Lecture 11. Continuous Time Markov Chain

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Motivation - You kinda already know this!

- Suppose there are total $n \geq 1$ customers in the 1-server post office.
  (1 in svc, $n-1$ in waiting)
- $T_A \sim exp(2)$ (arrival), $T_S \sim exp(3)$ (svc).

- (# of total customer) will become either [ ], or [ ].
  And we call this event "transition" here.
  - *Q)* Difference between "transition" [here] and [in DTMC]?
  - *A)*

- You already know this!
  - What is the math. expression for event transition to $n+1$?
  - What is the prob. of transition to $n+1$?
  - What is the prob. of transition to $n-1$?
  - What is the dist. of 'time to transition'?

- Draw DTMC type transition diagram.

- Transition in DTMC and here
  - In DTMC, $n = 1, 2, ..$ is when transition occurs.
  - In here, transition occurs when an event occurs.
    $\rightarrow$ We need description for 'time to transition'.
- What is CTMC?

- Suppose $S = \{0, 1, 2, 3, 4\}$ in post office. (max. 3 waiting)
  1. Draw DTMC type transition diagram; DTMC type transition matrix
  2. Add time to transition information
  3. Can you combine 1 and 2?
  4. Matrix representation

Stationary Distribution - by flow balance equation

Stationary Distribution - by cutting method

Stationary Distribution - by $\pi G = 0$

CTMC-Definition

Definition - CTMC (Continuous Time Markov Chain)

- Consider a continuous time stochastic process $\{X(t), t \geq 0\}$ in state space S. It is called a **Continuous Time Markov Chain (CTMC)** if for all $s, t \geq 0$ and $i, j \in S$ and $X(u) \in S$ for $0 \leq u < s$,
$$\mathbb{P}[X(t+s) = j | X(s) = i, X(u) = x(u), 0 \leq u < s] = \mathbb{P}[X(t+s) = j | X(s) = i]$$

- Continuous time version of Markov property.
- Key observations
  - Continuous time stochastic process that depends on the "most recent information".
  - History does not matter.
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CTMC transition matrix

- **Question**: Consider the first class example of CTMC. Suppose there are two customers now, what is the probability that there will be three customers in 6.5 minutes later?

- **Motivation**: We need "CTMC type of transition matrix" to answer this question.
- As opposed to DTMC transition matrix **P** that contains "one step transition probability", we need CTMC transition matrix $\mathbf{P}(t)$
- Specifically, CTMC transition matrix should be given as $[\mathbf{P}(t)]_{i,j} = \mathbb{P}[X(t) = j|X(0) = i]$
- The original question asks $[\mathbf{P}(6.5)]_{2,3} = \mathbb{P}[X(6.5) = 3|X(0) = 2]$

How to get $\mathbf{P}(t)$?

Claim, $\mathbf{P}(t) = e^{tG}$

pf)(very rough proof)
Remind Taylor's 1st order approximation of $f(t) \approx f(0) + t \cdot f'(0)$.
Similarly, $\mathbf{P}(t) \approx \mathbf{P}(0) + t \cdot \mathbf{P}'(0) = \mathbf{I} + t\mathbf{P}'(0)$.
For the differential equation $\mathbf{P}(t) = \mathbf{I} + t\mathbf{P}'(0)$, try $\mathbf{P}(t) = e^{tG}$ and get $e^{tG} = \mathbf{I} + tG$.
The last equation above holds in Taylor's 1st order approximation's sense.
($\because e^x \approx 1 + x$).
Therefore, $\mathbf{P}(t) = e^{tG}$ is verified.

Example

- So, what is $[\mathbf{P}(6.5)]_{2,3} = \mathbb{P}[X(6.5) = 3 | X(0) = 2]$ for the class example?

CTMC version of Chapman-Kolmogrov

- Remind in DTMC,
  - $\mathbf{P}^{n+m} = \mathbf{P}^n \mathbf{P}^m$.
  - $\mathbb{P}(X_{n+m} = j | X_0 = i) = \sum_{k \in S} \mathbb{P}(X_n = k | X_0 = i) \mathbb{P}(X_m = j | X_0 = k)$
- Now in CTMC,
  - $\mathbf{P}(s+t) = \mathbf{P}(s)\mathbf{P}(t)$
  - Even more, $\mathbf{P}(as+bt) = [\mathbf{P}(s)]^a [\mathbf{P}(t)]^b$

Stationary distribution

- Why is it possible to do $\pi G = 0$ to get stationary distribution?
- flow of proof)
  - Begin with $\pi = \pi \mathbf{P}(t)$ just like DTMC $\pi = \pi P$
  - Take derivative both side by t and evaluate at $t = 0$ to get...
  - $0 = \pi \mathbf{P}'(0) \Rightarrow 0 = \pi G$
- proof)

Revisit the first example (with $S = \{0, 1, 2, 3, 4\}$)

- Rate diagram & Rate matrix

- Stationary distribution
  $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = (\frac{81}{211}, \frac{54}{211}, \frac{36}{211}, \frac{24}{211}, \frac{16}{211})$.
- Questions of interest
  1. What is the long run fraction of time the system is empty?
  2. What is the long run fraction of time the server is busy?
  3. What is the probability that a customer is not accepted to system?
  4. What is the expected # of customer in the system?

  5. What is the expected # of customer in the queue?

More questions

- Revisit Little's Law: $L = \lambda W$ (LN , p)
- More questions of interest
  6. What is expected total time spent in the system for a customer? (Waiting + Service time)
  7. What is expected waiting time for a customer?
  8. What is expected service time for a customer?
  9. What is expected service time for a customer?
  10. What is TH(throughput)?

CTMC and Queuing Theory

- Revisit Kendall's notation: (LN , p)
- M/M/·/· queuing model (Both inter-arrival and service times are exponentially distributed) can be analysed by CTMC, where the state is defined as the number of customers in the system!
- Examples
  - M/M/1/3
  - M/M/1/∞
  - M/M/k/∞
  - M/M/∞/∞

$M/M/1/\infty$

- Arrival is PP(λ /minute) and service completion is PP(μ /minute).
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- Assume $\lambda < \mu$ for stability. ($\rho := \lambda/\mu < 1$)
- Let $X(t)$ be the number of customers in the system at time t. Then, $\{X(t), t \geq 0\}$ is CTMC with state space $S = \{0, 1, 2, ...\}$.
- Rate diagram

Stationary distribution (by cutting)

- Stationary distribution
  - $\pi_0 =?$

1. The long run average # of customers in the system (L_{sys})?

2. The expected time spent in the system by a customer (W_{sys})?

3. The long run average # of customers in the queue (L_q)?

4. The expected time spent in the queue by a customer (W_q)?

- Discussion on W_q

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Homework

- Consider M/M/∞/∞, M/M/k/∞. For each queuing model, respectively, do the followings:
  - Define the CTMC with arrival of PP(λ) and service of PP(μ).
  - Define state space
  - Draw rate diagram
  - Find the stationary distribution. While doing so, make necessary assumption so that the stationary distribution exists.
  - Find $L_{sys}, W_{sys}, L_q, W_q$.

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Discussion on stability and stationary distribution

- Example. Rate Diagram

- Is this system stable?

- What is being "stable"?

- Does this sound related to existence of stationary distribution?

- i.e. can unstable system have a stationary distribution?

- Review on random walk.

Stationary distribution in CTMC

- Remind that, *finite*, *irreducible*, *aperiodic* DTMC has a unique stationary distribution.
- Periodicity is not a concern in CTMC.
- Irreducibility is not a big concern because most of CTMC has only one class. Why?
- Thus, a finite state CTMC has a unique stationary distribution.
- How about the *infinite* state CTMC?

Conclusion: Stationary distribution in CTMC

- As long as irreducible,
  - Finite CTMC has a unique stationary distribution
  - Infinite CTMC has a unique stationary distribution if and only if it is stable.
- If you can calculate unique stationary distribution, then the CTMC is stable. (regardless of finite or infinite state)

M/M/2/1

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- Suppose arrival is PP(0.5/minutes) and each server's service time follows exponential distribution with mean 5 minutes. How would you define CTMC?
- State space
- Rate diagram

M/M/2/1 with different service rate

- Let arrival process be PP(0.5/minutes), same as the previous one.
  - Suppose the two servers (John and Mary) have service time exponentially distributed with mean 6 and 4 minutes, respectively.
- State space and rate diagram

Priority rule (Assignment rule)

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$M/M/k/\infty$

- Suppose $\lambda = 2/min$ and $\mu = 3/min$
- Stationary distribution

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$M/M/2/\infty$

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$M/M/\infty/\infty$

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"Learn from yesterday, live for today, hope for tomorrow. The important thing is
not to stop questioning. - Albert Einstein"
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