Abstract Algebra by Pinter, Chapter 23

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Abstract

Chapter 23 on Elements of Number Theory

Contents

1 A. Solving Single Congruences

1.1 Q1

1.1.1 a

$$60x \equiv 12 \pmod{24}$$

$$\gcd(60, 24) = 12/12$$

$$\implies 5x \equiv 1 \pmod 2$$
$$x \equiv 3 \pmod 2$$

1.1.2 b

$$\gcd(42, 30) = 6$$
$$7x \equiv 4 \pmod 5$$
$$x \equiv 2 \pmod 5$$

1.1.3 c

No solution because $\gcd(49, 25) = 1$ so equation cannot be reduced.

1.1.4 d

$$39 = 13 \times 3$$
$$52 = 13 \times 2^2$$
$$\gcd(39, 52) = 13 \nmid 14$$

1.1.5 e

$$\gcd(147, 98) = 49 \nmid 47$$

1.1.6 f

$$\gcd(39, 52) = 13$$
$$3x \equiv 2 \pmod 4$$
$$x \equiv 3$$

1.2 Q2

1.2.1 a

$$12x \equiv 7 \pmod{25}$$

Note that $12 \perp 25$

$$12k + 25l = 1$$
$$\implies k = -2, l = 1$$
$$\implies 12 \cdot (-2) \equiv 1 \pmod{25}$$
$$\implies 12 \cdot 23 \equiv 1 \pmod{25}$$
$$\implies 12 \cdot 23 \cdot 7 \equiv 7 \pmod{25}$$
$$\implies 12 \cdot 11 \equiv 7 \pmod{25}$$

1.2.2 b

$$35x \equiv 8 \pmod{12}$$
$$35 \perp 12$$
$$\implies 35 \cdot (-1) + 12 \cdot 3 = 1$$
$$\implies 35 \cdot (-1) \equiv 1 \pmod{12}$$
$$\implies 35 \cdot 11 \equiv 1 \pmod{12}$$
$$\implies 35 \cdot 88 \equiv 8 \pmod{12}$$
$$\implies 35 \cdot 4 \equiv 8 \pmod{12}$$

1.2.3 c

$$15x \equiv 9 \pmod{6}$$
$$15k + 6l = 1$$
$$15 = 6(2) + 3$$
$$6 = 3(2) + 0$$
$$\gcd(15, 6) = 3$$
$$5x \equiv 3 \pmod{2}$$
$$x \equiv 1 \pmod{2}$$
$$15(1) \equiv 9 \pmod{6}$$

1.2.4 d

$$42x \equiv 12 \pmod{30}$$
$$42k + 30l = \gcd(42, 30)$$
$$42 = 30(1) + 12$$
$$30 = 12(2) + 6$$
$$12 = 6(2) + 0$$
$$7x \equiv 2 \pmod{5}$$
$$2x \equiv 2 \pmod{5}$$
$$x \equiv 1 \pmod{5}$$
$$x \equiv 1 \pmod{30}$$

1.2.5 e

$$147x \equiv 49 \pmod{98}$$
$$1\bar{4}7 = \bar{4}9$$
$$\implies 49x \equiv 49 \pmod{98}$$
$$\implies x \equiv 1 \pmod{98}$$

1.2.6 f

$$39x \equiv 26 \pmod{52}$$
$$52 = 39(1) + 13$$
$$39 = 13(3) + 0$$
$$\implies \gcd(52, 39) = 13$$
$$\implies 3x \equiv 2 \pmod{4}$$
$$\implies x \equiv 2 \pmod{4}$$
$$\implies x \equiv 2 \pmod{52}$$

1.3 Q3

1.3.1 a

$$2x^2 \equiv 8 \pmod{10}$$
$$\implies 2x^2 - 8 = 10y$$

but $\gcd(2, 10) = 2$

$$\implies x^2 - 4 = 5y \in \langle 5 \rangle$$
$$\implies x^2 - 4 \equiv 0 \pmod{5}$$
$$\implies x^2 \equiv 4 \pmod{10}$$

1.3.2 b

$$1^2 \equiv 1 \pmod 5$$
$$2^2 \equiv 4 \pmod 5$$
$$3^2 \equiv 4 \pmod 5$$
$$4^2 \equiv 1 \pmod 5$$

1.4 Q4

1.4.1 a

$$6x^2 \equiv 9 \pmod{15} \implies 2x^2 \equiv 3 \pmod 5$$
$$\implies x = 2 \pmod 5$$

1.4.2 b

$$60x^2 \equiv 18 \pmod{24} \implies 10x^2 \equiv 3 \pmod 4$$
$$\implies 2x^2 \equiv 3 \pmod 4$$

$x \neq 2$ because $2 \times 2 \mid 4 \implies x^2 \equiv 0 \pmod 4$.

Likewise coefficient is 2 so for any n, $2n$ is either 2 or 0. No solution.

1.4.3 c

$$30x^2 \equiv 18 \pmod{24}$$
$$\implies 5x^2 \equiv 3 \pmod 4$$
$$\implies x^2 \equiv 3 \pmod 4$$

No solution.

1.4.4 d

$$4(x+1)^2 \equiv 14 \pmod{10}$$
$$\implies 4(x+1)^2 \equiv 4 \pmod{10}$$
$$x \equiv 0 \pmod{10}$$

1.4.5 e

$$4x^2 - 2x + 2 \equiv 0 \pmod 6$$
$$\implies 2x^2 - x + 1 \equiv 0 \pmod 3$$
$$\implies x = 2$$

1.4.6 f

$$3x^2 - 6x + 6 \equiv 0 \pmod{15}$$
$$\implies x^2 - 2x + 2 \equiv 0 \pmod 5$$
$$x = 3, 4 \pmod 5$$

1.5 Q5

1.5.1 a

$$x^4 \equiv 4 \pmod 6$$
$$x^4 \equiv (x^2)^2 \pmod 6$$

Let $y = x^2$

$$y^2 \equiv 4 \pmod 6$$
$$y \equiv 2 \pmod 6 \text{ or } 4 \pmod 6$$
$$x^2 = 2 \pmod 6 \text{ or } 4 \pmod 6$$
$$\implies x \equiv 2 \pmod 6$$

1.5.2 b

$$2(x-1)^4 \equiv 0 \pmod 8$$
$$\implies (x-1)^4 \equiv 0 \pmod 4$$
$$\implies (x-1)^2 \equiv 0, 2 \pmod 4$$

Let $y = x - 1$

$$\implies y^2 \equiv 0 \pmod 4$$
$$\implies y \equiv 0, 2 \pmod 4$$
$$\implies x \equiv 1, 3 \pmod 4$$

1.5.3 c

$$x^3 + 3x^2 + 3x + 1 \equiv 0 \pmod 8$$
$$(x+1)^3 \equiv 0 \pmod 8$$
$$\implies x + 1 \equiv 0, 2, 4, 6$$

(any factor of 2 since $2^3 \equiv 8 \equiv 0$)

$$\implies x \equiv 7, 1, 3, 5$$

1.5.4 d

$$x^4 + 2x^2 + 1 \equiv 4 \pmod 5$$
$$\implies (x^2+1) \equiv 4 \pmod 5$$
$$\implies x^2 + 1 \equiv 2, 3 \pmod 5$$
$$\implies x^2 \equiv 1, 2 \pmod 5$$
$$\implies x \equiv 1, 4 \pmod 5$$

1.6 Q6

1.6.1 a

$$14x + 15y = 11$$

Note that $14(-1) + 15(1) = 1$, thus

$$14(-1 \cdot 11) + 15(1 \cdot 11) = 11$$
$$x = -11, y = 11$$

1.6.2 b

$$4(-1) + 5(1) = 1$$

1.6.3 c

$21x + 10y$ is an ideal in $\mathbb{Z}$, with a least value t, such that $J = \langle t \rangle$ and therefore if $q \in J$ then $t \mid q$.

But the least value $t = 11$ and $11 \nmid 9$. So there is no solution.

1.6.4 d

$$30x^2 + 24y = 18$$
$$30x^2 \equiv 18 \pmod{24}$$
$$5x^2 \equiv 3 \pmod 4$$
$$x^2 \equiv 3 \pmod 4$$

2 B. Solving Sets of Congruences

2.1 Q1

2.1.1 a

$$x \equiv 7 \pmod 8 \qquad x \equiv 11 \pmod{12}$$
$$\gcd(8, 12) = 4$$
$$7 \pmod 4 \equiv 3 \equiv 11 \pmod 4$$

Solution exists.

$$\operatorname{lcm}(8, 12) = 8 \times 12/4 = 24$$

$$\begin{aligned} x &= 8q + 7 \\ \implies 8q + 7 &\equiv 11 \pmod{12} \\ 8q &\equiv 4 \pmod{12} \end{aligned}$$

$$q \equiv 5 \pmod{12}$$

$$\begin{aligned} x &= 8q + 7 \\ &= 8(12r + 5) + 7 \\ &= 96r + 47 \end{aligned}$$

$$\begin{aligned} x &\equiv 47 \pmod{24} \\ &\equiv 23 \pmod{24} \end{aligned}$$

2.1.2 b

$$x \equiv 12 \pmod{18} \qquad x \equiv 30 \pmod{45}$$
$$\gcd(18, 45) = 9$$
$$\operatorname{lcm}(18, 45) = 18 \times 45/9 = 90$$

$$x = 18q + 12$$

$$\begin{aligned} 18q + 12 &\equiv 30 \pmod{45} \\ 18q &\equiv 18 \pmod{45} \\ q &\equiv 1 \pmod{45} \end{aligned}$$

$$\begin{aligned} x &= 18(45r + 1) + 12 \\ &= 18 \times 45r + 30 \\ x &\equiv 30 \pmod{90} \end{aligned}$$

2.1.3 c

$$\begin{aligned} \gcd(15, 14) &= 1 \\ \operatorname{lcm}(15, 14) &= 210 \end{aligned}$$

$$\begin{aligned} 15q + 8 &\equiv 11 \pmod{14} \\ 15q &\equiv 3 \pmod{14} \\ q &\equiv 3 \pmod{14} \end{aligned}$$

$$x \equiv 53 \pmod{210}$$

2.2 Q2

2.2.1 a

$$10x \equiv 2 \pmod{12} \qquad 6x \equiv 14 \pmod{20}$$
$$\gcd(10, 12) = 2$$

$$\begin{aligned} 5x &\equiv 1 \pmod 6 \\ x &\equiv 5 \pmod 6 \end{aligned}$$

$$\begin{aligned} 6x &\equiv 14 \pmod{20} \\ 3x &\equiv 7 \pmod{10} && \gcd(6, 20) = 2 \\ x &\equiv 9 \pmod{10} \end{aligned}$$

$$\begin{aligned} \gcd(6, 10) &= 2 \\ 5 \pmod 2 = 1 &= 9 \pmod 2 \end{aligned}$$

has a solution.

$$\mathrm{lcm}(6, 10) = 30$$

solution is modulo 30.

$$x = 6q + 5$$

$$\begin{aligned} 6q + 5 &\equiv 9 \pmod{10} \\ 6q &\equiv 4 \pmod{10} \\ 3q &\equiv 2 \pmod 5 && [\text{from } \frac{a}{d}x \equiv \frac{b}{d} \pmod{\frac{c}{d}}] \\ q &\equiv 4 \pmod 5 \end{aligned}$$

$$q = 5r + 4$$

$$x = 6(5r + 4) + 5 = 30r + 29$$
$$x \equiv 29 \pmod{30}$$

2.2.2 b

$$4x \equiv 2 \pmod 6$$
$$9x \equiv 3 \pmod{12}$$

$$\gcd(4, 6) = 2$$
$$\therefore 4x \equiv 2 \pmod 6 \implies 2x \equiv 1 \pmod 3$$
$$x \equiv 2 \pmod 3$$

$$\gcd(9, 12) = 3$$
$$\therefore 9x \equiv 3 \pmod{12} \implies 3x \equiv 1 \pmod 4$$
$$x \equiv 3 \pmod 4$$
$$\gcd(3, 4) = 1$$
$$2 \pmod 1 = 0 \neq 3 \pmod 1$$

has no solution.

2.2.3 c

$$\begin{aligned} 6x &\equiv 2 \pmod{8} \\ 10x &\equiv 2 \pmod{12} \end{aligned}$$

$$\begin{aligned} \gcd(6,8) = 2 &\implies 3x \equiv 1 \pmod{4} \\ \gcd(10,12) = 2 &\implies 5x \equiv 1 \pmod{6} \end{aligned}$$

$$\implies x \equiv 3 \pmod{4}$$

$$\implies x \equiv 5 \pmod{6}$$

$$\gcd(4,6) = 2$$

$$3 \pmod{2} = 1 = 5 \pmod{2}$$

has a solution.

$$\operatorname{lcm}(4,6) = 12$$

$$x = 4q + 3$$

$$\begin{aligned} 4q + 3 &\equiv 5 \pmod{6} \\ 4q &\equiv 2 \pmod{6} \end{aligned}$$

$$\gcd(4,6) = 2$$

$$\implies 2q \equiv 1 \pmod{3}$$

$$q \equiv 2 \pmod{3}$$

$$q = 3r + 2$$

$$\begin{aligned} x &= 4(3r + 2) + 3 \\ &= 12r + 11 \\ x &\equiv 11 \pmod{12} \end{aligned}$$

2.3 Q3

See attached file `ch23b3.pdf`.

2.4 Q4

2.4.1 a

$$x \equiv 2 \pmod{3} \qquad x \equiv 3 \pmod{4} \qquad x \equiv 1 \pmod{5} \qquad x \equiv 4 \pmod{7}$$

All modulo are coprime so there is a solution.

$$\operatorname{lcm}(3,4,5,7) = 3 \times 4 \times 5 \times 7 = 420$$

recursively find a solution for each equation.

$$x = 3q + 2$$

$$\begin{aligned} 3q + 2 &\equiv 2 \pmod{4} \\ 3q &\equiv 1 \pmod{4} \\ q &\equiv 3 \pmod{4} \end{aligned}$$

$$q = 4r + 3$$

$$\begin{aligned} x &= 3(4r + 3) + 2 \\ &= 12r + 11 \\ x &\equiv 11 \pmod{12} \end{aligned}$$

but also $x \equiv 1 \pmod 5$

$$x = 12q' + 11$$

$$\begin{aligned}
12q' + 11 &\equiv 1 \pmod 5 \\
12q' &\equiv 0 \pmod 5 \\
q' &\equiv 0 \pmod 5 \\
q' &= 5r' \\
\implies x &= 11
\end{aligned}$$

this also fits the equation $x \equiv 4 \pmod 7$.

2.4.2 b

$$6x \equiv 4 \pmod 8 \qquad 10x \equiv 4 \pmod{12} \qquad 3x \equiv 8 \pmod{10}$$

$$\begin{gathered}
6x \equiv 4 \pmod 8 \implies 3x \equiv 2 \pmod 4 \implies x \equiv 2 \pmod 4 \\
10x \equiv 4 \pmod{12} \implies 5x \equiv 2 \pmod 6 \implies x \equiv 4 \pmod 6 \\
3x \equiv 8 \pmod{10} \implies x \equiv 6 \pmod{10}
\end{gathered}$$

$$\begin{gathered}
\gcd(4,6) = 2 \qquad 2 \pmod 2 = 0 = 4 \pmod 2 \\
\gcd(4,10) = 2 \qquad 2 \pmod 2 = 0 = 6 \pmod 2 \\
\gcd(6,10) = 2 \qquad 2 \pmod 2 = 0 = 6 \pmod 2
\end{gathered}$$

thus there is a solution x.

$$t = \operatorname{lcm}(4,6,10) = \operatorname{lcm}(\operatorname{lcm}(4,6),10) = \operatorname{lcm}(12,10) = 60$$

Solution is modulo $t = 60$.

$$\begin{aligned}
x &\equiv 2 \pmod 4 \\
x &\equiv 4 \pmod 6 \\
x &\equiv 6 \pmod{10}
\end{aligned}$$

$$x = 4q + 2$$

$$\begin{aligned}
4q + 2 &\equiv 4 \pmod 6 \\
4q &\equiv 2 \pmod 6 \\
q &\equiv 2 \pmod 6 \\
q &= 6r + 2
\end{aligned}$$

$$\begin{aligned}
x &= 4(6r + 2) + 2 \\
&= 24r + 10
\end{aligned}$$

$$\begin{aligned}
24r + 10 &\equiv 6 \pmod{10} \\
24r &\equiv -4 \pmod{10} \\
&\equiv 6 \pmod{10} \\
12r &\equiv 3 \pmod 5
\end{aligned}$$

$$\begin{aligned}
r &\equiv 4 \pmod 5 \\
r &= 5s + 4
\end{aligned}$$

$$\begin{aligned}
x &= 24(5s + 4) + 10 \\
&= 120s + 106 \\
x &= 106 \pmod{60} \\
&= 46 \pmod{60}
\end{aligned}$$

2.5 Q5

2.5.1 a

$$4x + 6y = 2 \implies 4x \equiv 2 \pmod 6$$
$$9x + 12y = 3 \implies 9x \equiv 3 \pmod{12}$$

$$4x \equiv 2 \pmod 6 \implies 2x \equiv 1 \pmod 3 \implies x \equiv 2 \pmod 3$$
$$9x \equiv 3 \pmod{12} \implies 3x \equiv 1 \pmod 4 \implies x \equiv 3 \pmod 4$$

$$x = 3q + 2$$

$$\begin{aligned} 3q + 2 &\equiv 3 \pmod 4 \\ 3q &\equiv 1 \pmod 4 \\ q &\equiv 3 \pmod 4 \end{aligned}$$

$$q = 4r + 3$$

$$\begin{aligned} x &= 3(4r + 3) + 2 \\ &= 12r + 11 \end{aligned}$$

$$t = \text{lcm}(6, 12) = 12$$

$$x \equiv 11 \pmod{12}$$

$$\begin{aligned} x &= 12s + 11 \\ &= -1 \\ y &= 1 \end{aligned}$$

2.5.2 b

$$\begin{aligned} 3x + 4y &= 2 \\ 5x + 6y &= 2 \\ 3x + 10y &= 8 \end{aligned}$$

$$\begin{aligned} 3x &\equiv 2 \pmod 4 \\ 5x &\equiv 2 \pmod 6 \\ 3x &\equiv 8 \pmod 10 \end{aligned}$$

From 23B4b, $x \equiv 46 \pmod{60}$

$$\begin{aligned} 6y &\equiv 2 \pmod 5 \\ y &\equiv 2 \pmod 5 \end{aligned}$$

$$\begin{aligned} 10y &\equiv 8 \pmod 3 \\ y &\equiv 2 \pmod 2 \end{aligned}$$

$$\begin{aligned} 4y &\equiv 2 \pmod 3 \\ y &\equiv 2 \pmod 3 \end{aligned}$$

$$t = \text{lcm}(3, 5) = 15$$

$$y = 5q + 2$$

$$\begin{aligned} 5q + 2 &\equiv 2 \pmod 3 \\ 2q &\equiv 0 \pmod 3 \end{aligned}$$

$$y = 2$$

but $x \equiv 46 \pmod{60}$

$$5(46) + 6(2) \equiv 50 + 12 \equiv 2 \pmod{60}$$
$$3(46) + 10(2) \equiv 18 + 20 \equiv 38 \not\equiv 8 \pmod{60}$$

so there's no solution.

3 C. Elementary Properties of Congruence

3.1 Q1

If $a \equiv b \pmod n$ and $b \equiv c \pmod n$, then $a \equiv c mod n$.

$$b - a = nq_1$$

$$b = nq_1 + a$$

$$b - c = nq_2$$

$$(nq_1 + a) - c = nq_2$$

$$\begin{aligned} a - c &= nq_2 - nq_2 \\ &= n(q_2 - q_1) \end{aligned}$$

$$\implies a \equiv c \pmod n$$

3.2 Q2

If $a \equiv b \pmod n$, then $a + c \equiv b + c \pmod n$.

$$a - b = nq$$

$$c - c = 0$$

$$a - b + (c - c) = nq$$

$$(a + c) - (b + c) = nq$$

$$\implies a + c \equiv b + c \pmod n$$

3.3 Q3

If $a \equiv b \pmod n$, then $ac \equiv bc \pmod n$.

$$a - b = nq$$

$$c(a - b) = cnq$$

$$ac - ab = n(qc)$$

$$ac \equiv bc \pmod n$$

3.4 Q4

$a \equiv b \pmod 1$.

$$a \equiv b \pmod n \iff n \mid (a-b)$$
$$1 \mid (a-b) \implies a \equiv b \pmod 1$$

3.5 Q5

If $ab \equiv 0 \pmod p$, where p is a prime, then $a \equiv 0 \pmod p$ or $b \equiv 0 \pmod p$.

$$\begin{aligned} ab \equiv 0 \pmod p &\implies ab = np \\ &\implies p \mid ab \end{aligned}$$

but p is prime so either $p \mid a$ or $p \mid b$.

If $p \mid a$ then $a \equiv 0 \pmod p$.

If $p \mid b$ then $b \equiv 0 \pmod p$.

3.6 Q6

If $a^2 \equiv b^2 \pmod p$, where p is a prime, then $a \equiv \pm b \pmod p$.

$$a^2 \equiv b^2 \pmod p$$
$$a^2 - b^2 = np$$
$$(a+b)(a-b) = np$$

Since p is prime then either $p \mid (a+b)$

If $p \mid (a+b)$ then $a \equiv -b \pmod p$.

If $p \mid (a-b)$ then $a \equiv b \pmod p$.

3.7 Q7

If $a \equiv b \pmod m$, then $a + km \equiv b \pmod m$, for any integer k. In particular, $a + km \equiv a \pmod m$.

$$a \equiv b \pmod m \implies a - b = mq_1$$

$$\begin{aligned} \implies (a+km) - b &= mq_1 + km \\ &= m(q_1 + k) \end{aligned}$$

$$\implies a + km \equiv b \pmod m$$
$$a - a = 0 = 0m \implies a \equiv a \pmod m$$
$$\implies a + km \equiv a \pmod m$$

3.8 Q8

If $ac \equiv bc \pmod n$ and $\gcd(c, n) \equiv 1$, then $a \equiv b \pmod n$.

\$$ac \equiv bc \pmod n \implies ac - bc = c(a-b) = nq$

So $n \mid c(a-b)$ but $\gcd(c,n) = 1 \implies n \mid (a-b) \implies a \equiv b \pmod n$.

3.9 Q9

If $a \equiv b \pmod{n}$, then $a \equiv b \pmod{m}$ for any m which is a factor of n.

$$n = rm$$

$$\begin{aligned} a - b &= nq = (rm)q \\ &= m(rq) \\ \implies& a \equiv b \pmod{m} \end{aligned}$$

4 D. Further Properties of Congruence

4.1 Q1

If $ac \equiv bc \pmod{n}$, and $\gcd(c, n) \equiv d$, then $a \equiv b \pmod{n}/d$.

$$\begin{gathered} ac - bc = nq \\ \gcd(c, n) = d \implies c = c_1 d, n = n_1 d \\ c_1 d(a - b) = n_1 dq \\ c_1(a - b) = n_1 q \end{gathered}$$

but $\gcd(c_1, n_1) = 1$ so $n_1 \nmid c_1 \implies n_1 \mid (a - b)$.

$$\begin{gathered} \implies a - b = n_1 k \\ n = n_1 d \implies n_1 = \frac{n}{d} \\ a - b = (\frac{n}{d})k \\ \implies a \equiv b \pmod{\frac{n}{d}} \end{gathered}$$

4.2 Q2

If $a \equiv b \pmod{n}$, then $\gcd(a, n) = \gcd(b, n)$.

$$\begin{gathered} a_1 d \equiv b \pmod{n_1 d} \\ a_1 d - b = n_1 dy \end{gathered}$$

$$\begin{gathered} b = a_1 d - n_1 dy \\ = d(a_1 - n_1 y) \\ \implies d \mid b \\ \gcd(a_1, n_1) = 1 \implies \gcd(b, n_1) = 1 \\ \implies \gcd(b, n) = d \end{gathered}$$

4.3 Q3

If $a \equiv b \pmod{p}$ for every prime p, then $a \equiv b$.

Assume $a \neq b$ and

$$\begin{aligned} a &= p_1 \cdots p_i p_{i+1} \cdots p_n \\ b &= p_1 \cdots p_i q_i \cdots q_m \end{aligned}$$

where $\gcd(a, b) = p_1 \cdots p_i$.

If $p \in \{p_1, \ldots, p_i\}$ then $p \mid a$ and $p \mid b$ and $a \pmod{p} \equiv 0 \equiv b \pmod{p}$.

If $p \in \{q_1, \ldots, q_m\}$ where $p \neq p_j$ such that $1 \leq j \leq n$ then $p \nmid a$ and $p \nmid b$ so $a \not\equiv b \pmod{p}$.

Likewise for $p = p_j : i \leq j \leq n$.

Therefore $a \equiv b \pmod{p}$ for all prime p implies they both share the exact same prime factors, and $a = b$.

4.4 Q4

If $a \equiv b \pmod{n}$*, then* $a^m \equiv bm \pmod{n}$ *for every positive integer* m*.*

$$(a-b) = nq$$

$$\begin{aligned}
a &= b + nq \\
a^m &= (b+nq)^m \\
&= b^m + \binom{1}{m} b^{m-1}(nq)^1 + \cdots + \binom{m-1}{m} b(nq)^{m-1} + (nq)^m
\end{aligned}$$

$$\implies a^m \equiv b^m \pmod{n}$$

4.5 Q5

If $a \equiv b \pmod{m}$ *and* $a \equiv b \pmod{n}$ *where* $\gcd(m,n) = 1$*, then* $a \equiv b \pmod{m} n$*.*

$$a - b = mx = ny$$

$$\implies n \mid (a-b) \text{ and } m \mid (a-b)$$

but $\gcd(m,n) = 1 \implies mn \mid (a-b)$

$$a \equiv b \mid mn$$

4.6 Q6

If $ab \equiv 1 \pmod{c}$*,* $ac \equiv 1 \pmod{b}$ *and* $bc \equiv 1 \pmod{a}$*, then* $ab + bc + ac \equiv 1 \pmod{a} bc$*. (Assume* $a, c > 0$*.)*

$$\begin{aligned}
ab - 1 &= cq_1 \\
ac - 1 &= bq_2 \\
bc - 1 &= aq_3
\end{aligned}$$

$$(ab-1)(ac-1)(bc-1) = (abc)(q_1 q_2 q_3)$$

$$(a^2bc - ab - ac + 1)(bc - 1) = a^2b^2c^2 - ab^2c - abc^2 + bc - a^2bc + ab + ac - 1$$

$$(a^2b^2c^2 - ab^2c - abc^2 - a^2bc) + bc + ab + ac \equiv 1 \pmod{abc}$$

$$\implies ab + bc + ac \equiv 1 \pmod{abc}$$

4.7 Q7

If $a^2 \equiv 1 \pmod{2}$*, then* $a^2 \equiv 1 \pmod{4}$*.*

$$a^2 - 1 \mid 2 \implies a^2 - 1 \mid 4$$

4.8 Q8

If $a \equiv b \pmod{n}$*, then* $a^2 + b^2 \equiv 2ab \pmod{n^2}$*, and conversely.*

$$a - b = nq$$

$$(a-b)^2 = a^2 - 2ab + b^2 = n^2q^2$$

$$\implies a^2 + b^2 = 2ab \pmod{n^2}$$

4.9 Q9

If $a \equiv 1 \pmod{m}$, then a and m are relatively prime.

$$a - 1 = mq$$
$$a - mq = 1$$

From 22c1 this implies $\gcd(a, b) = 1$.

5 E. Consequences of Fermat's Theorem

5.1 Q1

If p is a prime, find $\phi(p)$. Use this to deduce Fermat's theorem from Euler's theorem.

V_p is the set of all invertible elements in $\mathbb{Z}_p$.

V_p is thus a group with respect to multiplication.

Let $\bar{a} \in V_p$

$$\bar{s}\bar{a} = 1$$
$$\implies sa - 1 \in \langle n \rangle$$
$$\implies sa - 1 = tn$$
$$sa - tn = 1$$

So invertible elements a in $\mathbb{Z}_n \implies a$ and n are relatively prime, and vice versa.

All cosets of $\langle n \rangle$ (except $\langle n \rangle$ itself) have a gcd of 1.

$$\mathbb{Z}_p^* = \{\bar{1}, \bar{2}, \dots, p \bar{-} 1\}$$

So it follows that

$$\phi(p) = p - 1$$

5.2 Q2

If $p > 2$ is a prime and $a \not\equiv 0 \pmod{p}$, then

$$a(p-1)/2 \equiv \pm 1 \pmod{p}$$

$$a^{p-1} = 1 \pmod{p}$$
$$\implies a^{\frac{p-1}{2} \cdot 2} \equiv x^2 \equiv 1 \pmod{p}$$
$$x^2 \equiv 1 \pmod{p} \implies x \in \{-1, 1\}$$

$$a^{(p-1)/2} \equiv \pm 1 \pmod{p}$$

5.3 Q3

5.4 a

Let p be a prime > 2. If $p \equiv 3 \pmod 4$, then $(p-1)/2$ is odd.

$$
\begin{aligned}
p &\equiv 3 \pmod 4 \\
p - 1 &\equiv 2 \pmod 4 \\
\implies& 4 \mid [(p-1) - 2] \\
\implies& (p-1) - 2 = 4q \\
\implies& \frac{p-1}{2} - 1 = 2q \\
\implies& \frac{p-1}{2} \equiv 1 \pmod 2
\end{aligned}
$$

thus $\frac{p-1}{2}$ is odd.

5.5 b

Let $p > 2$ be a prime such that $p \equiv 3 \pmod 4$. Then there is no solution to the congruence $x^2 + 1 \equiv 0 \pmod p$.

$$
\begin{aligned}
x^2 &\equiv -1 \pmod p \\
x^{2 \cdot \frac{p-1}{2}} &\equiv (-1)^{\frac{p-1}{2}} \pmod p
\end{aligned}
$$

By Fermat's theorem

$$x^{p-1} \equiv 1 \pmod p$$

but since $(p-1)/2$ is odd, then $(-1)^{\frac{p-1}{2}} = -1$ so there is no solution to the congruence $x^2 + 1 \equiv 0 \pmod p$.

5.6 Q4

Let p and q be distinct primes. Then $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.

$$
\begin{aligned}
p^{q-1} &\equiv 1 \pmod q \\
q^{p-1} &\equiv 1 \pmod p
\end{aligned}
$$

$$
\begin{aligned}
p^{q-1} - 1 &= qn \\
q^{p-1} - 1 &= pm
\end{aligned}
$$

$$
\begin{aligned}
(p^{q-1} - 1)(q^{p-1} - 1) &= p^{q-1}q^{p-1} - p^{q-1} - q^{p-1} + 1 \\
&= (pq)(mn) \\
\implies p^{q-1} + q^{p-1} &\equiv 1 \pmod{pq}
\end{aligned}
$$

5.7 Q5

Let p be a prime.

5.7.1 a

If, $(p-1) \mid m$, then $a^m \equiv 1 \pmod p$ provided that $p \nmid a$.

$$
\begin{aligned}
(p-1) \mid m &\implies m = q(p-1) \\
a^m &= a^{q(p-1)} = (a^{p-1})^q
\end{aligned}
$$

$$
\begin{aligned}
a^{p-1} &\equiv 1 \pmod p \\
(a^{p-1})^q &\equiv 1^q \pmod p \\
a^m &\equiv 1 \pmod p
\end{aligned}
$$

5.7.2 b

If, $(p-1) \mid m$, then $a^m + 1 \equiv a \pmod{pq}$ for all integers a.

If $p \mid a$ then $a^x \equiv 0 \pmod{p}$ for any x so $a^{m+1} \equiv 0 \equiv a \pmod{p}$.

Otherwise $p \nmid a$ so $a^m \equiv 1 \pmod{p} \implies a^{m+1} \equiv a \pmod{p}$

5.8 Q6

Let p and q be distinct primes.

5.8.1 a

If $(p-1) \mid m$ and $(q-1) \mid m$, then $a^m \equiv 1 \pmod{pq}$ for any a such that $p \nmid a$ and $q \nmid a$.

$$a^m \equiv 1 \pmod{p}$$
$$a^m \equiv 1 \pmod{q}$$

$$\gcd(p, q) = 1 \implies p \text{ and } q \text{ share no divisors}$$

but $p \mid (a^m - 1)$ and $q \mid (a^m - 1) \implies pq \mid (a^m - 1)$

$$a^m - 1 \equiv 0 \pmod{pq}$$
$$a^m \equiv 1 \pmod{pq}$$

5.8.2 b

If $(p-1) \mid m$ and $(q-1) \mid m$, then $a^m + 1 \equiv a \pmod{pq}$ for integers a.

Let $p \mid a$ then $a \equiv 0 \pmod{p}$ and $a \equiv 1 \pmod{q}$.

$$\implies a^m(a-1) = (pq)(mn)$$
$$\implies a^{m+1} - a = (pq)(mn)$$
$$\implies a^{m+1} \equiv a \pmod{pq}$$

Likewise if $q \mid a$.

If both $p \mid a$ and $q \mid a$ then $pq \mid a$ and so $a \equiv 0 \pmod{pq}$ and $a^{m+1} \equiv 0 \pmod{pq}$.

Otherwise $p \nmid a$ and $q \nmid a$ so

$$a^m \equiv 1 \pmod{pq}$$
$$\implies a^{m+1} \equiv a \pmod{pq}$$

5.9 Q7

$$\forall i \in \{1, \ldots, n\}, (p_i - 1) \mid m$$
$$\implies a^{m+1} \equiv a \pmod{\prod_{i=1}^{n} p_i}$$

5.10 Q8

5.10.1 a

$$p = 7 \qquad q = 19 \qquad m = 18$$
$$(7-1) \mid 18 \qquad (19-1) \mid 18$$
$$\implies a^{18+1} = a \pmod{7 \times 19}$$
$$a^19 \equiv a \pmod{133}$$

5.10.2 b

$$a \in \langle 2\rangle, \langle 3\rangle, \langle 11\rangle$$

$$m = 10$$
$$q_1 = 2 \qquad q_2 = 3 \qquad q_3 = 11$$
$$\implies a^{10} = 1 \pmod{66}$$

5.10.3 c

$$q_1 = 5 \qquad q_2 = 17 \qquad q_3 = 3$$
$$m = 12$$
$$(5-1) \mid 12 \qquad (7-1) \mid 12 \qquad (3-1) \mid 12$$
$$\implies a^{13} \equiv a \pmod{105}$$

5.10.4 d

$$q_1 = 7 \qquad q_2 = 13 \qquad q_3 = 17$$
$$m = 48$$
$$(7-1) \mid 48 \qquad (13-1) \mid 48 \qquad (17-1) \mid 48$$
$$\implies a^{49} \equiv a \pmod{1457}$$

5.11 Q9

5.11.1 a

$$Q = \{2, 3, 5, 7\}$$
$$8^{38} = 8^{2\times 19} = (8^2)^1 9$$
$$\forall q \in Q, (q-1) \mid (19-1)$$
$$\implies a^{18+1} \equiv a \pmod{210}$$

where $a = 8^2$

$$\implies x = 8^2$$

5.11.2 b

$$p = 7 \qquad q = 19$$
$$7^{57} = (7^3)^1 9$$
$$m = 18$$
$$(7-1) \mid m \qquad (19-1) \mid m$$
$$a^{m+1} \equiv a \pmod{pq}$$
$$(7^3)^{18+1} \equiv 7^3 \pmod{7 \times 19}$$
$$x = 7^3$$

5.11.3 c

$$Q = \{2, 3, 11\}$$
$$72 = 2^3 3^2$$

73 is prime so $m \neq 73$ since there is no $(p-1) \mid m : p \in Q$.

Since $(p-1) \mid m$ then $m = 72$, if $p = 11$ then $(11-1) \nmid 72$ so $m \neq 72$.

Since 5 is a prime, and there are no factorizations of 73, this has no solution.

6 F. Consequences of Euler's Theorem

6.1 Q1

If $\gcd(a,n)=1$, *the solution modulo* n *of* $ax \equiv b \pmod{n}$ *is* $x \equiv a^{\phi(n)-1}b \pmod{n}$.

$\gcd(a,n)=1 \implies ax \equiv b \pmod{n}$ has a solution because it is equivalent to $\bar{a}\bar{x}=\bar{b}$ in $\mathbb{Z}_n$. By condition 4, $\bar{a}$ has a multiplicative inverse in $\mathbb{Z}_n$.

$$\bar{x} = \bar{a}^{-1}\bar{b}$$

$$\gcd(a,n)=1 \implies 1 - sa = tn \in \langle n \rangle \implies \bar{1} = \overline{sa}$$

Let V_n be the set of invertible elements in $\mathbb{Z}_n$. This is a group since inverses and products remain in V_n. From condition 4, $\bar{1}=\overline{sa} \implies 1-sa \in \langle n \rangle \implies \gcd(a,n)=1$. So $|V_n| = \phi(n)$ which is the number of relatively prime elements in V_n.

Since V_n is a group, the identity is $\bar{1}$ and for any $\bar{a} \in V_n$, $\bar{a}^{\phi(n)} = \bar{1}$. But we have $\bar{x} = \bar{a}^{-1}\bar{b}$ and it follows that

$$\begin{aligned}
\bar{a}^{-1} &= \bar{a}^{\phi(n)}\bar{a}^{-1} \\
&= \overline{a^{\phi(n)-1}} \\
\bar{x} &= \overline{a^{\phi(n)-1}}\bar{b} \\
\implies x &= a^{\phi(n)-1}b \pmod{n}
\end{aligned}$$

6.2 Q2

If $\gcd(a,n)=1$, *then* $a^{m\phi(n)} \equiv 1 \pmod{n}$ *for all values of* m.

$$\begin{aligned}
\gcd(a,n)=1 \implies a^{\phi(n)} &\equiv 1 \pmod{n} \\
(a^{\phi(n)})^m &\equiv 1^m \pmod{n} \\
a^{m\phi(n)} &\equiv 1 \pmod{n}
\end{aligned}$$

6.3 Q3

If $\gcd(m,n)=\gcd(a,mn)=1$, *then* $a^{\phi(m)\phi(n)} \equiv 1 \pmod{mn}$.

$$\begin{aligned}
a^{k\phi(m)} &\equiv 1 \pmod{m} \\
a^{l\phi(n)} &\equiv 1 \pmod{n}
\end{aligned}$$

$$\begin{aligned}
a^{\phi(m)\phi(n)} &\equiv 1 \pmod{m} \\
a^{\phi(m)\phi(n)} &\equiv 1 \pmod{n}
\end{aligned}$$

Since $\gcd(m,n)=1$, then by theorem 4 $t = \operatorname{lcm}(m,n) = mn$.

$$\begin{aligned}
m \mid (a^{\phi(m)\phi(n)}-1) \text{ and } n \mid (a^{\phi(m)\phi(n)}-1) &\iff t \mid (a^{\phi(m)\phi(n)}-1) \\
&\implies a^{\phi(m)\phi(n)} \equiv 1 \pmod{mn}
\end{aligned}$$

6.4 Q4

If p *is a prime,* $\phi(p^n) = p^n - p^{n-1} = p^{n-1}(p-1)$.

HINT: For any integer a, a and p^n have a common divisor $\neq \pm 1$ iff a is a multiple of p. There are exactly p^{n-1} multiples of p between 1 and p^n.

p is a prime and the only possible values for $\gcd(a,p^n)$ are $p, p^2, \ldots, p^n$.

Therefore $p \mid a$ and a is a multiple of p.

There are p^{n-1} multiples of p between 1 and p^n because there are p^{n-1} values in the sequence

$$p, 2p, 3p, \ldots, (p^{n-1})p$$

Therefore $\phi(p^n)$ is equal to the total number of values minus the total number of multiples of p (the only possible values that divide a).

$$\begin{aligned}\phi(p^n) &= p^n - p^{n-1}\\ &= p^{n-1}(p-1)\end{aligned}$$

6.5 Q5

For every $a \not\equiv 0 \pmod{p}$, $a^{p^n(p-1)}$ (? - malformed question), where p is a prime.

$$\begin{aligned}a \not\equiv 0 \pmod{p} &\implies \gcd(a,p) = 1\\ a^{\phi(p^n)} \equiv 1 \pmod{pn} &\implies \gcd(a,p^n) = 1\end{aligned}$$

but $\phi(p^n) = p^{n-1}(p-1)$ so $a^{\phi(p^n)} = a^{p^{n-1}(p-1)}$ but $a^{\phi(p^n)} \equiv 1 \pmod{p^n}$ so $a^{p^{n-1}(p-1)} \equiv 1 \pmod{p^n}$ and so also $(a^{p^{n-1}(p-1)})^p \equiv 1^p \pmod{p^n}$ or

$$a^{p^n(p-1)} \equiv 1 \pmod{p^n}$$

6.6 Q6

Under the conditions of part 3, if t is a common multiple of $\phi(m)$ and $\phi(n)$, then $a^t \equiv 1 \pmod{mn}$. Generalize to three integers l, m, and n.

$$\gcd(m,n) = \gcd(a,mn) = 1, \qquad a^{\phi(m)\phi(n)} \equiv 1 \pmod{mn}$$

$$\begin{aligned}\phi(mn) &= \phi(m)\phi(n)\\ \gcd(\phi(m),\phi(n)) \cdot \mathrm{lcm}(\phi(m),\phi(n)) &= \phi(m)\phi(n)\\ t = \mathrm{lcm}(\phi(m),\phi(n)) &= \frac{\phi(m)\phi(n)}{\gcd(\phi(m),\phi(n)}\end{aligned}$$

$$\begin{aligned}a^t &\equiv a^{\frac{\phi(m)\phi(n)}{\gcd(\phi(m),\phi(n))}} \equiv (a^{\phi(m)\phi(n)})^{\frac{1}{\gcd(\phi(m),\phi(n))}}\\ &\equiv 1 \pmod{mn}\end{aligned}$$

Likewise for l, m, n because $\gcd(\phi(l),\phi(m),\phi(n)) = \gcd(\phi(l),\gcd(\phi(m),\phi(n)))$ and the same for lcm.

6.7 Q7

6.7.1 a

$$180 = 2^2 3^2 5$$

$$\begin{aligned}\phi(180) &= \phi(2^2)\phi(3^2)\phi(5)\\ &= 2^{2-1}(2-1)3^{2-1}(3-1)(5-1)\\ &= (2)(3 \times 2)(4) = (2)(6)(4)\end{aligned}$$

Note $\gcd(2^2 3^2, 5) = 1$

$$\begin{aligned}a^{\mathrm{lcm}(\phi(2^2 3^2),\phi(5))} &\equiv 1 \pmod{180}\\ a^{\mathrm{lcm}(12,4)=12} &\equiv 1 \pmod{180}\end{aligned}$$

6.7.2 b

$$a^42 \equiv 1 \pmod{1764}$$
$$1764 = 2^2 3^2 7^2$$
$$\gcd(2^2, 3^2, 7^2) = 1$$
$$\operatorname{lcm}(\phi(2^2), \phi(3^2), \phi(7^2)) = 42$$
$$a^42 \equiv 1 \pmod{1764}$$

6.7.3 c

$$1800 = 2^3 3^2 5^2$$
$$\gcd(2^3, 3^2, 5^2) = 1$$
$$\operatorname{lcm}(\phi(2^3), \phi(3^2), \phi(5^2)) = 60$$
$$a^{60} \equiv 1 \pmod{1800}$$

6.8 Q8

If $\gcd(m, n) = l$*, prove that* $n^{\phi(m)} + m^{\phi(n)} \equiv 1 \pmod{mn}$*.*

$$n^{\phi(m)} \equiv 1 \pmod{m} \implies n^{\phi(m)} - 1 = mq_1$$
$$m^{\phi(n)} \equiv 1 \pmod{n} \implies m^{\phi(n)} - 1 = nq_2$$

$$\begin{aligned}(n^{\phi(m)} - 1)(m^{\phi(n)} - 1) &= (mn)(q_1q_2) \\ &= n^{\phi(m)}m^{\phi(n)} - n^{\phi(m)} - m^{\phi(n)} + 1\end{aligned}$$
$$n^{\phi(m)}m^{\phi(n)} \equiv 1 \pmod{mn}$$

6.9 Q9

If l, m, n *are relatively prime in pairs, prove that* $(mn)^{\phi(l)} + (ln)^{\phi(m)} + (lm)^{\phi(n)} \equiv 1 \pmod{lmn}$*.*

$$(mn)^{\phi(l)} \equiv 1 \pmod{mn}$$
$$(lm)^{\phi(n)} \equiv 1 \pmod{lm}$$
$$(ln)^{\phi(m)} \equiv 1 \pmod{ln}$$

$$\begin{aligned}[(mn)^{\phi(l)} - 1][(lm)^{\phi(n)} - 1][(ln)^{\phi(m)} - 1] &= (l^2m^2n^2)(q_1q_2q_3) \\ &= [(mn)^{\phi(l)}(lm)^{\phi(n)} - (lm)^{\phi(n)} - (mn)^{\phi(l)} + 1][(ln)^{\phi(m)} - 1] \\ &= (mn)^{\phi(l)}(lm)^{\phi(n)}(ln)^{\phi(m)} - (lm)^{\phi(n)}(ln)^{\phi(m)} \\ &\quad - (ln)^{\phi(m)}(mn)^{\phi(l)} + (ln)^{\phi(m)} - (mn)^{\phi(l)}(lm)^{\phi(n)} \\ &\quad + (lm)^{\phi(n)} + (mn)^{\phi(l)} - 1\end{aligned}$$
$$(mn)^{\phi(l)} + (ln)^{\phi(m)} + (lm)^{\phi(n)} \equiv 1 \pmod{lmn}$$

7 G. Wilson's Theorem, and Some Consequences

7.1 Q1

Prove that in $\mathbb{Z}_p$*,* $\overline{2} \cdot \overline{3} \cdots \overline{p-2} = \overline{1}$*.*

Firstly note $x^2 \equiv 1 \pmod{p} \implies x = \pm 1$ or that $x = \overline{1}$ or $x = \overline{p-1}$.

So the remaining nonzero integers in $\mathbb{Z}_p$ have a multiplicative inverse since $\mathbb{Z}_p$ is an integral domain having the cancellation property.

7.1.1 Every Finite Integral Domain is a Field

We show a typical element $a \neq 0$ has a multiplicative inverse.

Consider $a, a^2, a^3, \ldots$ Since there are finite elements, the group is cyclic so we must have $a^m \equiv a^n \pmod{p}$ for some $m < n$. So $0 \equiv a^m - a^n \equiv a^m(1 - a^{n-m}) \pmod{p}$.

Since there are no zero divisors $a^m \not\equiv \pmod{p}$ and hence $1 - a^{n-m} \equiv 0 \pmod{p}$

$$a(a^{n-m-1}) \equiv 1 \pmod{p}$$

7.1.2 Remaining Elements Product is Unity

For any $x \in \mathbb{Z}_p : x \neq \pm 1$, there is a multiplicative inverse $y \in \mathbb{Z}_p : y \neq \pm 1$. This is the set $\{\overline{2}, \overline{3}, \ldots, p \bar{-} 2\}$, which has exactly p pairs, where $xy = \overline{1}$, and so the product of all these pairs is 1.

$$\overline{2} \cdot \overline{3} \cdots \overline{p-2} = \overline{1}$$

7.2 Q2

Prove $(p-2)! \equiv 1 \pmod{p}$ *for any prime number* p.

$$(p-2)! = 2 \cdot 3 \cdots (p-2)$$

From the previous question $\overline{2} \cdot \overline{3} \cdots \overline{p-2} = \overline{1}$ in $\mathbb{Z}_p$.

But also $\overline{2} \cdot \overline{3} \cdots \overline{p-2} = \overline{2 \cdot 3 \cdots (p-2)} = \overline{(p-2)!}$ and so $\overline{(p-2)!} = \overline{1}$. Both terms are in the same coset for $\langle p \rangle \implies p \mid [(p-2)! - 1]$.

$$\implies (p-2)! \equiv 1 \pmod{p}$$

7.3 Q3

Prove $(p-1)! + 1 \equiv 0 \pmod{p}$ *for any prime number* p. *This is known as Wilson's theorem.*

$$(p-1) \equiv -1 \pmod{p}$$
$$(p-2)! \equiv 1 \pmod{p}$$

$$(p-1)! = (p-2)!(p-1)$$

$$\begin{aligned}(p-1)! &\equiv (p-2)!(p-1) \pmod{p} \\ &\equiv (1)(-1) \pmod{p} \\ &\equiv -1 \pmod{p}\end{aligned}$$

$$(p-1)! + 1 \equiv 0 \pmod{p}$$

7.4 Q4

Prove that for any composite number $n \neq 4, (n-1)! \equiv 0 \pmod{n}$.

Any prime factor p of n will be a divisor of $(n-1)!$ because $p < n$ since $p \mid n$.

$$(n-1)! = (n-1) \cdots p \cdots 3 \cdot 2 \cdot 1$$

This also applies to all prime powers p^k in n, and so n itself is a factor of $(n-1)!$. Since n is composite (product of 2 or more integers).

$$(n-1)! \equiv 0 \pmod{n}$$

7.5 Q5

Prove that $[(p-1)/2]!^2 \equiv (-1)^{(p+1)/2} \pmod{p}$ *for any prime* $p > 2$.

$$(p-1)! + 1 \equiv 0 \pmod{p}$$
$$(p-1)! \equiv (-1)^{(p-1)/2}\left(1 \cdot 2 \cdots \frac{p-1}{2}\right)^2 \pmod{p}$$
$$(-1)^{(p-1)/2}\left(1 \cdot 2 \cdots \frac{p-1}{2}\right)^2 \equiv -1 \pmod{p}$$

Multiply both sides by $(-1)^{(p-1)/2}$, noting that

$$((-1)^{(p-1)/2})^2 = (-1)^{p-1} = 1 \text{ for any prime } p > 2$$

(as p was specified in the question).

$$\left(1 \cdot 2 \cdots \frac{p-1}{2}\right)^2 \equiv -1 \cdot (-1)^{(p-1)/2} \pmod{p}$$
$$\left(\frac{p-1}{2}\right)!^2 \equiv (-1)^{(p+1)/2} \pmod{p}$$

7.6 Q6

Prove that if $p \equiv 1 \pmod{4}$ *then* $(p+1)/2$ *is odd. Conclude that* $\left(\frac{p-1}{2}\right)!^2 \equiv -1 \pmod{p}$.

$$p - 1 = 4q$$
$$p + 1 = 4q + 2$$
$$\frac{p+1}{2} = 2q + 1$$

therefore $\frac{p+1}{2}$ is odd, so $(-1)^{(p+1)/2} = -1$.

7.7 Q7

Prove that if $p \equiv 3 \pmod{4}$ *then* $(p+1)/2$ *is even. Conclude that* $\left(\frac{p-1}{2}\right)!^2 \equiv 1 \pmod{p}$.

$$p - 3 = 4q$$
$$p + 1 = 4q + 4$$
$$\frac{p+1}{2} = 2q + 2$$

So $\frac{p+1}{2}$ is even and $(-1)^{(p+1)/2} = 1$.

7.8 Q8

Prove that when $p > 2$ *is a prime, the congruence* $x^2 + 1 \equiv 0 \pmod{p}$ *has a solution if* $p \equiv 1 \pmod{4}$.

$$p \equiv 1 \pmod{4} \implies 4 \mid (p-1)$$

From 23G6

$$\left(\frac{p-1}{2}\right)!^2 \equiv -1 \pmod{p}$$
$$\therefore x = \left(\frac{p-1}{2}\right)!$$

7.9 Q9

Prove that for any prime $p > 2$, $x^2 \equiv -1 \pmod p$ has a solution iff $p \not\equiv 3 \pmod 4$.

From 23E3b, there is no solution to $x^2 + 1 \equiv 0 \pmod p$ when $p \equiv 3 \pmod 4$.

From 23G8, there is a solution when $p \equiv 1 \pmod p$.

8 H. Quadratic Residues

8.1 Q1

Let $h : \mathbb{Z}_p^* \to \mathbb{Z}_p^*$ be defined by $h(\overline{a}) = \overline{a}^2$. To show this is a homomorphism, let $\overline{x}, \overline{y} \in \mathbb{Z}_p^*$, then $h(\overline{x}\,\overline{y}) = h(\overline{xy}) = \overline{xy}^2 = (\overline{x}\,\overline{y})^2 = \overline{x}^2\overline{y}^2 = h(\overline{x})h(\overline{y})$. The kernel is $\{\pm\overline{1}\}$ because $h(\pm\overline{1}) = \overline{1}$ which is the identity element.

8.2 Q2

$$|\mathbb{Z}_p^\times| = p - 1$$

For any $\overline{a} \in \mathbb{Z}_p^\times, h(\overline{a}) = h(\overline{-a}) = \overline{a}^2$, so the range of h is $(p-1)/2$ elements.

$$\operatorname{ran} h = R$$

The kernel of h is $\{\pm 1\}$ and $h(\pm\overline{1}) = \overline{1}$. So R contains the identity element. Secondly for any $\overline{x}^2, \overline{y}^2 \in R$, then $\overline{x}^2\overline{y}^2 = \overline{xy}^2 \in R$, so R is a subgroup of $\mathbb{Z}_p^\times$.

By the orbit-stabilizer theorem, the number of cosets is $\frac{|\mathbb{Z}_p^\times|}{|R|} = 2$.

Finally if there is an $\overline{x}$ such that there is no $\overline{a} \in R : \overline{a}^2 = \overline{x}$, then $\overline{x} \neq R$, but $\overline{x} = Rx$. Since $1 \in R$ and $1 \cdot x = x \in Rx$.

8.3 Q3

Question is wrong. Maybe it's asking about Euler's criterion?

8.4 Q4

$$\left(\frac{17}{23}\right) = -1$$

$$\left(\frac{3}{29}\right) = -1$$

$$\left(\frac{5}{11}\right) = 1$$

$$\left(\frac{8}{13}\right) = -1$$

$$\left(\frac{2}{23}\right) = 1$$

8.5 Q5

Prove if $a \equiv b \pmod p$ then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$. In particular, $\left(\frac{a+kp}{p}\right) = \left(\frac{a}{p}\right)$.

$$a + kp \equiv a \pmod p, x^2 \equiv a \pmod p$$
$$\implies x^2 \equiv a + kp \pmod p$$
$$\implies \left(\frac{a+kp}{p}\right) = \left(\frac{a}{p}\right)$$
$$a \equiv b \pmod p \implies \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$$

8.6 Q6

8.6.1 a

Show the Legendre symbol is homomorphic.

$$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$$

If $a, b \in R$, then $ab \in R$, and $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right) = 1$.

Otherwise if $a \in R, b \notin R$, then $ab \notin R \implies ab \in R \cdot -1$, so $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right) = -1$ and vice versa.

Finally if $a, b \notin R$, then $a, b \in R \cdot -1$ and $ab \in R$, so $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right) = 1$.

8.6.2 b

Malformed question.

$$\left(\frac{a}{p}\right)\left(\frac{a}{p}\right) = \left(\frac{a^2}{p}\right)$$

8.7 Q7

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ -1 & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

From G8 and G9.

$x^2 \equiv -1 \pmod p$ has a solution if $p \equiv 1 \pmod 4$.

$x^2 \equiv -1 \pmod p$ has no solution if $p \equiv 3 \pmod 4$.

8.8 Q8

8.8.1 $\left(\frac{30}{101}\right)$

$$\left(\frac{30}{101}\right) = \left(\frac{3}{101}\right)\left(\frac{5}{101}\right)\left(\frac{2}{101}\right)$$

$$\left(\frac{101}{3}\right) = \left(\frac{2}{3}\right) = -1$$

$$101 \equiv 1 \pmod 4 \implies \left(\frac{3}{101}\right) = -1$$

$$\left(\frac{101}{5}\right) = \left(\frac{1}{5}\right) = 1 \implies \left(\frac{5}{101}\right) = 1$$

Cannot use reciprocity rule because only works for prime > 2.

$$\left(\frac{2}{101}\right) = -1$$

$$\therefore \left(\frac{30}{101}\right) = 1$$

8.8.2 $\left(\frac{10}{151}\right)$

$$\left(\frac{10}{151}\right) = \left(\frac{2}{151}\right)\left(\frac{5}{151}\right)$$
$$5 \equiv 1 \pmod 4 \implies \left(\frac{5}{151}\right) = \left(\frac{151}{5}\right) = \left(\frac{1}{5}\right) = 1$$
$$\left(\frac{2}{151}\right) = 1$$
$$\therefore \left(\frac{10}{151}\right) = 1$$

8.8.3 $\left(\frac{15}{41}\right)$

$$\left(\frac{15}{41}\right) = \left(\frac{3}{41}\right)\left(\frac{5}{41}\right)$$
$$41 \equiv 1 \pmod 4 \implies \left(\frac{3}{41}\right) = \left(\frac{41}{3}\right) = \left(\frac{2}{3}\right) = -1, \left(\frac{5}{41}\right) = \left(\frac{41}{5}\right) = \left(\frac{1}{5}\right) = 1$$
$$\therefore \left(\frac{15}{41}\right) = -1$$

8.8.4 $\left(\frac{14}{59}\right)$

$$\left(\frac{14}{59}\right) = \left(\frac{2}{59}\right)\left(\frac{7}{59}\right)$$

Both $59 \equiv 7 \equiv 3 \pmod 4 \implies \left(\frac{7}{59}\right) = -\left(\frac{59}{7}\right) = -\left(\frac{3}{7}\right) = -(-1) = 1$.

$$\frac{2}{59} = -1$$
$$\frac{14}{59} = -1$$

8.8.5 $\left(\frac{379}{401}\right)$

$$401 \equiv 1 \pmod 4 \implies \left(\frac{379}{401}\right) = \left(\frac{401}{379}\right) = \left(\frac{22}{379}\right) = 1$$

8.8.6 Is 14 a quadratic residue modulo 59

No

8.9 Q9

$x^2 \equiv 30 \pmod{101}$ is solvable. The other two are not solvable.

9 I. Primitive Roots

Recall that V_n is the multiplicative group of all the invertible elements in $\mathbb{Z}_n$. If V_n happens to be cyclic, say $V_n = \langle m \rangle$, then any integer $a \equiv m \pmod n$ is called a primitive root of n.

9.1 Q1

Prove that a is a primitive root of n iff the order of $\overline{a}$ in V_n is $\phi(n)$.

$$\operatorname{ord}(\overline{a}) = \phi(n) \implies \overline{a}^{\phi(n)} = \overline{1} \text{ in } V_n$$

This means there are $\phi(n)$ distinct powers of $\overline{a}$, which generate all the invertible elements of $\mathbb{Z}_n$\$, that is $a \equiv m \pmod n$ and $V_n = \langle a \rangle$.

9.2 Q2

Prove that every prime number p has a primitive root. (HINT: For every prime p, $\mathbb{Z}_p^\times$ is a cyclic group. The simple proof of this fact is given as Theorem 1 in Chapter 33.)

For every prime number, $\mathbb{Z}_p^\times = \{\overline{1}, \overline{2}, \ldots, \overline{p-1}\}$ is a group with order $p - 1$.

Thus $\forall x \in \mathbb{Z}_p^\times, \overline{x}^{p-1} = \overline{1}, V_p = \langle x \rangle$.

9.3 Q3

Find primitive roots of the following integers (if there are none, say so): 6, 10, 12, 14, 15.

9.3.1 6

$$n = 6, \phi(6) = 2, \mathbb{Z}_6^\times = \{1, 5\}$$

```
1: 1
5: 5, 1
```

Primitive root of 6 is 5

9.3.2 10

$$n = 10, \phi(10) = 4, \mathbb{Z}_{10}^\times = \{1, 3, 7, 9\}$$

```
   x  x^2 x^3 x^4
1: 1
3: 3,   9,  7,  1
7: 7,   9,  3,  1
8: 9,   1
```

Primitive roots of 10 is 3 and 7

9.3.3 12

$$n = 12, \phi(12) = 4, \mathbb{Z}_{12}^\times = \{1, 5, 7, 11\}$$

```
1:  1
5:  5, 1
7:  7, 1
11: 11, 1
```

No primitive root of 12.

9.3.4 14

$$n = 14, \phi(14) = 6, \mathbb{Z}_{14}^\times = \{1, 3, 5, 9, 11, 13\}$$

```
1:   1
3:   3,  9, 13, 11, 5, 1
5:   5, 11, 13,  9, 3, 1
9:   9, 11, 1
11: 11,  9, 1
13: 13,  1
```

14 has primitive roots 3 and 5

9.3.5 15

$$n = 15, \phi(15) = 8, \mathbb{Z}_{15}^\times = \{1, 2, 4, 7, 8, 11, 13, 14\}$$

```
1: 1
2: 2, 4, 8, 1
4: 4, 1
7: 7, 4, 13, 1
8: 8, 4, 2, 1
11: 11, 1
```

```
13: 13, 4, 7, 1
14: 14, 1
```

There are no primitive roots modulo 15.

9.4 Q4

Suppose a is a primitive root of m. Prove: If b is any integer which is relatively prime to m, then $b \equiv a^k \pmod{m}$ for some $k \geq 1$.

$$\begin{aligned} \gcd(b,m) = 1 &\implies \bar{b} \in V_m = \langle a \rangle \\ &\implies \bar{b} = \bar{a}^k \text{ in } \mathbb{Z}_m \\ &\implies b = a^k \pmod{m} \end{aligned}$$

9.5 Q5

Suppose m has a primitive root, and let n be relatively prime to $\phi(m)$. (Suppose $n > 0$.) Prove that if a is relatively prime to m, then $x^n \equiv a \pmod{m}$ has a solution.

$\mathbb{Z}_m^\times$ is a multiplicative group with a cyclic subgroup V_m of invertible elements.

$$\forall x \in \mathbb{Z}_m^\times : \gcd(a,m) = 1 \iff a \in V_m$$

Thus $V_m = \langle g \rangle$, so $\overline{a} = \overline{g}^l$. So we want to find an $\overline{x} \in V_m$ or $\overline{x} = \overline{g}^k$ such that $\overline{x}^n = (\overline{g}^k)^n = \overline{g}^l$

$$(g^k)^n \equiv g^l \pmod{m}$$

This is equivalent to writing

$$\begin{aligned} kn &\equiv l \pmod{\phi(m)} \\ &\implies \phi(m) \mid (kn - l) \\ &\implies kn - l = q\phi(m) \end{aligned}$$

But note that since $\gcd(n, \phi(m)) = 1$ then

$$cn + d\phi(m) = 1 \text{ for some } c \text{ and } d$$

Returning to our previous statement, we have

$$kn - q\phi(m) = l$$

Since l is a linear combination of n and $\phi(m)$, then l is a multiple of the ideal J generated by $\gcd(n, \phi(m)) = 1$. Since J is the entire group of $\mathbb{Z}_p^\times$, so $l \in J$ and exists as a linear combination of n and $\phi(m)$.

Thus there is an $\overline{x} = \overline{g}^k$ such that $x^n \equiv a \pmod{m}$.

9.6 Q6

Let $p > 2$ be a prime. Prove that every primitive root of p is a quadratic nonresidue, modulo p. (HINT: Suppose a primitive root a is a residue; then every power of a is a residue.)

$$V_m = \langle a \rangle$$

but if a is a quadratic residue then

$$a^2 \equiv a \pmod{p}$$

So a cannot be a primitive root of p and a quadratic residue since it can only generate even powers of a.

Also there are $\phi(p)/2$ quadratic residues from 23H3, but $\phi(p)$ elements in V_m.

So a is not a quadratic residue.

9.7 Q7

A prime p of the form $p = 2^m + 1$ is called a Fermat prime. Let p be a Fermat prime. Prove that every quadratic nonresidue mod p is a primitive root of p.

Number of quadratic residues in $\mathbb{Z}_p^\times$ is $(p-1)/2$ but $p = 2^m + 1$

$$\frac{p-1}{2} = \frac{(2^m + 1) - 1}{2} = 2^{m-1}$$

The number of primitive roots are the coprimes in $\mathbb{Z}_p^\times$ which equals $\phi(\phi(p)) = \phi(p-1) = \phi((2^m+1)-1) = \phi(2^m)$. Since 2 is prime

$$\phi(2^m) = 2^{m-1}(2-1) = 2^{m-1}$$

From 23I6, we know every primitive root is a quadratic non-residue. Since both groups are the same size, we thus conclude that every quadratic non-residue is a primitive root.