Linear Programming
↳ planning
↳ all functions in the model are linear (1st order)
⇒ Simplex method

General LP problems — a_{ij}?
- m resources → b_i $(i=1 \cdots m)$ ⇒ $\sum_j a_{ij} x_j \le b_i$
- n activities → x_j $(j=1 \cdots n)$
- obj $Z = \sum_j c_j x_j$, c_j = contribution of unit activity j to obj

other form
- Min obj
- = constraints
- ≥ constraints
- x_j can be negative.

Max $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
subject to

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned}\right\} \quad x_1, x_2, \cdots x_n \ge 0$$

* Example Problem

company { prod #1 (\$3): 1 hour @ F_1, 3 hours @ F_3 ; prod #2 (\$5): 2 hours @ F_2, 2 hours @ F_3 } ⇒ { #prod 1: x_1 ; #prod 2: x_2 } decision variable

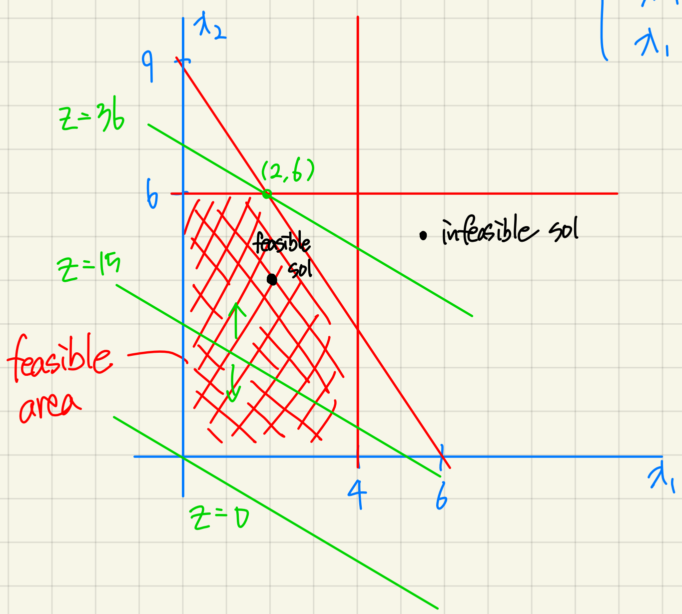
↓

3 Factories
- F_1 4 hours
- F_2 12 hours
- F_3 18 hours

$Z = \underbrace{\max(3x_1 + 5x_2)}_{\text{objective function}}$ with constraints

$$\left\{\begin{aligned} x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1, x_2 &\ge 0. \end{aligned}\right.$$

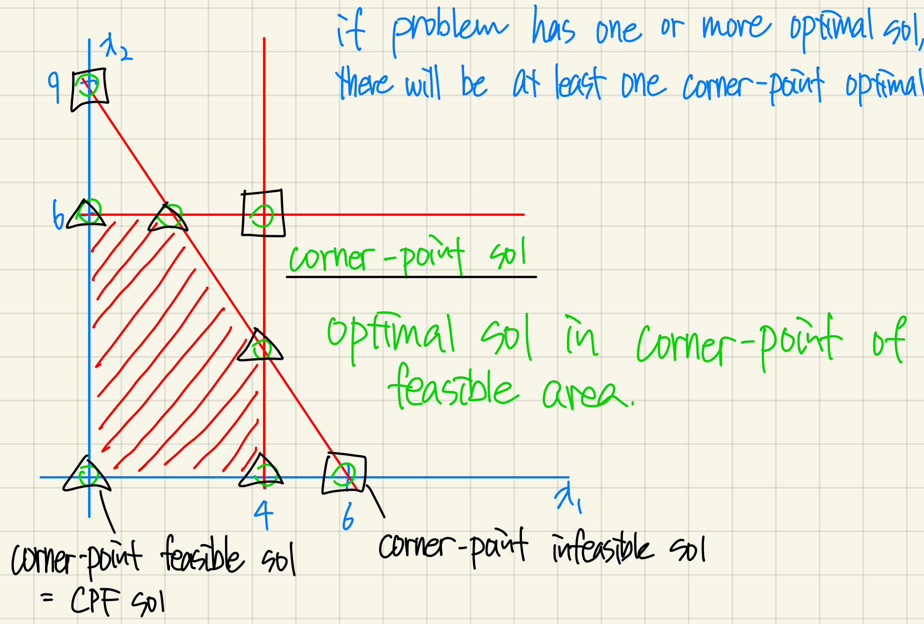
functional / structural constraints (first three); non-negativity constraints (last).



cases
1. one optimal sol
2. multiple optimal sol
3. infeasible
4. unbounded

(3), (4): No optimal sol

if problem has one or more optimal sol, there will be at least one corner-point optimal sol.



* LP assumption.
- proportionality
- additivity
- divisibility : x_1 & x_2 can be real number
- certainty

* Description.

a patient with malicious tumor
two radiations can be used together
hurst both cells
(malicious & normal)
how to mix radiation will change the effect of treatment

	$x_1 \ge 0$ rad ①	$x_2 \ge 0$ rad ②		
normal	4	5		
major organs	3	1	≤ 27	$3x_1 + x_2 \le 27$
whole tumor	5	5	$= 60$	$5x_1 + 5x_2 = 60$
core tumor	6	4	≥ 60	$6x_1 + 4x_2 \ge 60$

Minimize effect on normal cell $Z = 4x_1 + 5x_2$

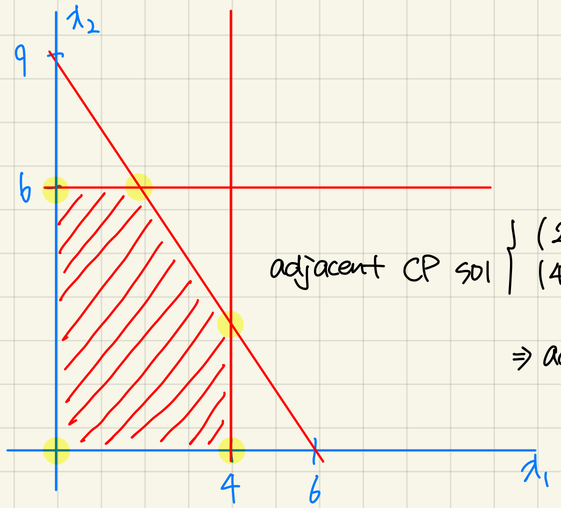
Max $Z = 3x_1 + 5x_2$

(s.t) $x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$

$x_1, x_2 \ge 0$



adjacent CP sol $\begin{cases} (2,6) & 3x_1+2x_2=18 \ \& \ 2x_2=12 \\ (4,3) & 3x_1+2x_2=18 \ \& \ x_1=4 \end{cases}$

⇒ adjacent CP sol share n-1 constraints (as equal)
have 1 different constraints (as equal)

n variables → CP sol will satisfy n constraints as equality

$x_1 = 2$ $\quad S_1 = 2$ (remaining hours of F_1)

$$\Rightarrow \left.\begin{aligned} x_1 \quad \boxed{+S_1 \qquad\qquad} &= 4 \\ 2x_2 \quad \boxed{\quad +S_2 \quad} &= 12 \\ 3x_1+2x_2 \quad \boxed{\qquad\quad +S_3} &= 18 \end{aligned}\right\} \text{augmented form}$$

slack var ≥ 0 ($\because x_1, x_2 \ge 0$)

$(x_1, x_2) \to (x_1, x_2, S_1, S_2, S_3)$

$(2,6) \to (2,6,2,\boxed{0},\boxed{0})$

$(0,6) \to (0,6,4,\boxed{0},6)$

$(4,3) \to (4,3,0,6,\boxed{0})$

$(0,0) \to (0,0,4,12,18)$

adjacent basic sol
⇒ share (n-1) NBVs $=0$
have 1 different NBVs & BVs $\ne 0$

x_2: entering variable (fix $x_1 = 0$)

$S_1 = 4$

leaving variable ⋯ $S_2 = 12 - 2x_2 \ge 0$ $\quad x_2 \to 6$ $\left(= \frac{12}{2}\right)$

$S_3 = 18 - 2x_2 \ge 0$ $\quad x_2 \to 9$ $\left(= \frac{18}{2}\right)$ $\Big\}\ x_2 \to 6$

minimum ratio test (MRT)

next solution $(0,6,4,0,6)$ ⇒ check optimal sol or not.

$$\left.\begin{aligned} &\text{BVs}: S_1, x_2, S_3 \\ &\underset{=0}{\text{NBVs}}: x_1, S_2 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 \quad + S_1 \qquad\qquad &= 4 \\ x_2 \quad + \tfrac{1}{2}S_2 \qquad &= 6 \\ 3x_1 \qquad - S_2 + S_3 &= 6 \end{aligned}$$

② $(0,6,4,0,6)$

⇒ $Z = 30$

$$\begin{aligned} Z &= 3x_1 + 5x_2 \\ -30 &= \quad -5x_2 - \tfrac{5}{2}S_2 \\ \hline Z-30 &= 3x_1 - \tfrac{5}{2}S_2 \end{aligned}$$

$\Rightarrow Z = \underbrace{\overset{③?}{3x_1} - \tfrac{5}{2}S_2}_{\text{NBVs}=0} + \overset{①}{30}$ ⇒ Max $Z = 30$.

original	augmented
CP solution ⇒	basic solution
CPF solution ⇒	basic feasible solution (BFS)

$(x_1, x_2) = (0,0)$
⇒ $S_1 = 4$, $S_2 = 12$, $S_3 = 18$

Non-basic variables (NBVs) = $\overset{m+n}{(\# \text{ all vars}} - \overset{m}{\# \text{ structural constraints})}$

Basic variables (BVs) $= \underset{m}{\#}$ structural constraints

x_1: entering variable

$S_1 = 4 - x_1$

$x_2 + \frac{1}{2}S_2 = 6$

leaving var ⋯ $S_3 = 6 - 3x_1$ ⋯ $x_1 \to 2$ $\Big)$ MRT

$$\left.\begin{aligned} x_1 \quad + S_1 \qquad\qquad &= 4 \\ x_2 \quad + \tfrac{1}{2}S_2 \qquad &= 6 \\ x_1 \quad -\tfrac{1}{3}S_1 \qquad + \tfrac{1}{3}S_3 &= 2 \end{aligned}\right] \Rightarrow S_1 + \tfrac{1}{3}S_2 - \tfrac{1}{3}S_3 = 2$$

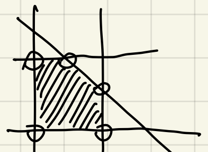
BFS $= (2,6,2,0,0)$

$$\begin{aligned} Z &= 3x_1 - \tfrac{5}{2}S_2 + 30 \\ 6 &= 3x_1 + S_2 + S_3 \\ \hline Z-6 &= -\tfrac{7}{2}S_2 - S_3 + 30 \end{aligned}$$

$\Rightarrow Z = \underbrace{-\tfrac{3}{2}S_2 - S_3}_{\text{NBVs}=0} + 36$

Maximize $3x_1+5x_2$

(s.t.)
$$\begin{cases} x_1 \le 4 \\ 2x_2 \le 12 \\ 3x_1+2x_2 \le 18 \\ x_1, x_2 \ge 0 \end{cases} \rightarrow \boxed{\begin{array}{l} S_1 = 4-x_1 \\ S_2 = 12-2x_2 \\ S_3 = 18-3x_1-2x_2 \end{array}} \rightarrow$$



Maximize $3x_1+5x_2$
$$\begin{cases} x_1 + S_1 = 4 \\ 2x_2 + S_2 = 12 \\ 3x_1+2x_2 + S_3 = 18 \\ x_1, x_2, S_1, S_2, S_3 \ge 0 \end{cases}$$

eq, $m=3$.

$(x_1, x_2, S_1, S_2, S_3)$

$\Rightarrow (0, 0, \underbrace{4, 12, 18}_{m\ \text{variables}})$, $(0, 6, 4, 0, 6)$

CPF sol.   CPF sol.

Basic Feasible (BF) sol.

$x_1, x_2 \Rightarrow$ non-basic var (NBV)

$S_1, S_2, S_3 \Rightarrow$ basic var (BV)

$Z = 3x_1+5x_2$ efficient to max Z

$(0, 0, 4, 12, 18)$

x_1 x_2 S_1 S_2 S_3

x_2: entering var. (increase to 6), S_2: leaving var. (decrease to 0)

$S_2 = 12-2x_2 \ge 0 \Rightarrow x_2 \le 6$

$S_3 = 18-2x_2 \ge 0$ ($\because x_1=0$) $\Rightarrow x_2 \le 9$

intersection $\therefore x_2 \le 6$

Pivot Operations (Gaussian elimination)

BV: x_2 S_1 S_3

NBV: x_1 S_2

$Z = 3x_1+5x_2+0S_1+0S_2+0S_3$
$$\begin{cases} x_1 + S_1 = 4 \\ -5x_2 - \frac{5}{2}S_2 = -30 \\ 3x_1 - S_2 + S_3 = 6 \end{cases}$$

$3x_1+5x_2 = Z$

$+ \quad -5x_2 - \frac{5}{2}S_2 = -30$

$3x_1 - \frac{5}{2}S_2 = Z-30.$

$\therefore Z = 3x_1 - \frac{5}{2}S_2 + 30.$ $\Rightarrow (\boxed{0}, 6, 4, 0, 6)$, $Z=30$.

$\Rightarrow$ increase x_1, Z increase more

Simplex table

BV	Z	x_1	x_2	S_1	S_2	S_3	RHS
Z	1	-3	-5	0	0	0	0
S_1	0	1	0	1	0	0	4
S_2	0	0	2	0	1	0	12
S_3	0	3	2	0	0	0	18

pivot row   pivot column   pivot num (elem)

MRT

$12/2 = 6$

$18/2 = 9$

select min as leaving var

entering var

BV	Z	x_1	x_2	S_1	S_2	S_3	RHS
Z	1	-3	0	0	5/2	0	30
S_1	0	1	0	1	0	0	4
x_2	0	0	1	0	1/2	0	6
S_3	0	3	0	0	-1	1	6

$S_1=4$

$x_2=6$

$x_3=6$

$4/1$ or $6/3$ MR

all positive coef. ⋯ done!

BV	Z	x_1	x_2	S_1	S_2	S_3	RHS
Z	1	3-3	0	0	5/2-1	0+1	30+6
S_1	0	1-1	0	1	0+1/3	0-1/3	4-2
x_2	0	0	1	0	1/2	0	6
x_1	0	1	0	0	-1/3	1/3	2

optimal sol

$$\boxed{\begin{array}{l} S_1=2 \\ S_2=S_3=0,\ x_2=6 \\ x_1=2 \end{array}}$$

$\text{Max } Z = 3x_1 + 5x_2$.

$$\left.\begin{aligned} x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &= 18 \\ x_1, x_2 &\ge 0. \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 + S_1 &= 4 \\ 2x_2 + S_2 &= 12 \\ 3x_1 + 2x_2 + a_1 &= 18 \\ x_1, x_2, S_1, S_2, a_1 &\ge 0 \end{aligned}$$

$\Rightarrow (0, 0, 4, 12, 18)$

Minimize $W = \Sigma a_i = a_1 \Rightarrow \text{Max } -W = -a_1$, $-W + a_1 = 0$

BV	W	x_1	x_2	S_1	S_2	a_1	RHS
W	-1					1 $\to 0$	0
S_1	0	1		1			4
S_2	0		2		1		12
a_1	0	3	2			1	18

EV

BV	W	x_1	x_2	S_1	S_2	a_1	RHS	
W	-1	-3	-2				-18	
S_1	0	1		1			4	4/1 ⋯ MR
S_2	0		2		1		12	
a_1	0	3	2			1	18	18/3

BV	W	x_1	x_2	S_1	S_2	a_1	RHS
W	-1		-2	3			-6
x_1		1		1			4
S_2			2		1		12
a_1			2	-3		1	6

$(4, 3, 0, 6, 0)$

BV	W	x_1	x_2	S_1	S_2	a_1	RHS
W	-1					1	0
x_1		1		1			4
S_2				3	1	-1	6
x_2			1	-3/2		1/2	3

$Z - 3x_1 - 5x_2 = 0$

BV	Z	x_1	x_2	S_1	S_2	RHS
Z	1	-3	-5 $\Rightarrow 0$			
x_1		1		1		4
S_2				3	1	6
x_2			1	-3/2		3

EV

BV	Z	x_1	x_2	S_1	S_2	RHS	
Z	1			-9/2		27	
x_1		1		1		4	4/1
S_2				3	1	6	6/3 ⋯ MR
x_2			1	-3/2		3	–

EV

BV	Z	x_1	x_2	S_1	S_2	RHS
Z	1				3/2	36
x_1		1			-1/3	2
S_2				1	1/3	2
x_2			1		1/2	6

$Z = 36$
$x_1 = 2$
$S_1 = 2$
$x_2 = 6$
$S_2 = 0$

2 phase - Simplex method.

1-phase. introduce artificial var.
No std. form LP.
→ solve sub-problem. Min $W = \Sigma a_i$

2-phase. use og objective by starting sol. of phase 1.

* Std. LP : Max Z ...

* Not std. form
① Equality constraints. e.g., $3x_1 + 5x_2 = 18 \to 3x_1 + 5x_2 + a_1 = 18$
② Minimization Z = maximization $(-Z)$ i.e., $\min \Sigma c_i x_i = \max \Sigma -c_i x_i$
③ Negative RHS constraints. e.g., $-x_1 + x_2 \ge -2 \Rightarrow x_1 - x_2 \le 2 \Rightarrow x_1 - x_2 + S_1 = 2$
④ $\ge$ constraints e.g., $-x_1 + x_2 \ge 2 \Rightarrow -x_1 + x_2 - S_2 = 2 \Rightarrow -x_1 + x_2 - S_1 + a_1 = 2$ ($\because$ coef (-1) cannot be BV.)

Minimize $Z = 4x_1 + 5x_2$

$$\left.\begin{aligned} 3x_1 + x_2 &\le 27 \\ 5x_1 + 5x_2 &= 60 \\ 6x_1 + 4x_2 &\ge 60 \\ x_1, x_2 &\ge 0 \end{aligned}\right\} \Rightarrow$$

Maximize $-Z = -4x_1 - 5x_2 \Rightarrow W = a_1 + a_2$
$\Rightarrow \text{Max } -W = -a_1 - a_2$, $-W + a_1 + a_2 = 0$.

$$\begin{aligned} 3x_1 + x_2 + S_1 &= 27 \\ 5x_1 + 5x_2 + a_1 &= 60 \\ 6x_1 + 4x_2 - S_2 + a_2 &= 60 \\ x_1, x_2, S_1, a_1, S_2, a_2 &\ge 0 \end{aligned}$$

phase 1.

BV	W	x_1	x_2	S_1	a_1	S_2	a_2	RHS
W	-1				1		1	0
S_1		3	1	1				27
a_1		5	5		1			60
a_2		6	4			-1	1	60

EV

BV	W	x_1	x_2	S_1	a_1	S_2	a_2	RHS	
W	-1	-11	-9			1		-120	
S_1		3	1	1				27	27/3 ⋯ MR
a_1		5	5		1			60	60/5
a_2		6	4			-1	1	60	60/6

EV

BV	W	x_1	x_2	S_1	a_1	S_2	a_2	RHS	
W	-1		-16/3	11/3		1		-21	
x_1		1	1/3	1/3				9	9/(1/3)
a_1			10/3	-5/3	1			15	15/(10/3)
a_2			2	-2		-1	1	6	6/2

EV

BV	W	x_1	x_2	S_1	a_1	S_2	a_2	RHS	
W	-1			-5/3		-5/3	8/3	-5	
x_1		1		2/3		1/6	-1/6	8	8/(2/3)
a_1				5/3	1	5/3	-5/3	5	5/(5/3)
x_2			1	-1		-1/2	-1/2	3	

BV	W	x_1	x_2	S_1	a_1	S_2	a_2	RHS
W	-1				1		1	0
x_1		1			-2/5	-1/2	1/2	6
S_1				1	3/5	1	-1	3
x_2			1		3/5	1/2	-1/2	6

BFS
$x_1 = 6$
$x_2 = 6$
$S_1 = 3$
$S_2 = 0$
$a_1, a_2 = 0$

phase 2.

Minimize $Z = 4x_1 + 5x_2 \to$ Maximize $-Z = -4x_1 - 5x_2 \to -Z + 4x_1 + 5x_2 = 0$

BV	Z	x_1	x_2	S_1	S_2	RHS
Z	-1	4	5	0	0	0
x_1	0	1	0	0	-1/2	6
S_1	0	0	0	1	1	3
x_2	0	0	1	0	1/2	6

BV	Z	x_1	x_2	S_1	S_2	RHS
Z	-1	0	0	0	-1/2	-54
x_1	0	1	0	0	-1/2	6
S_1	0	0	0	1	1	3
x_2	0	0	1	0	1/2	6

BV	Z	x_1	x_2	S_1	S_2	RHS
Z	-1	0	0	1/2	0	-105/2
x_1	0	1	0	1/2	0	15/2
S_2	0	0	0	1	1	3
x_2	0	0	0	-1/2	0	9/2

$x_1 = 15/2$
$x_2 = 9/2$
$S_1 = 0$
$S_2 = 3$

Max $-Z = -105/2$
Min $Z = 105/2$

if $W = \Sigma a_i \ne 0$, $a_i \ne 0$.
artificial variable is left.
$\Rightarrow$ Not BFS. infeasible.

$$\text{Max } Z = 3X_1 + 5X_2$$

$$\begin{aligned}
Z - 3X_1 - 5X_2 &= 0 \\
X_1 + S_1 &= 4 \\
2X_2 + S_2 &= 12 \\
3X_1 + 2X_2 + S_3 &= 18
\end{aligned}$$

BV	Z	X_1	X_2	S_1	S_2	S_3	RHS	
Z	1	-3	-5				0	
S_1	0	1		1			4	
S_2	0		2		1		12	12/2 ⋯ MR
S_3	0	3	2			1	18	18/2

pivot operation ① make pivot elem to 1. ② make other values to 0.

Simplex Method
① initialization
② check optimality
③ if not, repeat ⓐ entering var ⓑ leaving var ⓒ pivot operation.

$$\text{Min } Z = 4X_1 + 5X_2 \quad \begin{cases} 3X_1 + X_2 \le 27 \\ 5X_1 + 5X_2 = 60 \\ 6X_1 + 4X_2 \ge 60 \\ X_1, X_2 \ge 0 \end{cases} \Rightarrow \begin{cases} -Z + 4X_1 + 5X_2 = 0 \\ 3X_1 + X_2 + S_1 = 27 \\ 5X_1 + 5X_2 + a_1 = 60 \\ 6X_1 + 4X_2 - S_2 + a_2 = 60 \\ X_1, X_2, S_1, a_1, S_2, a_2 \ge 0 \end{cases} \Rightarrow \text{Max } (-Z)$$

if artificial variables exist,
Minimize $W = a_1 + a_2$
→ Maximize $-W = -a_1 - a_2$, $-W + a_1 + a_2 = 0$

BV	Z	X_1	X_2	S_1	a_1	S_2	a_2	RHS
W	-1	0	0	0	1	0	1	0
Z	-1	4	5					0
S_1	0	3	1	1				27
a_1	0	5	5		1			60
a_2	0	6	4			-1	1	60

$$\text{Max } Z = 3X_1 + 5X_2, \quad X_1 \le 4, \quad 2X_2 \le 12, \quad 3X_1 + 2X_2 \le 18, \quad X_1, X_2 \ge 0$$

$$C = [3, 5] \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad X_S = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 12 \\ 8 \end{bmatrix}$$

$$\Rightarrow \text{Max } CX, \quad AX \le b, \quad X \ge 0$$

$$Z - 3X_1 - 5X_2 = 0 \Rightarrow 1 \cdot Z - CX + 0 \cdot X_S = 0$$

$$\underbrace{\begin{pmatrix} X_1 \\ 2X_2 \\ 3X_1 + 2X_2 \end{pmatrix}}_{AX} \underbrace{\begin{pmatrix} +S_1 \\ +S_2 \\ +S_3 \end{pmatrix}}_{IX_S} \begin{matrix} = 4 \\ = 12 \\ = 18 \end{matrix} \quad = b$$

$$[A \mid I]\begin{bmatrix} X \\ X_S \end{bmatrix} = b$$

$$\Rightarrow \quad 1 \cdot Z - CX = 0, \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = b \quad \Rightarrow \quad [1 \; -C]\begin{bmatrix} Z \\ X \end{bmatrix} = 0, \quad [A \; I]\begin{bmatrix} X \\ X_S \end{bmatrix} = b$$

(first three columns: A; last three columns: I)

BV	Z	X_1	X_2	S_1	S_2	S_3	RHS
Z	1	-3	-5				0
S_1	0	1		1			4
S_2	0		2		1		12
S_3	0	3	2			1	18

init BVs

$$B^{-1}[A \; I]\begin{bmatrix} X \\ X_S \end{bmatrix} = B^{-1}b$$

$$X_B = \begin{bmatrix} S_1 \\ X_2 \\ S_3 \end{bmatrix} \to B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \to B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad C_B = [0 \; 5 \; 0] \quad B^{-1}b = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

	Z	X_1	X_2	S_1	S_2	S_3	RHS
S_1		1	0	1	0	0	4
X_2		0	1	0	1/2	0	6
S_3		3	0	0	-1	1	6

$B^{-1}A$, B^{-1}

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} \cdot [A \; I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 3 & 0 & 0 & -1 & 1 \end{bmatrix} = [B^{-1}A \;\; B^{-1}]$$

BV	Z	X_1	X_2	S_1	S_2	S_3	RHS
Z	1	-3	-5				0
S_1	0	1		1			4
S_2	0		2		1		12
S_3	0	3	2			1	18

⇒ initial tableau

$$Z - CX = 0.$$

	Z	X	X_S	RHS
Z	1	$-C$	0	0
X_B (iter 0)	0	A	I	b

	Z	X	X_S	RHS
Z	1	$C_B B^{-1} A - C$	$C_B B^{-1}$	$C_B B^{-1} b$
X_B (iter i)	0	$B^{-1}A$	B^{-1}	$B^{-1}b$
		$C_B B^{-1} A$	$C_B B^{-1}$	$C_B B^{-1} b$

→ B, B^{-1}, C_B

iter 0. $X_B = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B^{-1}$ $C_B = [0 \; 0 \; 0]$

optimality check. $C_B B^{-1} A - C = [0 \; 0 \; 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} - [3 \; 5] = [-3 \; -5]$

(NBV) $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

iteration EV = X_2

LV $\begin{cases} \text{i) EV is original var} : B^{-1}b \,/\, B^{-1}A \text{ (of EV)} \\ \text{ii) EV is slack var} : B^{-1}b \,/\, B^{-1} \end{cases}$

Sensitivity Analysis

	BV	Z	X_1	X_2	S_1	S_2	S_3	RHS
	Z	1	-3	-5				0
	S_1		1		1			4
init	S_2			2		1		12
	S_3		3	2			1	18
	Z	1				3/2	1	36
final	S_1				1	1/3	-1/3	2
(old)	X_2			1		1/2		6
	X_1		1			-1/3	1/3	2

$C_1 = 4$

$a_{31} = 2$

$b_2 = 24$

New problem

$\bar{C} = [4\ 5\ 0\ 0\ 0]$

$\bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}$ $\quad \bar{b} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$

$B^{-1} = \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix}$

$C_B B^{-1} = [0\ \ 3/2\ \ 1]$

$B^{-1} b = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}$

$C_B B^{-1} \bar{b} = 54$

$B^{-1} \bar{A} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \\ 2/3 & 0 \end{bmatrix}$

$C_B B^{-1} \bar{A} = [2\ 5]$

$C_B B^{-1} \bar{A} - C = [-2\ 0]$

Sensitivity Analysis

$C = [C_1, C_2, C_3, \cdots]$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \Rightarrow \begin{array}{ll} \text{Max} & CX \\ \text{(s.t.)} & AX \le b \\ & X \ge 0 \end{array}$$

$$X_s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix} \quad I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \in \mathbb{R}^{m\times m}$$

Max $z = 3X_1 + 5X_2$
(s.t.)
$X_1 \le 4$
$2X_2 \le 12$
$3X_1 + 2X_2 \le 18$
$X_1, X_2 \ge 0$

$$C = [3, 5] \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

BV	Z	x_1	x_2	s_1	s_2	s_3	RHS
Z	1	-3	5				
s_1		1		1			4
s_2			2		1		12
s_3		3	2			1	18
					C_BB^{-1}		
Z	1				3/2	1	36
s_1				1	1/3	-1/3	2
x_2			1		1/2		6
x_1		1			-1/3	1/3	2

B^{-1}

Max $z = CX + 0 \cdot X_s$
(s.t.) $[A \; I]\begin{bmatrix} X \\ X_s \end{bmatrix} = b$, ($X \ge 0$ $X_s \ge 0$)

(initial)

BV	Z	X	X_s	RHS
Z	1	-C	0	0
X_s	0	A	I	b

After iteration

BV	Z	X	X_s	RHS
Z	1	$C_BB^{-1}A - C$	C_BB^{-1} y^*	$C_BB^{-1}b$
X_B	0	$B^{-1}A$	B^{-1} s^*	$B^{-1}b$

$\underbrace{\text{original objective row}}_{[-C,0]} + C_B[B^{-1}AB^{-1}]$

$RHS = [0] + C_B \cdot [B^{-1}b]$

$\times B^{-1}$

$[A \; I]\begin{bmatrix} X \\ X_s \end{bmatrix} = b$

$\Rightarrow [B^{-1}A \,\vdots\, B^{-1}]\begin{bmatrix} X \\ X_s \end{bmatrix} = B^{-1}b$

* Change in b

$b_2 = 12 \rightarrow 24$,

$$\bar{b} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

BV	Z	x_1	x_2	s_1	s_2	s_3	RHS
Z							$C_BB^{-1}\bar{b} = 54$
s_1							$B^{-1}\bar{b}$
x_2							
x_1							

$$B^{-1}\bar{b} = \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}$$

$$\bar{b} = \begin{bmatrix} 4 \\ 12+\theta \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} + \begin{bmatrix} 0 \\ \theta \\ 0 \end{bmatrix}$$

$$B^{-1}\bar{b} = B^{-1}b + B^{-1}\begin{bmatrix} 0 \\ \theta \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/3\theta \\ 1/2\theta \\ -1/3\theta \end{bmatrix} = \begin{bmatrix} 2+1/3\theta \\ 6+1/2\theta \\ 2-1/3\theta \end{bmatrix} \ge 0. \Rightarrow -6 \le \theta \le 6$$ allowable range of b_2

* Change in coefficient of NBV (x_j): $C_j \rightarrow \bar{C}_j$, $A_j \rightarrow \bar{A}_j$

$C_1 = 3 \rightarrow \bar{C}_1 = 4$

$a_{13} = 3 \rightarrow \bar{a}_{13} = 2$

$$\Rightarrow A_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \rightarrow \bar{A}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Max $3X_1 + 5X_2$
(s.t)
$X_1 \le 4$
$2X_2 \le 24$
$3X_1 + 2X_2 \le 18$
$X_1, X_2 \ge 0$

BV	x_1	x_2	s_1	s_2	s_3	RHS
Z	9/2				5/2	45
s_1	1		1			4
x_2	3/2	1			1/2	9
x_1	-3			1	-1	6

C_BB^{-1}

B^{-1}

BV	Z	X_1	X_2	X_s			RHS
Z	1	1	0	0	0	5/2	45
s_1	0	1	0	1	0	0	4
X_2	0	1	1	0	0	1/2	9
S_2	0	-1	0	0	1	-1	6

$B^{-1}\bar{A} =$

*	1	2	3	4	S_i
1	8	6	10	9	35
2	9	12	13	7	50
3	14	9	16	5	40
d_j	45	20	30	30	

Northwest Corner Method: ↓ →

source

m

	1	2	3	4	S_i
1	35[1]				0
2	10[2]	20[3]	20[4]		0
3			10[5]	30[6]	0
d_j	0	0	0	0	

destination n.

$m+n-1$ basic variables.

Vogel's approximation method.

	1	2	3	4	S_i	Δ	
1	8	6	10	9	35	8-6	
2	9	12	13	7	50	9-7	max
3	14	9	16	$\boxed{5}$	40	9-5	4
d_j	45	20	30	30	$x_{34}=30$		
Δ	9-8	9-6	13-10	7-5	remove col 4		

	1	2	3	S_i	Δ	
1	8	6	10	35	8-6	
2	9	12	13	50	12-9	max
3	14	$\boxed{9}$	16	10	14-9	5
d_j	45	20	30	$x_{32}=10$		
Δ	9-8	9-6	13-10	remove row 3		

	1	2	3	S_i	Δ
1	8	$\boxed{6}$	10	35	2
2	9	12	13	50	3
d_j	45	10	30	$x_{12}=10$	
Δ	1	6	3	remove col 2	

	1	3	S_i	Δ	
1	8	10	25	2	
2	$\boxed{9}$	13	50	4	4
d_j	45	30	$x_{21}=45$		
Δ	1	3	remove col 1		

	3	S_i
1	10	25
2	13	5
d_j	30	$x_{13}=25$, $x_{23}=5$

$\Rightarrow$

	1	2	3	4	S_i
1		10	25		35
2	45		5		50
3		10		30	40
d_j	45	20	30	30	

Transportation Simplex Method.

	1	2	3	4	S_i	u_i
1	${}^{8}\,35$	${}^{6}\,-5$	${}^{10}\,2$	${}^{9}\,8$	35	-1
2	${}^{9}\,10$	${}^{12}\,20$	${}^{13}\,20$	${}^{7}\,5$	50	0
3	${}^{14}\,2$	${}^{9}\,-6$	${}^{16}\,10$	${}^{5}\,30$	40	3
d_j	45	20	30	30		
v_j	9	12	13	2		

For basic cells, $C_{ij}-u_i-v_j=0$.

For non-basic cells $C_{ij}-u_i-v_j$

Optimality check:
if $C_{ij}-u_i-v_j<0$. for any non-basic cells, current solution is not optimal

Simplex iteration)

① entering variable selection
$\Rightarrow x_{32}$ $\because$ most negative.

② leaving variable selection.
ⓐ finding cycle :
$x_{32}\to+\theta$
$x_{22}\to 20-\theta\geq 0$
$x_{23}\to 20+\theta$
$x_{34}\to 10-\theta\geq 0$. $\therefore$ leaving var

ⓑ select smallest donor cell & θ
$\theta=10$.

③ New BFS

re-calculate u_i & v_j & NBV ✓

	1	2	3	4	S_i	u_i
1	${}^{8}\,35$	${}^{6}\,-5$	${}^{10}\,-2$	${}^{9}\,2$	35	-1
2	${}^{9}\,10$	${}^{12}\,10$	${}^{13}\,30$	${}^{7}\,-1$	50	0
3	${}^{14}\,8$	${}^{9}\,10$	${}^{16}\,6$	${}^{5}\,30$	40	-3
d_j	45	20	30	30		
v_j	9	12	13	8		

ⓐ Entering x_{12}

ⓑ
$x_{12}\to\theta$
$x_{11}\to 35-\theta\geq 0$
$x_{21}\to 10+\theta$
$x_{22}\to 10-\theta\geq 0$

$\therefore\ \theta=10$. Leaving x_{22}

④ update

	1	2	3	4	S_i	u_i
1	${}^{8}\,25$	${}^{6}\,10$	${}^{10}\,-2$	${}^{9}\,7$	35	0
2	${}^{9}\,20$	${}^{12}\,5$	${}^{13}\,30$	${}^{7}\,4$	50	1
3	${}^{14}\,7$	${}^{9}\,10$	${}^{16}\,1$	${}^{5}\,30$	40	3
d_j	45	20	30	30		
v_j	8	6	12	2		

ⓐ Entering x_{13}

ⓑ
$x_{13}\to\theta$
$x_{23}\to 30-\theta\geq 0$
$x_{21}\to 20+\theta$
$x_{11}\to 25-\theta\geq 0$

$\therefore\ \theta=25$. Leaving x_{11}

⑤ update

	1	2	3	4	S_i	u_i
1	${}^{8}\,2$	${}^{6}\,10$	${}^{10}\,25$	${}^{9}\,7$	35	0
2	${}^{9}\,45$	${}^{12}\,3$	${}^{13}\,5$	${}^{7}\,2$	50	3
3	${}^{14}\,5$	${}^{9}\,10$	${}^{16}\,3$	${}^{5}\,30$	40	3
d_j	45	20	30	30		
v_j	6	6	10	2		

$\Rightarrow$ all pos in table $\therefore$ optimal

* Assignment

Tasks

Asignee	1	2	3	4
1	13	16	12	11
2	15	M	13	20
3	5	7	10	6
4(D)	0	0	0	0

=

Hungarian Algorithm :

<equivalent cost table>

	1	2	3	4
1	13	16	12	11
2	15	M	13	20
3	0	2	5	1
4(D)	0	0	0	0

	1	2	3	4
1	2	4	1	0
2	2	M	0	7
3	0	2	5	1
4(D)	0	0	0	0

1. Row reduction. (subtract min of each row)
2. Column reduction

	1	2	3	4	5
1a	820	810	840	960	0
1b	820	810	840	960	0
2a	800	870	M	920	0
2b	800	870	M	920	0
3	740	900	810	840	M

⇒

	1	2	3	4	5
1a	80	0	30	120	0
1b	80	0	30	120	0
2a	60	60	M	80	0
2b	60	60	M	80	0
3	0	90	0	0	M

$3 < 5$ ∴ not optimal

repeat
3. optimality check
(min # lines cover all zero.
= # rows = # cols)

4. additional zeros:
- subtract min of uncovered elements
- maintain covered elements
- add that min value to intersection.

	1	2	3	4	5
1a	50	0	0	90	0
1b	50	0	0	90	0
2a	30	60	M	50	0
2b	30	60	M	50	0
3	0	120	0	0	M

	1	2	3	4	5
1a	50	0	0	90	30
1b	50	0	0	90	30
2a	0	30	M	20	0
2b	0	30	M	20	0
3	0	120	0	0	M

∴ 810+840+800+0+840

=

	1	2	3	4	5	
1a	80	0	30	120	0	0
1b	80	0	30	120	0	0
2a	60	60	M	80	0	0
2b	60	60	M	80	0	0
3	0	90	0	0	M	0
	740	810	810	840	0	

	1	2	3	4	5	
1a	50	0	0	90	0	0
1b	50	0	0	90	0	0
2a	30	60	M	50	0	0
2b	30	60	M	50	0	0
3	0	120	0	0	M	-30
	770	810	840	870	0	