On the calculation of the ramified Siegel series

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2022 早稲田整数論研究集会
(Day 3 2022.3.25)

Eisenstein series

- Eisenstein series: For $\tau \in \mathbb{H}$, $k \geq 2$,

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m+n\tau)^{2k}}$$

An essential example of modular forms

- Fourier coefficients of the Eisenstein series

$$G_{2k}(\tau) = 2\zeta(2k)\left(1 + \frac{(2\pi\sqrt{-1})^{2k}}{(2k-1)!\zeta(2k)}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n\right)$$

where σ: divisor function, $q = \exp(2\pi\sqrt{-1}\tau)$

Siegel Eisenstein series (1)

- Siegel Eisenstein series: a generalization of the Eisenstein series to matrix variables
  For $n \geq 2$, $G = \mathrm{Sp}_n(\mathbb{Z})$,

$$E^n_{k,\ell,\psi}(Z) = \sum_{\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_0(\ell)} \psi(\det D) \frac{1}{\det(CZ+D)^{2k}}$$

where ψ: Dirichlet character modulo ℓ, $Z \in \mathbb{H}^n$ and

$$\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \,\middle|\, C = 0 \right\}$$

$$\Gamma_0(\ell) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \,\middle|\, C \equiv 0 \bmod \ell \right\}$$

Siegel Eisenstein series (2)

- Fourier coefficients of the Siegel Eisenstein series

$$E^n_{k,\ell,\psi}(Z) = \sum_{N \in S_h^+} C(N) \exp(2\pi\sqrt{-1}\mathrm{tr}(NZ))$$

(S_h^+: the set of half-integral, symmetric, positive semidefinite maritces)
Then, for N: positive definite matrix,

$$C(N) = \frac{2^{-\frac{n(n-1)}{2}}(-2\pi\sqrt{-1})^{nk}}{\pi^{\frac{n(n-1)}{4}}\prod_{i=0}^{n-1}\Gamma(k-\frac{i}{2})}(\det N)^{k-\frac{n+1}{2}}\prod_p S_n^p(\psi, N, k)$$

We call $S_n^p(\psi, N, k)$ the ramified Siegel series when p is a prime factor of ℓ.

Ramified Siegel series and Degenerate Whittaker Function

- The caluculation of the ramified Siegel series $S_n^p(\psi, N, k)$ is reduced to the calculation of the following integral;

$$S_0(B,s)^\chi = \int_{\mathrm{Sym}_n(F)} f_0\left(w_n\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}\right)\psi(\mathrm{tr}(-BX))dX$$

where $w_n = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $f_0 \in I_n(\omega, s-\frac{n+1}{2})^{\Gamma,\omega}$
(The definition of f_0 is on the following pages)

Notations (1)

- F: non-archimedean non-dyadic local field
- $\mathfrak{o}, \mathfrak{p}$: ring of integers, prime ideal of F
- π: a fixed prime element ($\mathfrak{p} = \pi\mathfrak{o}$)
- $q = \#\mathfrak{o}/\mathfrak{p}$ (assumed to be odd)
- $G = \mathrm{Sp}_n(F)$: Symplectic group of degree n
- $P = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \,\middle|\, C = 0 \right\}$: Siegel parabolic subgroup
- $K = \mathrm{Sp}_n(\mathfrak{o})$: maximal compact subgroup
- $\Gamma = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K \,\middle|\, C \equiv 0 \bmod \mathfrak{p} \right\}$
- $w_i = \left(\begin{array}{cc|cc} 1_{n-i} & & & \\ & & & -1_i \\ \hline & & 1_{n-i} & \\ & 1_i & & \end{array}\right)$ a complete set of representatives of $P\backslash G/\Gamma$ $(0 \le i \le n)$

Notations (2); Induced Representations

- ω: a character of $F^\times$ satisfying $\omega^2 = 1$
- χ: a non-trivial ramified character of $F^\times$ satisfying $\chi^2 = 1$
- $I_n(\omega, s) = \mathrm{Ind}_P^G(\omega \circ |\det|^s)$: the space of C^∞- functions on G satisfying

$$f\left(\begin{pmatrix} A & * \\ 0 & {}^tA^{-1} \end{pmatrix} g\right) = \omega(\det A)|\det A|^{s+\frac{n+1}{2}} f(g)$$

- $I_n(\omega, s)^{\Gamma,\omega} =$
$\left\{ f \in I(\omega, s) \mid f(gk) = \omega(\det D) f(g) \text{ for all } k = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \right\}$
- $f_i \in I_n(\omega, s)^{\Gamma,\omega}$ $(0 \leq i \leq n)$: the functions satisfying $f_i(w_j) = \delta_{ij}$ (the Kronecker delta)

We note that $I_n(\omega, s)^{\Gamma,\omega} = \bigoplus_i \mathbb{C} f_i$

Degenerate Whittaker Function (General Definition)

- G: a reductive algebraic group over a non-archiemedean local field F
- P: a parabolic subgroup of G with a Levi subgroup M.
- P': another parabolic subgroup of G with Levi subgroup M' of unipotent part N'
- π: an admissible representation of G
- ψ: a one-dimensional non-trivial character of N'
- $\mathrm{Wh}_\psi(\pi) = \mathrm{Hom}_{N'}(\pi, \psi)$: The space of degenerate Whittaker functional
- $W_{v_0}(g) = \Phi(\pi(g)v_0)$: The degenerate Whittaker function ($\Phi \in \mathrm{Wh}_\psi(\pi)$, $g \in G$, $v_0 \in \mathrm{Ind}_P^G \tau$, τ: character of M)

Siegel Series and Degenerate Whittaker Funciton

- We call (ramified) Siegel series

$$S_t(B,s)^{\chi} = \int_{\mathrm{Sym}_n(F)} f_t\left(w_n\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}\right)\psi(\mathrm{tr}(-BX))dX,$$

which corresponds to Whittaker function

$$W_{v_0}(g) = \Phi(\pi(g)v_0)$$

- We calculated the Siegel series from the viewpoint of Whittaker function and representation theory

Previous Work

The Fourier coefficients of $E^n_{k,\ell,\psi}$ is known by

- H. Katsurada, 1999, n: any interger, $\ell = 1$
- Y. Mizuno, 2009, $n = 2$, ℓ: square-free odd, ψ: primitive
- S. Takemori, 2012, $n = 2$, ℓ: any integer, ψ: primitive
- S. Takemori, 2015, n: any integer, ℓ: odd, $\psi = \prod_p \psi_p$ where $\psi_p \neq \chi_p$ (the quadratic character modulo p)
- K. Gunji, 2019, $n = 3$, ℓ: odd prime, ψ: primitive
- K. Gunji, 2022, all n, ℓ: odd prime, ψ: primitive

For the Whittaker Function,

- T. Ikeda, 2015, calculated the coefficients of the functional equation

Main Results

- The explicit calculation of the ramified Siegel series (Theorem 1, 2)
- The calculation of the intertwining operators (Theorem 3)
- Notations
- The result of Sato and Hironaka (Theorem 4, 5, 6)
- The proof of idea of Theorem 1, 2
- The proof of idea of Theorem 3

An Explicit Formula for the Siegel Series (1)

We calculate the Siegel series

$$S_t(B,s)^\chi = \int_{\mathrm{Sym}_n(F)} f_t\left(w_n \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}\right) \psi(\mathrm{tr}(-BX))dX$$

where $0 \leq t \leq n$ and $f_t \in I_n(\chi, s - \frac{n+1}{2})^{\Gamma,\chi}$.

Theorem 1 (the case $\varphi = f_0$; Gunji(2022), W.)

$$S_0(B,s)^\chi = \alpha_\psi(\pi)^n \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma^2=1}} (1-q^{-1})^{c_2(\sigma)} \sum_{I=I_0\cup\cdots\cup I_r} q^{-c_2(\sigma)-\tau(\{I_i\})-t(\sigma,\{I_i\})}$$

$$\times \sum_{\{\nu\}} \prod_{\ell=0}^{r} \chi(\pi)^{\nu_\ell n^{(\ell)}} q^{\frac{1}{2}\nu_\ell n^{(\ell)}(2s-1-n^{(\ell)})+\tilde{\rho}_{\ell,\nu_0+\cdots+\nu_\ell}(\sigma;B)} \prod_{\substack{i\in I_\ell \\ \sigma(i)=i}} \xi_{i,\nu_0+\cdots+\nu_\ell}(B)_\chi$$

An Explicit Formula for the Siegel Series (2)

Theorem 2 (the case $\varphi = f_t \quad (0 \leq t \leq n)$)

$$S_t(B,s)^{\chi} = \alpha_{\psi}(\pi)^{n-t} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma^2=1}} (1-q^{-1})^{c_2(\sigma)} q^{-c_2(\sigma)} \sum_{\substack{I=I_0\cup\cdots\cup I_r \\ n^{(k)}=t}} q^{-\tau(\{I_i\})-t(\sigma,\{I_i\})}$$

$$\times \frac{(1-q^{-1})^{\sum_{\ell=k}^{r} c_1^{(\ell)}(\sigma)} q^{n(k)}}{\prod_{\ell=k}^{r}(q^{n(\ell)}-1)}$$

$$\times \sum_{\{\nu\}_k^t} \prod_{\ell=0}^{k-1} \chi(\pi)^{\nu_\ell(n^{(\ell)}-n^{(k)})} q^{\nu_\ell((sn^{(\ell)}-n(\ell))-(sn^{(k)}-n(k)))+\tilde{\rho}_{\ell,\nu_0+\cdots+\nu_\ell}(\sigma;B)}$$

$$\times \prod_{\substack{i\in I_\ell \\ \sigma(i)=i}} \xi_{i,\nu_0+\cdots+\nu_\ell}(B)_{\chi}$$

The action of the intertwining operators (1)

Notation

- $M_{w_n}^{(s)} : I_n(\omega, s) \to I_n(\omega, n+1-s)$: intertwining operator

$$M_{w_n}^{(s)}(f)(g) = \int_{X \in \mathrm{Sym}_n(F)} f\left(w_n \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} g\right) dX$$

- There is a square matrix $E_n^{(s),\omega} = ((E_n^{(s),\omega})_{ij})_{0 \le i,j \le n}$ of size $n+1$ satisfying

$$M_{w_n}^{(s)} \begin{bmatrix} f_0^{(s)} & f_1^{(s)} & \cdots & f_n^{(s)} \end{bmatrix} = \begin{bmatrix} f_0^{(n+1-s)} & f_1^{(n+1-s)} & \cdots & f_n^{(n+1-s)} \end{bmatrix} ((E_n^{(s),\omega})_{ij})$$

We calculate the component of $E_n^{(s),\chi}$ when $n = 1$ and 2.

The action of the intertwining operators (2)

Theorem 3

When $n = 1$, we have

$$\begin{cases} M_{w_1}^{(s)}(f_0^{(s)}) = \chi(-1)q^{-1} f_1^{(2-s)} \\ M_{w_1}^{(s)}(f_1^{(s)}) = f_0^{(2-s)} \end{cases}$$

When $n = 2$, we have

$$M_{w_2}^{(s)}(f_1^{(s)}) = \chi(-1)q^{-1}\frac{1-q^{-1-2s}}{1-q^{-2s}} f_1^{(3-s)}$$

$$M_{w_2}^{(s)}\begin{pmatrix} f_0^{(s)} \\ f_2^{(s)} \end{pmatrix} = \begin{pmatrix} \chi(-1)q^{-1}(1-q^{-1})\frac{q^{-2s}}{1-q^{-2s}} & q^{-3} \\ 1 & \chi(-1)q^{-1}\frac{1-q^{-1}}{1-q^{-2s}} \end{pmatrix} \begin{pmatrix} f_0^{(3-s)} \\ f_2^{(3-s)} \end{pmatrix}$$

Notations (3); Using In Proofs

- $\mathfrak{S}_n$: the symmetric group of degree n
  (later we only consider the permutation $\sigma^2 = 1$)
- $I = \{1, 2, \cdots, n\}$
- $I = I_0 \cup \cdots \cup I_r$: σ-stable partition

$$c_2(\sigma) = \frac{1}{2}\#\{i \in I \mid \sigma(i) \neq i\}$$

$$c_1^{(k)}(\sigma) = \#\{i \in I_k \mid \sigma(i) = i\}$$

$$t(\sigma, \{I_i\}) = \sum_{l=0}^{r} \#\{(i,j) \in I_l \times I_l \mid i < j < \sigma(i),\ \sigma(j) < \sigma(i)\}$$

$$\tau(\{I_i\}) = \sum_{l=1}^{r} \#\{(i,j) \in I_l \times (I_0 \cup \cdots \cup I_{l-1}) \mid j < i\}$$

$$n_i = \#(I_i),\ \ n^{(i)} = \#(I^{(i)}) = n_i + \cdots + n_r$$

Notations (4)

Let $B = \mathrm{diag}(v_1\pi^{\beta_1}, v_2\pi^{\beta_2}, \cdots, v_n\pi^{\beta_n})$ $(v_i \in \mathfrak{o}^\times,\ \beta_i \in \mathbb{Z})$

$$e_{\sigma,i,k} = \begin{cases} 0 & (k \le i,\ k \le \sigma(i)) \\ 1 & (\sigma(i) < k \le i \ \text{ or } \ i < k \le \sigma(i)) \\ 2 & (i < k,\ \sigma(i) < k) \end{cases}$$

$$\tilde{\rho}_{l,\lambda}(\sigma; B) = \frac{1}{2}\sum_{i \in I_l}\sum_{k=1}^{n} \min\{\beta_k + e_{\sigma,i,k} + \lambda, 0\}$$

$$B_i(\lambda) = \{k \mid 1 \le k \le i-1,\ \beta_k + \lambda < 0,\ \beta_k \not\equiv \lambda \bmod 2\}$$
$$\cup\ \{k \mid i+1 \le k \le n,\ \beta_k + \lambda + 2 < 0,\ \beta_k \not\equiv \lambda \bmod 2\}$$

$$\xi_{i,\lambda}(B)_\chi = \prod_{k \in B_i(\lambda)} \chi(v_k) \times \begin{cases} 0 & \beta_i + \lambda \ge 0,\ \#B_i(\lambda) : \text{even} \\ (1-q^{-1})\chi(-1)^{[\#B_i(\lambda)/2]+1} & \beta_i + \lambda \ge 0,\ \#B_i(\lambda) : \text{odd} \\ \chi(v_i)\chi(-1)^{[\#B_i(\lambda)/2]+1} & \beta_i + \lambda = -1,\ \#B_i(\lambda) : \text{even} \\ -q^{-1/2}\chi(-1)^{[\#B_i(\lambda)/2]+1} & \beta_i + \lambda = -1,\ \#B_i(\lambda) : \text{odd} \end{cases}$$

Notations (5)

$$b_l(\sigma, B) = \min[\{\beta_i \mid i \in I_l, \sigma(i) > i\} \cup \{\beta_i + 1 \mid i \in I_l, \sigma(i) \leq i\}]$$

$\{\nu\}$ runs through the finite set

$$\{(\nu_0, \nu_1, \cdots, \nu_r) \in \mathbb{Z} \times \mathbb{Z}^r_{>0} \mid -b_l(\sigma, B) \leq \nu_0 + \nu_1 + \cdots + \nu_l \leq -1 \ (0 \leq l \leq r)\}$$

$\{\nu\}^t_k$ for $k \geq 1$ runs through the finite set

$$\Big\{(\nu_0, \nu_1, \cdots, \nu_{k-1}) \in \mathbb{Z} \times \mathbb{Z}^{k-1}_{>0} \mid -b_l(\sigma, B) \leq \nu_0 + \nu_1 + \cdots + \nu_l \leq -1 \, (0 \leq l \leq k-1) \, , n^{(k)} = t\Big\}$$

- Weil constant $\alpha_\psi(a)$ ($a \in F^\times$): a complex number satisfying

$$\int_F \phi(x)\psi(ax^2)dx = \alpha_\psi(a)|2a|^{-\frac{1}{2}} \int_F \hat{\phi}(x)\psi\left(-\frac{x^2}{4a}\right) dx$$

for any $\phi \in \mathcal{S}(F)$: locally constant function and $\hat{\phi}(x) = \int_F \phi(y)\psi(xy)dy$: its Fourier transform.

The Result of Sato and Hironaka (1)

- $S_n(F) = \{X \in \mathrm{Sym}_n(F) \mid \det X \neq 0\}$
- $\Gamma_0 := \{\gamma = (\gamma_{ij}) \in \mathrm{GL}_n(\mathfrak{o}) \mid \gamma_{ij} \in \mathfrak{p}\ (i > j)\}$
  which acts on $S_n(F)$ by $Y \mapsto \gamma \cdot Y = \gamma Y {}^t\gamma$
- $I = \{1, 2, \cdots, n\}$

Theorem 4 (Sato, Hironaka)

The set

$$\{S_{\sigma,e,\varepsilon} := (s_{ij}),\ \ s_{ij} = \varepsilon_i \pi^{e_i} \delta_{i,\sigma(j)} \mid (\sigma, e, \varepsilon) \in \Lambda_n\}$$

gives the complete set of representatives of Γ_0-equivalence classes in $S_n(F)$, where Λ_n is the collection of $(\sigma, e, \varepsilon) \in \mathfrak{S}_n \times \mathbb{Z}^n \times \{1, \eta\}^n$ ($\{1, \eta\} = \mathfrak{o}^\times / \mathfrak{o}^{\times 2}$) satisfying

$$\sigma^2 = 1,\ \ e_{\sigma(i)} = e_i\ (i \in I),\ \ \varepsilon_i = 1\ (i \in I, \sigma(i) \neq i).$$

The Result of Sato and Hironaka (2)

Theorem 5 (Sato, Hironaka)

Let $Y_0 \in S_n(F)$ then the following integral formula holds for any continuous function f on $\Gamma_0 \cdot Y_0$:

$$\int_{\Gamma_0 \cdot Y_0} f(Y)dY = \alpha(\Gamma_0; Y_0)^{-1} \int_{\Gamma_0} f(\gamma Y_0 {}^t\gamma) d\gamma$$

where

$$\alpha(\Gamma_0; S_{\sigma,e,\varepsilon}) = 2^{n-2c_2(\sigma)}(1-q^{-1})^{c_2(\sigma)} q^{c(\sigma,e,\varepsilon)}$$

and

$$c(\sigma, e, \varepsilon) = -\frac{n(n-1)}{2} + \tau(\{I_i\}) + t(\sigma, \{I_i\}) + c_2(\sigma) + \sum_{l=0}^{r} \nu_l n(l).$$

The Result of Sato and Hironaka (3)

Let $\mathscr{G}_{\Gamma_0}(Y,T)=\int_{\Gamma_0}\psi(-\mathrm{tr}(Y\cdot T[\gamma]))d\gamma.\ \ \left(T[\gamma]={}^t\gamma T\gamma\right)$

$I(a)=\int_{\mathfrak{o}}\psi(ax^2)dx,\ \ I^*(a)=\int_{\mathfrak{o}^\times}\psi(ax^2)dx=I(a)-\frac{1}{q}I(a\pi^2).$

Theorem 6 (Sato, Hironaka)

Let $T=\mathrm{diag}(v_1\pi^{\beta_1},v_2\pi^{\beta_2},\cdots,v_n\pi^{\beta_n})$ ($v_i\in\mathfrak{o}^\times$, $\beta_i\in\mathbb{Z}$). For $(\sigma,e,\varepsilon)\in\Lambda_n$, the character sum $\mathscr{G}_{\Gamma_0}(S_{\sigma,e,\varepsilon},T)$ vanishes unless $e_i\geq\begin{cases}-\beta_i-1 & \text{if } \sigma(i)\leq i\\ -\beta_i & \text{if } \sigma(i)>i\end{cases}$ for any $i\in I$. When the condition above is satisfied, we have

$$\begin{aligned}\mathscr{G}_{\Gamma_0}(S_{\sigma,e,\varepsilon},T)=&(1-q^{-1})^{2c_2(\sigma)}q^{-\frac{n(n-1)}{2}+d(\sigma,e,\beta)}\\ &\times\prod_{\substack{i=1\\ \sigma(i)=i}}^{n}\left\{I^*(-\varepsilon_i v_i\pi^{e_i+\beta_i})\prod_{k=1}^{i-1}I(-\varepsilon_i v_k\pi^{e_i+\beta_k})\prod_{k=i+1}^{n}I(-\varepsilon_i v_k\pi^{e_i+\beta_k+2})\right\}\end{aligned}$$

where

$$d(\sigma,e,\beta)=\sum_{\substack{i=1\\ \sigma(i)>i}}^{n}\left\{\sum_{k=1}^{i-1}\min\{e_i+\beta_k,0\}+\sum_{k=i+1}^{\sigma(i)-1}\min\{e_i+\beta_k+1,0\}+\sum_{k=\sigma(i)+1}^{n}\min\{e_i+\beta_k+2,0\}\right\}.$$

The Idea of Proofs of the Theorem 2

We calculate the integral

$$S_t(B,s)^{\chi} = \int_{\mathrm{Sym}_n(F)} f_t\left(w_n\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}\right)\psi(\mathrm{tr}(-BX))dX$$

- Divide the domain of integration by its Γ_0-orbits (Theorem 4) (note that the measure $\mathrm{Sym}_n(F)\backslash S_n(F)$ is 0)
- Calculation of each orbits reduced to the integration on Γ_0 (Theorem 5)
- Calculation on Γ_0 (Theorem 6)

Note that Gunji proofs this theorem "basically" in the same way.

The Idea of Proofs of the Theorem 3 (1)

- The first part comes from the calculation of the integral
- To prove the second part, we extend the space of induced representation as

$$I_2(\chi, s) \subset \mathrm{Ind}_{B_2}^{G}(\chi|\cdot|^{s-\frac{1}{2}}, \chi|\cdot|^{s+\frac{1}{2}})$$

where $\mathrm{Ind}_{B_2}^{G}(\chi|\cdot|^{t_1}, \chi|\cdot|^{t_2})$ is the space of C^∞ functions on G satisfying

$$f(bg) = \chi(a_1)|a_1|^{t_1+2}\chi(a_2)|a_2|^{t_2+1}f(g)$$

for any $b = \left(\begin{array}{cc|cc} a_1 & * & * & * \\ & a_2 & * & * \\ \hline & & a_1^{-1} & \\ & & * & a_2^{-1} \end{array}\right) \in B_2$.

The Idea of Proofs of the Theorem 3 (2)

- The Weyl group $B_2\backslash G/B_2$ is generated by

$$w_1^{B_2} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} \textit{ and } \quad w_2^{B_2} = \begin{pmatrix} 1 & & & \\ & & & -1 \\ & & 1 & \\ & 1 & & \end{pmatrix}.$$

- Note that $\#B_2\backslash G/B_2 = 8$.

The Idea of Proofs of the Theorem 3 (3)

- The intertwining operator relating to $w_1^{B_2}$ and $w_2^{B_2}$ is represented by using $E_1^{(s),\chi}$ and $E_1^{(s),\chi^2}$
- If $l(w_1 w_2) = l(w_1) + l(w_2)$, then $M_{w_1 w_2} = M_{w_1} M_{w_2}$
- Calculate the product of three square matrices of size 8

The matrix of intertwining operator when $n \geq 3$

- The representation matrix $E_3^{(s),\chi}$ is

$$\begin{pmatrix} 0 & \chi(-1)q^{-2s}\frac{1-q^{-3}}{1-q^{-2s+1}} & 0 & 1 \\ q^{-2s-3}\frac{1-q^{-1}}{1-q^{-2s+1}} & 0 & \chi(-1)q^{-1}\frac{1-q^{-2s-1}}{1-q^{-2s+1}} & 0 \\ 0 & q^{-3}\frac{1-q^{-2s-1}}{1-q^{-2s+1}} & 0 & \chi(-1)q^{-1}\frac{1-q^{-1}}{1-q^{-2s+1}} \\ \chi(-1)q^{-6} & 0 & q^{-2}\frac{1-q^{-3}}{1-q^{-2s+1}} & 0 \end{pmatrix}$$

- We note that

$$(E_n^{(s),\chi})_{ij} = 0 \quad (i+j+n : \text{odd})$$

so the non-zero element of the matrix is a chekkered pattern. (by multiplying the long root n times, the parity of i and j changes n times)

- The elements of $E_n^{(s),\chi}$ when $n \geq 4$ are more complicated

The Eigenvectors of the matrix $E_n^{(s),\chi}$

- The matrix $E_n^{(s),\chi}$ can be diagonalized (as a vector with the coefficients the rational functions of q)
  ($\leftarrow$ multiplicity free when it is considered to the representation on Sp_n)
- We fix a complex number β such that $\beta^2 = \chi(-1)q^{-1}$

The eigenvectors of $E_n^{(s),\chi}$ when $n = 1, 2, 3$ are as follows:

- $n = 1$ $[1, \pm\beta]$
- $n = 2$ $[1, \pm\beta, \beta^2]$, $\quad [1, 0, -\beta^2 q^{-1}]$
- $n = 3$ $[1, \pm\beta, \beta^2, \pm\beta^3]$,
  $[1 + q + q^2, \pm\beta q, -\beta^2 q^{-1}, \mp\beta^3 q^{-2}(1 + q + q^2)]$

In $n \geq 4$ case, the coefficient are more complicated q-rational function.

Hecke ring (1)

We put $G = \mathrm{Sp}_n(\mathbb{F}_q)$.
The multiplication of the Hecke ring $\mathscr{H}(G, P, \chi)$(generalized by $f_0, \cdots, f_n$ as vector space) is defined by the convolution

$$f_i * f_j(g) = \sum_{h \in G} f_i(h) f_j(h^{-1} g).$$

We can calculate $n \geq 2$ by the reduction of the calculation of rank 1
We note that f_0 is the identity element in this Hecke ring.

Hecke ring (2)

- $n = 1$

$$f_1 * f_1 = \chi(-1) q f_0$$

- $n = 2$

$$\begin{aligned} f_1 * f_1 &= \chi(-1) q(q+1) f_0 + (q+1) f_2 \\ f_1 * f_2 &= \chi(-1) q^2 f_1 \\ f_2 * f_2 &= q^3 f_0 + \chi(-1) q(q-1) f_2 \end{aligned}$$

- $n = 3$

$$\begin{aligned} f_1^2 &= \chi(-1) q(q^2+q+1) f_0 + (q+1) f_2 \\ f_1 * f_2 &= \chi(-1) q^2 (q+1) f_1 + (q^2+q+1) f_3 \\ f_1 * f_3 &= \chi(-1) q^3 f_2 \\ f_2^2 &= q^3(q^2+q+1) f_0 + \chi(-1) q(q^3+q^2+q-1) f_2 \\ f_2 * f_3 &= q^5 f_1 + \chi(-1) q(q-1)(q^2+q+1) f_3 \\ f_3^2 &= \chi(-1) q^6 f_0 + q^4 (q-1) f_2 \end{aligned}$$

Hecke ring (3)

From these calculations, the primitive idempotents of the Hecke ring can be calculated. They contain the information of eigenvectors. The following are some calculations.

- $n = 1$

$$e^{\pm}_{1,1} = \frac{1}{2}(f_0 \pm \beta f_1)$$

- $n = 2$

$$e^{\pm}_{2,1} = \frac{1}{2(q+1)}(f_0 \pm \beta f_1 + \beta^2 f_2)$$
$$e_{2,2} = \frac{q}{q+1}(f_0 - \beta^2 q^{-1} f_2)$$

- $n = 3$ $\quad ([a_0, a_1, a_2, a_3] := a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3)$

$$e_{1,\pm} = \frac{1}{2(q+1)(q^2+1)}[1, \pm\beta, \beta^2, \pm\beta^3]$$
$$e_{2,\pm} = \frac{1}{2q^{-1}(q+1)(q^2+1)}[(1+q+q^2), \pm\beta q, -\beta^2 q^{-1}, \mp\beta^3(1+q^{-1}+q^{-2})]$$

Future development (1)

- The calculation of then ramified Siegel seires on the function
$$e^{\pm}_{n,1} = \frac{1}{2\mathrm{vol}(P)} \left(\sum_{i=0}^{n} (\pm\beta)^i f_i \right)$$
(which is expected to be the idempotent of all n) and calculate the functional equations (which are accomplished on the unramified case)
- In order to show $e^{\pm}_{n,1}$ is an eigenvector of the intertwining operator for all n, we use the Weil representation
$$\omega_Q : \mathcal{O}_Q \times \mathrm{Sp}_n \to \mathcal{S}(V^n)$$

Future development (2)

where (Q, V) is a nondegenerate quadratic forms of rank m over $\mathbb{F}_q$.
We conjecture $\mathcal{S}(V^n)^{\mathcal{O}_Q}$ and $\mathrm{Ind}_P^{\mathrm{Sp}_n}\chi^m$ are isomorphic when Q is isotropic, then we need to show that $e_{n,1}^{\pm}$ is an invariant element of the Weil representation. (This is in progress)

Reference

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Thank you for your attention.