Meca 482: Final project

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MECA 482

Furuta Pendulum

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1. Introduction

The Furuta Pendulum was invented by Katsuhisa Furuta in 1992 and has since been a widely used tool for teaching the principles of control systems design. The device consists of a motor, which is mounted to a base, that rotates about the vertical axis, and an arm which is free to rotate about the horizontal axis. The arm and motor are connected by a horizontal rod that protrudes from the motor. The goal of this project is to devise a system capable of maintaining the arm balanced upright by controlling the rotation of the motor about the vertical axis.

2. Modeling

Parameter	Description
α	Angular position of pendulum
θ	Angular position of rotary arm
m_p	Pendulum mass
L_p	Pendulum length
L_r	Rotary arm Length
J_p	Moment of inertia of pendulum
J_r	Moment of inertia of rotary arm
τ	Applied torque to rotary arm

The picture shown in Figure 1 below shows the rotary inverted pendulum model used for the control system design. The rotary arm, which is connected to the motor, has an arm length of L_r

and an angle of θ. As the motor rotates in a CCW direction, the angle θ increases. The moment of inertia of the rotary arm is designated by the variable J_r. The motor turns in the CCW direction whenever the control voltage is greater than 0.

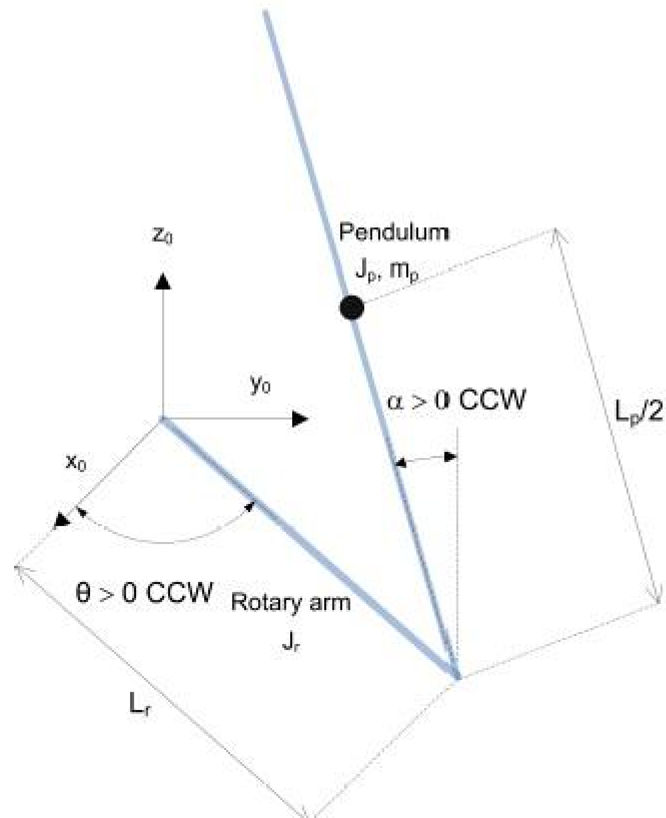


Figure 1. Rotary inverted pendulum conventions

3. Sensor Calibration

The focus of this project is to design a functional control system, and simulate the results. Since there is no requirement for implementation of physical hardware, there is no sensor, or sensor calibration.

4. Controller Design

a. Equations of Motion

The Equations of Motion (EOM) section covers the derivation of equations that describe the motion of both the pendulum and rotary arms with respect to the motor voltage. The full derivation of these equations begins with the Euler-Lagrange equation, seen below in Eq.1.

$$\frac{\delta^2 L}{\delta t \delta \dot{q}_i} - \frac{\delta L}{\delta q_i} = Q_i \qquad \textbf{Eq. 1}$$

In this equation, the variables q_i represent what are called generalized coordinates. Because of the nature of our system, it is logical to use polar coordinates. Therefore, the equation for the generalized coordinates is written as

$$q(t) = [\theta(\mathrm{t})\ \alpha(\mathrm{t})] \qquad \textbf{Eq. 2}$$

Where $\theta(t)$ is the angle of the rotary arm and $\alpha(t)$ is the angle of the inverted pendulum (see Figure 1.) Please note that Eq.2 is presented with respect to time and that from this point on the nomenclature will be simplified to exclude the (t) term in all variables that are with respect to time. That is to say that $\theta=\theta(t)$ and $\alpha=\alpha(t)$.

By combining equations 1 and 2, equations 3 and 4 are derived and shown below. Q_1 and Q_2 represent generalized forces acting on each arm.

$$\frac{\delta^2 L}{\delta t \delta \dot{\theta}} - \frac{\delta L}{\delta \theta} = Q_1 \qquad \textbf{Eq. 3}$$

$$\frac{\delta^2 L}{\delta t \delta \dot{\alpha}} - \frac{\delta L}{\delta \alpha} = Q_2 \qquad \textbf{Eq. 4}$$

The Lagrangian of the system is equal to the total kinetic energy of the system minus the total potential energy of the system. It is denoted by the variable L in equations 1-4. The generalized forces acting on each arm, Q_1 and Q_2, are equated to the non-conservative forces acting on each arm. Equations 5 and 6 show this relationship.

$$Q_1 = \tau - B_r \dot{\theta} \qquad \textbf{Eq. 5}$$

$$Q_2 = - B_p \dot{\alpha} \qquad \textbf{Eq. 6}$$

The control variable for this system is the voltage input to the servo motor, V_m. As a torque is applied to the servo, it is opposed by a viscous damping force from both arms. The viscous damping forces for the rotary and pendulum arms are B_r and B_p, respectively. After completing the process of obtaining the Lagrangian from the kinetic and potential energies and computing the various derivatives necessary to calculate the EOM's, the nonlinear equations of motion are found for both arms. The EOM for the rotary arm is shown in Eq. 7 and the EOM for the pendulum arm is shown in Eq. 8.

$$(m_p L_p^2 + \frac{1}{4} m_p L_p^2 - \frac{1}{4} m_p L_p^2 cos(\alpha)\ddot{\theta} - (\frac{1}{2} m_p L_p L_r cos(\alpha))\ddot{\alpha}$$

$$+ (\frac{1}{2} m_p L_p^2 sin(\alpha) cos(\alpha))\dot{\theta}\dot{\alpha} + (\frac{1}{2} m_p L_p L_r sin(\alpha))\dot{a}^2 = \tau - B_r \dot{\theta} \qquad \textbf{Eq. 7}$$

$$-\frac{1}{2}m_pL_pL_r cos(\alpha)\ddot{\theta} + (J_p + \frac{1}{4}m_pL_p^2)\ddot{\alpha} - \frac{1}{4}m_pL_p^2 cos(\alpha)sin(\alpha)\dot{\theta}^2$$

$$-\frac{1}{2}m_pL_p g sin(\alpha) = -B_p\dot{\alpha} \quad \textbf{Eq. 8}$$

The torque provided by the servo motor is applied at the base of the motor itself. Eq.9, shown below, describes the torque in terms of the servo motor input voltage V_m, among others.

$$\tau = \frac{\eta_g K_g \eta_m k_t (V_m - K_g k_m \dot{\theta})}{R_m} \quad \textbf{Eq. 9}$$

The typical form of an EOM is shown in Eq.10, where τ_1 is the scale value of torque applied to the rotary arm via the servo motor, $g(x)$describes the gravitational function affecting the system, $b$is the damping coefficient, and J is the moment of inertia of the system.

$$J\ddot{x} + b\dot{x} + g(x) = \tau_1 \quad \textbf{Eq. 10}$$

Eq. 10 is used to generalize the coordinate vector q into the matrix form seen in Eq.11 below,

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \quad \textbf{Eq.11}$$

where the inertial matrix is represented by D, the damping matrix is represented by C, the gravitational vector is represented by $g(q)$**,** and the applied torque is represented by τ.

Finally, the nonlinear EOM's shown in Eq.7 and Eq.8 can be placed into the matrix form derived above in Eq. 11.

b. Linearization Method

In order to linearize the EOM's for both arms, Eq. 10 is used,

$$f_{lin} = f\left(z_0\right) + \left.\frac{\partial f(z)}{\partial z_1}\right|_{z=z_0} \left(z_1 - a\right) + \left.\frac{\partial f(z)}{\partial z_2}\right|_{z=z_0} \left(z_2 - b\right) \qquad \textbf{Eq.12}$$

where

$$z^T = \left[z_1\, z_2\right]$$

and

$$z_0^T = [a\, b]$$

c. Linear State Space

In the equations for linear state space provided below, *A, B, C* and *D* denote the state space matrices while *x* denotes the state, and *u* denotes the control input to the system.

$$x = A\dot{x} + Bu \qquad \textbf{Eq. 13}$$

$$y = Cx + Du \qquad \textbf{Eq. 14}$$

The state of the rotary pendulum system then becomes

$$x = [\theta \ \ \alpha \ \ \dot{\theta} \ \ \dot{\alpha}] \qquad \textbf{Eq. 15}$$

And the output of the system becomes

$$y = [x_1 \ \ x_2] \qquad \textbf{Eq. 16}$$

Since only the positions of the servo and the angle of the arms are taken into account in terms of measured variables, the C and D matrices corresponding to the output equation are as follows:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Eq. 17

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Eq. 18

By taking the derivatives and filtering the results through a high-pass filter using the digital controller the velocities of both the servo and pendulum can be found.

d. Linearization of EOM

For the linearization of Eq.7 and Eq.8, the initial conditions of all the variables are set to equal 0. Under these conditions, the linearized equations Eq.19 and Eq. 20 are derived.

$$\tau - B_r\dot{\theta} = \ddot{\theta}(m_pL_r^2 + J_r) - \frac{\ddot{\alpha}}{2}\left(L_rL_pm_p\right)$$ **Eq.19**

$$B_p\dot{\alpha} = \frac{\alpha}{2}\left(gL_pm_p\right) + \frac{\ddot{\theta}}{2}\left(L_rL_pm_p\right) - \ddot{\alpha}\left(J_p + \frac{1}{4}m_pL_p^2\right)$$ **Eq.20**

The following equations display the matrix and determinant needed to calculate the angular acceleration of the rotary and pendulum arm.

$$\begin{bmatrix} J_r + m_pL_p^2, & -\frac{m_pL_pL_r}{2} \end{bmatrix}\begin{bmatrix} \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \tau - B_r(\dot{\theta}) \end{bmatrix}$$ **Eq.21**

$$\left[-\frac{m_p L_p L_r}{2}, J_p + \frac{m_p L_p^2}{4}\right]\left[\ddot{\alpha}\right] = \left[\frac{m_p L_p g}{2} - B_p \alpha\right]$$

$$Det = \left(J_p m_p J_r^2 + \frac{J_r m_p L_p^2}{4} + J_r J_p\right) \quad \textbf{Eq.22}$$

As aforementioned, the above equations are used to calculate the angular accelerations of both arms. These are shown in Eq. 23 and Eq.24 below.

$$\ddot{\theta} = \frac{1}{Det}\left(\left(J_p + \frac{m_p L_p^2}{4}\right)\tau + \frac{m_p^2 L_p^2 L_r^2 g\alpha}{4} - \frac{m_p L_p L_r B_p \dot{\alpha}}{2} - \left(J_p + \frac{m_p L_p^2}{4}\right)B_r \dot{\theta}\right) \quad \textbf{Eq.23}$$

$$\ddot{\alpha} = \frac{1}{Det}\left(\frac{m_p L_p L_r \tau}{2} + \frac{m_p L_p g}{2}\left(J_r + m_p L_r^2\right)\alpha + \frac{m_p L_p L_r B_r \dot{\theta}}{2} - B_p \dot{\alpha}\left(J_r + m_p L_r^2\right)\right) \quad \textbf{Eq.24}$$

By substituting x3 in for θ and x4 in for α into the above equations, we get the following,

$$\ddot{x3} = \frac{1}{Det}\left(\left(J_p + \frac{m_p L_p^2}{4}\right)\tau + \frac{m_p^2 L_p^2 L_r^2 g x4}{4} - \frac{m_p L_p L_r B_p \dot{x4}}{2} - \left(J_p + \frac{m_p L_p^2}{4}\right)B_r \dot{x3}\right) \quad \textbf{Eq.25}$$

$$\ddot{x4} = \frac{1}{Det}\left(\frac{m_p L_p L_r \tau}{2} + \frac{m_p L_p g}{2}\left(J_r + m_p L_r^2\right)x4 + \frac{m_p L_p L_r B_r \dot{x3}}{2} - B_p \dot{x4}\left(J_r + m_p L_r^2\right)\right) \quad \textbf{Eq.26}$$

Finally, the matrices *A* and *B* from the state space Eq. **X** is shown below in Eq.27.

$$A = (Det)^{-1} \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{L_p^2 m_p^2 L_r g}{4} & -B_r\left(J_p + \frac{m_p L_p^2}{4}\right) & \frac{-m_p L_p L_r B_p}{2} \\ 0 & \frac{m_p L_p g\left(L_r^2 m_p + J_r\right)}{2} & \frac{m_p L_p L_r B_p}{2} & -B_p\left(m_p L_r^2 + J_r\right) \end{matrix}$$

$$B = (Det)^{-1} \begin{matrix} 0 \\ 0 \\ \frac{m_p L_p^2}{4} + J_p \\ \frac{m_p L_p L_r}{2} \end{matrix}$$

5. Simulations

a. Diagrams

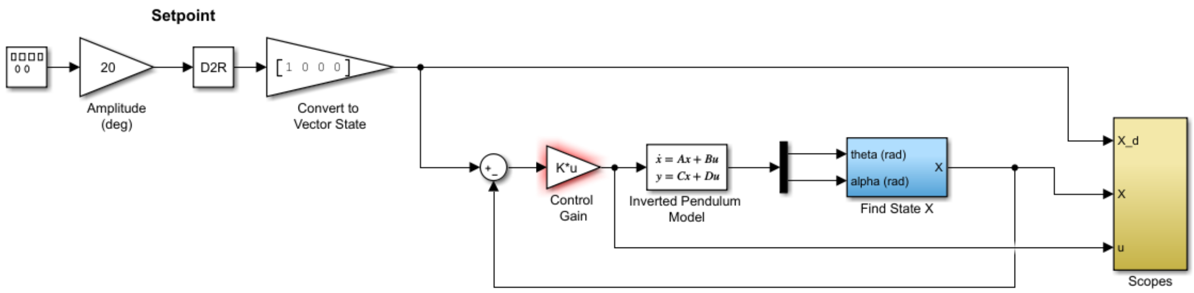


Figure. 2

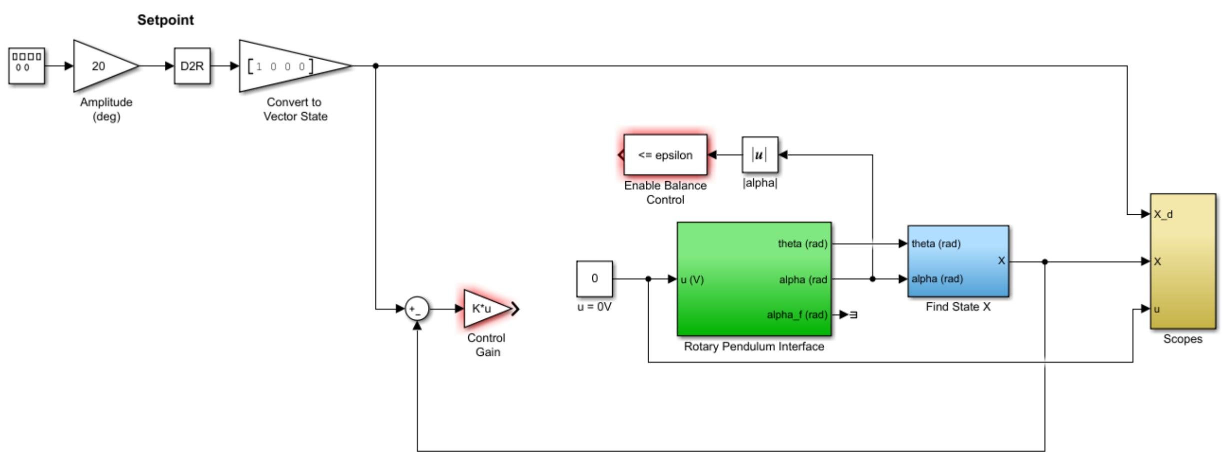
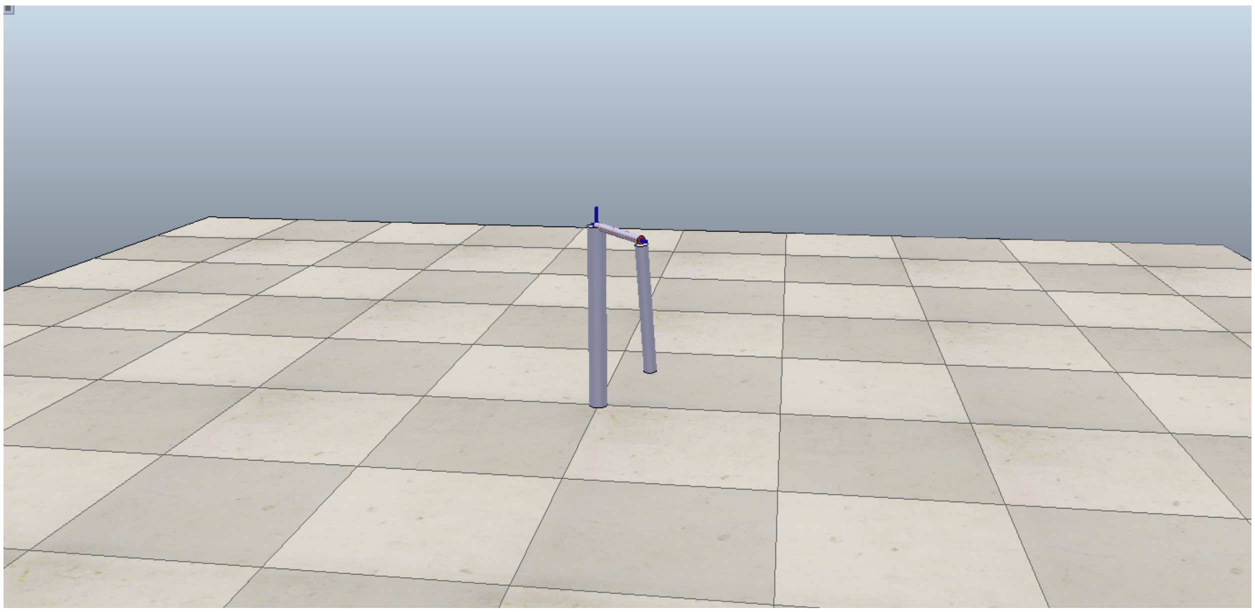


Figure. 3

6. Conclusion

The goal of this project was to create a control model that was designed to drive the Furuta Pendulum system. Ideally, the controller designed would be capable of driving the servo motor of the system to keep the pendulum arm balanced. This was accomplished by linearizing the equations of motion derived from the Euler-Lagrange equation. The linearized equations were then represented in state space and moved to a simulation in matlab.

7. Appendix: Simulation Code



8. References

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