- Prototype Example
 - Products: doors, windows
 - Facilities: Plant 1, Plant 2, Plant 3
 - Given/Assumptions
 - ✓ Each plant has a fixed number of available hours per week
 - ✓ Making a door or a window requires different plants and time.
 - Goal: Maximize total profit
 - ✓ Information about profit: \$300 per door, \$500 per door

Factory	Time/door	Time/window	Available hours
1	1	0	4
2	0	2	12
3	3	2	18

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- Prototype Example

- Definition of decision variables

- ✓ x_1 : weekly production of doors

- ✓ x_2 : weekly production of windows

- Objective function:

$$\max Z = 3x_1 + 5x_2 \text{ (unit: \$100)}$$

- Constraints (convert the available hours in each plant into conditions)

- ✓ Plant 1: $1x_1 + 0x_2 \leq 4$

- ✓ Plant 2: $0x_1 + 2x_2 \leq 12$

- ✓ Plant 3: $3x_1 + 2x_2 \leq 18$

- These three constraints are called functional (or structural) constraints

- ✓ Non-negativity constraints: $x_1, x_2 \geq 0$

Profit: \$300 per door, \$500 per window

Factory	Time/door	Time/window	Available hours
1	1	0	4
2	0	2	12
3	3	2	18

Linear Programming

- Standard Form of LP
 - In LP, the standard form is designed so that algorithms (e.g., the Simplex method) work properly.
 - 1. The objective function is “Maximize”
 - 2. All constraints are in the form of “ \leq ”
 - 3. All variables have non-negativity constraints

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- Standard Form of LP – Generalization

- Setting

- ✓ m resources → the limitation of the i -th resource b_i ($i=1, \dots, m$)
 - The amount of resource i consumed by 1 unit of activity j : a_{ij}
 - ✓ n activities → the quantity of the j -th activity x_j ($j=1, \dots, n$)
 - Contribution of 1 unit of activity j to total performance Z : c_j
 - ✓ Total performance measure: Z

- Formulation

- ✓ Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$
 - ✓ (subject to)

- $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$
 - \dots
 - $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$
 - $x_1, x_2, \dots, x_n \geq 0$

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- Other Forms
 - Minimization objective function
 - ✓ Minimize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$
 - Greater-than-or-equal-to constraints
 - ✓ $a_1x_1 + a_2x_2 + \dots + a_nx_n \geq b_i$
 - Equality constraints
 - ✓ $a_1x_1 + a_2x_2 + \dots + a_nx_n = b_i$
 - Variables without non-negativity restrictions
- Although we introduced management science as allocating limited resources among competing activities, not all problems necessarily fall into this framework.

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- Assumptions of LP
 - Since an LP model is a simplified mathematical representation of real-world problems, several assumptions are made.
 - 1. Proportionality
 - ✓ Changes in variable values cause proportional changes in the objective function and constraints. (e.g., if producing 1 unit of a product requires 5 kg of material, then producing 2 units requires 10kg).
 - ✓ Violation in real-world: bulk purchase discounts, economies of scale
 - 2. Additivity
 - ✓ The total resource usage or total profit is assumed to be the simple sum of each variable's contribution.
 - ✓ Violation in real-world: when there is interaction between variables (synergy or interference).
 - When one plant produce two types of products, the efficiency decreases.

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- Assumptions of LP
 - 3. Divisibility
 - ✓ Variables are assumed to take continuous values (no integer restrictions).
 - For example, we assume that producing 2.5 units of a product is possible.
 - ✓ Violation in real-world: a person cannot work in half-unit increments
 - → requires Integer Programming
 - 4. Certainty
 - ✓ Coefficients in objective function or constraints, and resource availability are assumed to be known with certainty.
 - ✓ Violation in real-world: fluctuations in raw material prices, variations in production time
 - → Requires Stochastic Programming

Graphical Solution

- Graphical Method
 - The graphical method finds solutions visually when there are two variables.
 - It is useful for intuitively understanding the basic structure of LP and the nature of the optimal solution.
- Key Concepts of the Graphical Method
 - Graphing the constraints
 - ✓ Each constraint is a line, and the feasible side of the line is identified.
 - Example: $x_1 + x_2 \leq 4$ corresponds to the line $x_1 + x_2 = 4$ and the region below the line satisfies the constraint.
 - Feasible region
 - ✓ The region that satisfies all constraints simultaneously.
 - ✓ The solution of the LP problem always lies within this region.
 - ✓ The region forms a convex polygon.

Graphical Solution

- Key Concepts of the Graphical Method
 - Objective function line (Iso-profit or Iso-cost line)
 - ✓ The line representing all points with the same value of the objective function $Z = c_1x_1 + c_2x_2$
 - ✓ By moving this line in a parallel manner, the last point where it touches the feasible region is the optimal solution.
 - Property of the optimal solution
 - ✓ In LP, the optimal solution always occurs at one of the corner-points of the feasible region.
 - ✓ This property forms the foundation of the Simplex method.

Graphical Solution

- Prototype Example
 - Determine the mix of two products that maximizes profit under limited resources.
 - ✓ x_1 : the number of doors produced
 - ✓ x_2 : the number of windows produced
 - Objective function: Maximize $Z = 3x_1 + 5x_2$
 - Constraints
 - ✓ $x_1 \leq 4$
 - ✓ $x_2 \leq 6$
 - ✓ $3x_1 + 2x_2 \leq 18$
 - ✓ Non-negativity constraints: $x_1, x_2 \geq 0$

Graphical Solution

- Procedure of the Graphical Method in Prototype Example
 - 1. Draw the constraints
 - ✓ $x_1 = 4$
 - ✓ $x_2 = 6$
 - ✓ $3x_1 + 2x_2 = 18$
 - 2. Find the feasible region
 - ✓ Shade the area that satisfies each constraint
 - ✓ The region that satisfies all constraints simultaneously is the feasible region.
 - 3. Move the objective function line
 - ✓ From $Z = 3x_1 + 5x_2$, choose an arbitrary value and draw the line.
 - ✓ Move this line in parallel until it reaches the boundary of the feasible region (the last contact point is the optimal solution)
 - 4. Calculate the corner points
 - ✓ $(0, 0), (0, 6), (4, 0), (4, 3), (2, 6)$
 - 5. Evaluate the objective function at each corner point
 - ✓ $(0, 0) = 0, (0, 6) = 30, (4, 0) = 12, (4, 3) = 27, (2, 6) = 36$ (최대값)

Linear Programming

- Special Cases: Infeasible, Multiple Optimal Solutions, Unbounded
 - Infeasible solution
 - ✓ Occurs when there is no point that satisfies all constraints simultaneously
 - ✓ Diagnosis: in the graph, no overlapping shaded region exists; in modeling, conflicting (contradictory) constraints may be present.
 - ✓ Response: review or relax constraints, check data for errors, or adjust the objective
 - Multiple optimal solutions
 - ✓ Occurs when the objective function line is parallel to one edge of the feasible region.
 - ✓ Diagnosis: if two different corner points give the same optimal value, then every points on the line segment between them is also optimal.
 - ✓ When there are multiple optimal solutions, they act as alternatives each other, so increase flexibility.
 - ✓ Caution: in sensitivity analysis, allowable increase/decrease values and slack may be subtle.
 - Unbounded solution
 - ✓ Occurs when the feasible region is open in one direction, and that direction aligns with the improvement direction of the objective function.
 - ✓ Diagnosis: in the graph, moving the objective function line in the improvement direction never reaches a boundary.
 - ✓ Response: in practice, an unbounded solution usually indicates that some constraint is missing from the model.

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- Additional Example: Radiation Therapy Design
 - Background: Designing a treatment plan using two types of radiation beams for a patient's tumor.
 - ✓ Deliver a sufficient high dose of radiation to kill malignant tumor cells.
 - ✓ Minimize damage to healthy tissue cells.
 - ✓ Satisfy allowable radiation dose limits for major organs.
 - Decision variables: amounts of the two types of radiation beams x_1, x_2
 - Objective function: minimize the average radiation delivered to healthy tissue
 - Constraints
 - ✓ Average radiation to critical organs ≤ 27
 - ✓ Average radiation to the entire tumor = 60
 - ✓ Radiation to the tumor center ≥ 60

Target	Radiation 1	Radiation 2
Healthy tissue	4	5
Critical organs	3	1
Entire tumor	5	5
Tumor center	6	4

Linear Programming

- Additional Example: Radiation Therapy Design
 - Objective function: Minimize the average radiation delivered to healthy tissue
 - Constraints
 - ✓ Average radiation to critical organs ≤ 27
 - ✓ Average radiation to the entire tumor = 60
 - ✓ Radiation to the tumor center ≥ 60
 - Mathematical formulation
 - ✓ Minimize $Z = 4x_1 + 5x_2$
 - ✓ (subject to)
 - $3x_1 + x_2 \leq 27$
 - $5x_1 + 5x_2 = 60$
 - $6x_1 + 4x_2 \geq 60$

Target	Radiation 1	Radiation 2
Healthy tissue	4	5
Critical organs	3	1
Entire tumor	5	5
Tumor center	6	4